Curvature Functionals,

Kähler Metrics, &

the Geometry of 4-Manifolds IV

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Our Focus. *If (M^4, J) is a compact complex surface, when does M^4 admit an Einstein metric g (unrelated to J)?*

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$$h = \sum_{j,k=1}^{m} h_{j\bar{k}} \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right]$$

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where $[h_{j\bar{k}}]$ Hermitian matrix at each point.

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$[\omega] \in H^2(M, \mathbb{R})$ called the Kähler class.

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Kähler if the 2-form

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is closed:

$$d\omega = 0.$$

But we do not assume this!

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Only two metrics arise in non-Kähler case!

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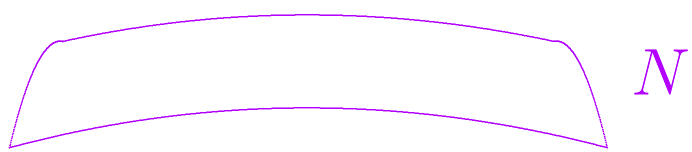

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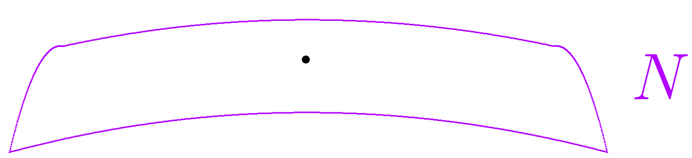

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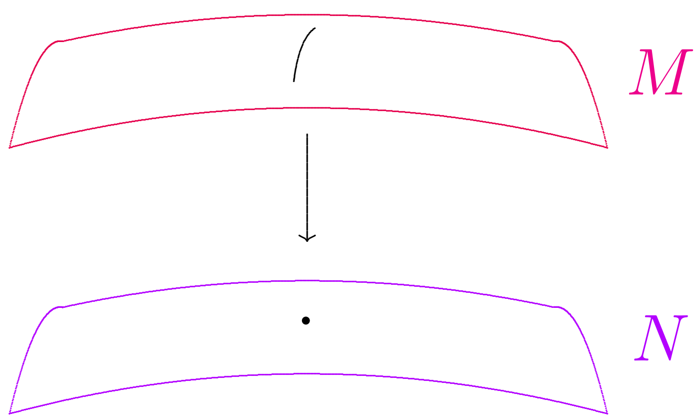

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in which new $\mathbb{CP}_1$ has self-intersection -1.

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Similarly when M symplectic instead of complex.

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Remark. When $n = 2m = 4$, such M are necessarily minimal complex surfaces of general type. Among such surfaces, exactly those s.t.

$$\nexists \mathbb{CP}_1 \overset{\mathcal{O}}{\hookrightarrow} M$$

of homological self-intersection -2.

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$K3$ admits Ricci-flat Kähler metrics.

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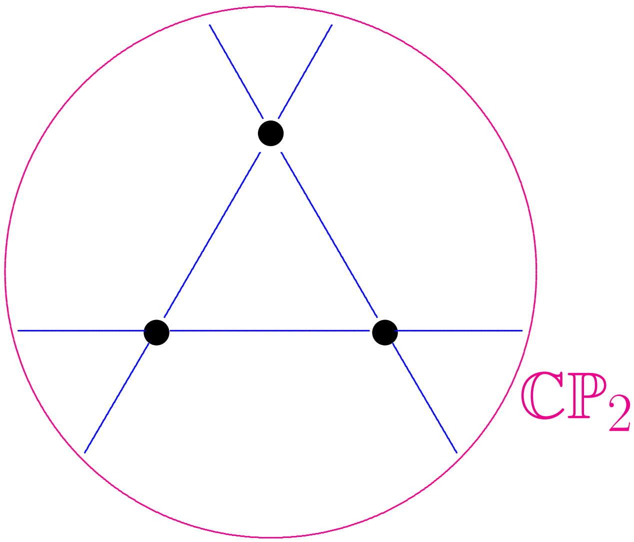

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Exceptions: $\mathbb{CP}_2$ blown up at 1 or 2 points.

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Derdziński ('83) then discovered that this metric is conformally Kähler, and proved fundamental structure theorems concerning conformally Kähler, Einstein metrics.

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Note both of above Einstein metrics are Hermitian.

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Will describe a second proof (L '12) which contains much more information.

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among all Kähler metrics on M*. Consequently,* h *is an absolute minimizer of the functional*

$$h \longmapsto \int_M |W|_h^2 \, d\mu_h.$$

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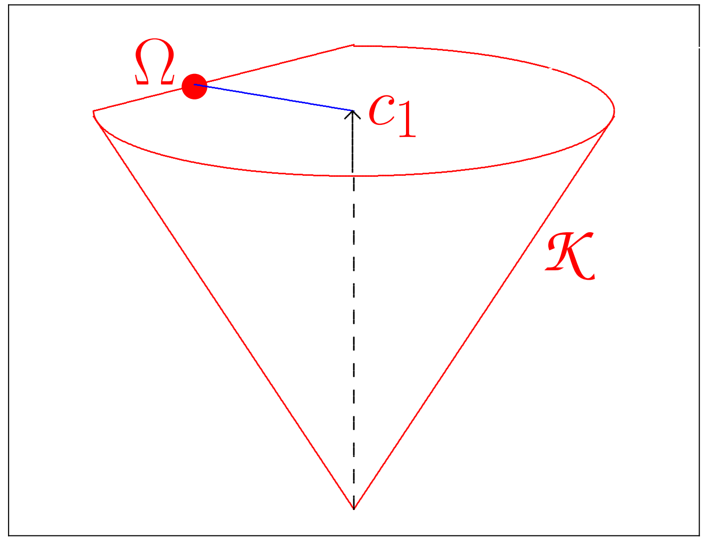

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Similarly for $\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$, though less interesting. . .

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X.X. Chen: always minimizers.

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Donaldson/Mabuchi/Chen-Tian:
unique modulo bihomorphisms.

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^-	$\mathring{r}$	$W_- + \frac{s}{12}$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature *(conformally invariant)*

W_- = anti-self-dual Weyl curvature ″

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1-parameter family of metrics

$$g_t := g + t\dot{g} + O(t^2)$$

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$$\frac{d}{dt}\mathcal{W}_+(g_t)\Big|_{t=0} = -\int \dot{g}^{ab} B_{ab} \, d\mu_g$$

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$$\nabla^a B_{ab} = 0$$

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Conformally Einstein $\Longrightarrow$ $B = 0$

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$$|W_{+}|^2 = \frac{s^2}{24}$$

Restriction of $\mathcal{W}_+$ to Kähler metrics?

On Kähler metrics,

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In fact, for Kähler metrics,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\Big]$$

where Hess_0 denotes trace-free part of $\nabla\nabla$.

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- $g_t = g + tB$ *is Kähler metric for small* t*.*

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So the critical metrics of restriction of $\mathcal{W}_+$ to $\{$Kähler metrics$\}$ are Bach-flat Kähler metrics.

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Lemma. *For any extremal Kähler g on any Del Pezzo M, scalar curvature $s > 0$ everywhere.*

Explicit lower bound:

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$$\frac{1}{32\pi^2}\int s^2 d\mu_g \geq \mathcal{A}([\omega])$$

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where $\mathcal{F}$ is Futaki invariant.

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Lemma. *For all $[\omega]$ on any Del Pezzo M,*

$$\mathcal{B}([\omega]) < \frac{1}{4}$$

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Then there is an extremal Kähler metric g *on* M *with Kähler form* $\omega \in [\omega]$.

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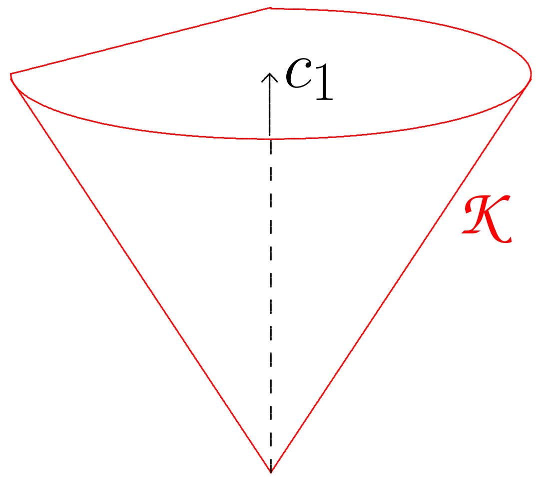
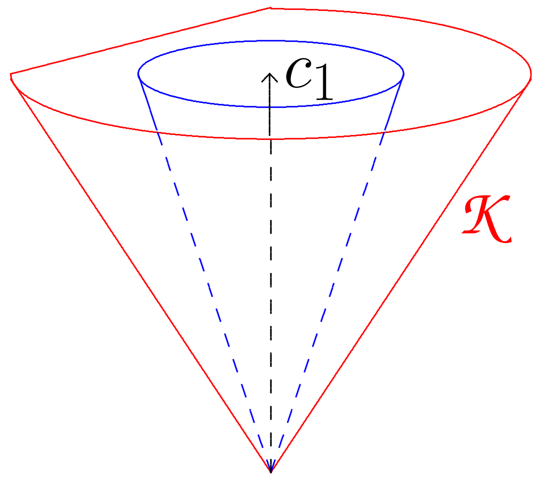
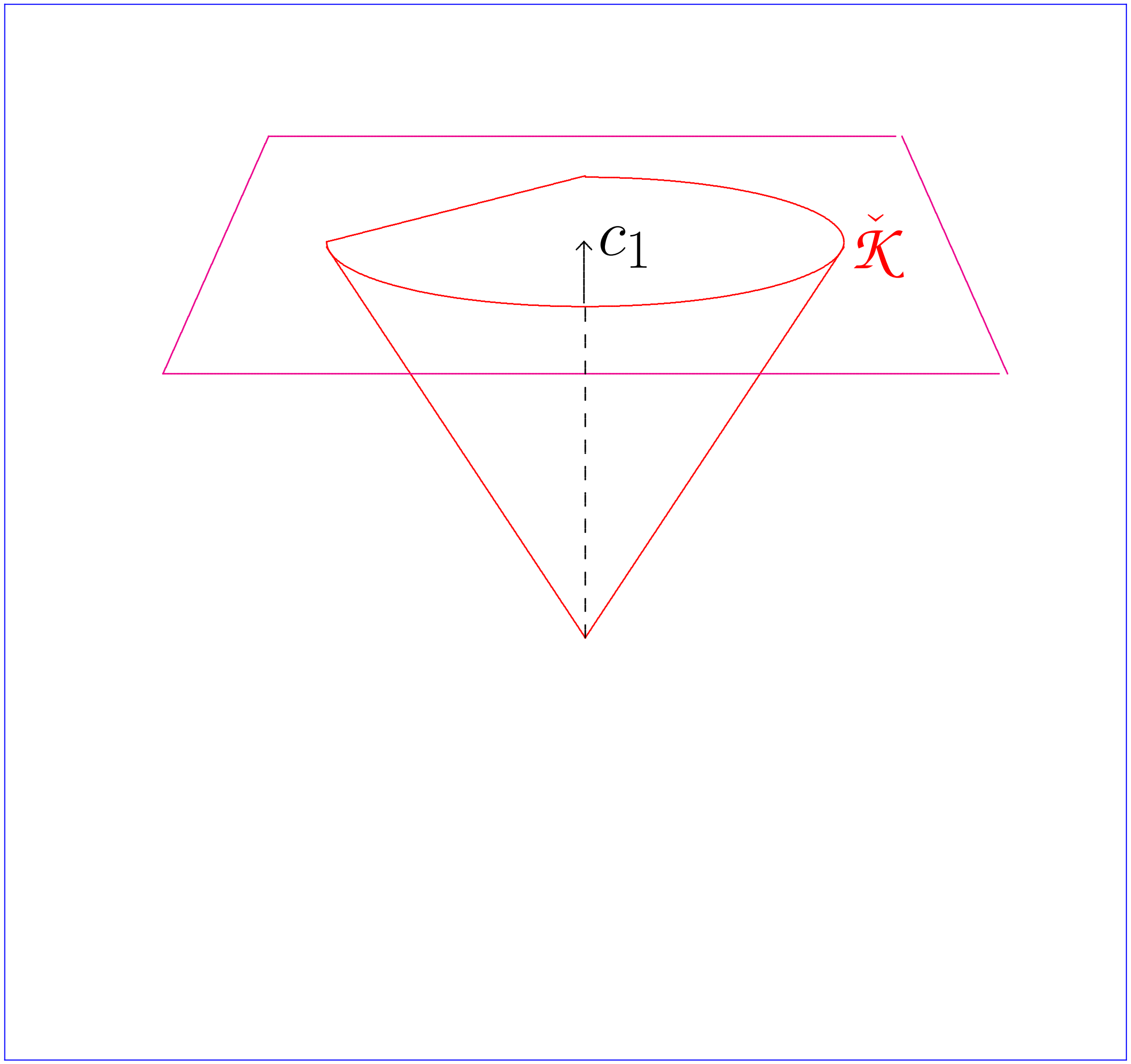


$$H^{1,1}(M, \mathbb{R}) = H^2(M, \mathbb{R})$$



$$\mathcal{T}([\omega]) = \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} \leq const$$



$$\check{\mathcal{K}} = \mathcal{K}/\mathbb{R}^+$$

Next time:

- Prove existence of these extremal Kähler metrics;

- Use it to prove existence of Einstein metrics; and

- discuss uniqueness of Einstein metrics.

End, Part IV