The Geometry of 4-Manifolds:

Curvature in the Balance

II

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Pisa, Italia. Il 9 giugno 2022.

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Riemann $=$ Weyl $\oplus$ trace-free Ricci $\oplus$ scalar

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where

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But $\exists u$ such that $\hat{\mathring{r}} = 0$ at any given $p \in M$.

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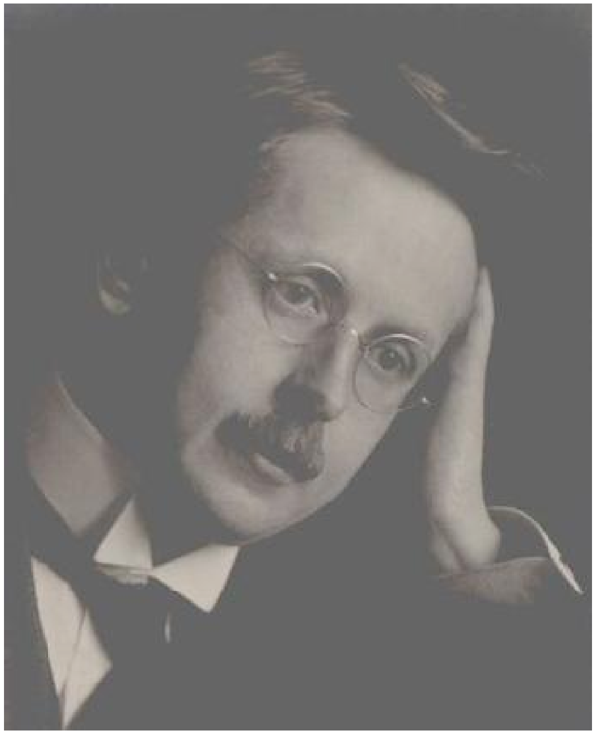

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Cotton tensor $C = \nabla \wedge (\mathring{r} - \frac{s}{12}g)$ obstruction.

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Of course, conformally Einstein good enough!

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when $\ell > 0$, because $\mathscr{W} \propto \mathrm{Vol}(T^\ell)$!

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$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

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What? Why?

What's So Special About Dimension Four?

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

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splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^-	$\mathring{r}$	$W_- + \frac{s}{12}$

Riemann curvature of g

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because $\nabla J = 0$.

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$$W_{+} = \begin{pmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{pmatrix}$$

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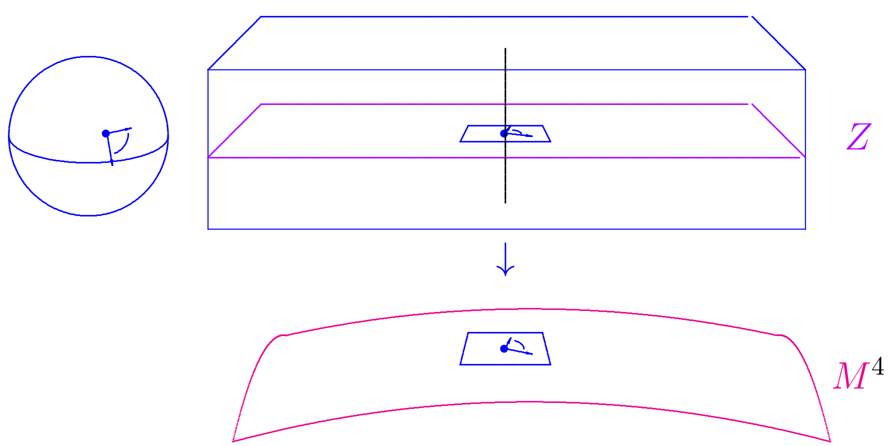

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Violate Hitchin-Thorpe, so $\nexists$ Einstein on such M.

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Twistor-related ideas $\rightsquigarrow$ understood long before cscK.

If not Ricci-flat Kähler, then

rational or ruled, with $(2\chi + 3\tau)(M) < 0$.

L-Singer '93, Kim-L-Pontecorovo '97 Any rational/ruled (M, J) has blow-ups admitting SFK.

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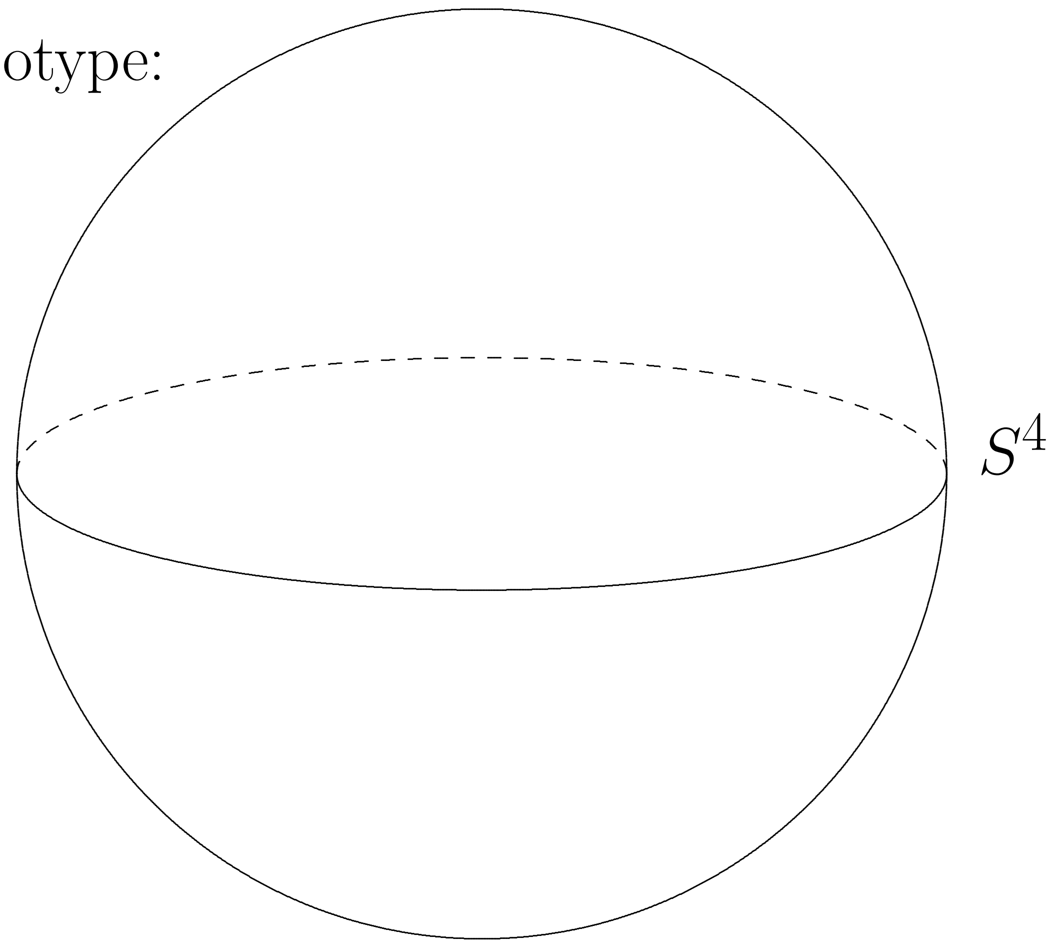

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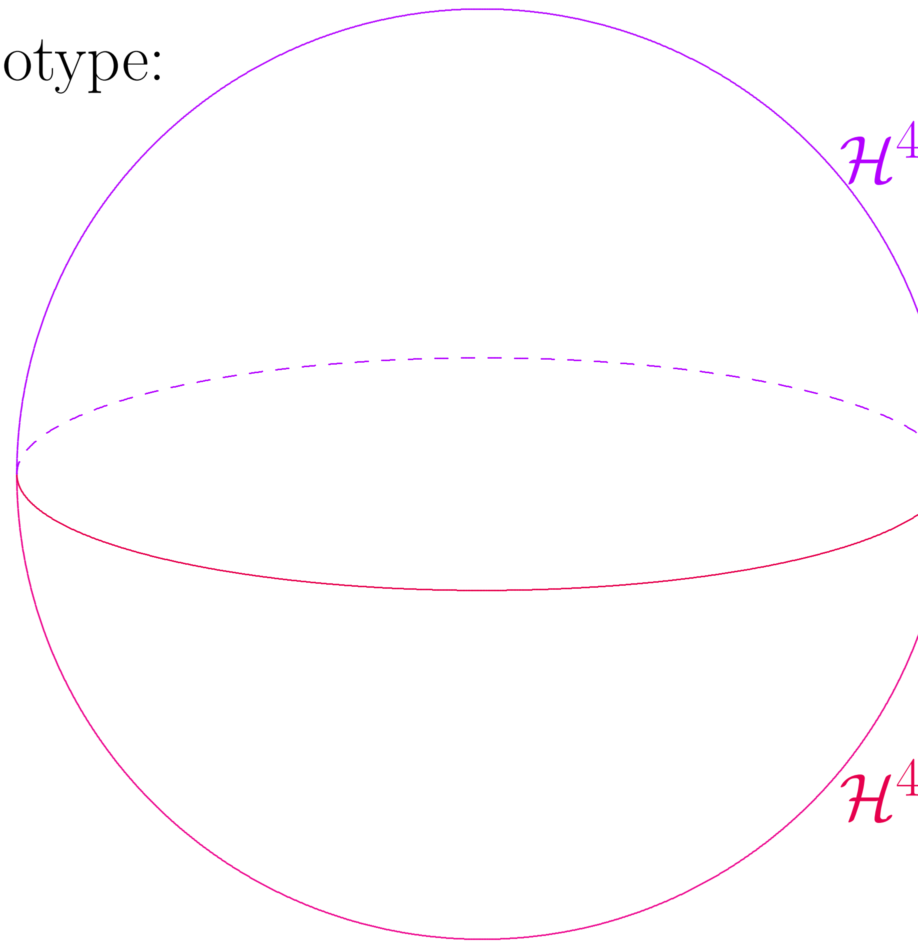

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S^4 is also Einstein, ASD.

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But $\exists$ genuine examples that aren't.

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But no compact counter-examples are known!

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Bach-flat Kähler

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X.X. Chen: always minimizers.

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Donaldson et al: unique modulo bihomorphisms.

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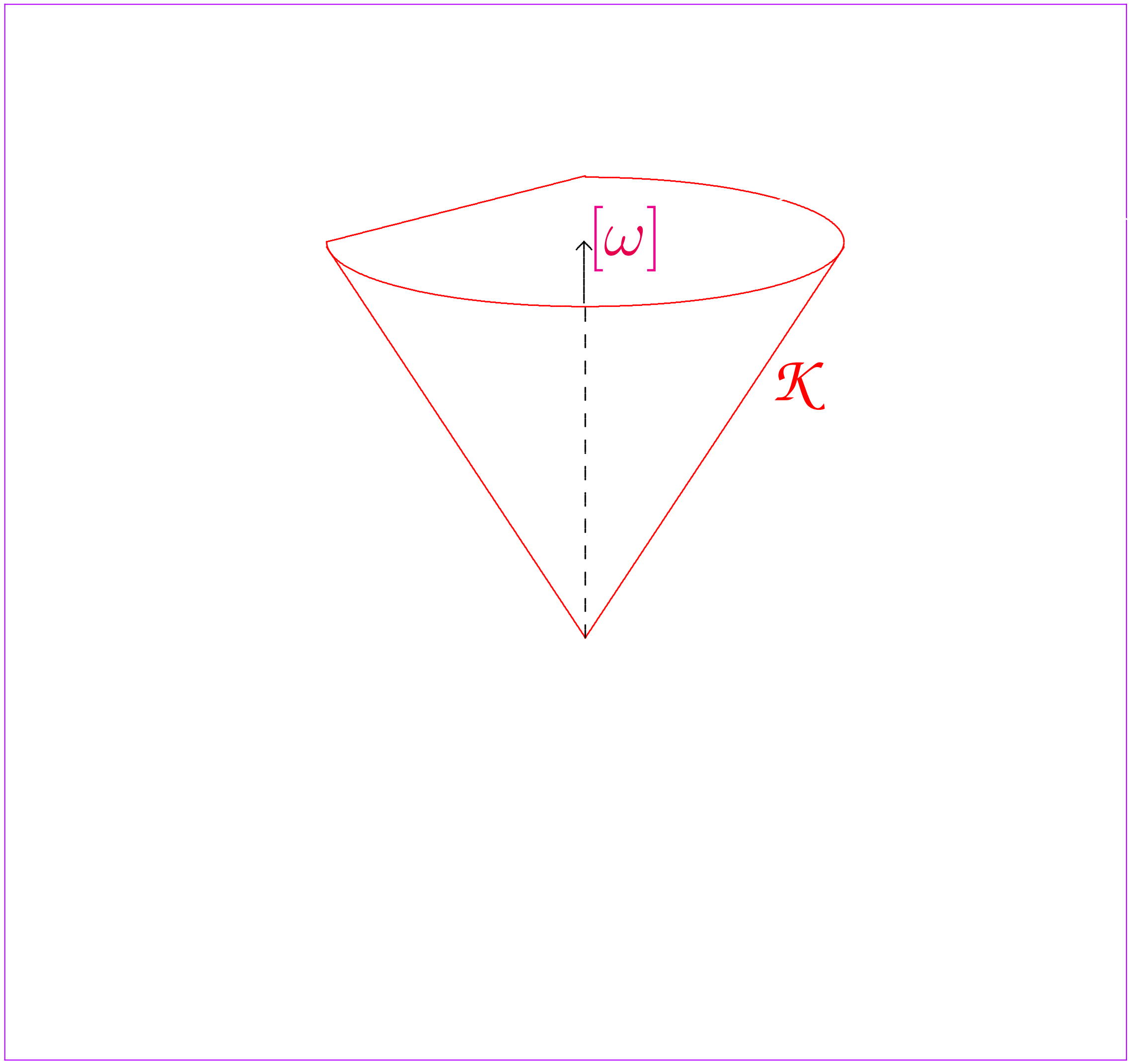
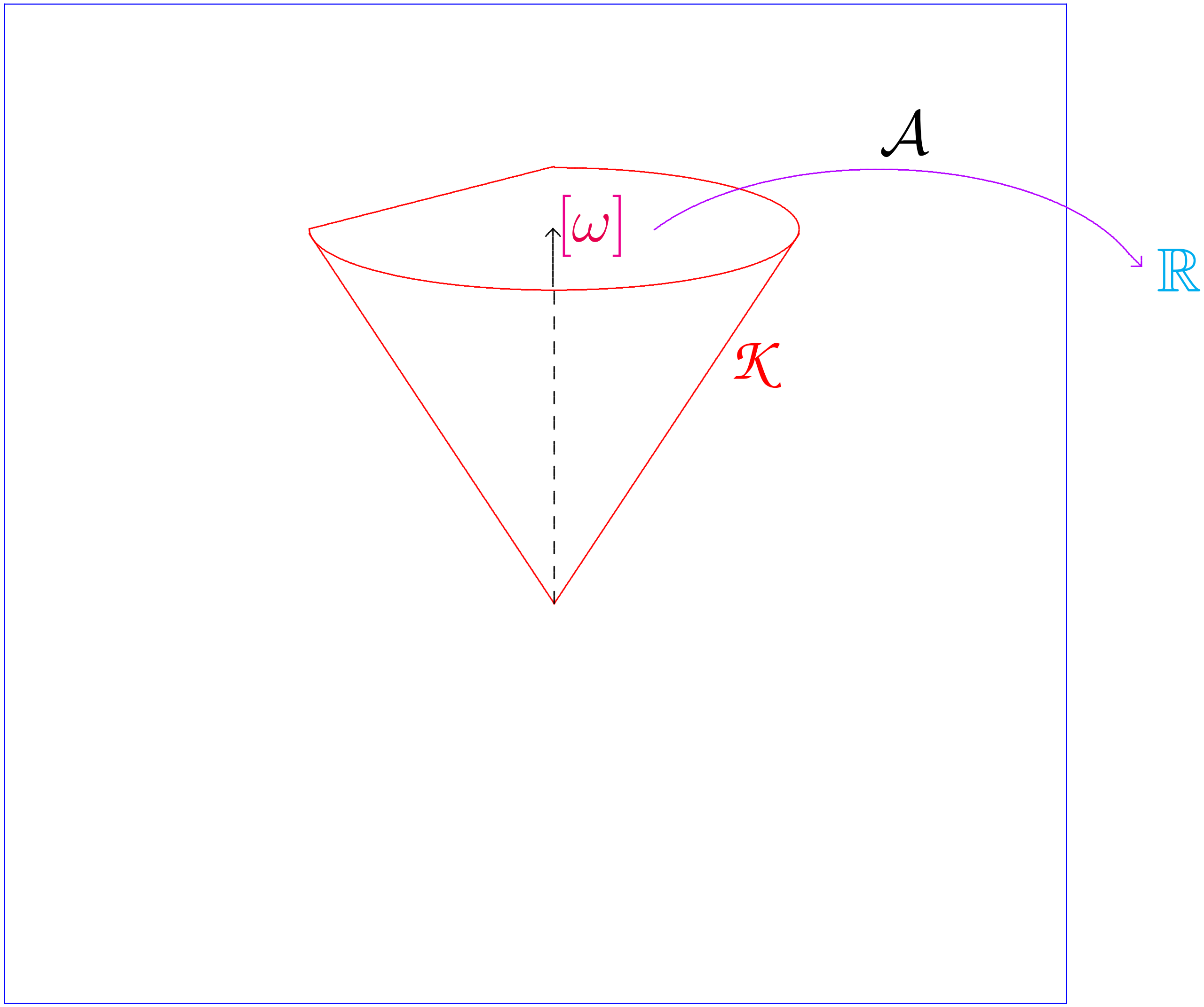


$$\mathcal{K} \subset H^{1,1}(M, \mathbb{R}) \subset H^2(M, \mathbb{R})$$



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Action Function on Kähler Cone

For any extremal Kähler (M^4, g, J),

$$\frac{1}{32\pi^2}\int s^2 d\mu_g = \frac{(c_1\cdot[\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2}\|\mathcal{F}_{[\omega]}\|^2$$
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where $\mathcal{F}$ is Futaki invariant.

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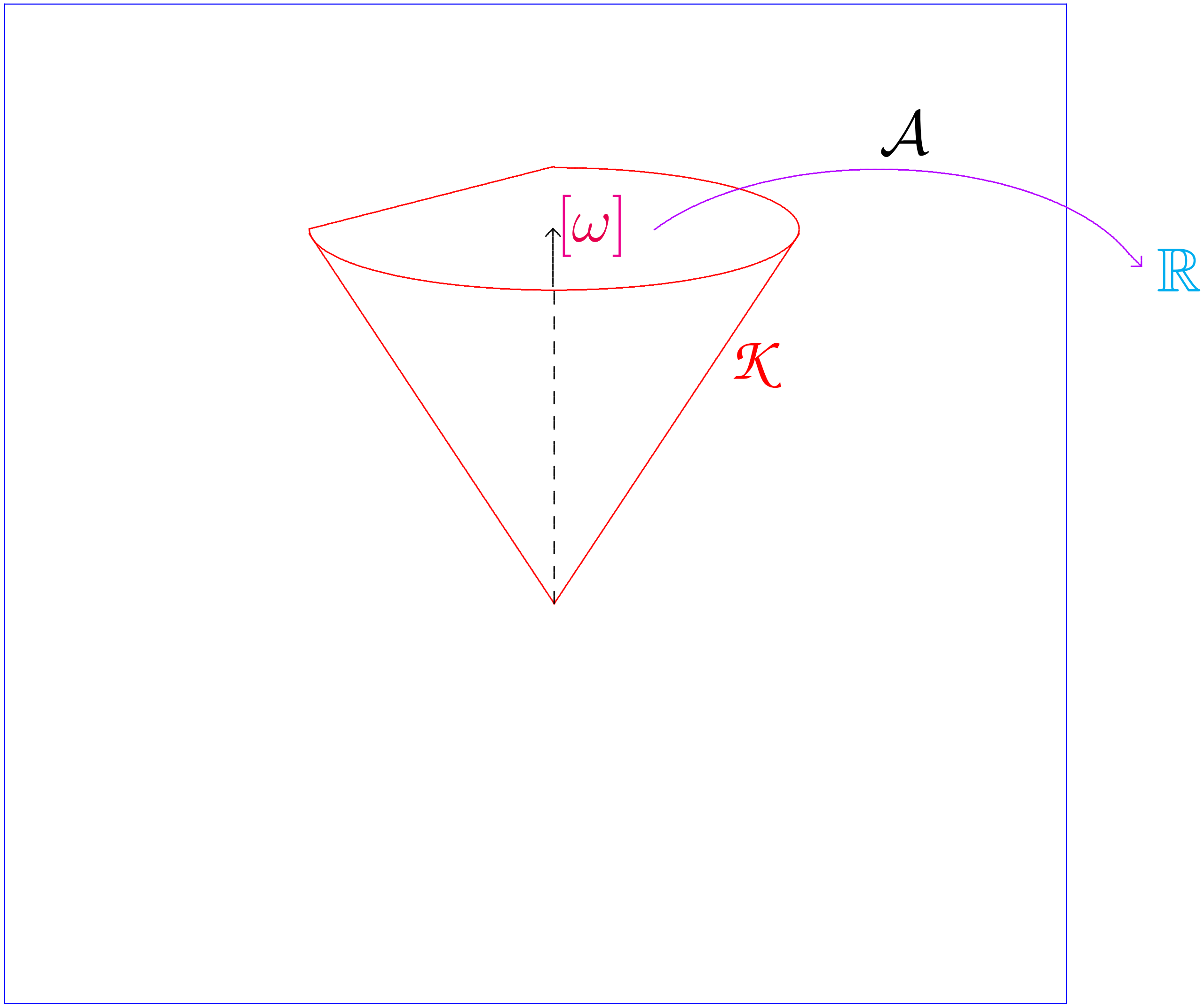

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I. $s > 0$ *everywhere. Then*

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If not Kähler-Einstein:

I. s *is positive. Then*

$(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. s *is zero. Then*

(M, g, J) *SFK, but not Ricci-flat.*

III. s *changes sign. Then*

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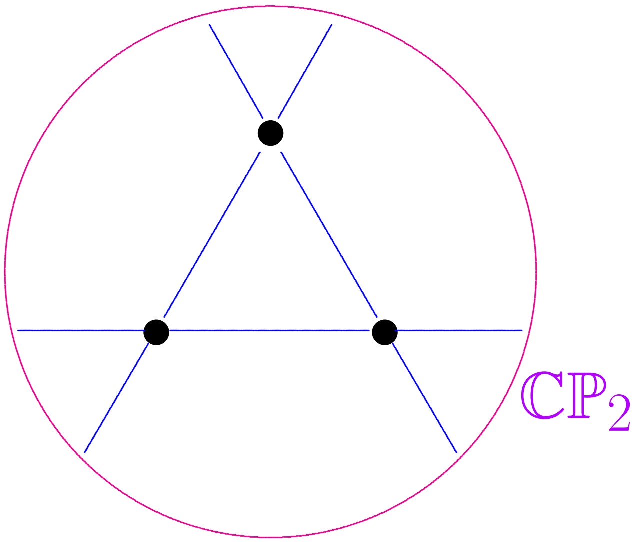

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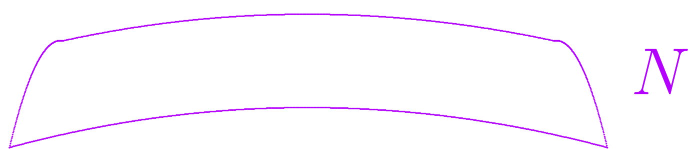

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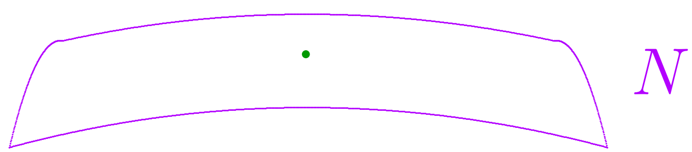

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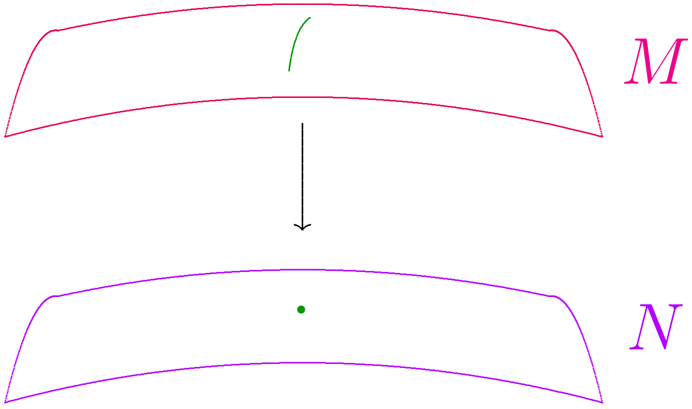

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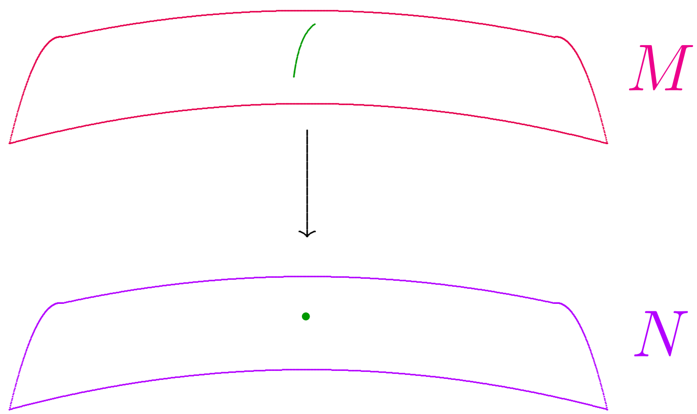
$$M \approx N \# \overline{\mathbb{CP}}_2$$



Conventions:

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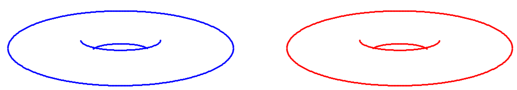

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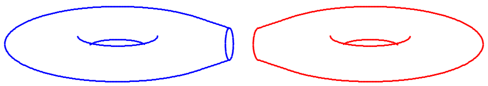

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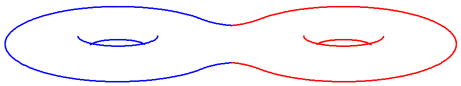

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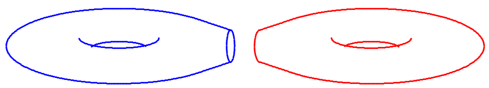

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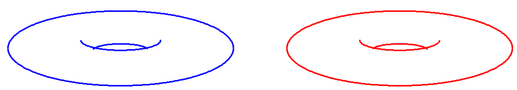

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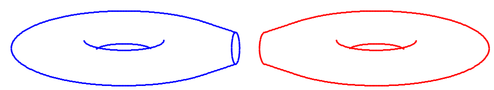

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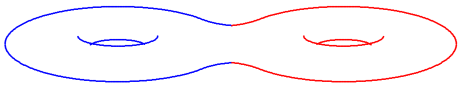

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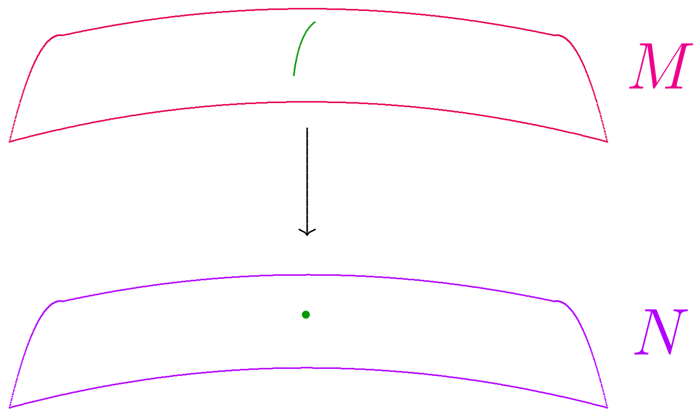

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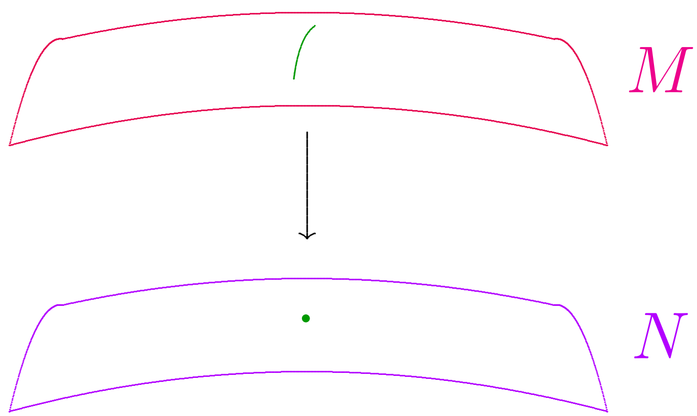

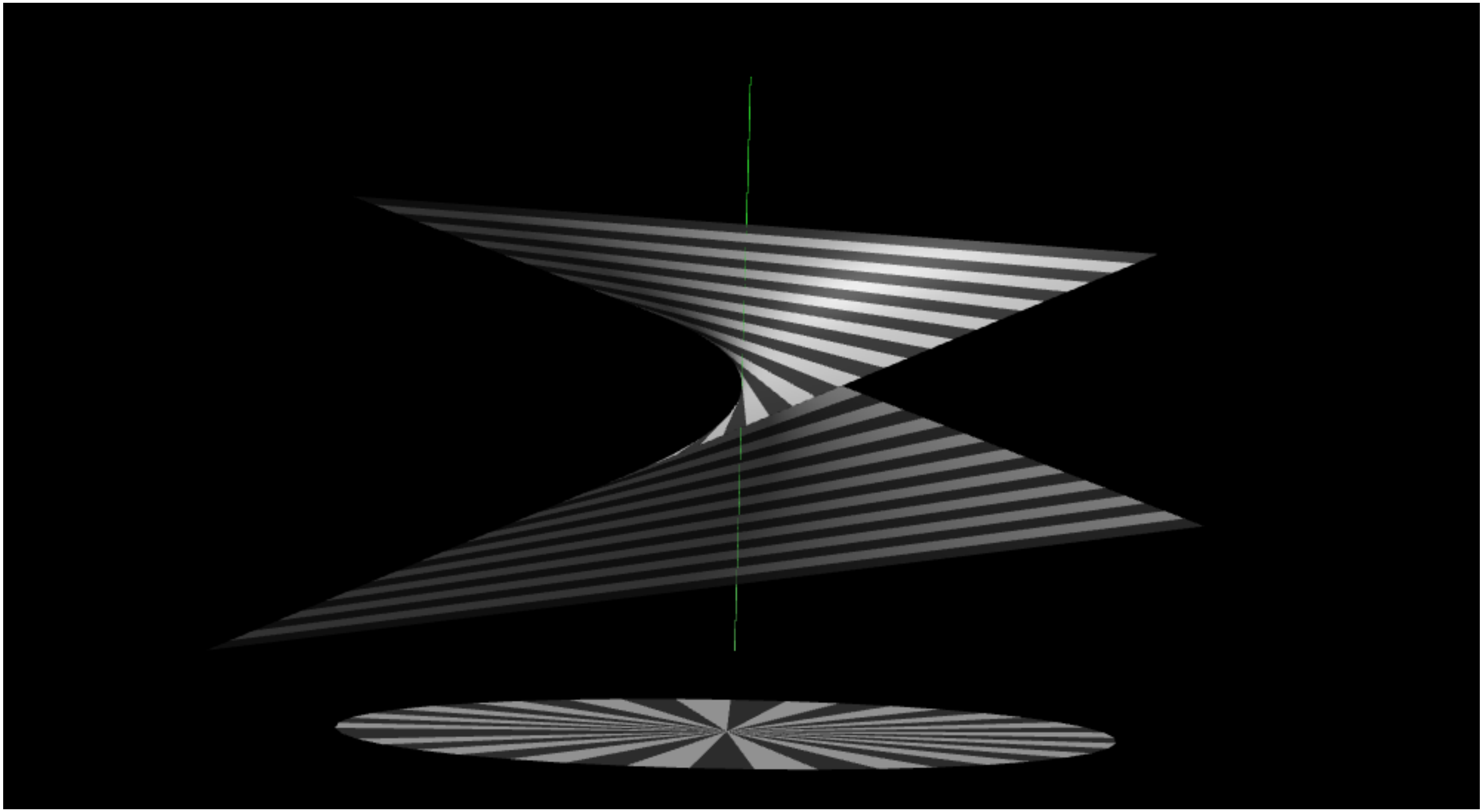

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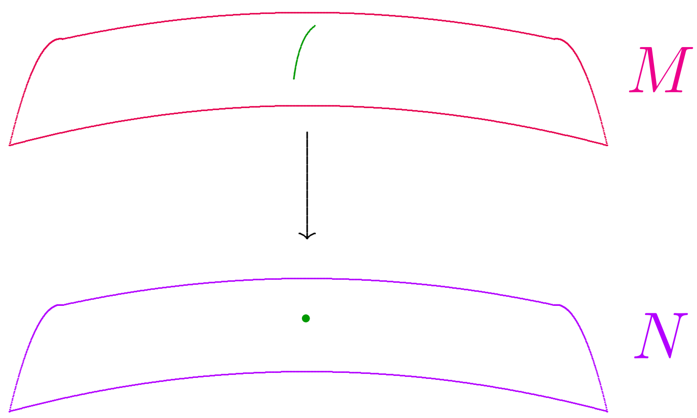

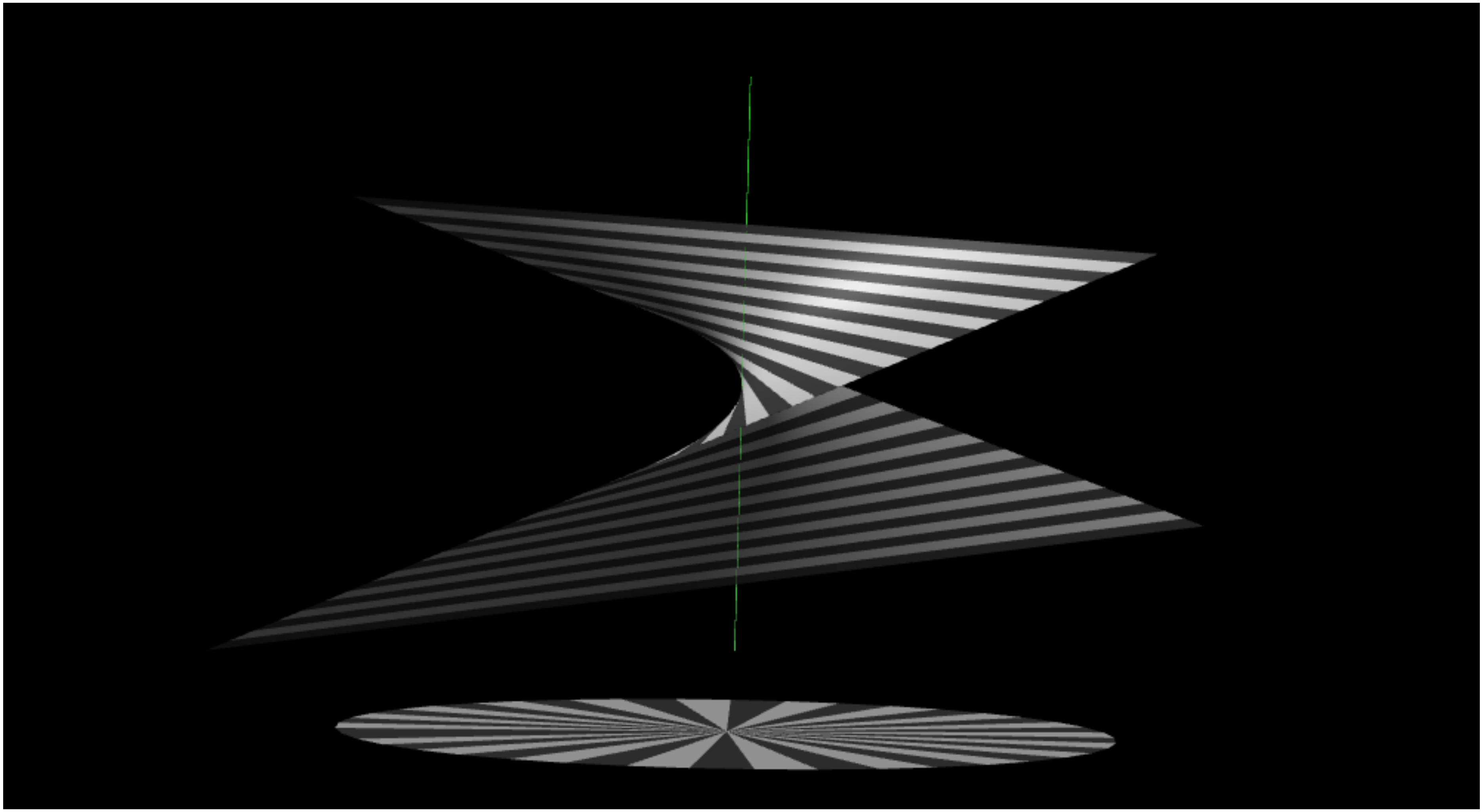

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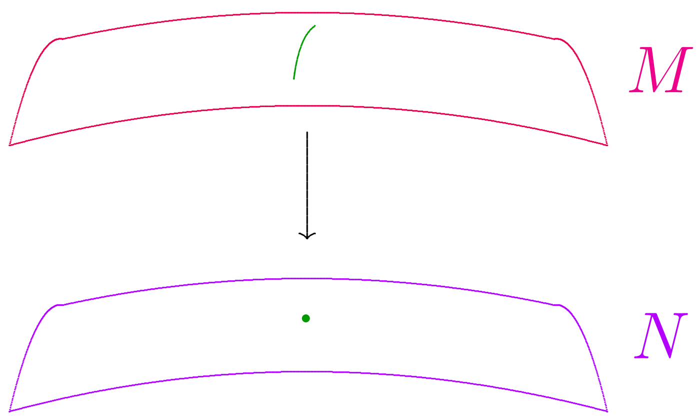

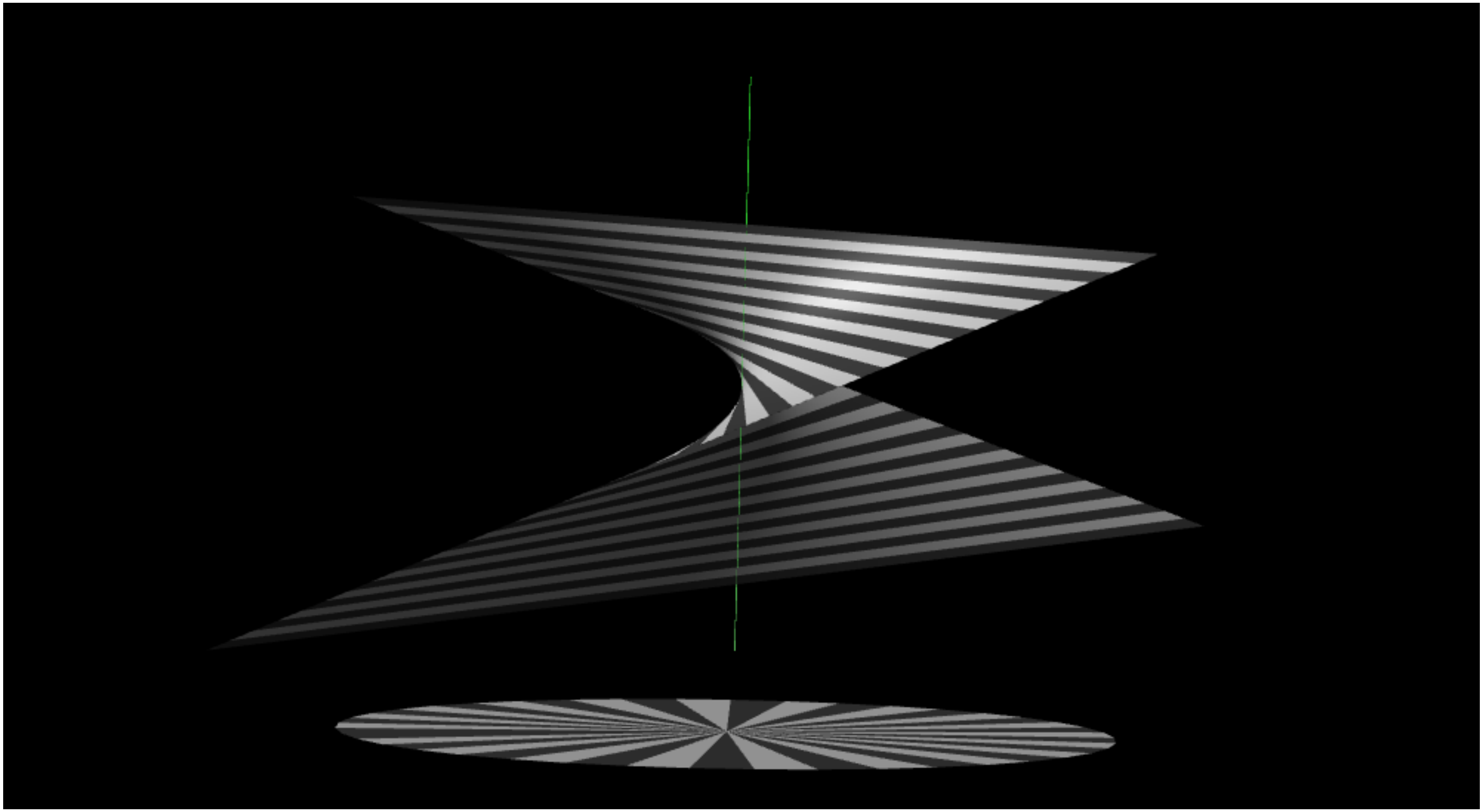

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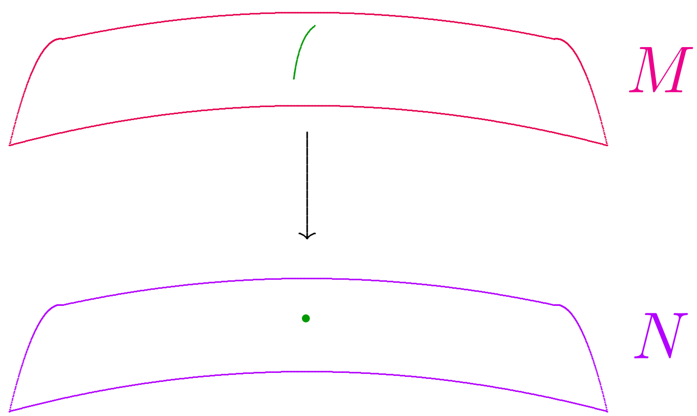

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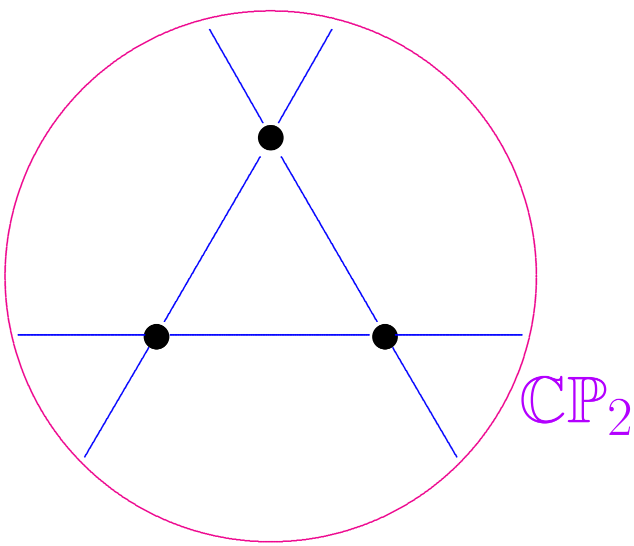

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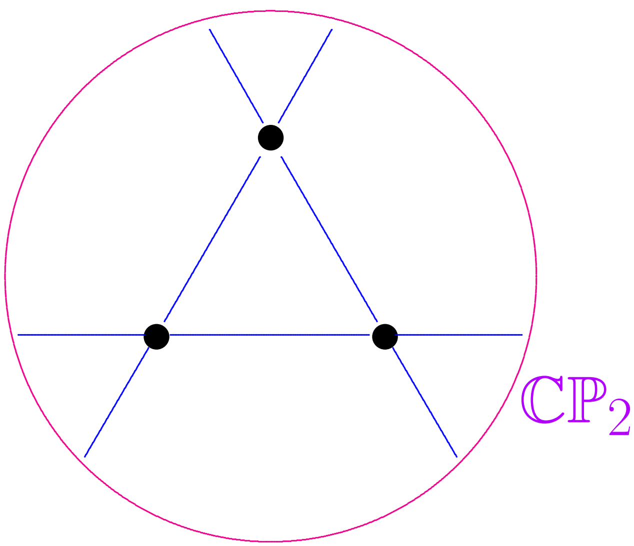

No 3 on a line,

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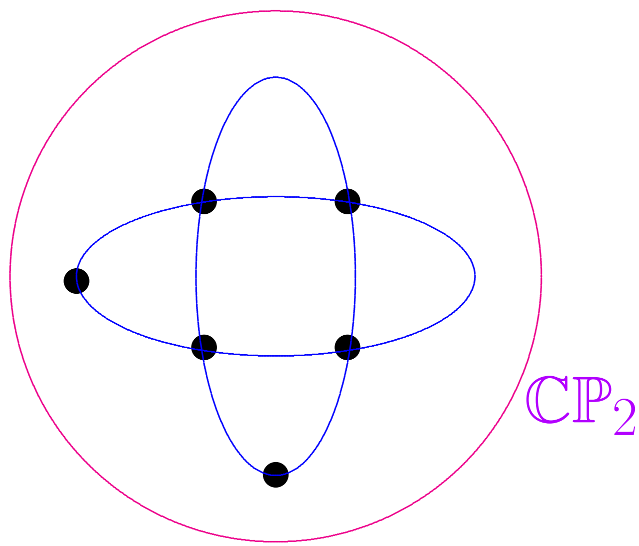

No 3 on a line, no 6 on conic,

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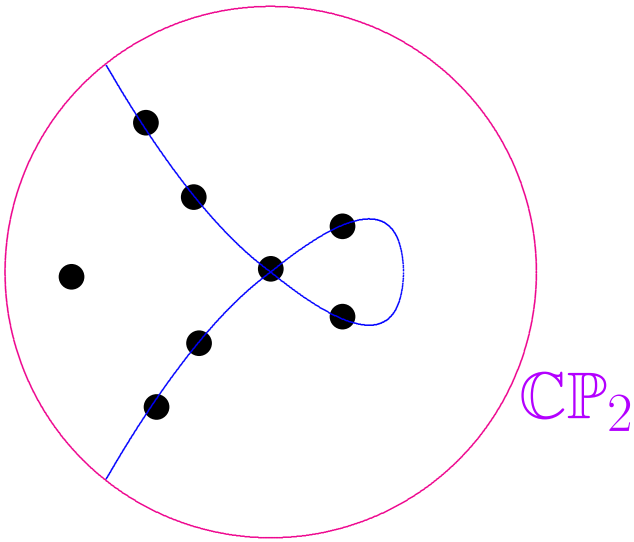

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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Existence: Page

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Existence: Page-Derdziński,

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Existence: Page-Derdziński, Siu,

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Existence: Page-Derdziński, Siu, Tian-Yau,

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian,

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun,

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One reason this seems satisfying. . .

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But this is not needed in above result.

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(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

This happens $\iff$ $c_1 > 0$.

$\iff$ (M^4, J) is a Del Pezzo surface.

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Convention: $\dim \varnothing := -\infty$.

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Examples of (b): Hwang-Simanca, Tønnesen-Friedman

A few words about the proof...

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$$\Delta s \neq 0 \quad \text{at } \min s \text{ and } \max s.$$

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Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3,$$

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Lemma. *The function κ is constant, and has the same sign $(+, -, 0)$ as* $\min s$.

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Otherwise $\kappa = (s^2 - 6\Delta s)s$ at $\min s$.

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On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

Lemma. *The function κ is constant, and has the same sign $(+, -, 0)$ as* $\min s$. *On set where $s \neq 0$, the constant $\kappa =$ scalar curvature of $s^{-2}g$.*

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$$s(p) = 0 \implies \mathrm{Hess}_0(s)\Big|_p = 0.$$

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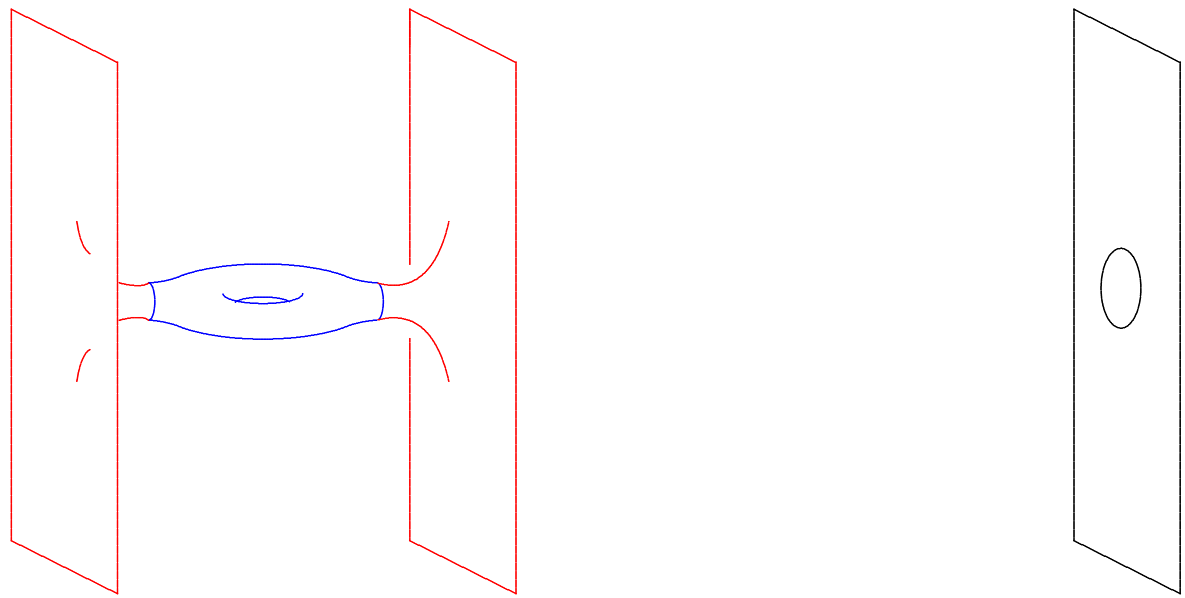

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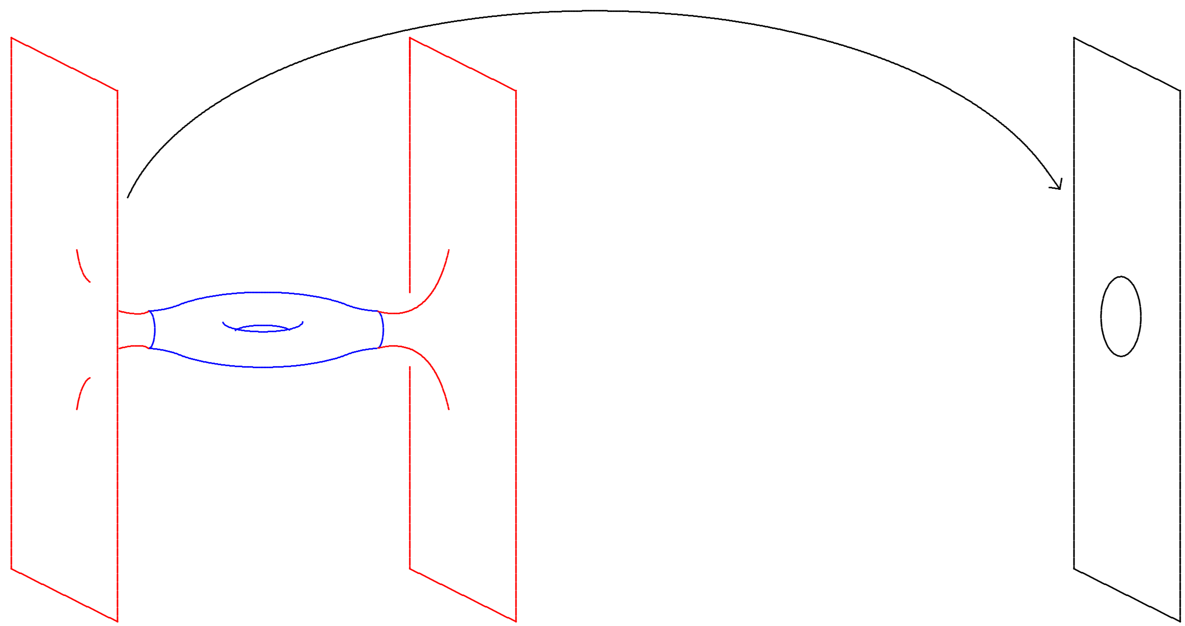

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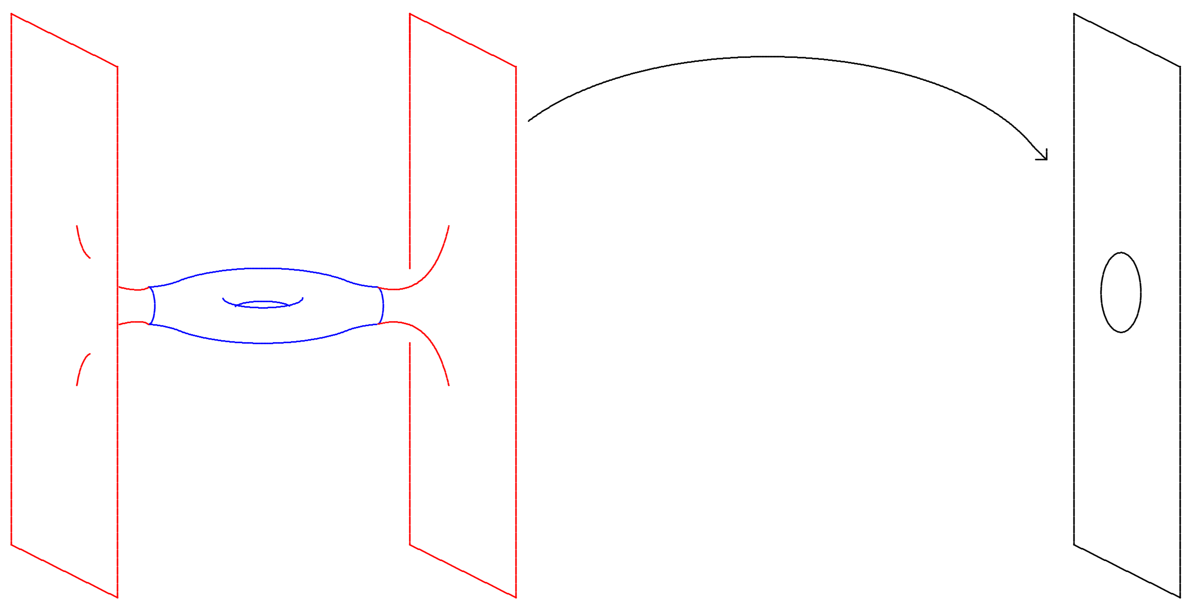

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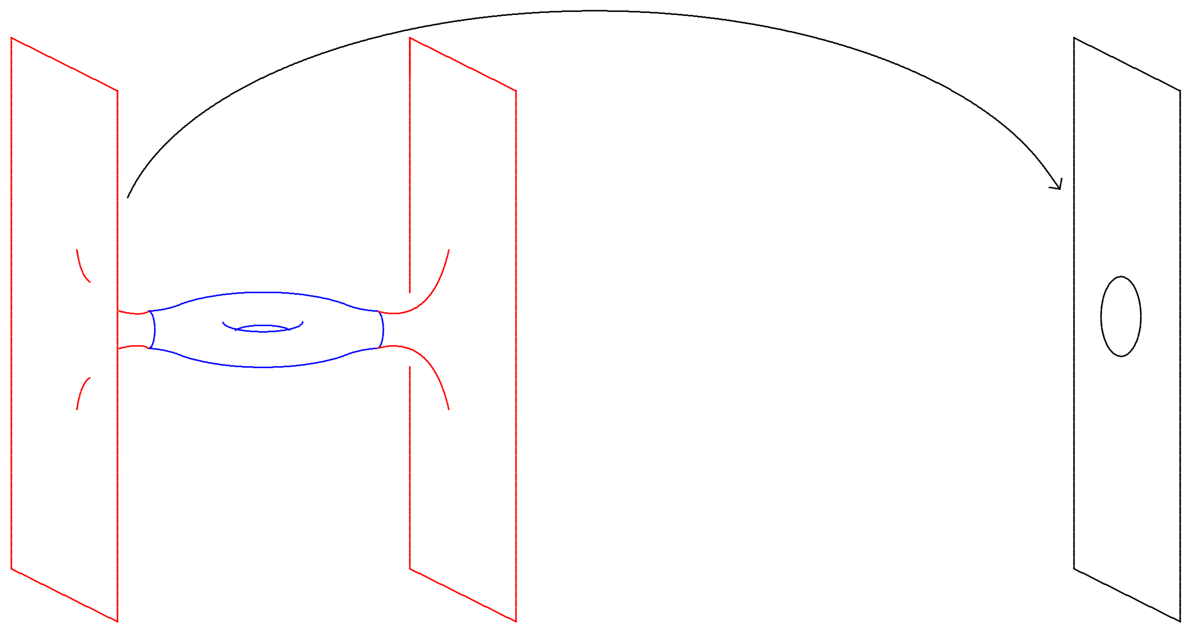

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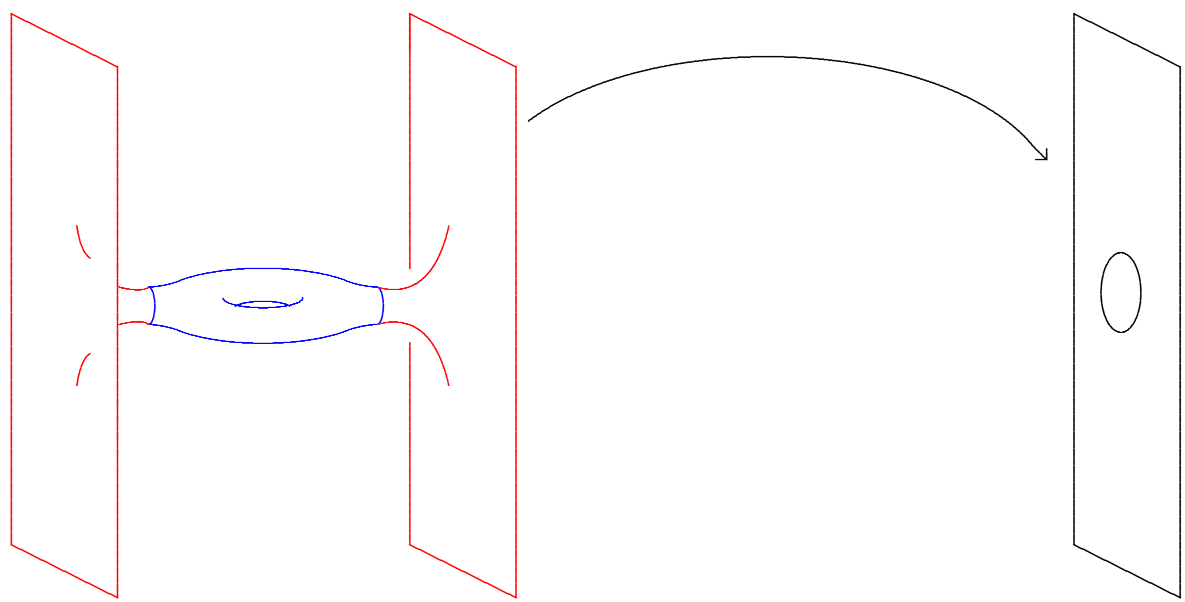

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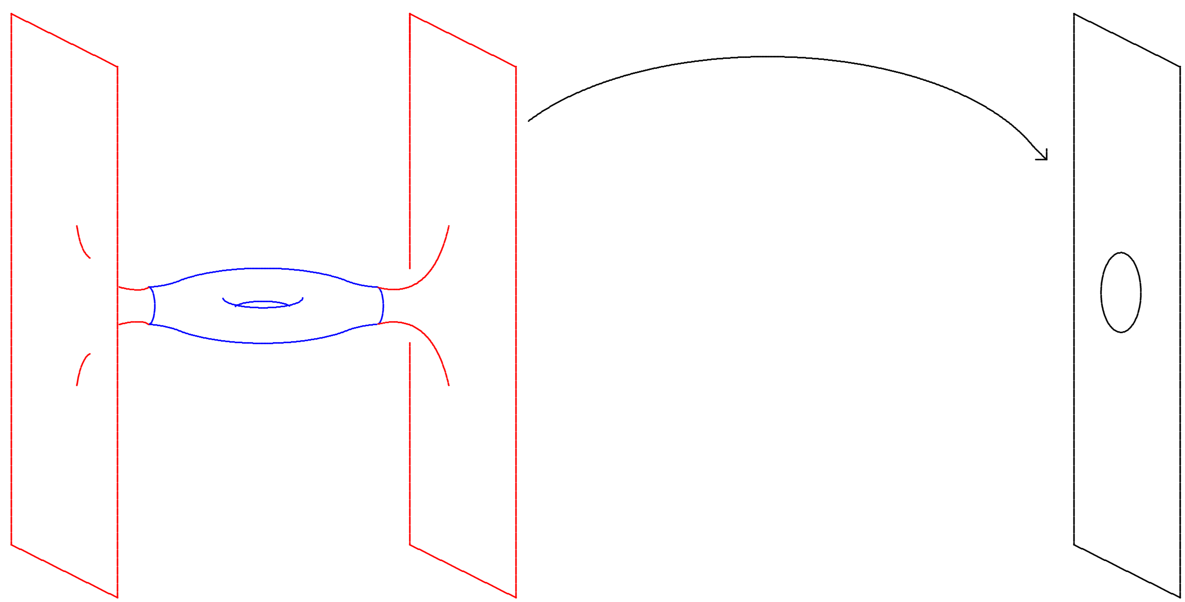

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Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

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(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

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Similarly, hypersurface $s = 0$ connected, too.

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Is this true for some a priori reason?

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A version of Alice's uniqueness of fill-in question!

End, Part II