Curvature Functionals,

Kähler Metrics, &

the Geometry of 4-Manifolds I

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Stony Brook University

IHP, December 3, 2012

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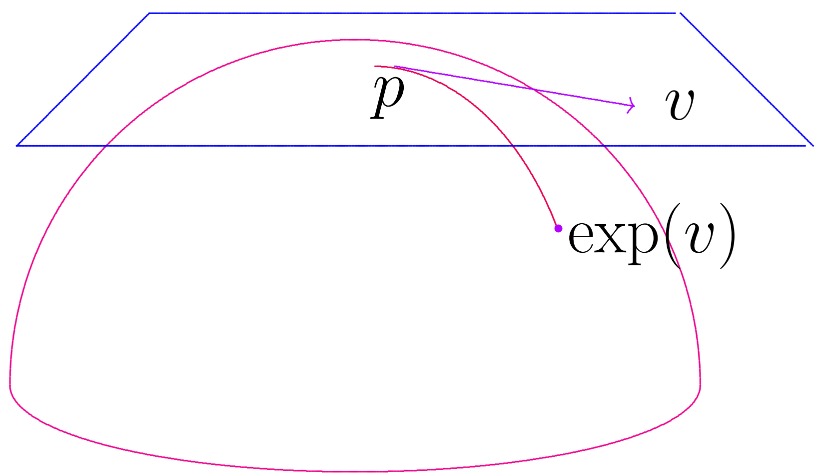

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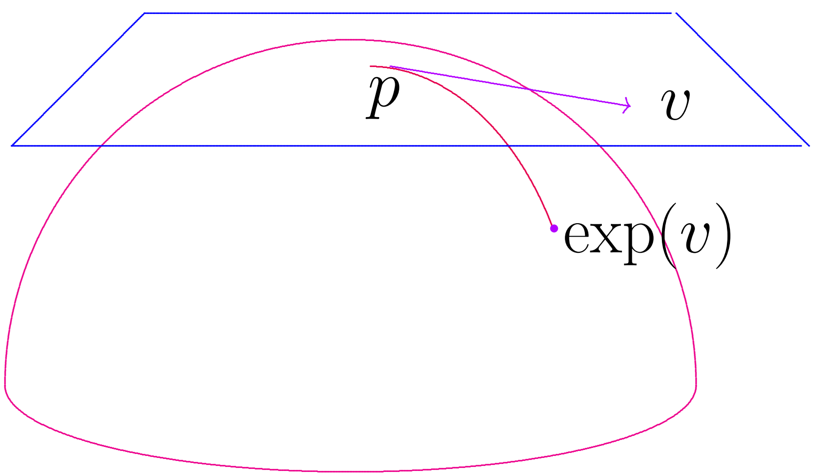

Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

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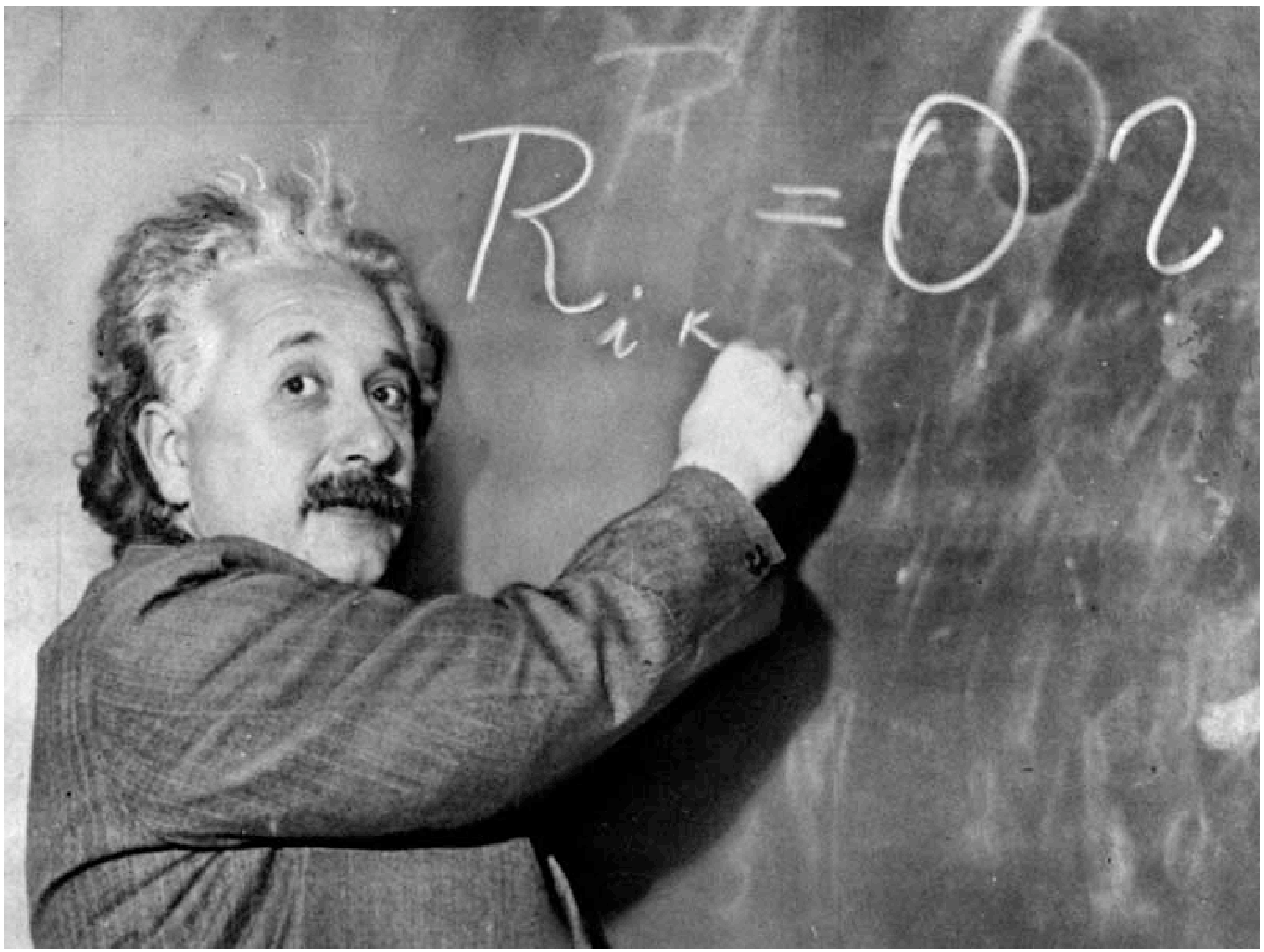

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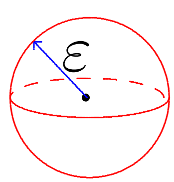

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Proof. Bianchi identity $\Longrightarrow \nabla \cdot \mathring{r} = (\frac{1}{2} - \frac{1}{n}) ds$.

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Try to find Einstein metrics by minimizing?

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By contrast, high-dimensional Einstein metrics too common; have little to do with geometrization.

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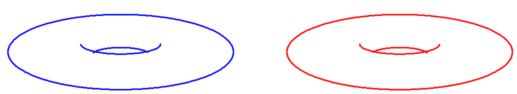

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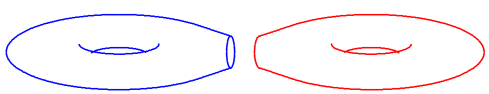

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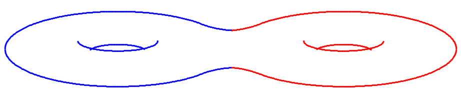

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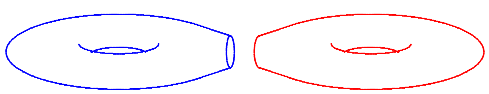

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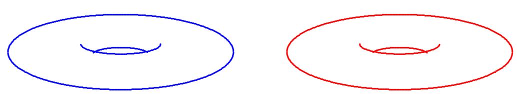

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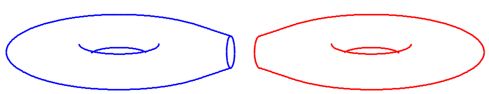

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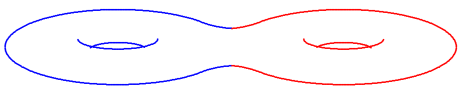

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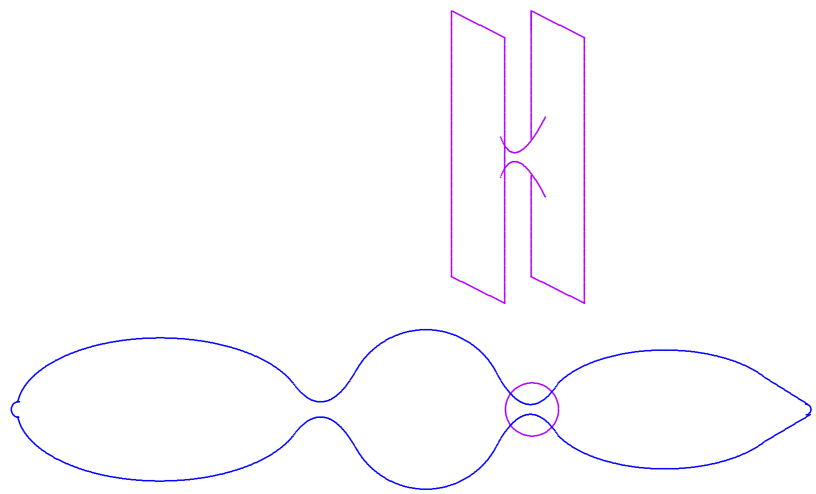

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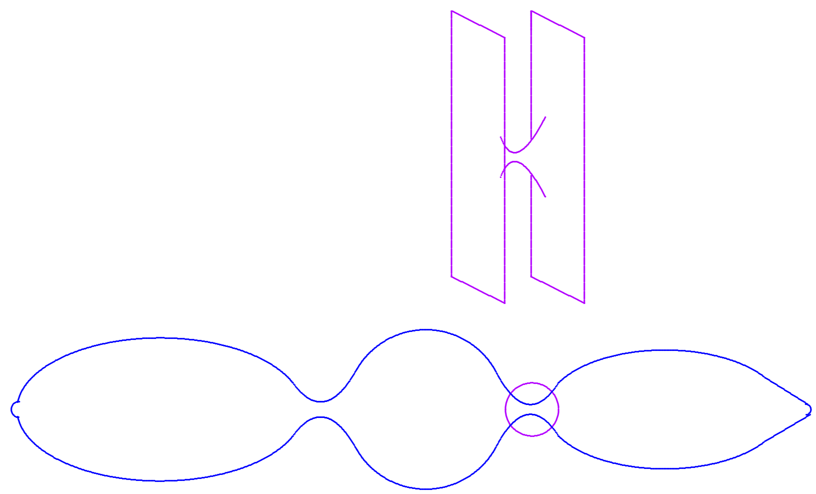

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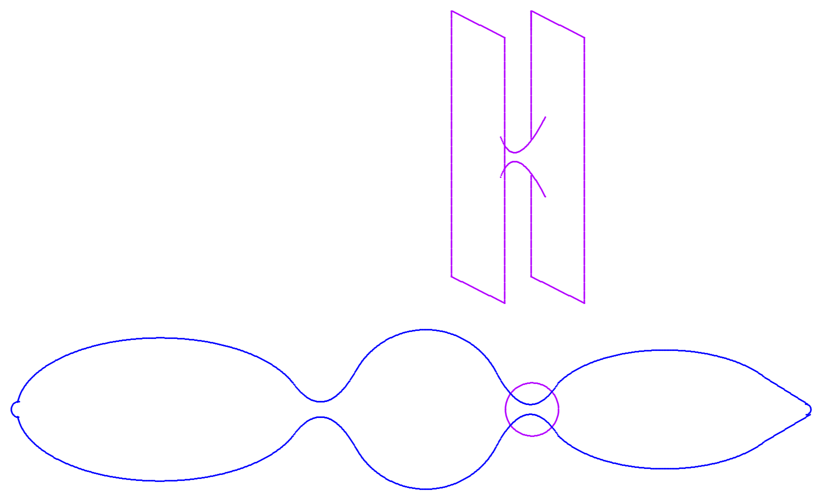

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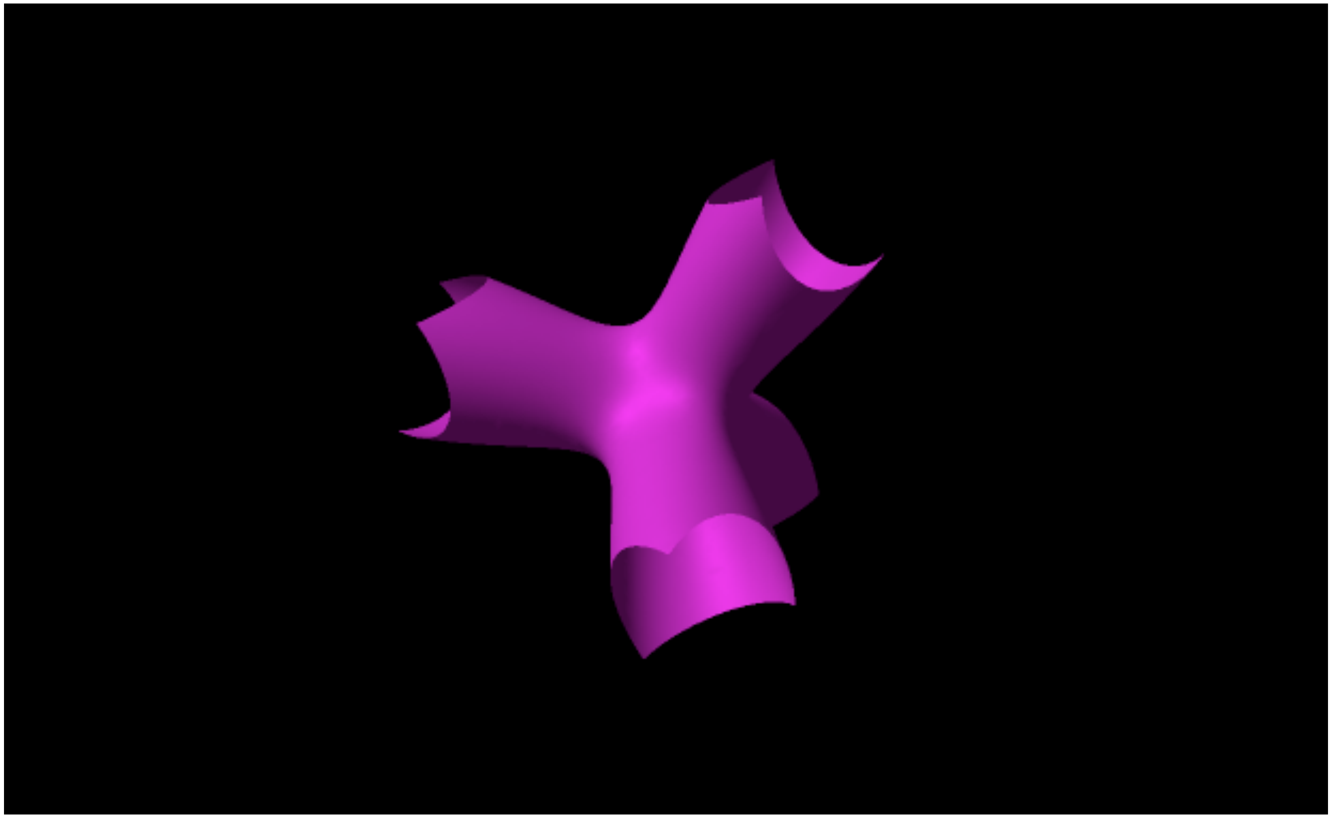

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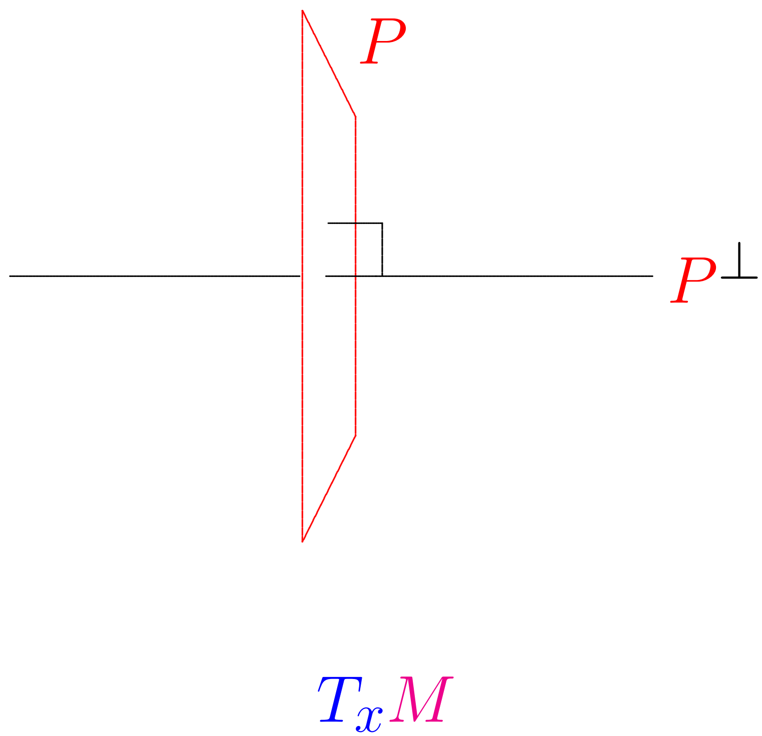

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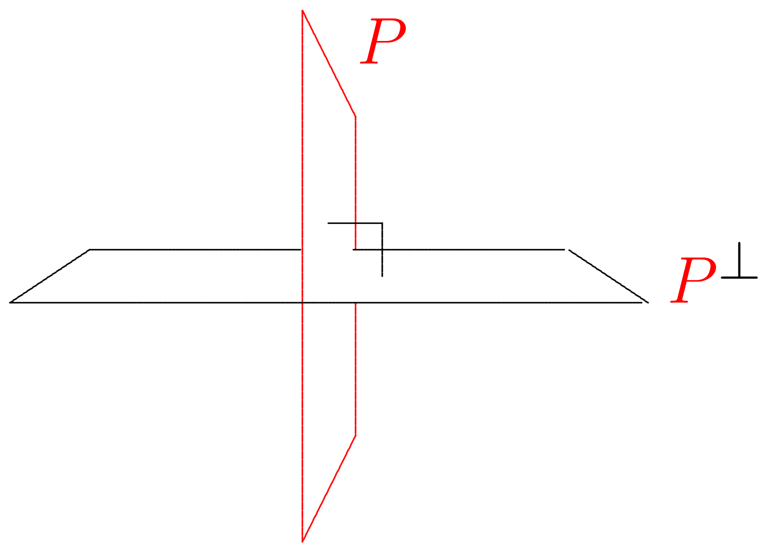

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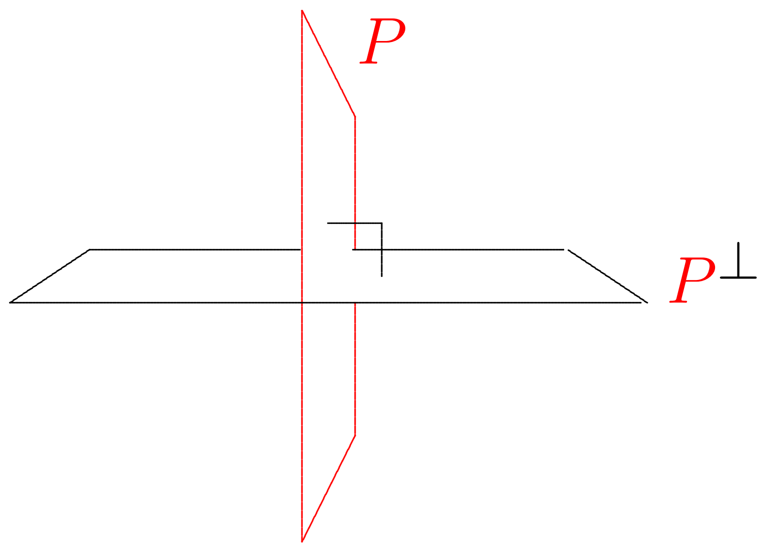

$$K(P) = K(P^\perp)$$

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But also natural and interesting to consider

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Integrals give four scale-invariant functionals.

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However, these are not independent!

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for Euler-characteristic $\chi(M) = \sum_j (-1)^j b_j(M)$.

4-dimensional signature formula

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 \qquad\qquad\qquad \right) d\mu$$

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Here $b_\pm(M) = \max\dim$ subspaces $\subset H^2(M, \mathbb{R})$ on which intersection pairing

$$\begin{array}{ccc} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\,[\varphi]\,,\,[\psi]\,) & \mapsto & \int_M \varphi \wedge \psi \end{array}$$

is positive (resp. negative) definite.

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Similarly for any quadratic curvature functional which is not conformally invariant.

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"Bach-flat" metrics. Conformally invariant!

Main Question. *Which smooth compact* 4*-manifolds* M^4 *admit Einstein metrics?*

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Our Focus. *If* (M^4, J) *is a compact complex surface, when does* M^4 *admit an Einstein metric* g *(unrelated to* J*)?*

(M^n, g): holonomy

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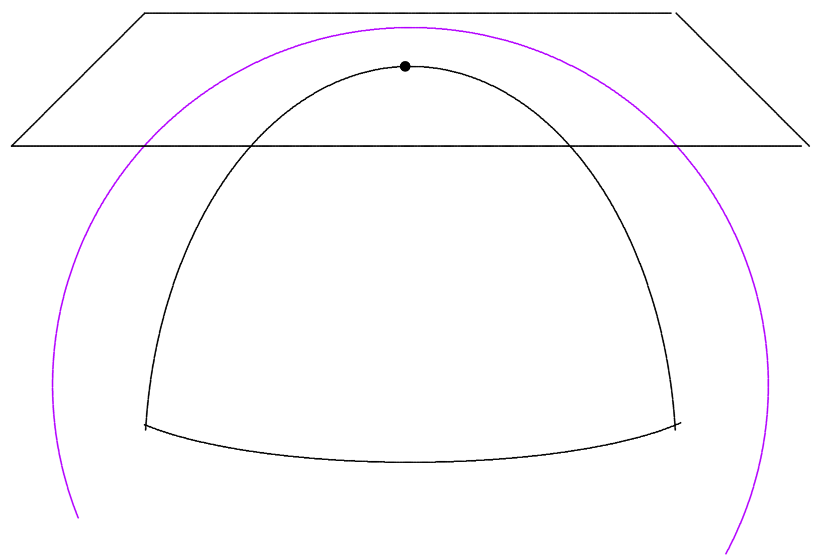

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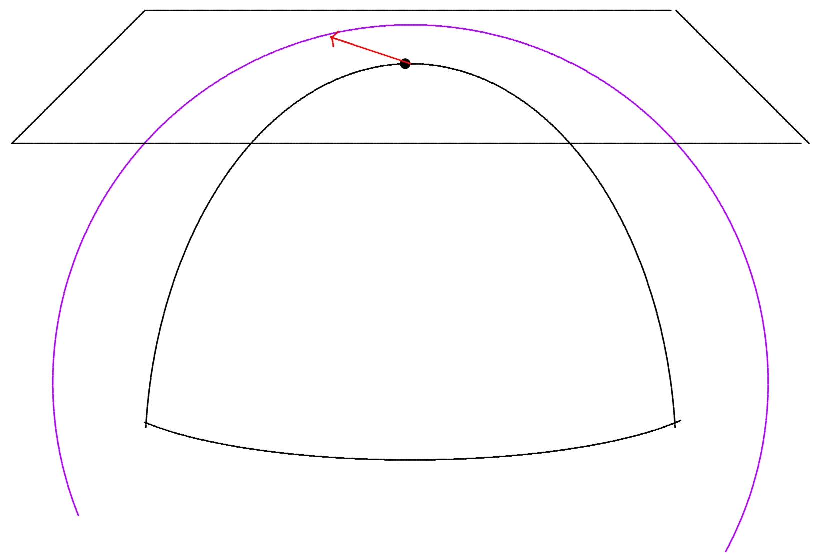

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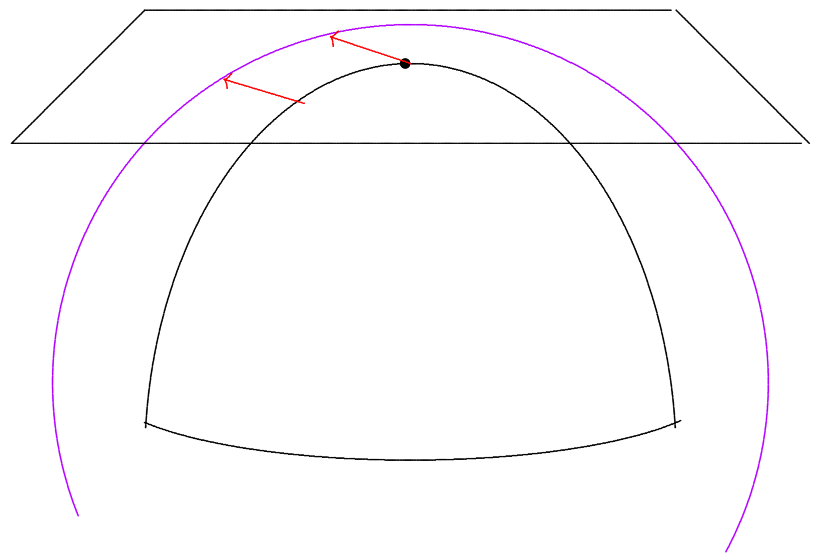

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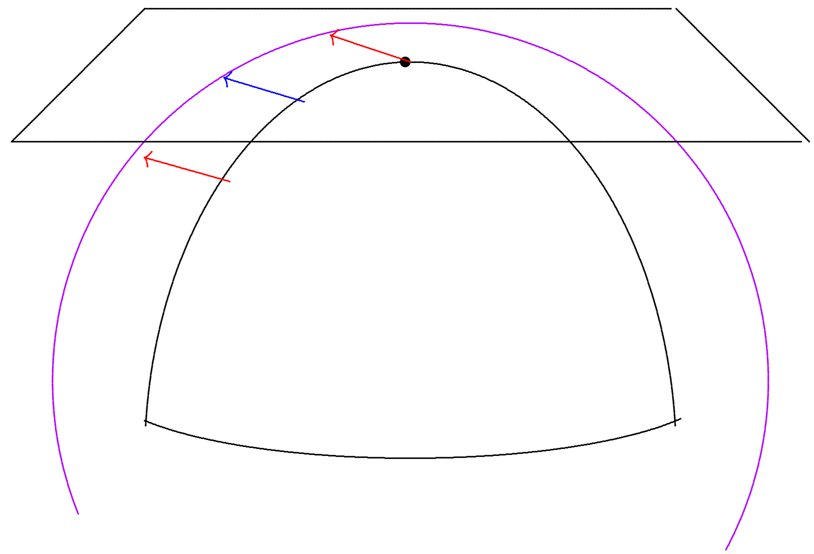

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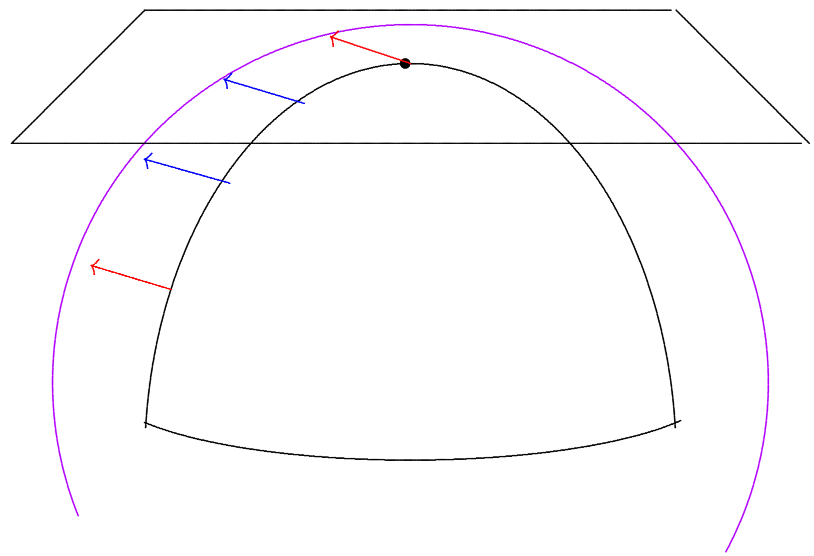

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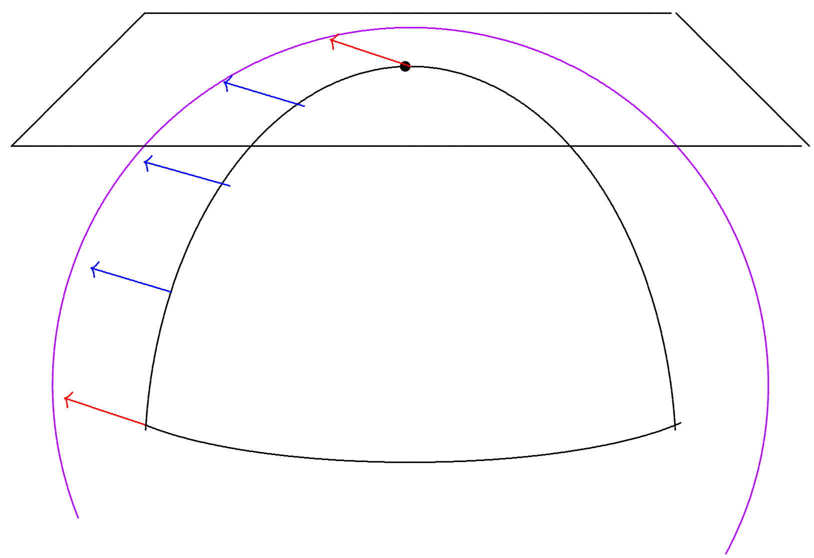

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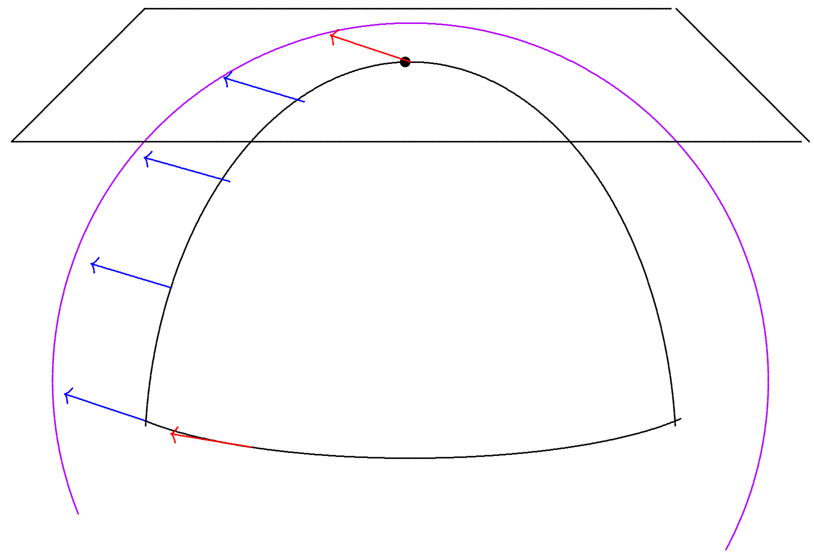

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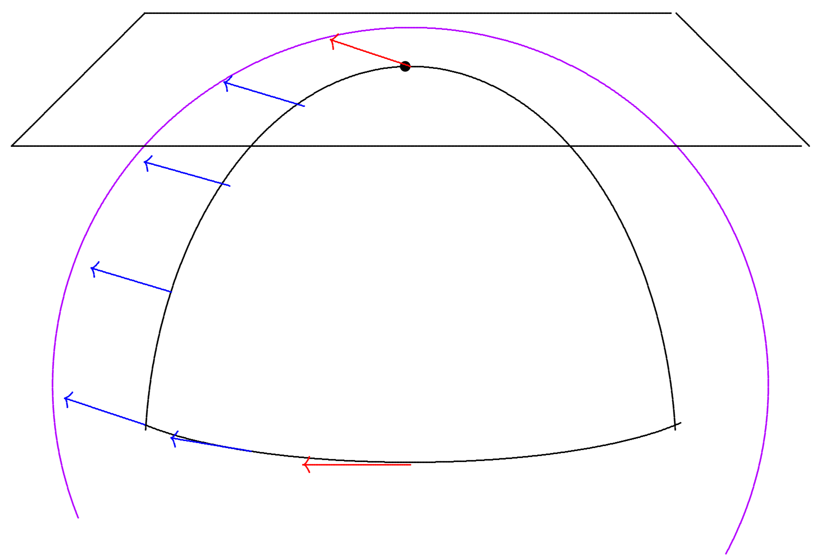

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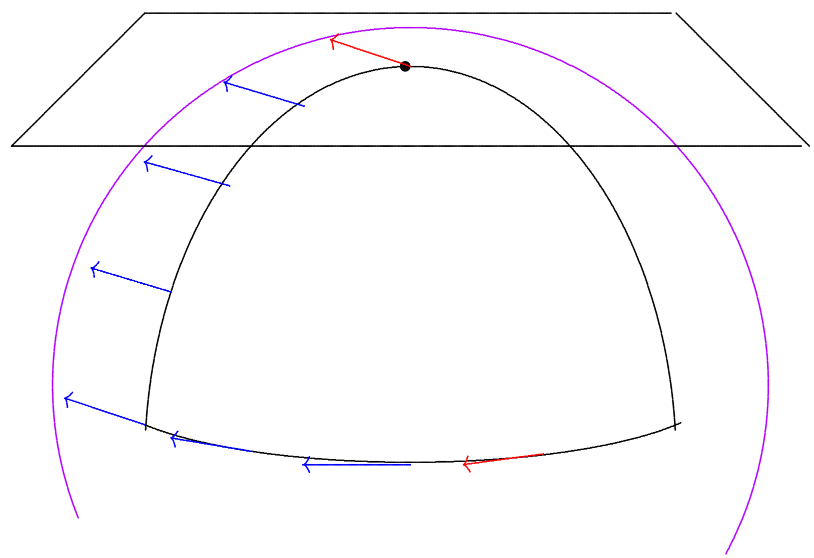

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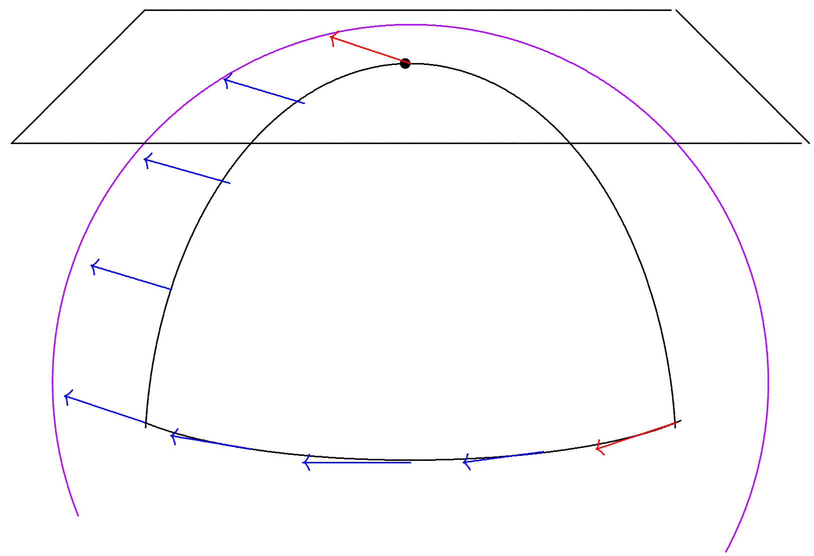

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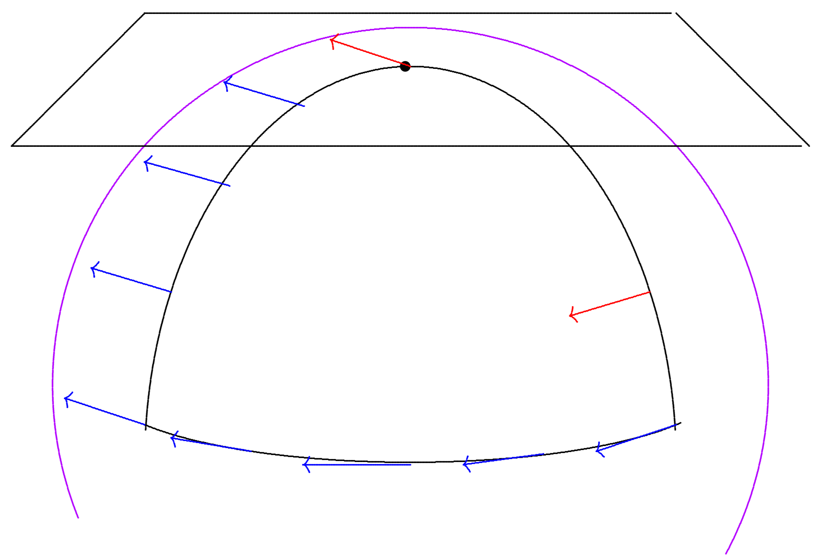

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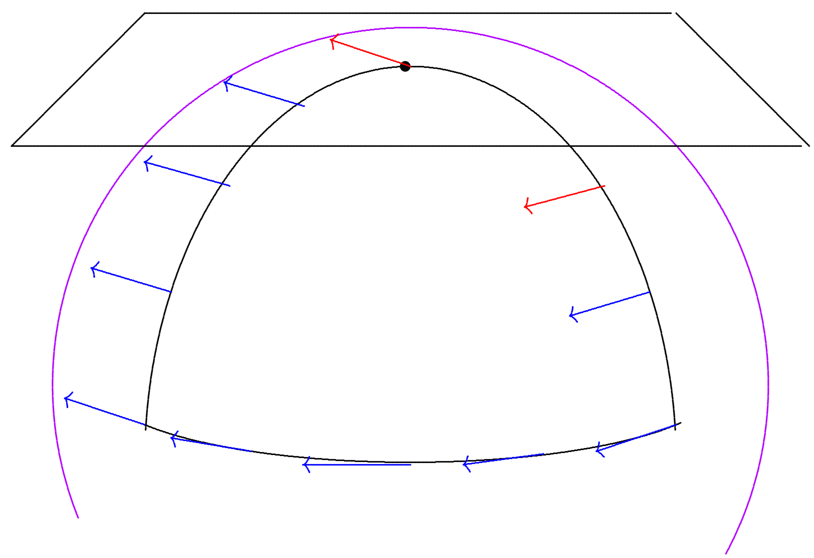

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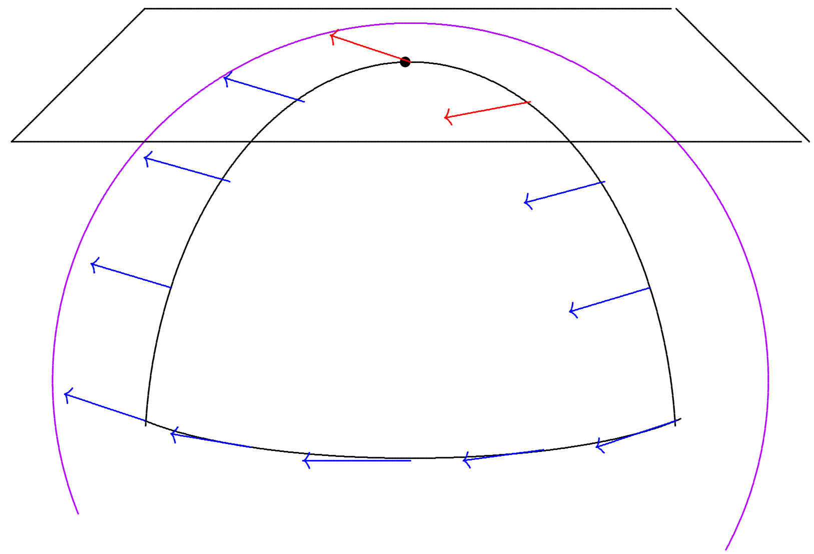

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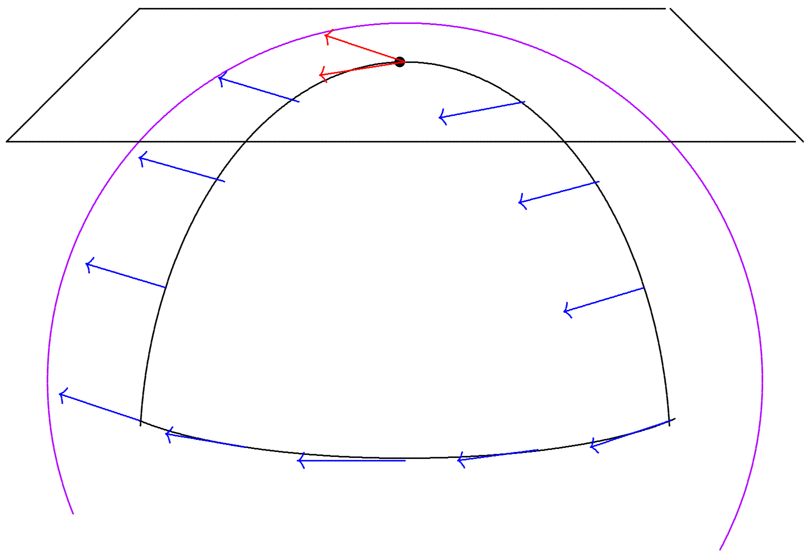

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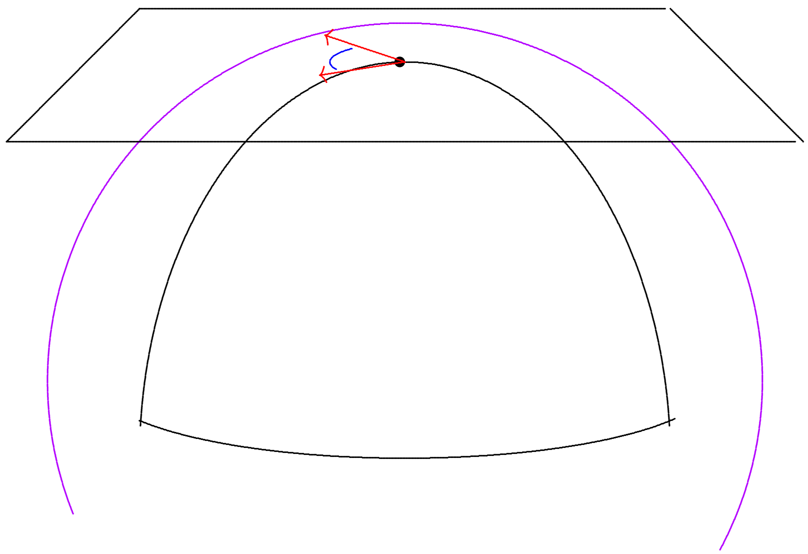
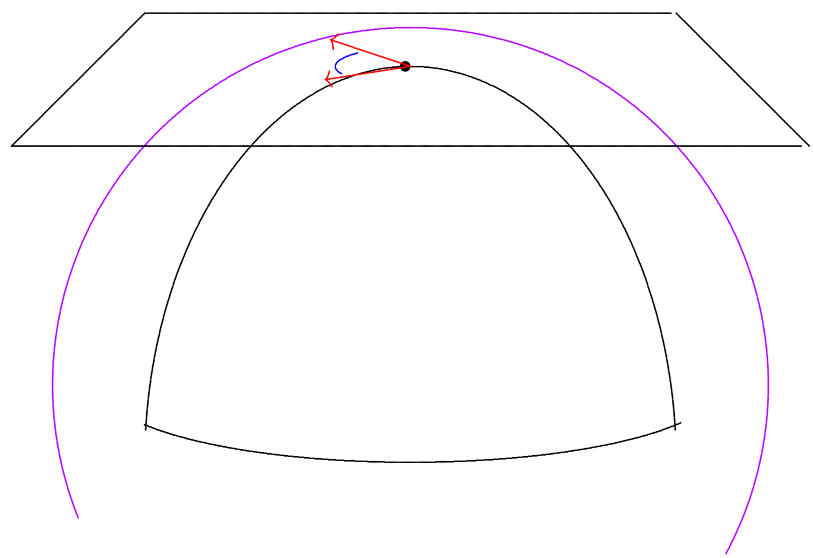
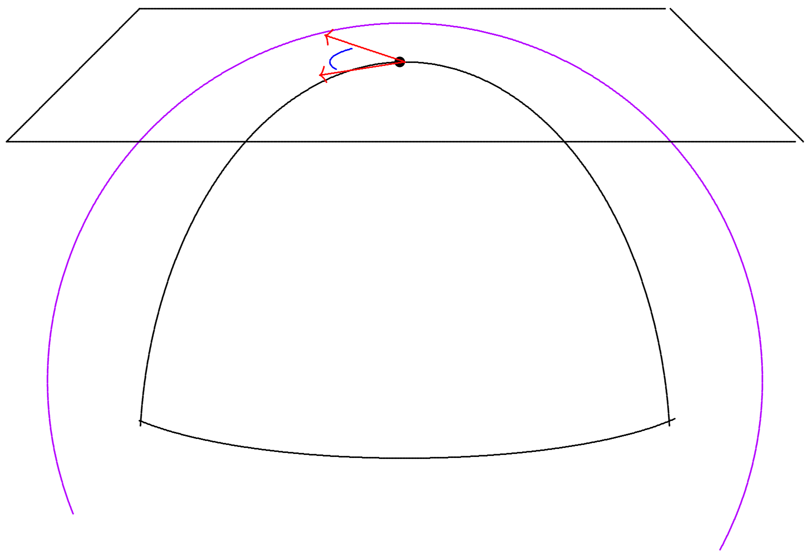


(M^n, g): holonomy $\subset \mathbf{O}(n)$



Kähler metrics:

(M^{2m}, g): holonomy



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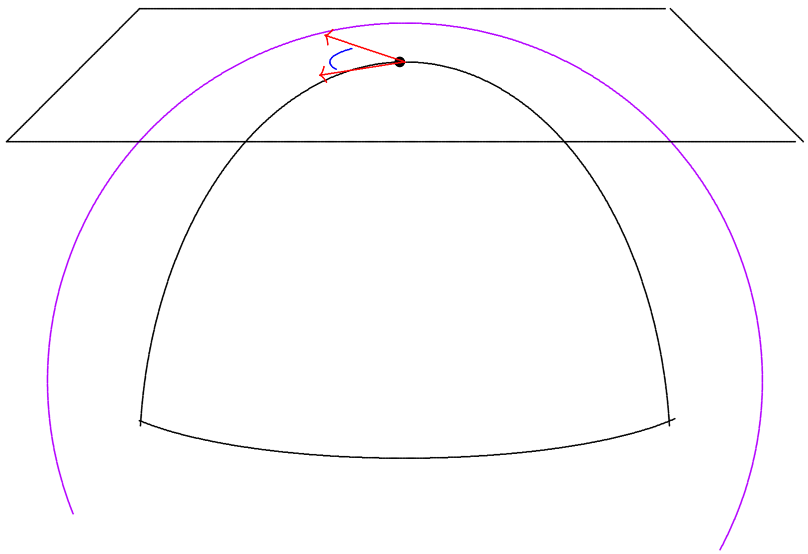

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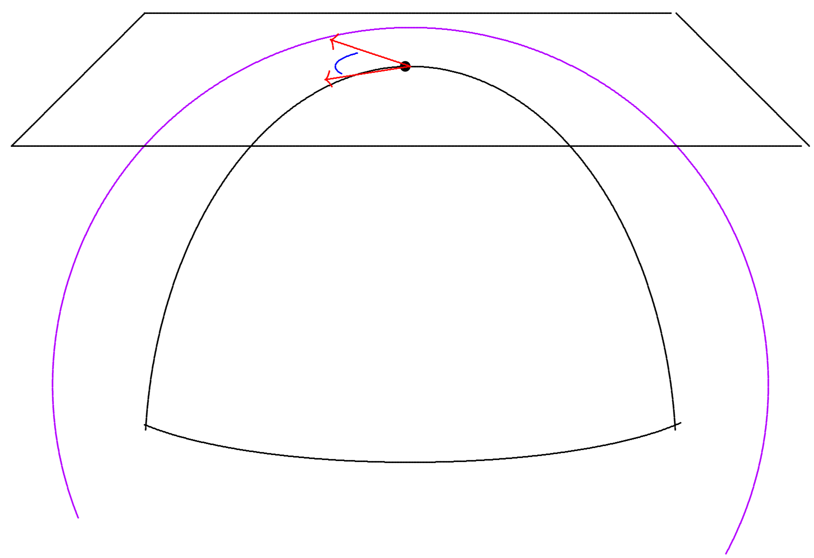

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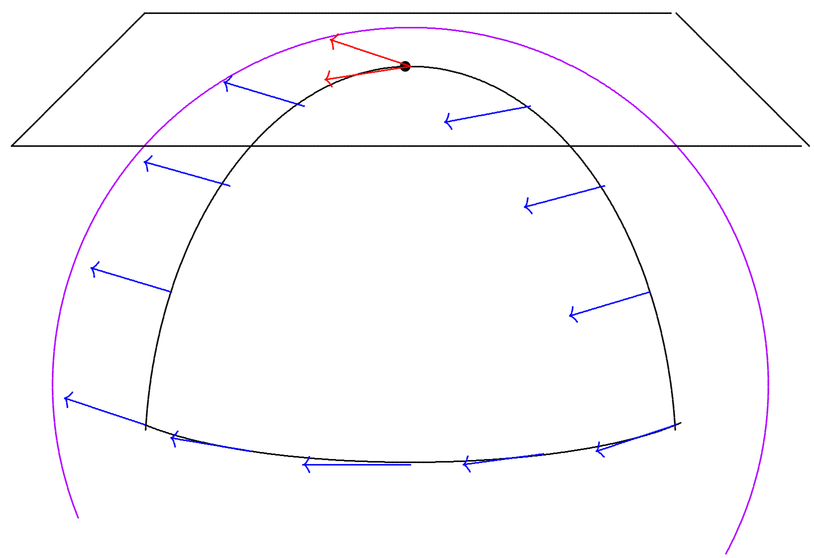

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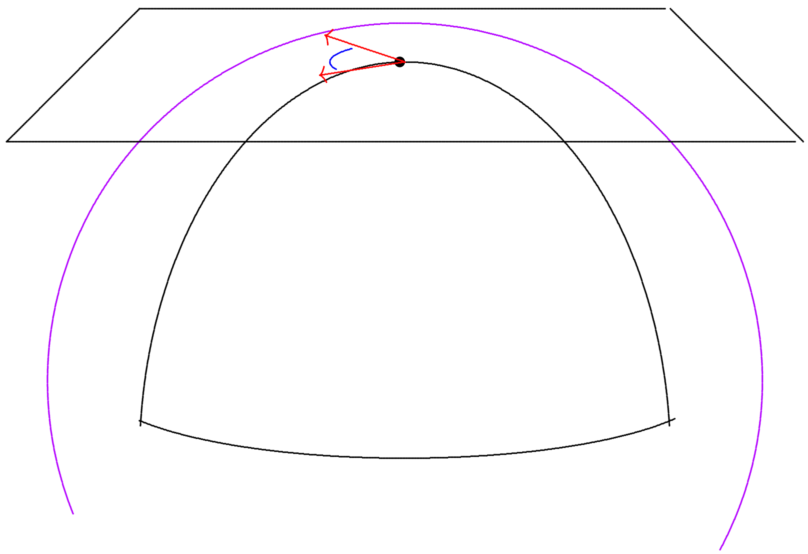

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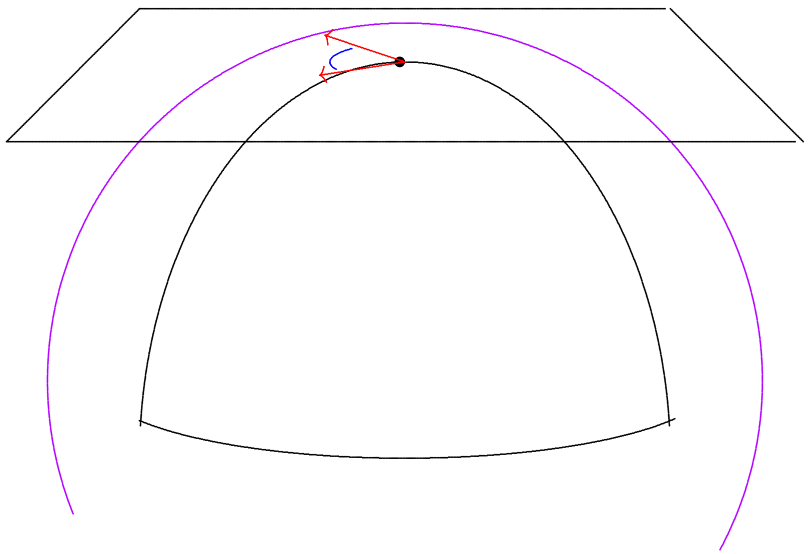
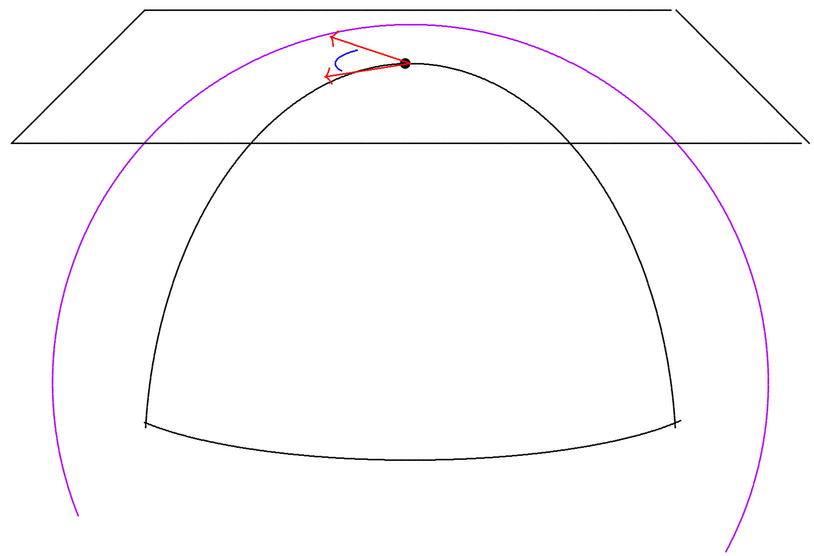


Hyper-Kähler metrics:

$(M^{4\ell}, g)$ Hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(\ell)$



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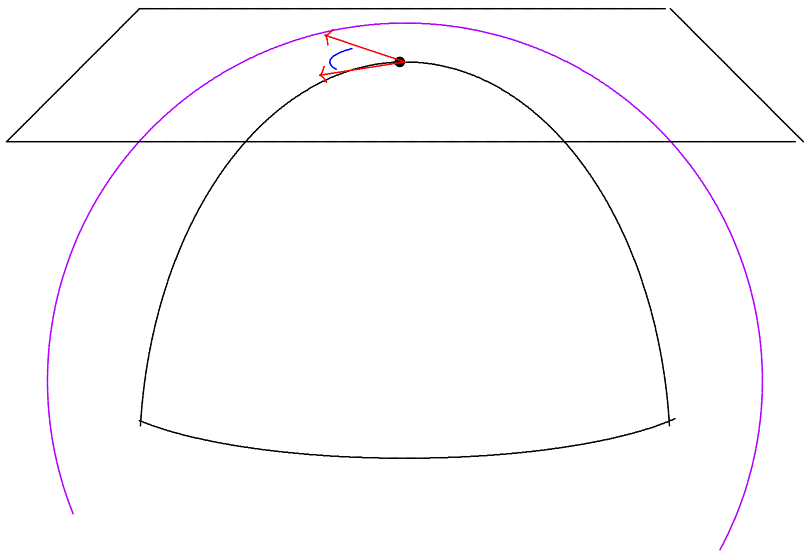

Kähler, for many different J's.

$\mathbf{Sp}(\ell) = \mathbf{O}(4\ell) \cap \mathbf{GL}(\ell, \mathbb{H})$

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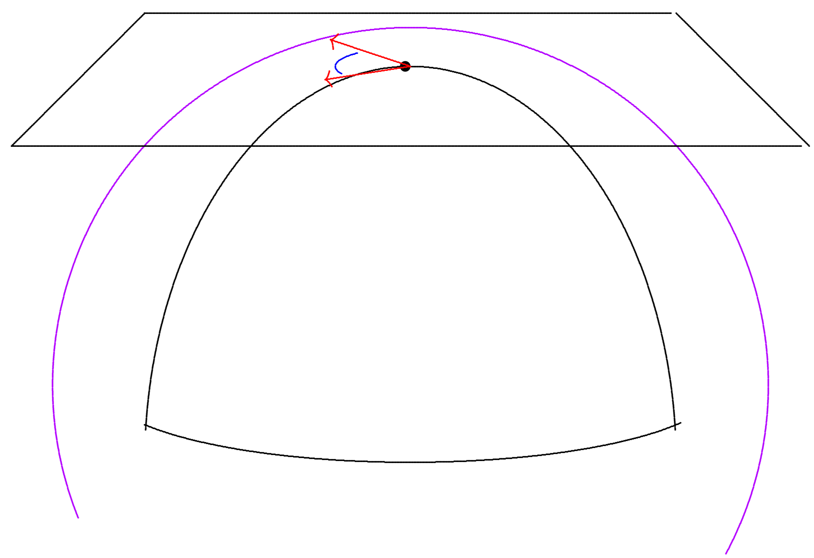
(M^4, g) Hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



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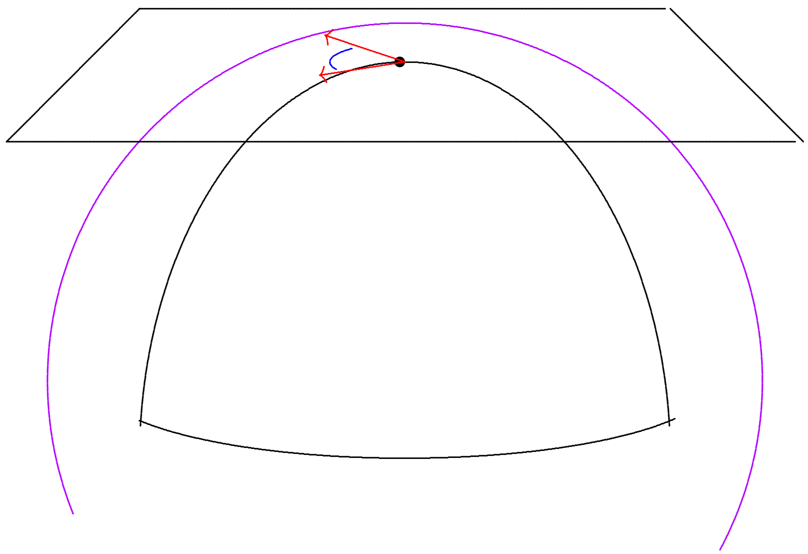

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$$0 \to \mathbb{Z}_2 \to \mathbf{SO}(4) \to \mathbf{SO}(3) \times \mathbf{SO}(3) \to 0$$

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Simply connected complex surface with $c_1 = 0$.

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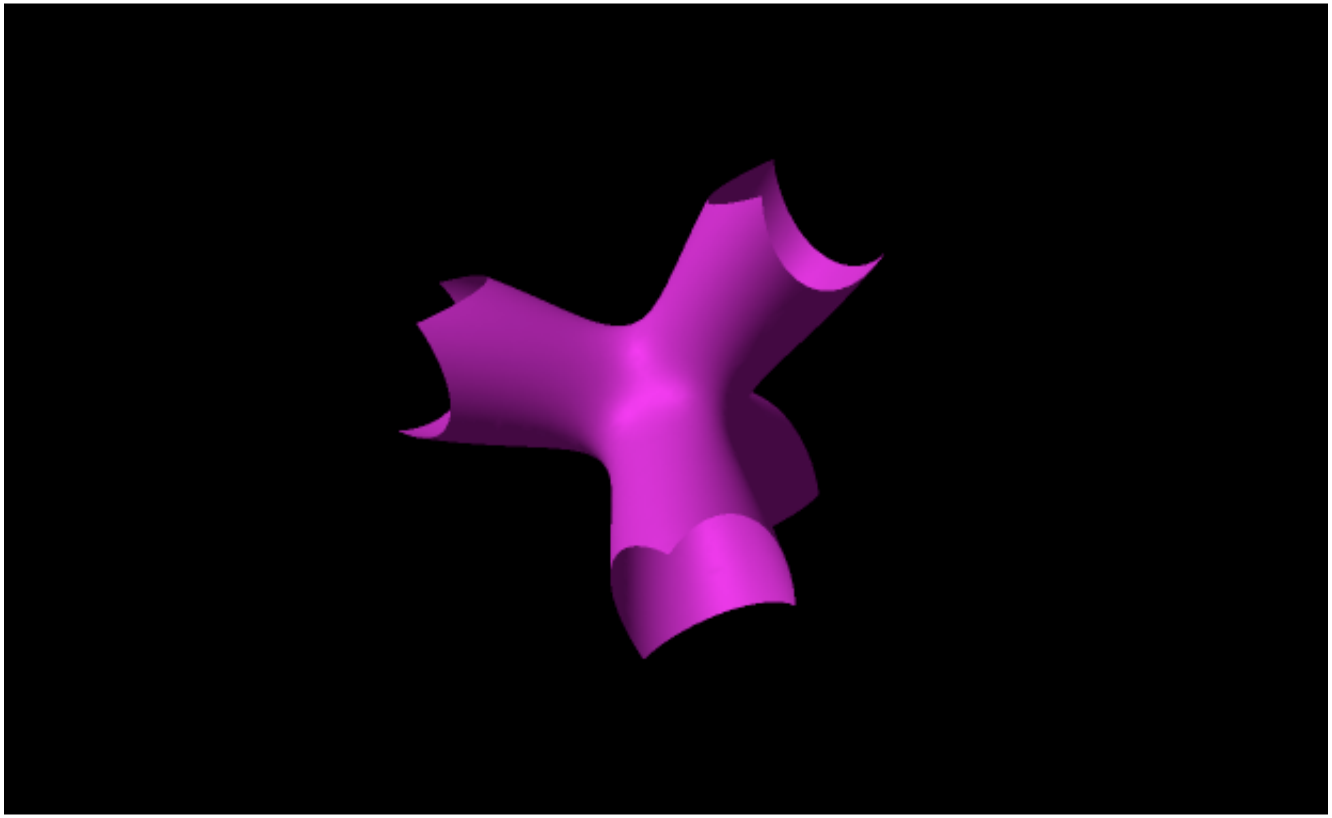
Diffeomorphic to quartic in $\mathbb{CP}_3$

$$x^4 + y^4 + z^4 + w^4 = 0$$



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Yau: Conversely, any K3 admits such metrics.

Berger's Inequality:

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

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Theorem (Berger Inequality). *If smooth compact M^4 admits Einstein g, then*

$$\chi(M) \geq 0,$$

with equality only if (M, g) flat, and finitely covered by $T^4 = \mathbb{R}^4/\Lambda$.

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Theorem (Hitchin-Thorpe Inequality). *If smooth compact oriented M^4 admits Einstein g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality only if (M, g) is locally hyper-Kähler. The latter case happens only if M finitely covered by flat T^4 or $K3$.

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Next lecture:

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Next lecture: Obtaining such estimates, using Seiberg-Witten theory.

End, Part I