The Einstein-Maxwell Equations

and

Conformally Kähler Geometry

Claude LeBrun
Stony Brook University

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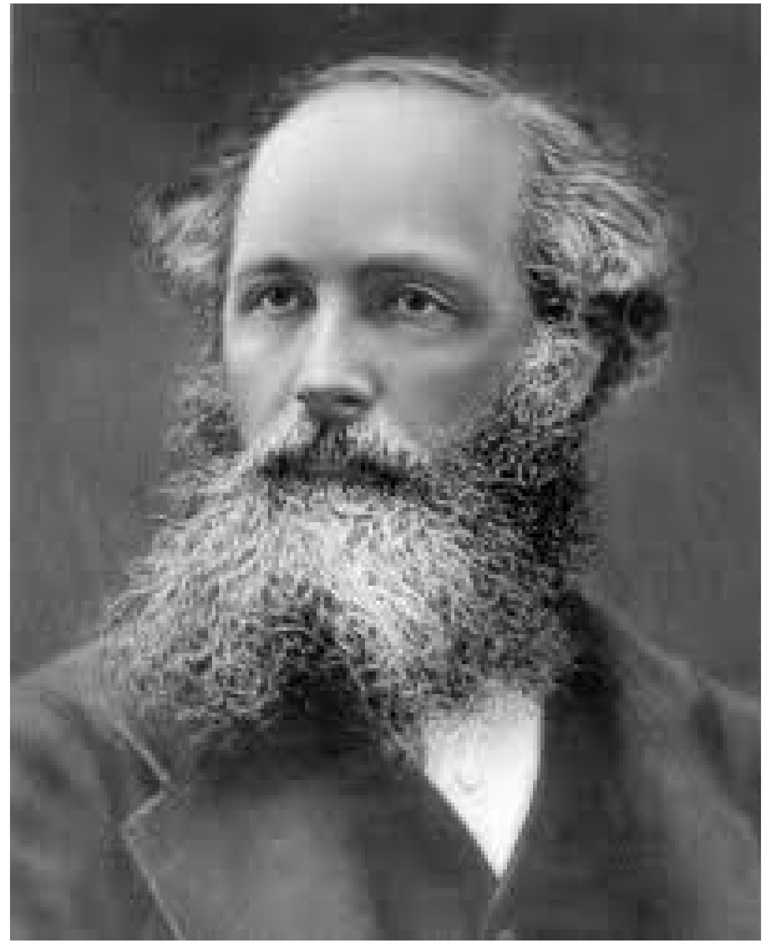

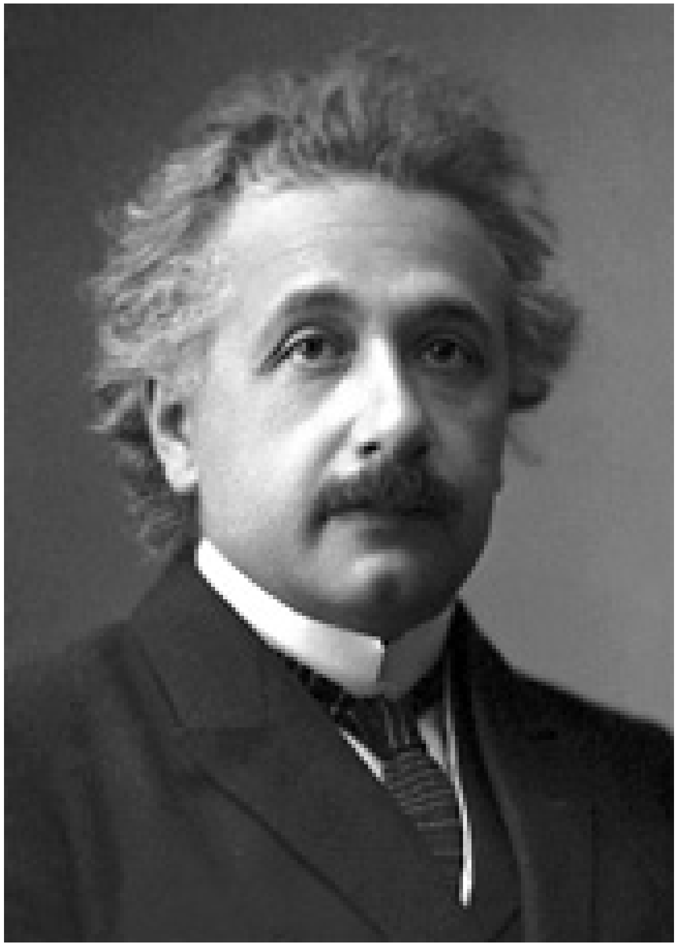

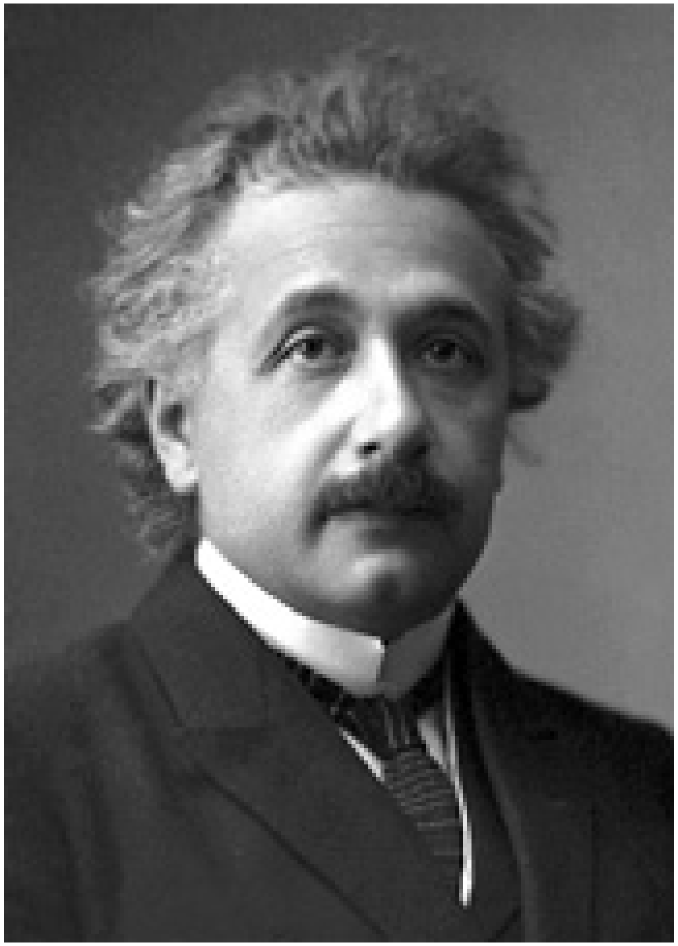
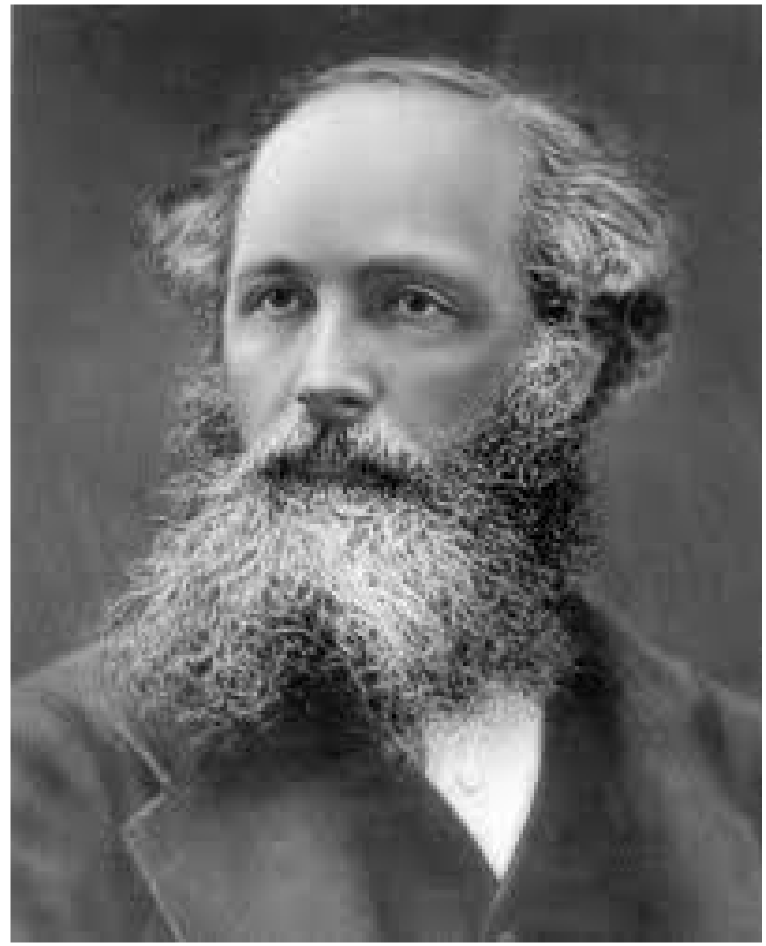

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Purely 4-dimensional phenomenon.

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Henceforth, assume M compact.

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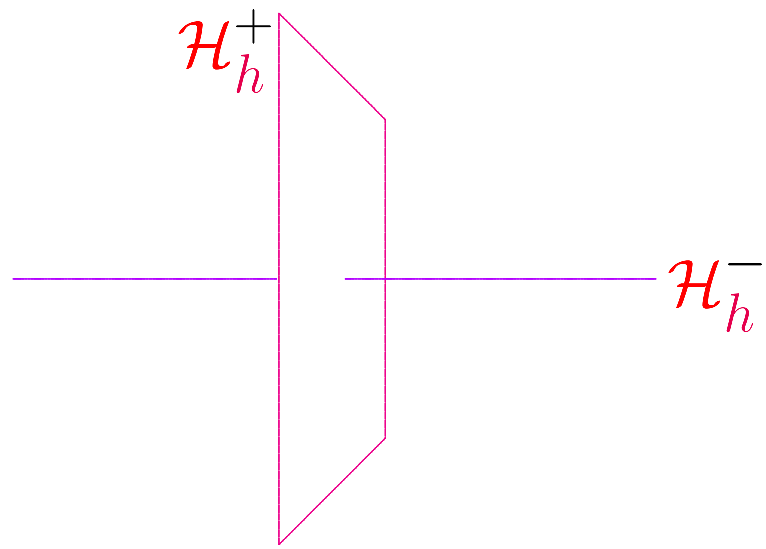
Decomposition is conformally invariant.

The numbers

$$b_\pm(M) = \dim \mathcal{H}^\pm_h.$$

are important homotopy invariants of M.



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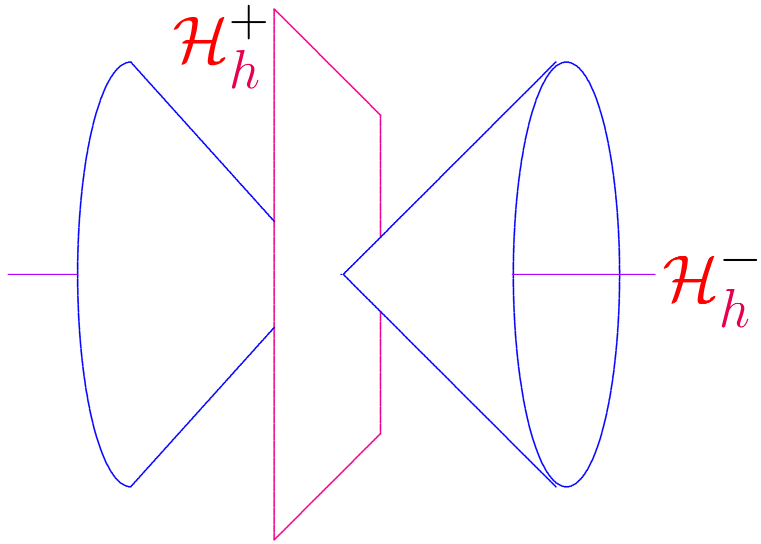

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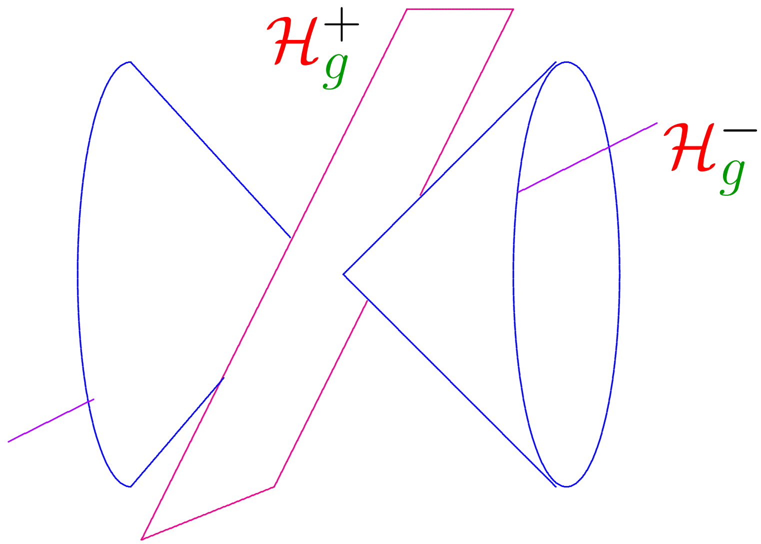

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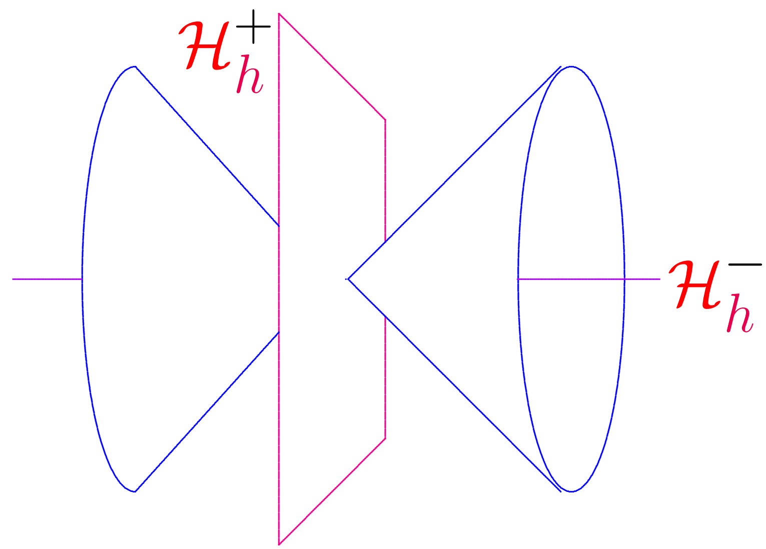

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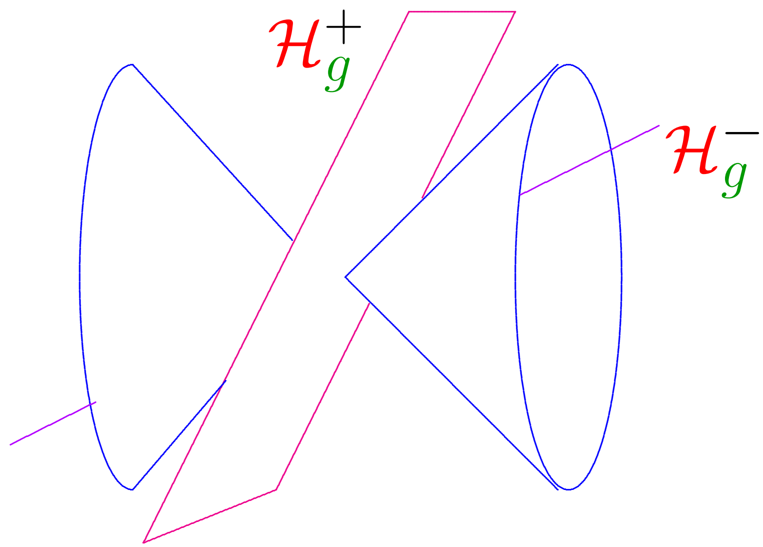

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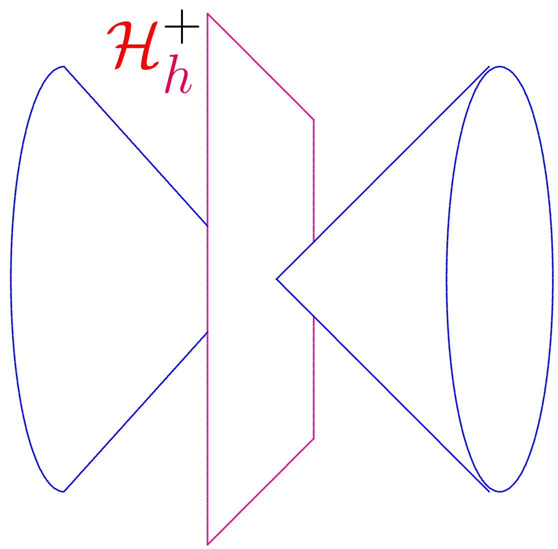

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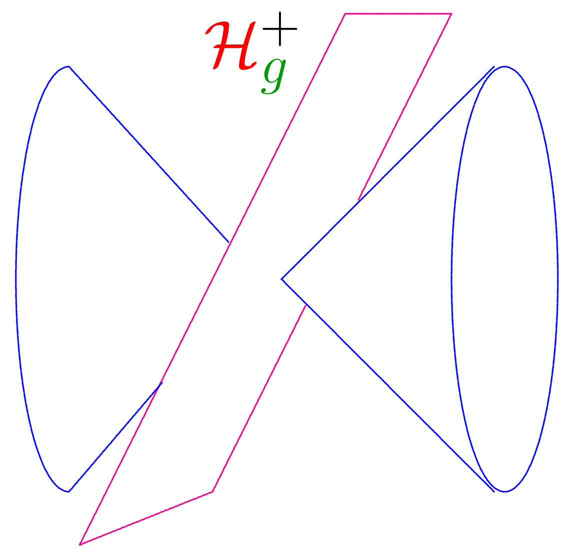

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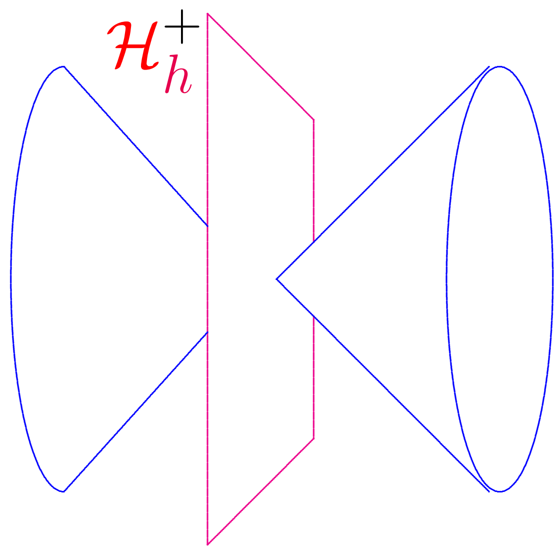

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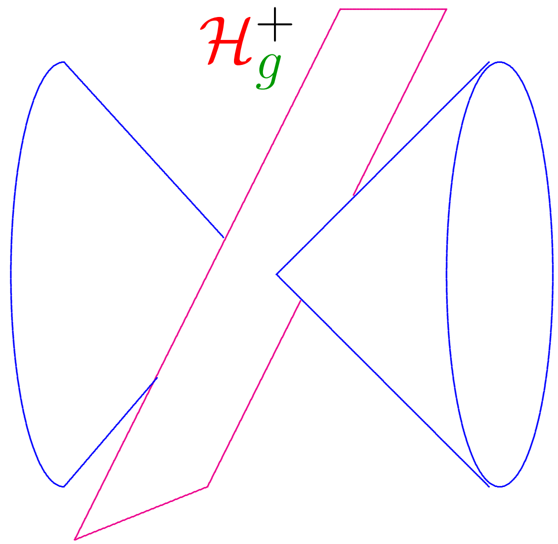

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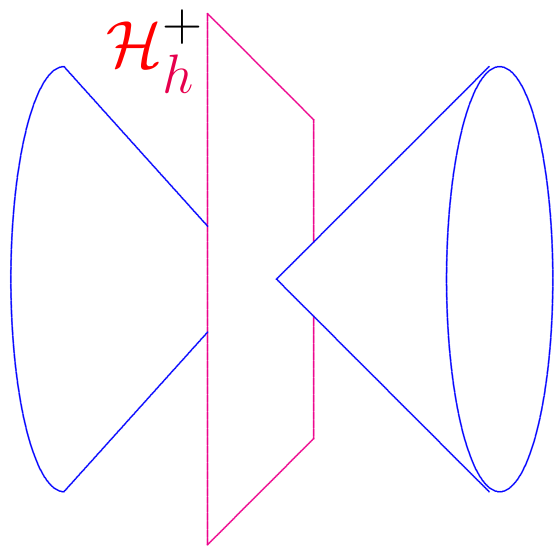

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Remark Notice, however, that

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for any $\lambda \in \mathbb{R}^\times$. Moreover, $\mathcal{G}_\Omega$ invariant under $\mathrm{Diff}_0(M)$ and conformal rescalings.

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Einstein-Hilbert action functional

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Previously saw…

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Let (M^4, h, J) be cscK:

Kähler surface with

$$\boxed{s = \text{constant.}}$$

Set

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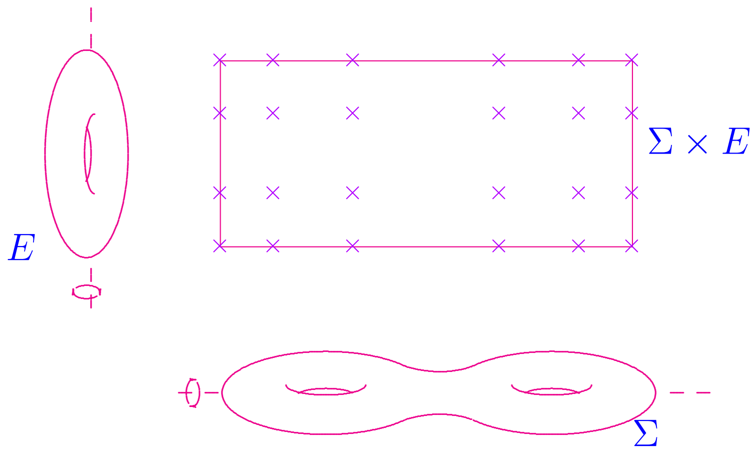
Typical example of Kodaira dimension 1:



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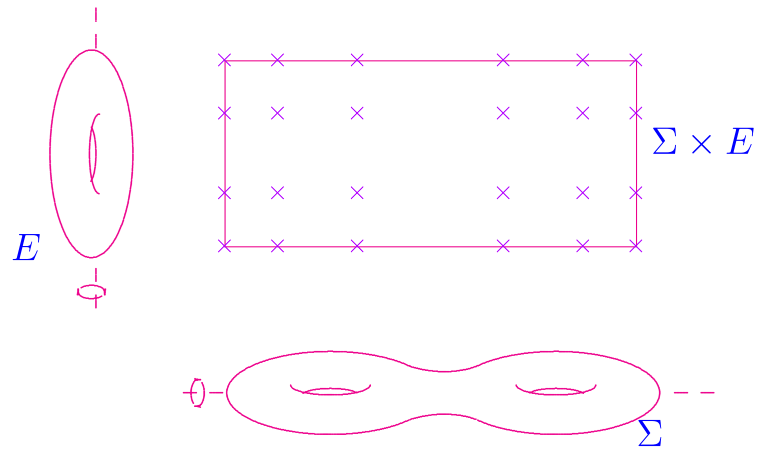

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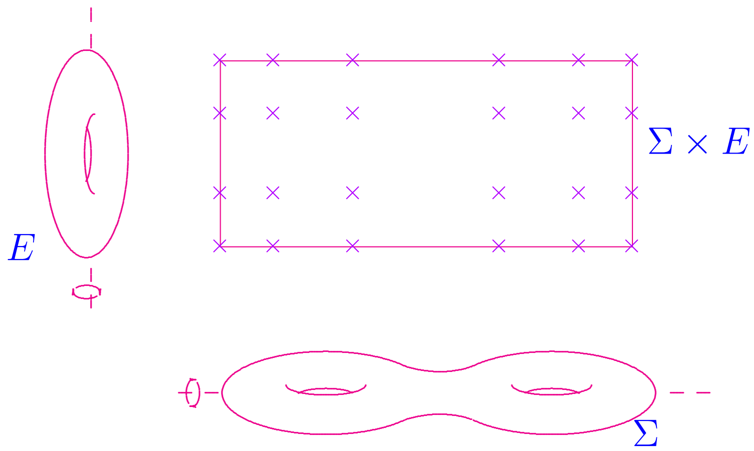

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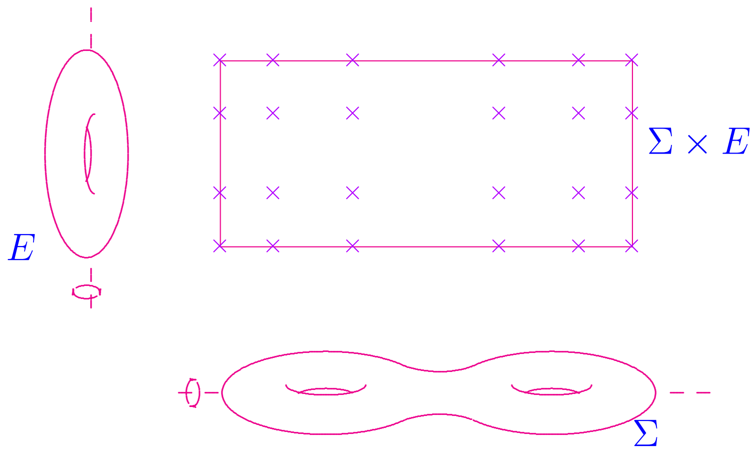

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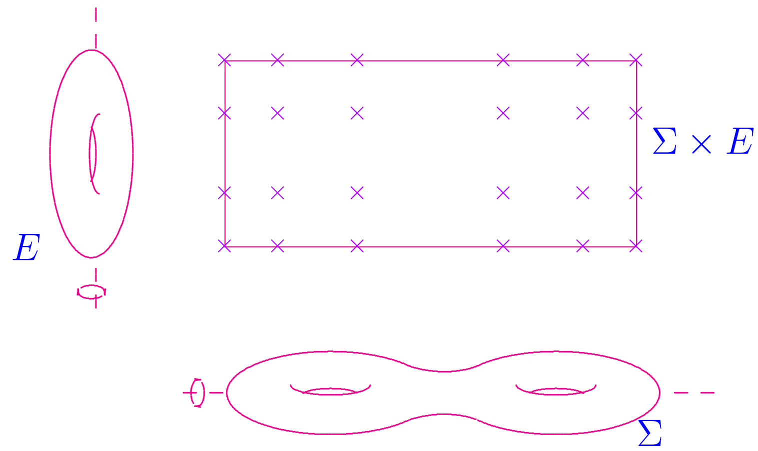

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Systematic study: Yujen Shu's thesis.

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We will show this using yet other Kählerian ideas.

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Constructions & Proofs

Prototype:

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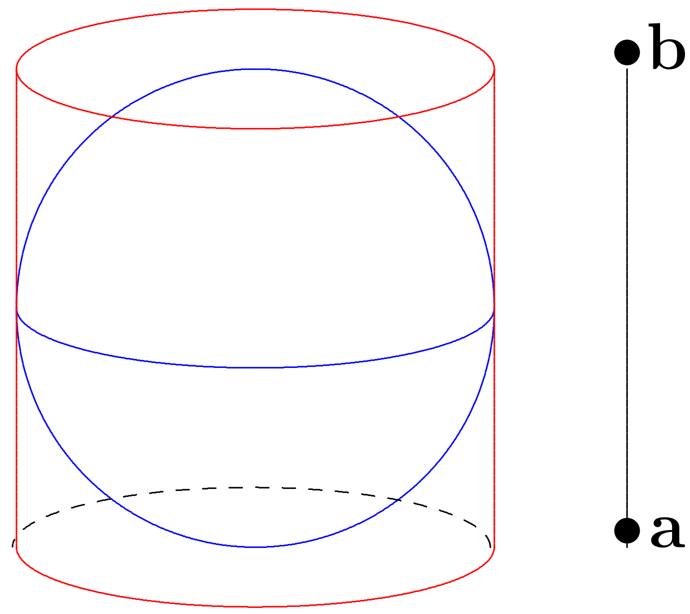

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$$\Longrightarrow \quad \Phi(t) = At^4 + Bt^3 + \frac{\mathbf{c}}{2}t^2 - \frac{\mathbf{d}}{12}$$

Global solution:

$$\Phi(\mathbf{a}) = \Phi(\mathbf{b}) = 0, \quad \Phi'(\mathbf{a}) = -\Phi'(\mathbf{b}) = 2, \quad \Phi'(0) = 0.$$

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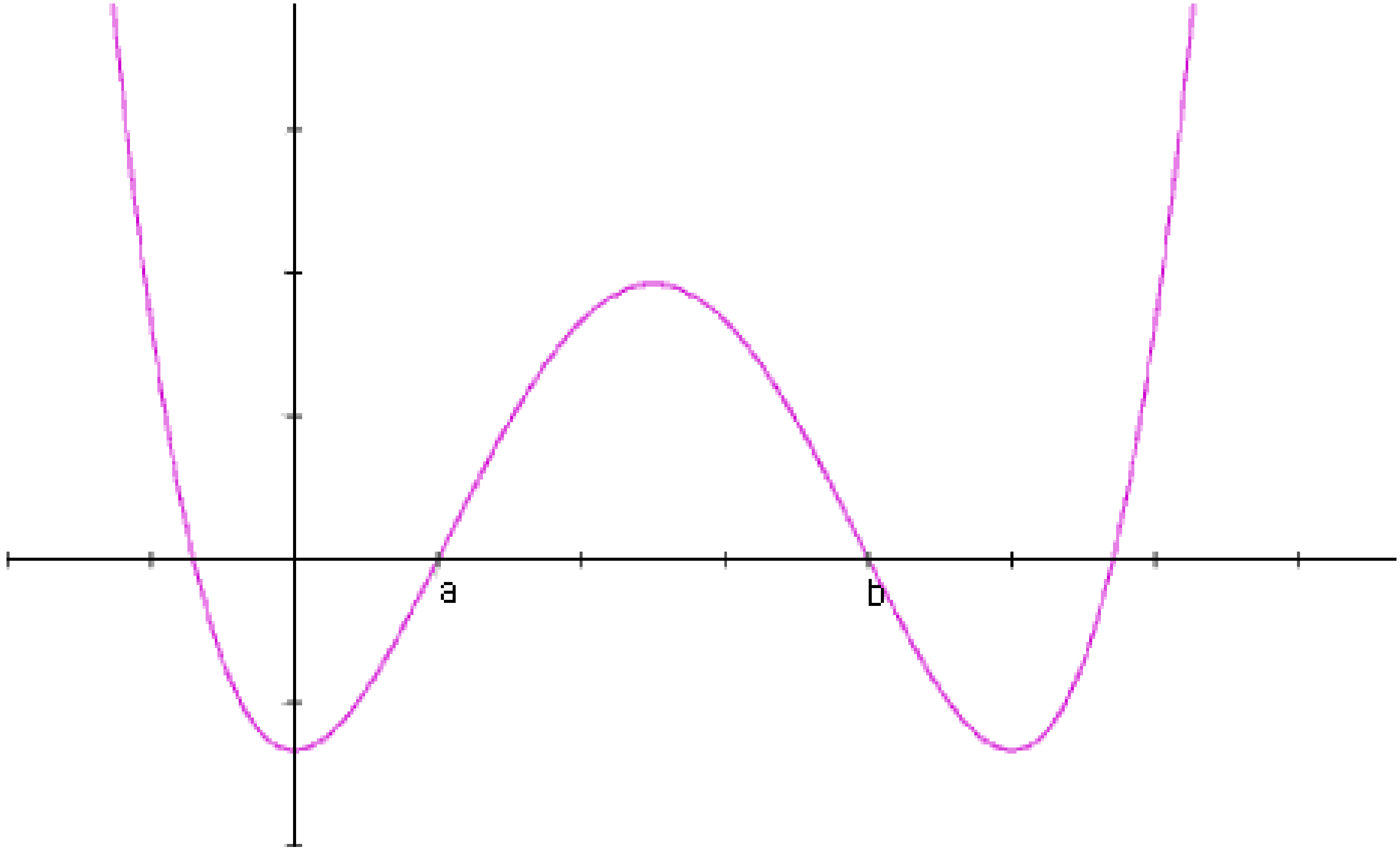

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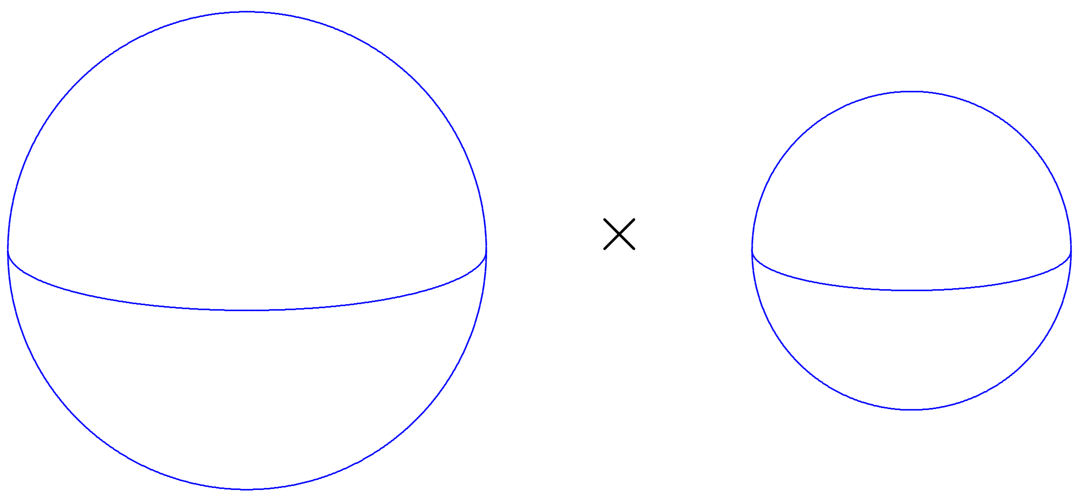

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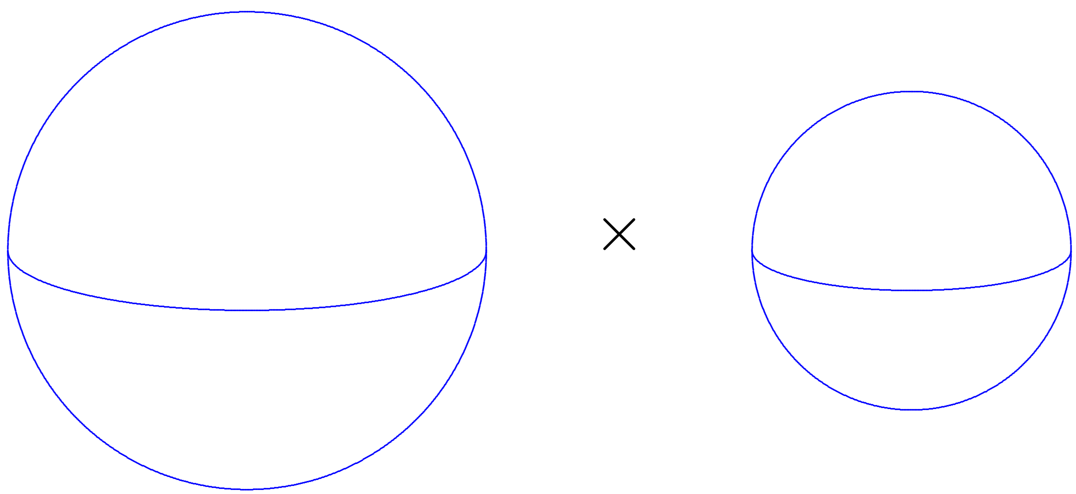

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Are any of these metrics Yamabe?

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Ann. Glob. An. Geom. 50 (2016) 29–46.

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  Futaki-Ono 2017: Variational approach.

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  Apostolov-Maschler 2016:

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  Also for higher-dimensional problem:

  $h = f^{-2}g$ conformally Kähler,

  $s =$ constant, $\qquad r(J\cdot, J\cdot) = r$.

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  cKEM: "conformally Kähler Einstein-Maxwell."

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  Apostolov-Maschler 2016:
  Generalized Futaki invariant.
  Also for higher-dimensional problem:
  cKEM: "conformally Kähler Einstein-Maxwell."
  Although only Einstein-Maxwell in dimension 4!

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  Futaki-Ono 2017:

  Reinterpreted Apostolov-Maschler in terms of constrained Einstein-Hilbert variational problem.

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- General 4-manifolds?