Curvature Functionals,

Kähler Metrics, &

the Geometry of 4-Manifolds II

Claude LeBrun
Stony Brook University

IHP, December 4, 2012

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Much more flexible in higher dimensions.

Why is Dimension Four Exceptional?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\mathring{r}$
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where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature *(conformally invariant)*

W_- = anti-self-dual Weyl curvature ″

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Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

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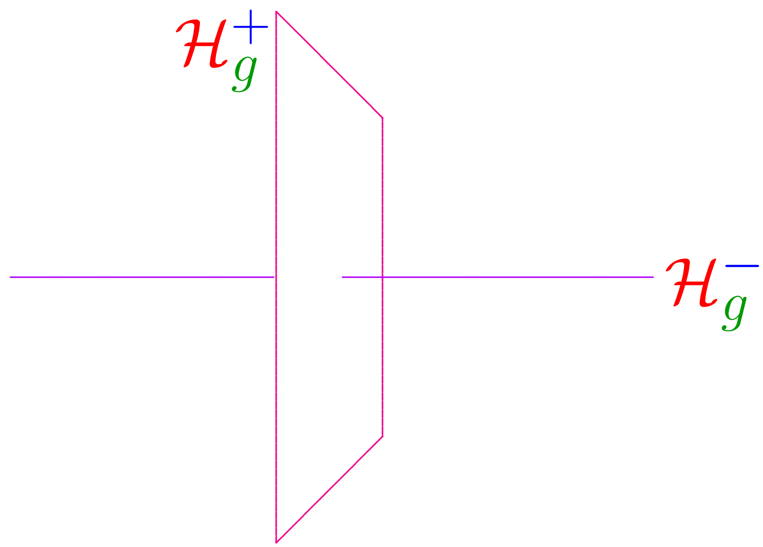
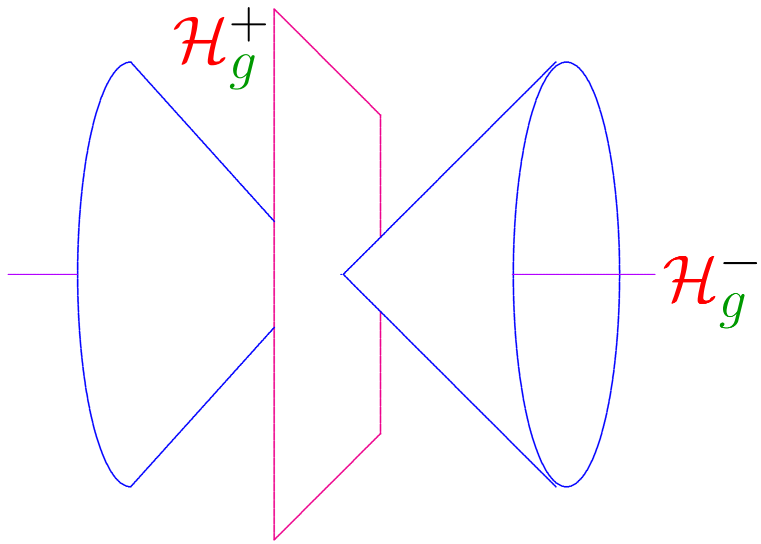
$$b_\pm(M) = \dim \mathcal{H}_g^\pm.$$



$$H^2(M,\mathbb{R})$$



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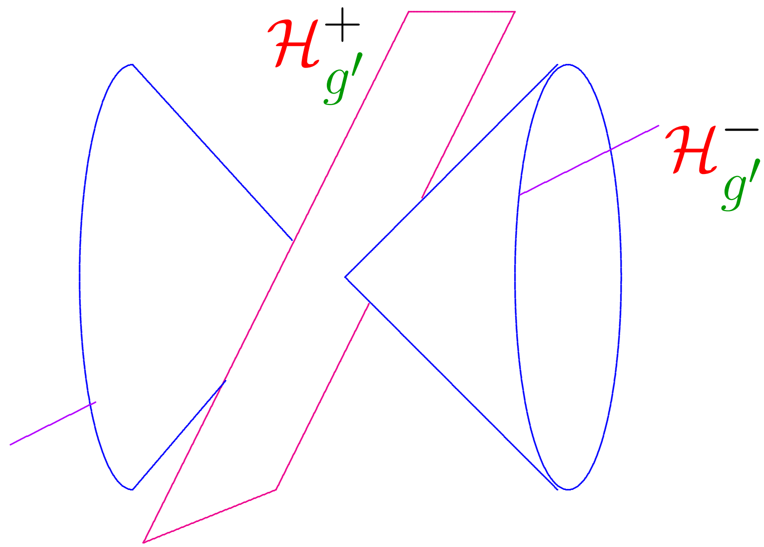

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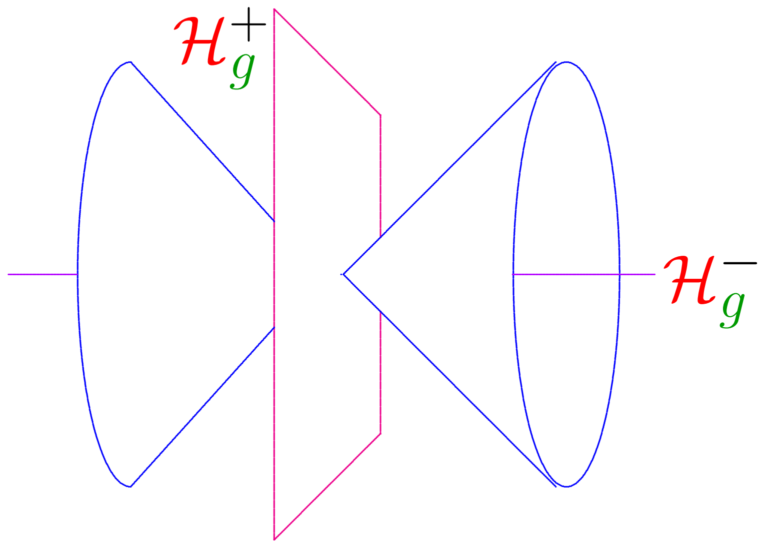

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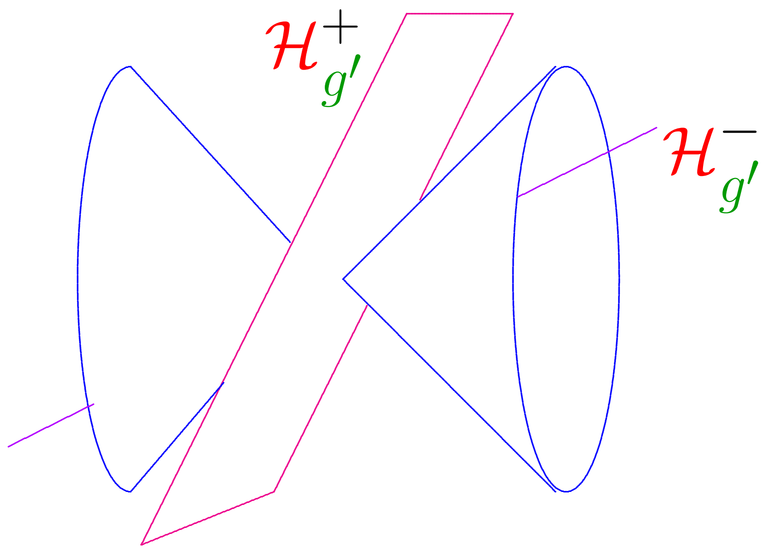

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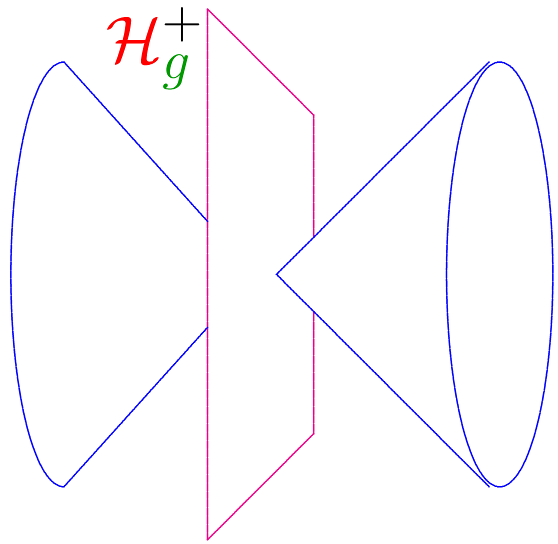

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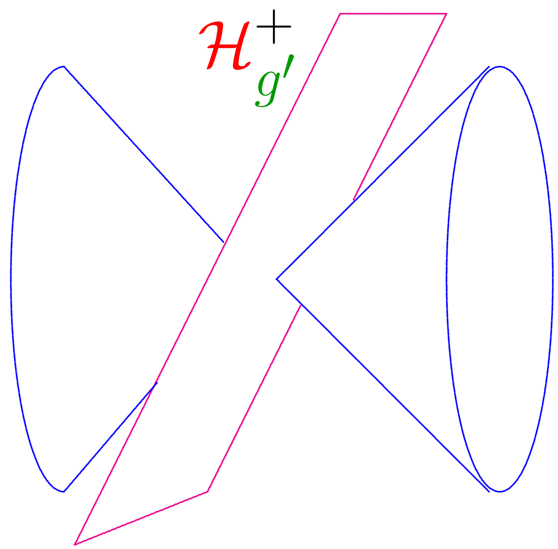

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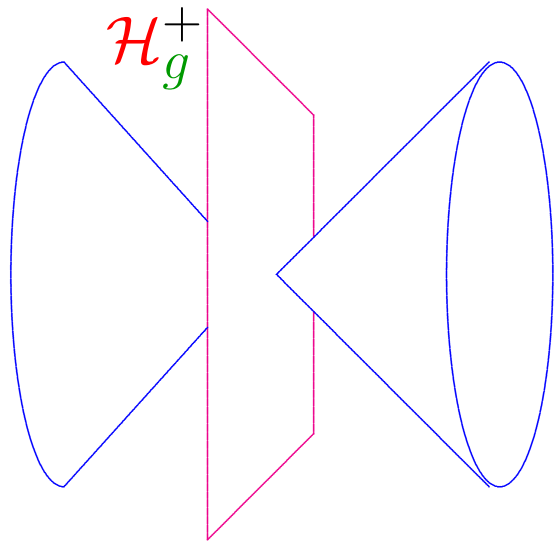

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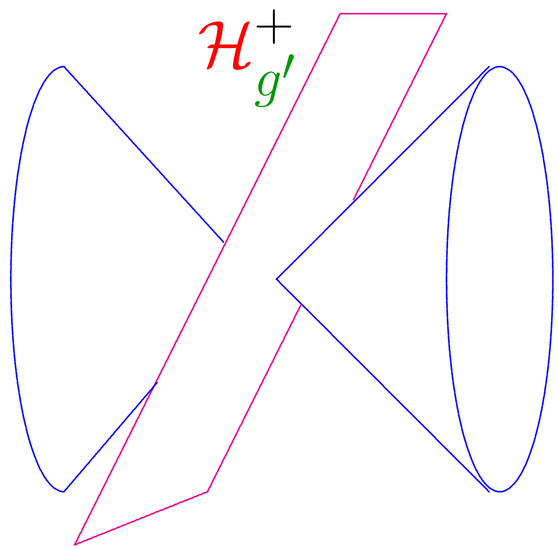

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Invariant under $\mathit{Diff}_0(M)$ & conformal rescalings.

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Simply connected case:

$$\begin{aligned} \chi &= 2 + b_+ + b_- \\ \tau &= b_+ - b_- \end{aligned}$$

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Typically, one homeotype $\longleftrightarrow$ ∞ many diffeotypes.

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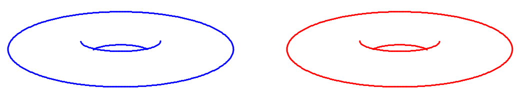

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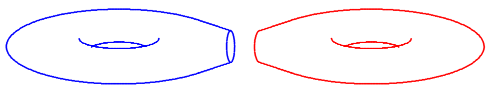

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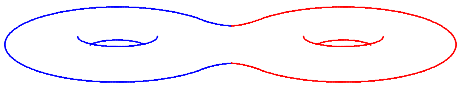

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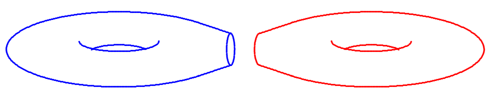

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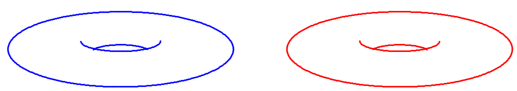

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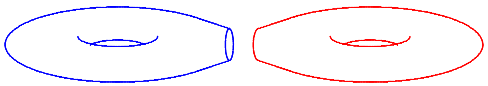

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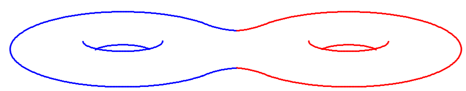

Corollary. *Any smooth compact* simply connected *non-spin* 4*-manifold* M *is homeomorphic to a connect sum*

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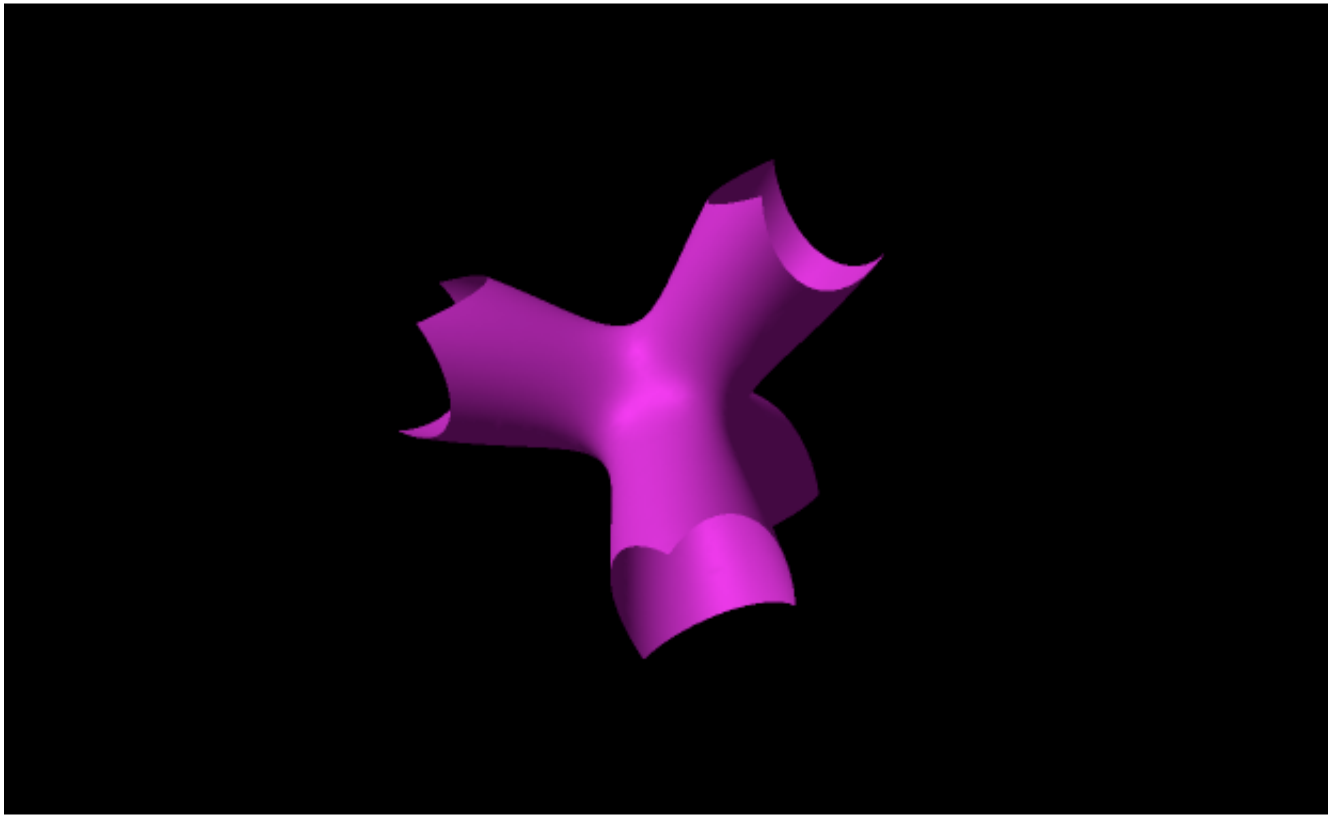
Conjecture (11/8 Conjecture)**.** *Any smooth compact* simply connected *spin 4-manifold M is (un-orientedly) homeomorphic to either S^4 or a connected sum* $jK3\#k(S^2 \times S^2)$.

$K3$ = Kummer-Kähler-Kodaira manifold.

Diffeomorphic to quartic in $\mathbb{CP}_3$

$$x^4 + y^4 + z^4 + w^4 = 0$$



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Proof posted on ArXiv yesterday by Stefan Bauer.

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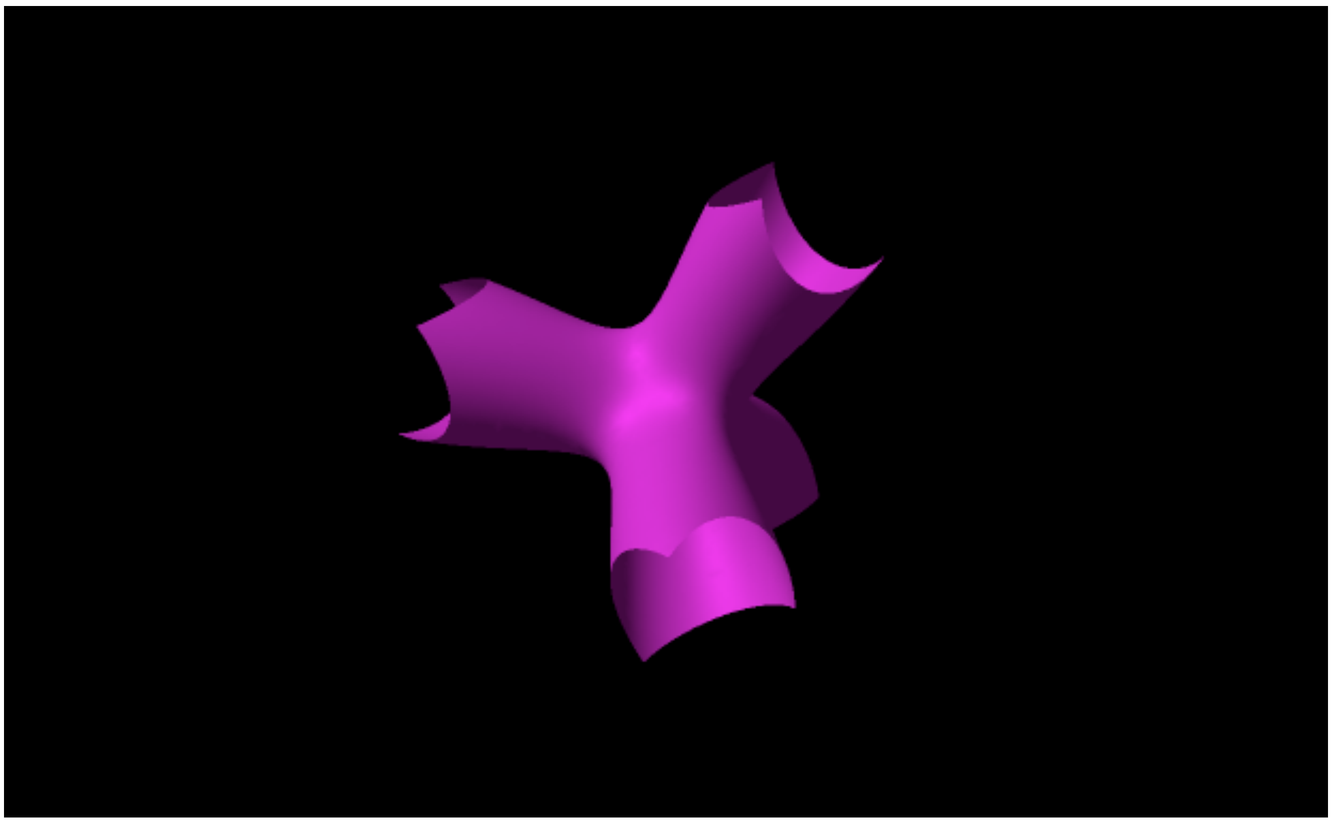

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Our Focus. *If* (M^4, J) *is a compact complex surface, when does* M^4 *admit an Einstein metric* g *(unrelated to* J*)?*

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Then $\mathrm{Kod}(M, J) \in \{-\infty, 0, 1, 2\}$ is exactly

$$\max \ \dim_{\mathbb{C}} \ \mathrm{Image}(M \dashrightarrow \mathbb{CP}_N)$$

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Most important invariant: Kodaira dimension.

Given (M^4, J) compact complex surface, set

$$\mathrm{Kod}(M) = \limsup_{\ell \to +\infty} \frac{\log \dim \Gamma(M, \mathcal{O}(K^{\otimes \ell}))}{\log \ell}$$

where $K = \Lambda^{2,0}$ is canonical line bundle.

Then $\mathrm{Kod}(M, J) \in \{-\infty, 0, 1, 2\}$ is exactly

$$\max \ \dim_{\mathbb{C}} \ \mathrm{Image}(M \dashrightarrow \mathbb{CP}_N)$$

over maps defined by holomorphic sections of $K^{\otimes \ell}$.

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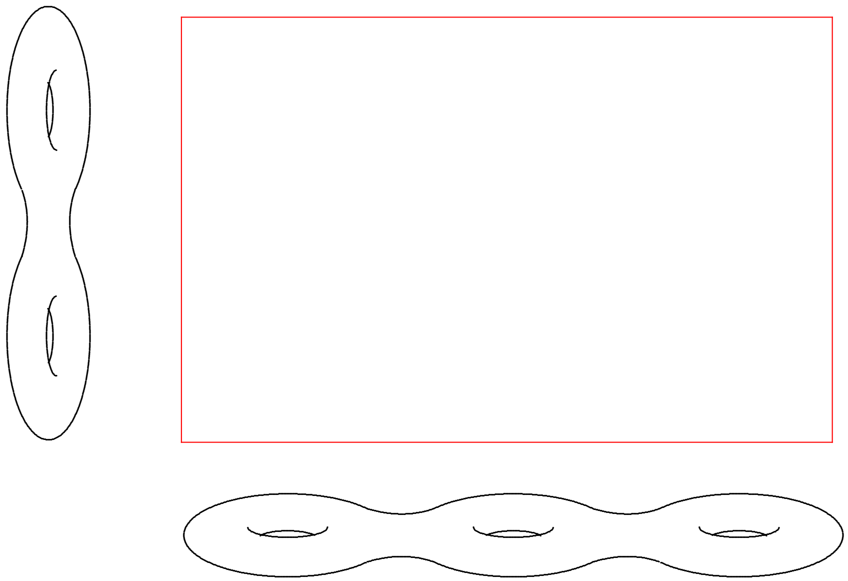

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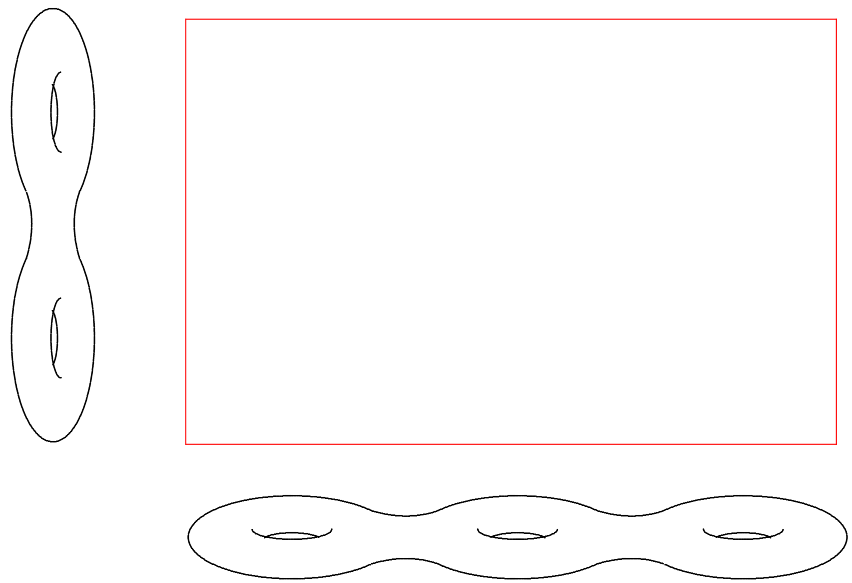

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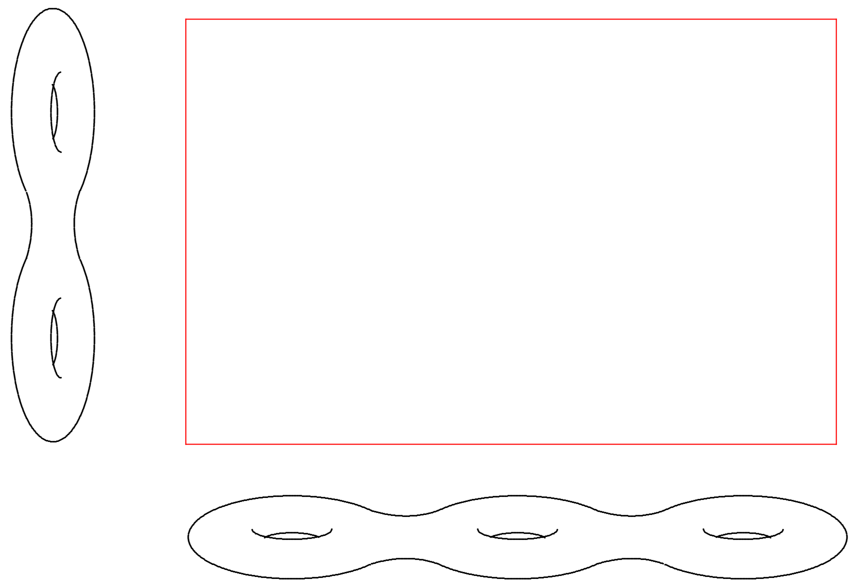

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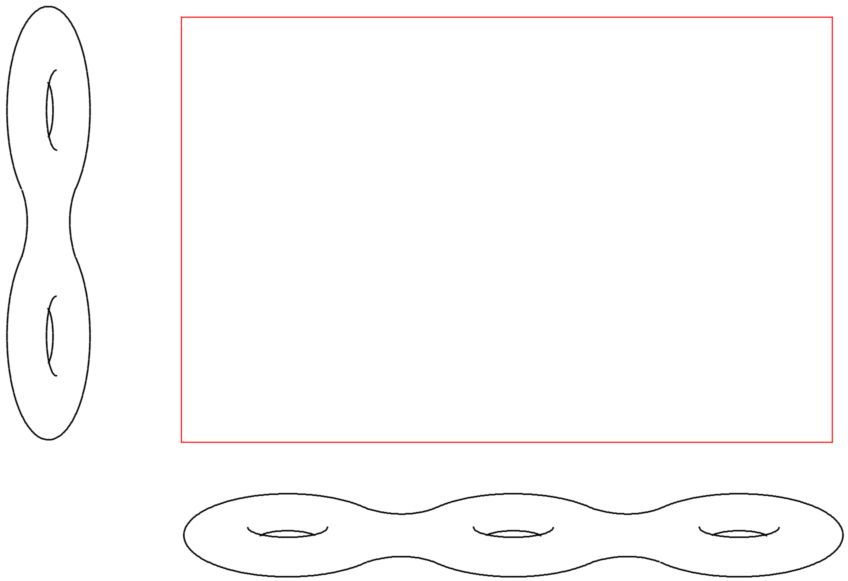

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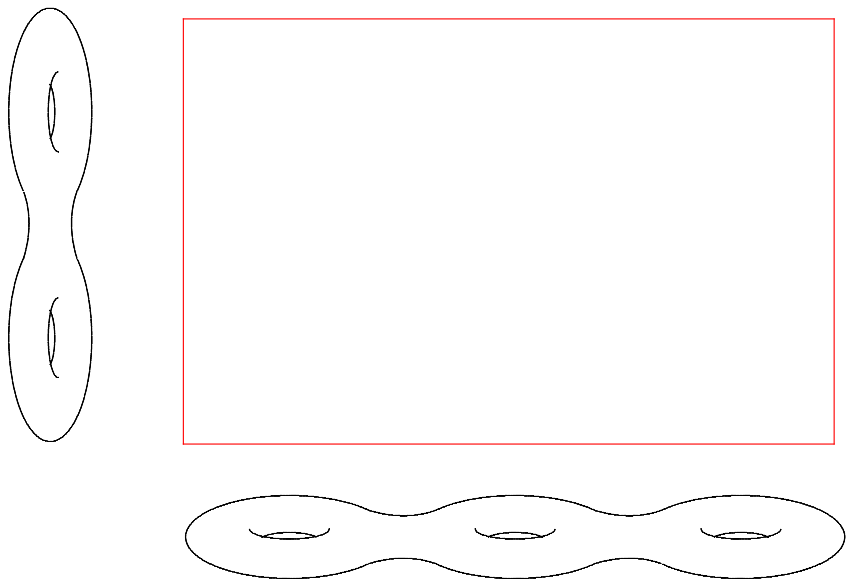

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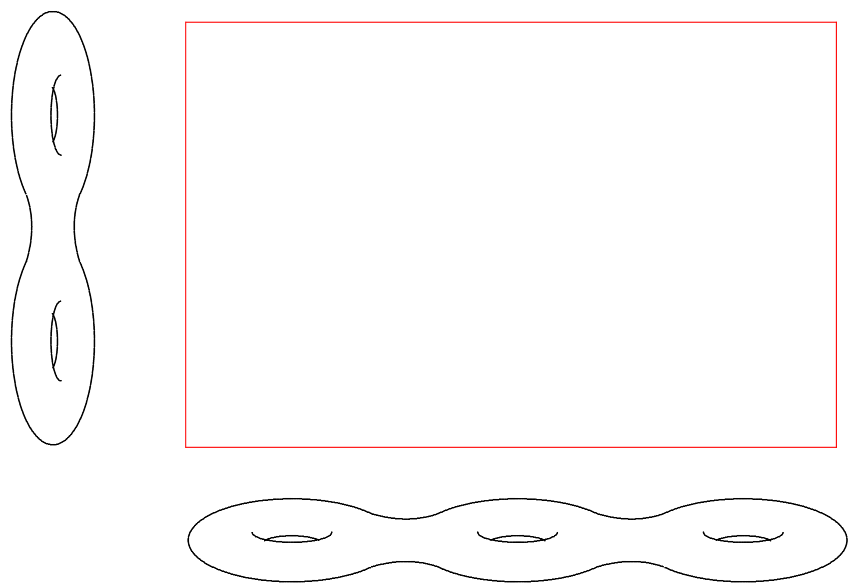

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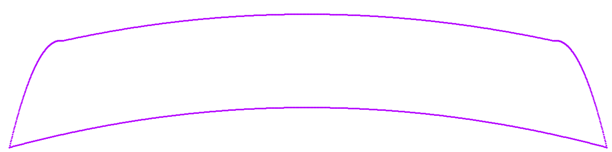

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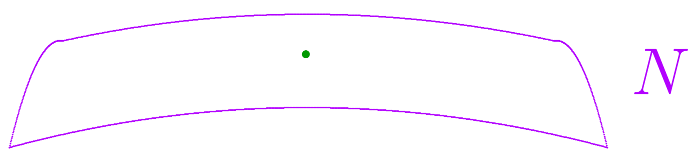

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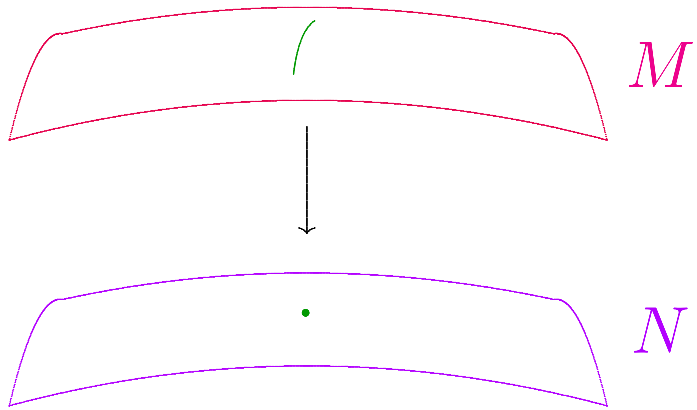

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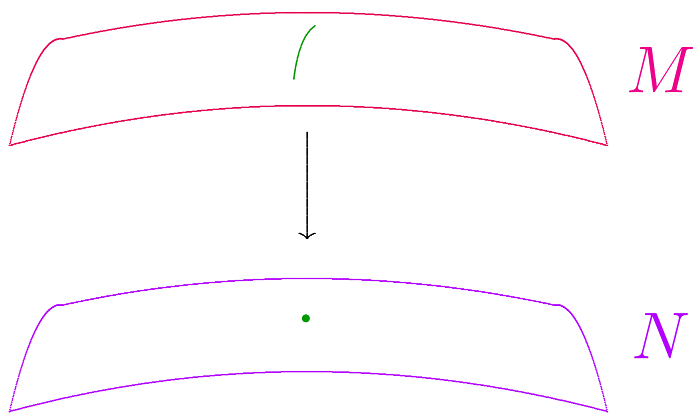

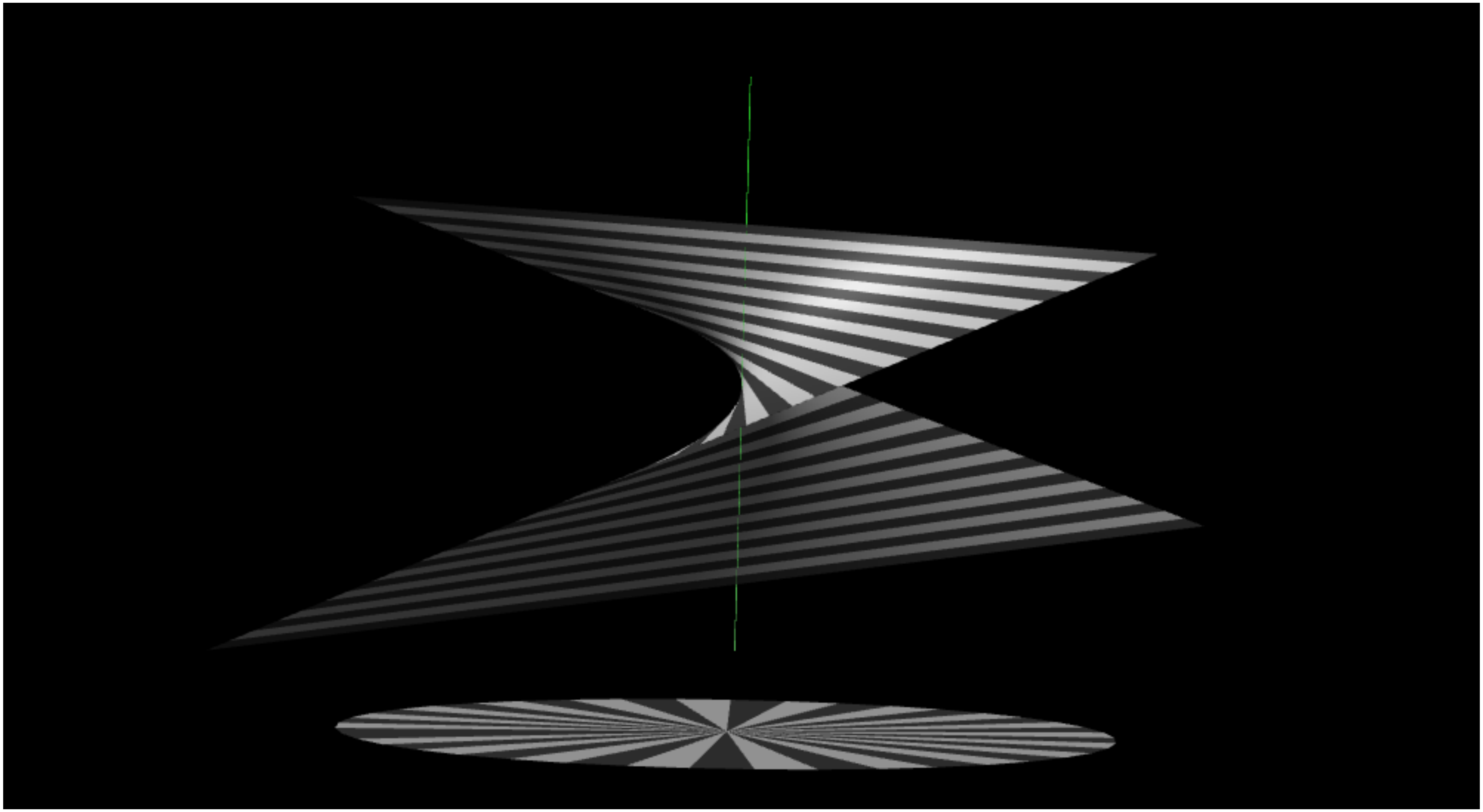

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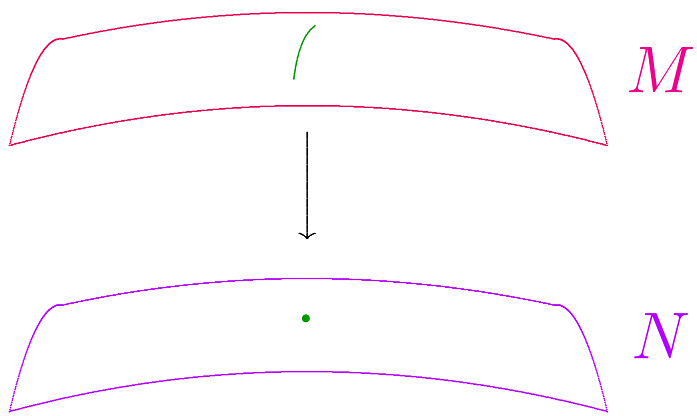

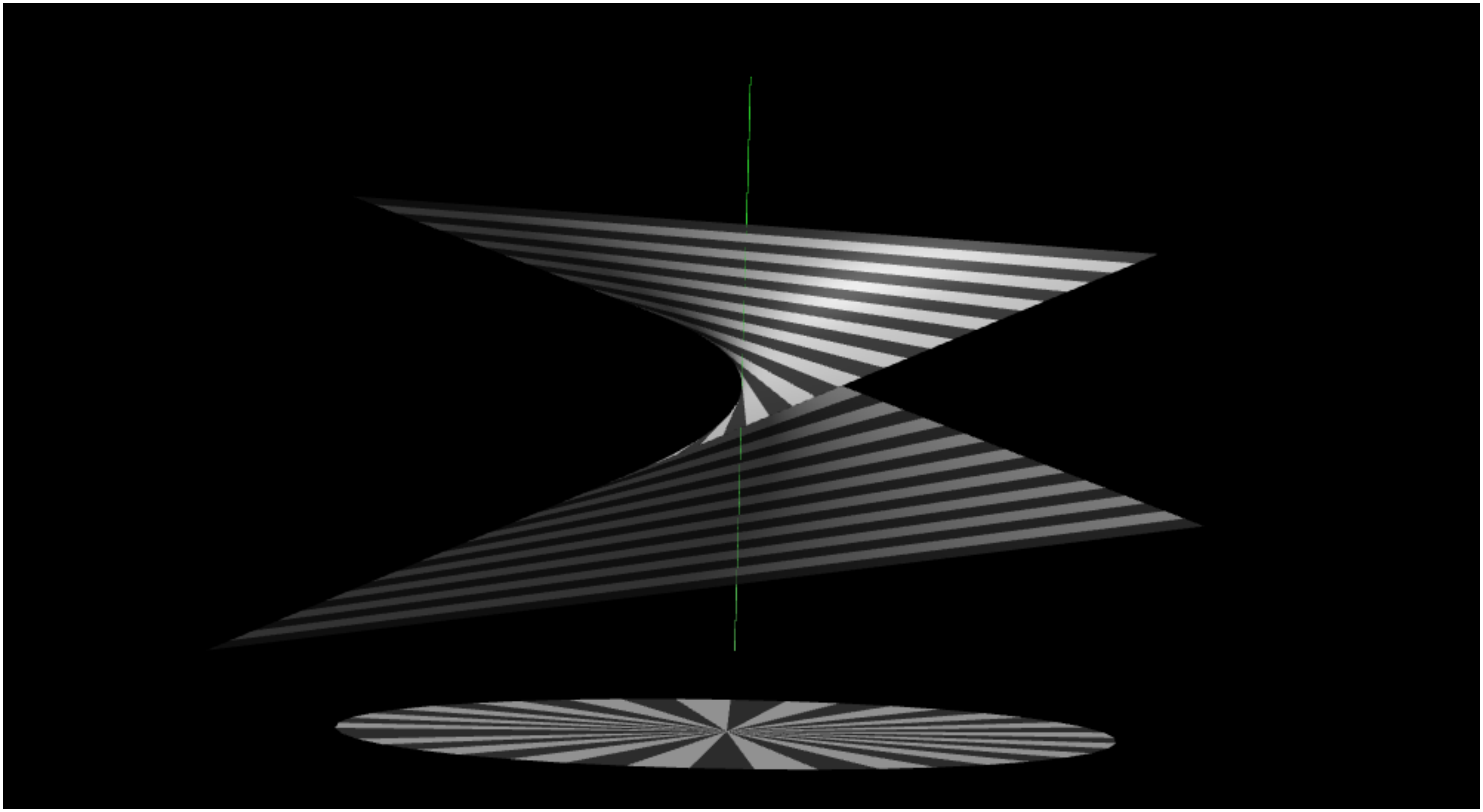

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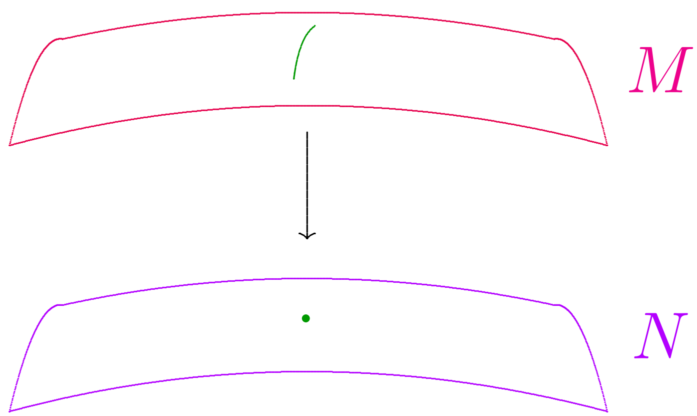

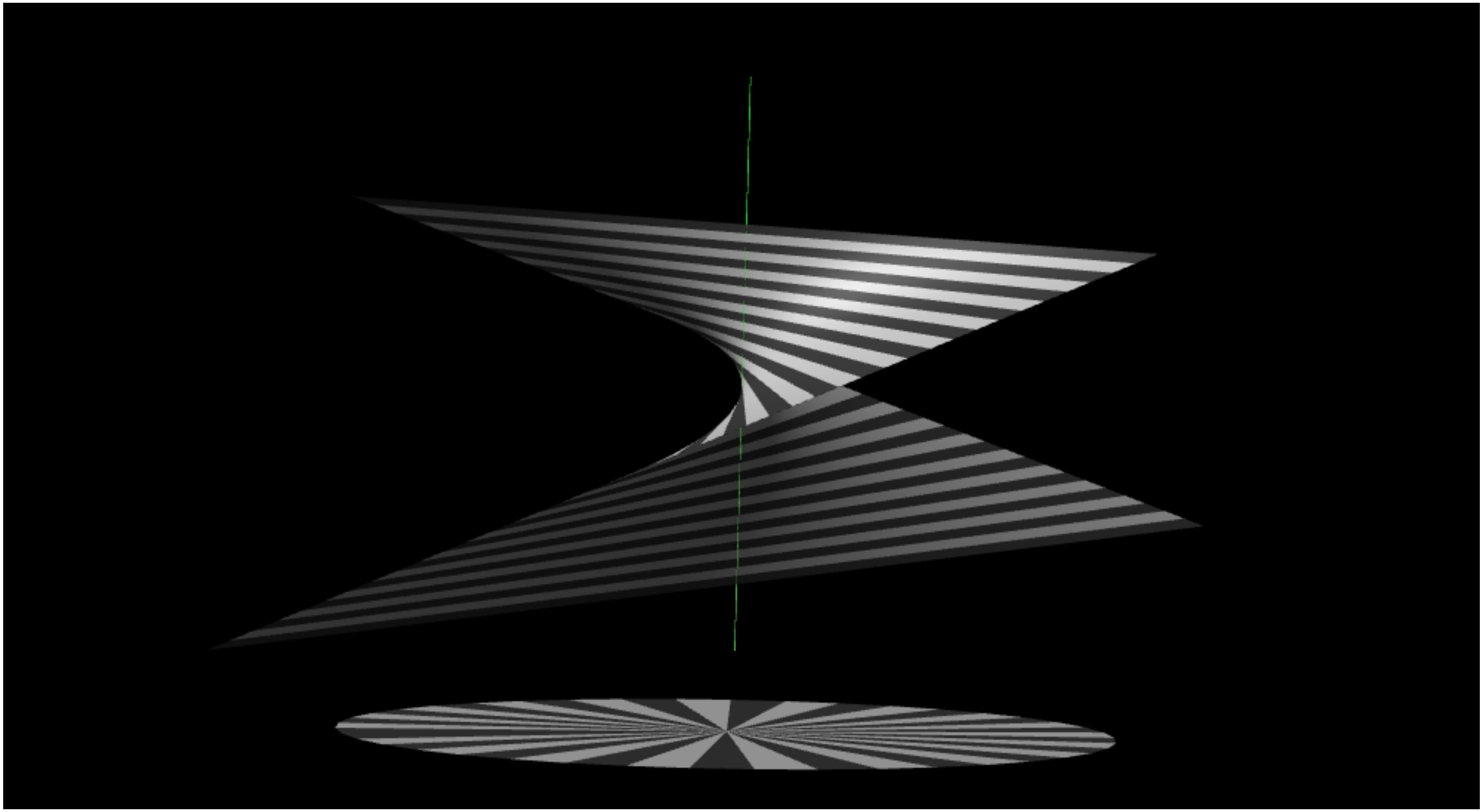

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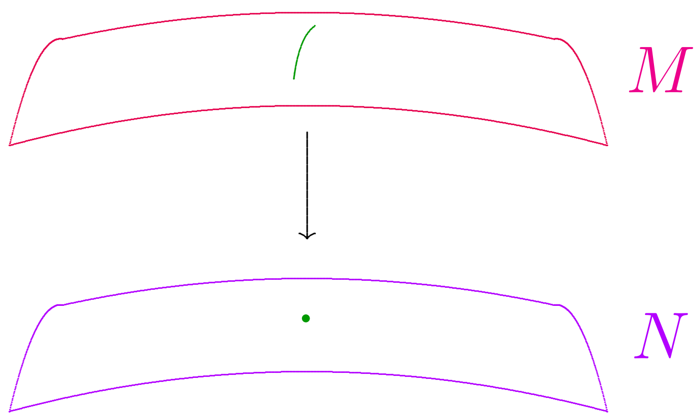

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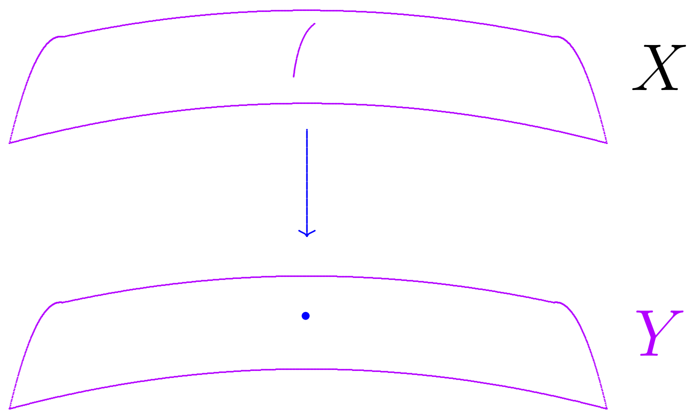

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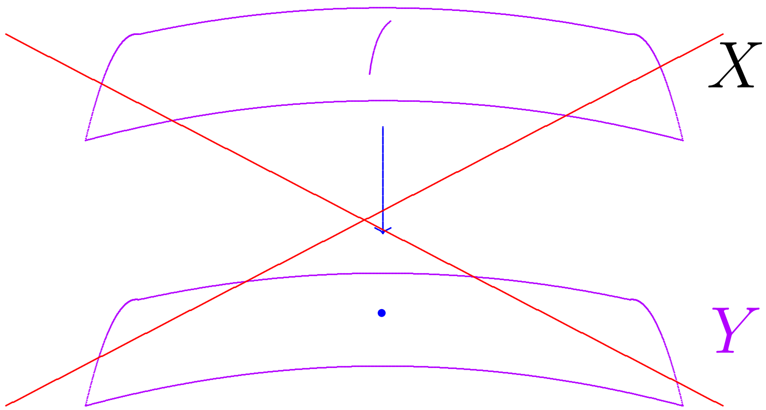

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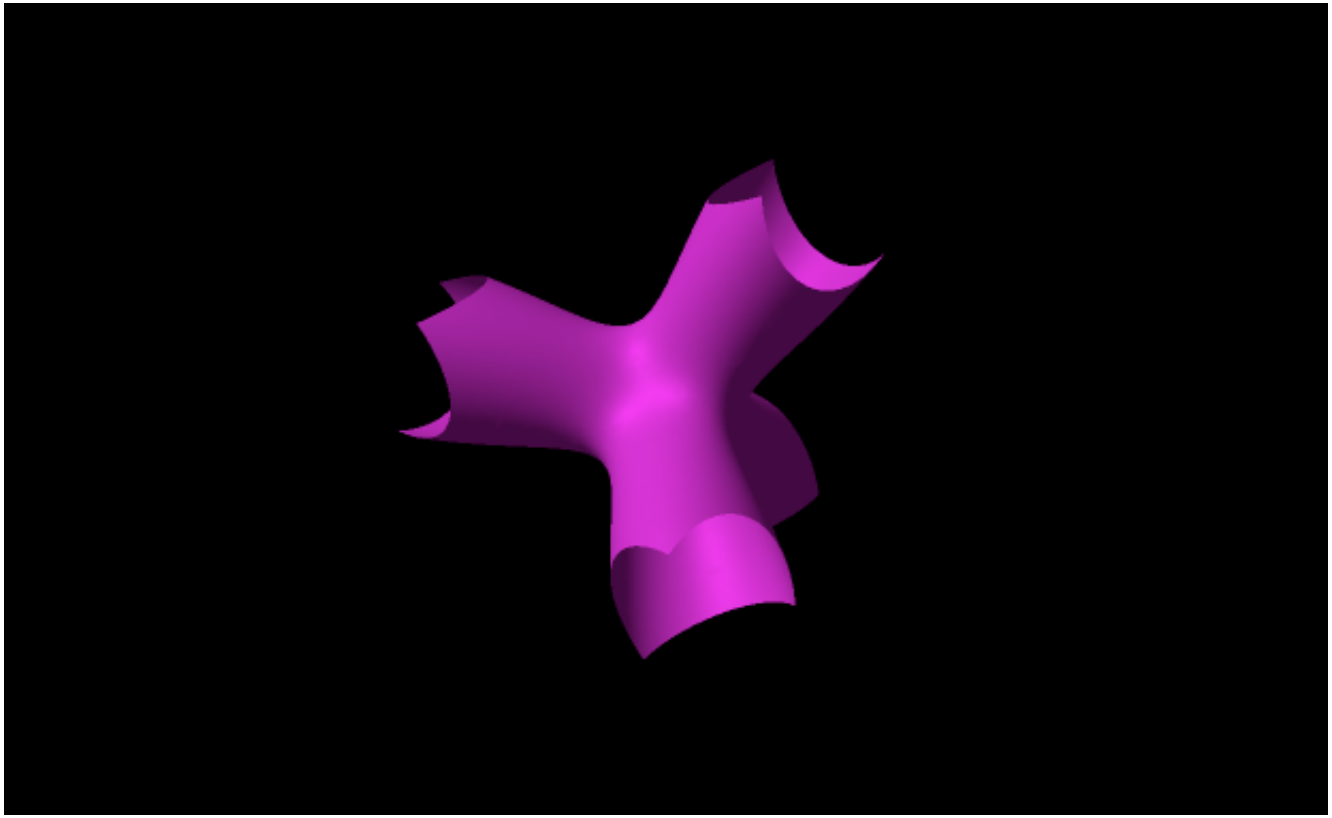

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Notice that $c_1^2 > 0 \Longrightarrow \mathrm{Kod}(X) \in \{-\infty, 2\}$, and that X must be of Kähler type.

Grauert: happens because

$$\begin{aligned}
\chi(X, \mathcal{O}(K^{\otimes \ell})) &= h^0(\mathcal{O}(K^{\otimes \ell})) - h^1(\mathcal{O}(K^{\otimes \ell})) + h^2(\mathcal{O}(K^{\otimes \ell})) \\
&= h^0(\mathcal{O}(K^{\otimes \ell})) - h^1(\mathcal{O}(K^{\otimes \ell})) + h^0(\mathcal{O}(K^{\otimes (1-\ell)})) \\
&= \ell(\ell - 1)\frac{c_1^2}{2} + \chi(X, \mathcal{O})
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Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

Einstein $\Rightarrow$ $$= \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g$$

Theorem (Hitchin-Thorpe Inequality). *If smooth compact oriented M^4 admits Einstein g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality only if (M, g) is locally hyper-Kähler. The latter case happens only if M finitely covered by flat T^4 or $K3$.

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Proposition. *If (M, J) compact complex surface, and if M admits Einstein metric g (unrelated to J) with $\lambda \neq 0$, then*

$$Kod(M, J) \in \{-\infty, 2\},$$

and M admits a symplectic structure.

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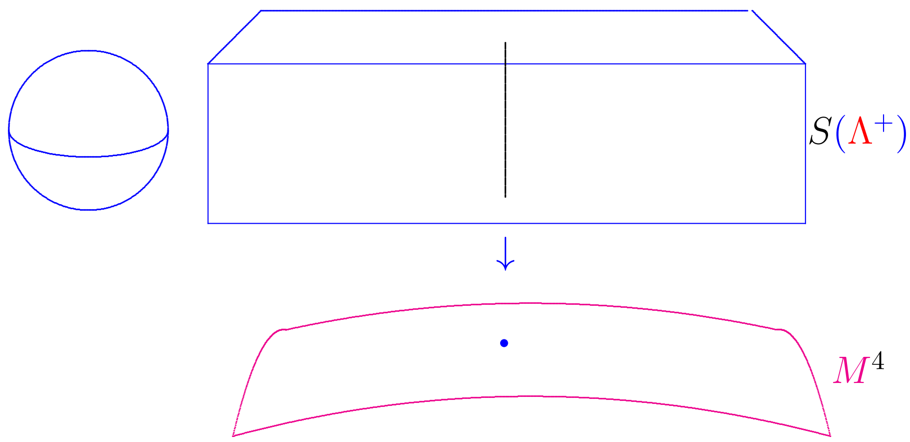

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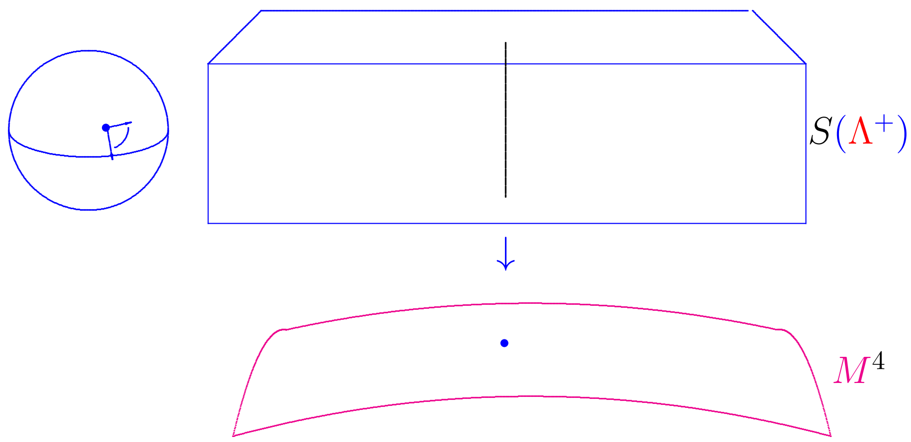

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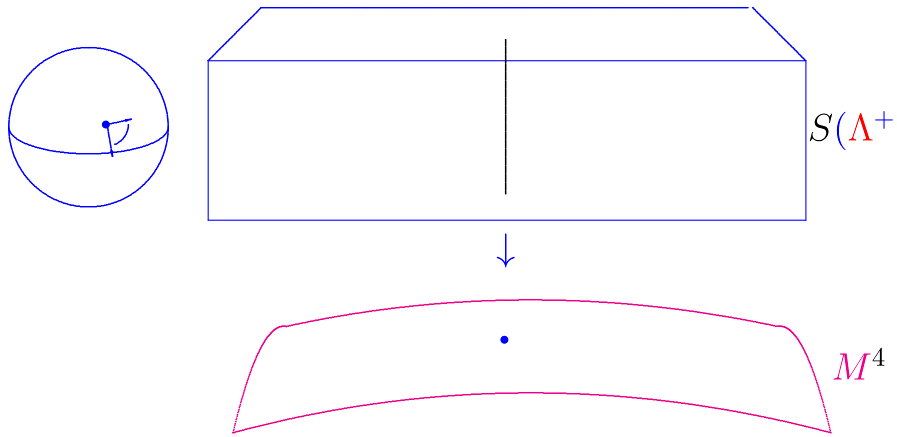

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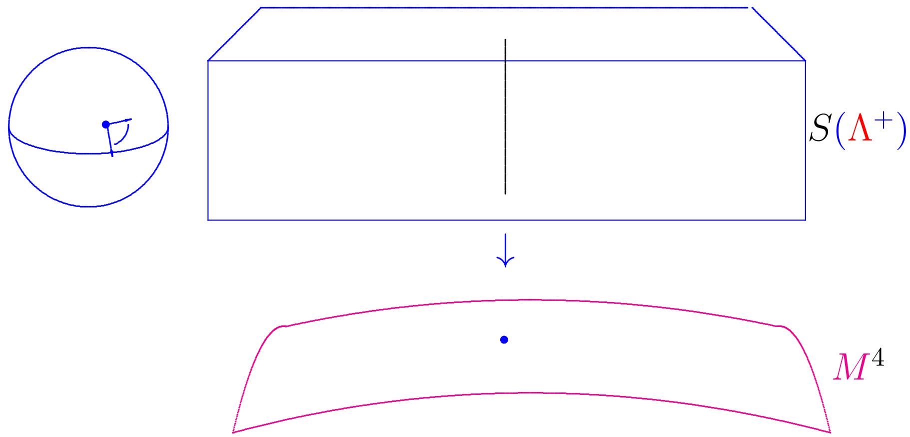

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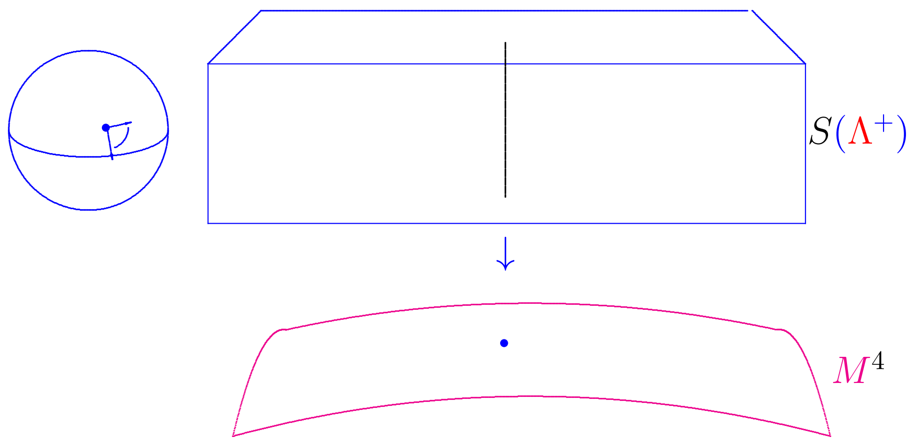

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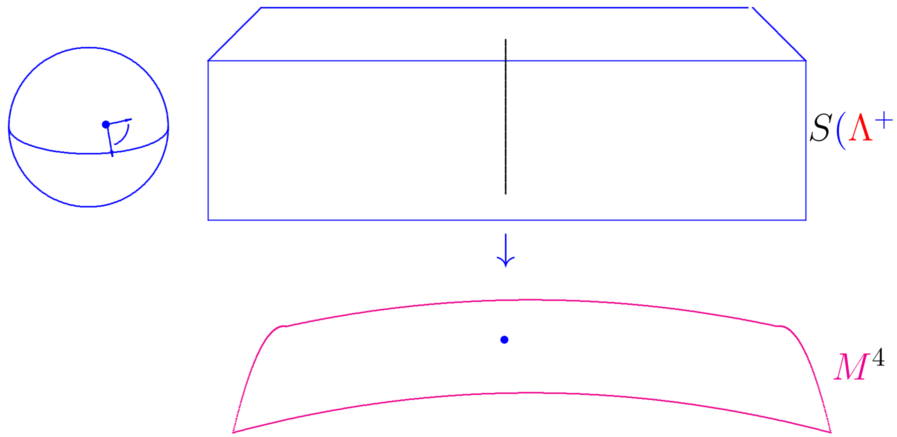

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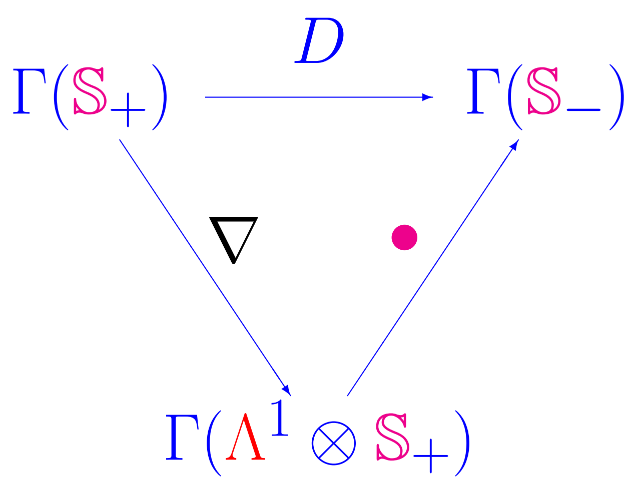

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Example. $\nexists$ metric of $s > 0$ on $K3$.

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Every unitary connection A on L induces spin^c Dirac operator

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generalizing $\bar\partial + \bar\partial^*$.

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where $F_A{}^+ =$ self-dual part curvature of A, and $\sigma : \mathbb{V}_+ \to \Lambda^+$ is a natural real-quadratic map,

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Non-linear, but elliptic once 'gauge-fixing'

$$d^*(A - A_0) = 0$$

imposed to eliminate automorphisms of $L \to M$.

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Implies non-existence of metrics g for which $s > 0$.

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When $b_+(M) = 1$, theory is more complicated, but one can still do something. For complex surfaces, shows that $s > 0$ metrics can only exist when $\mathrm{Kod}(M) = -\infty$.

End, Part II