Mass in

Kähler Geometry

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Stony Brook University

New Horizons in Twistor Theory
Oxford, January 5, 2017

Joint work with

Joint work with

Hans-Joachim Hein
University of Maryland

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Fordham University

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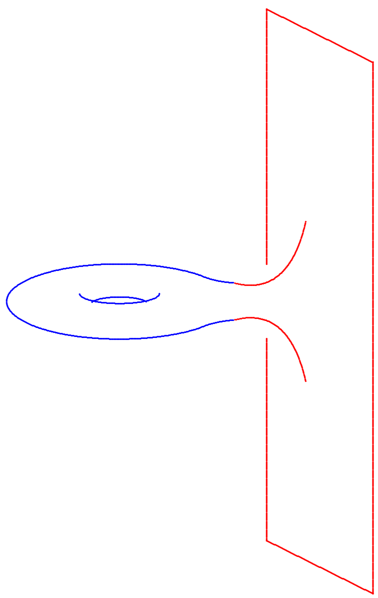
Comm. Math. Phys. 347 (2016) 621–653.

Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g)



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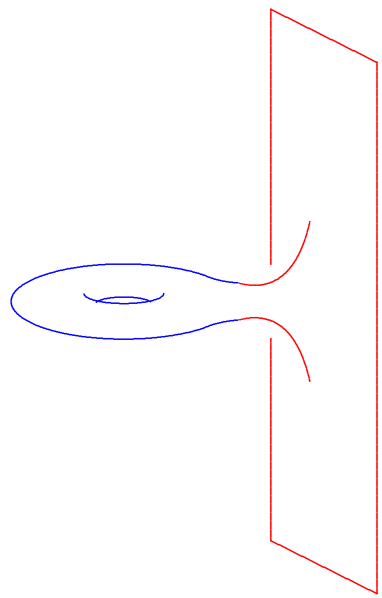

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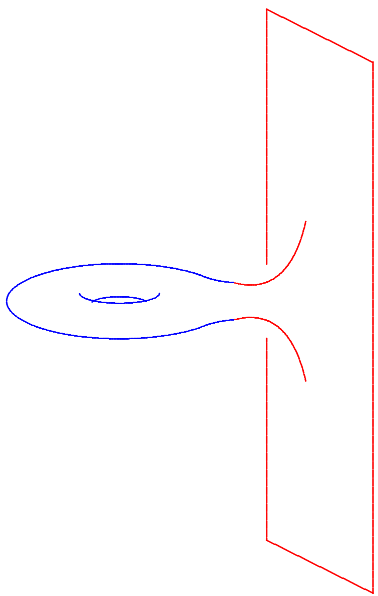

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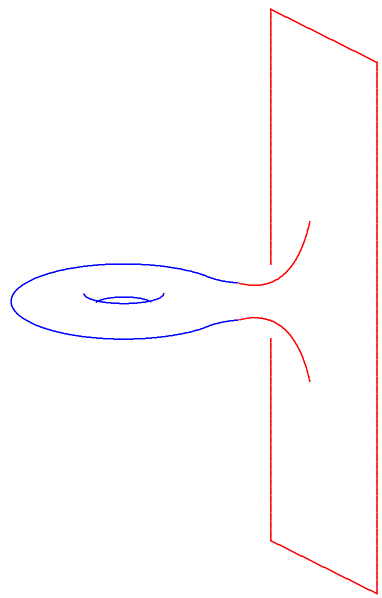

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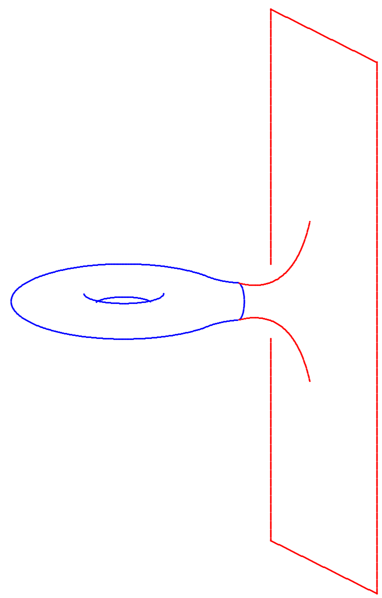

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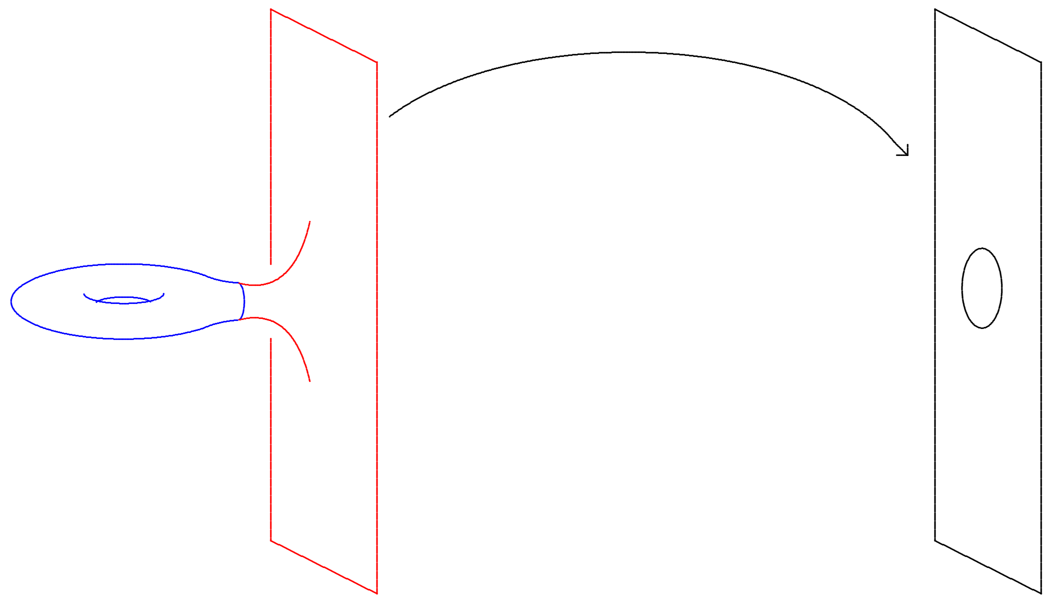

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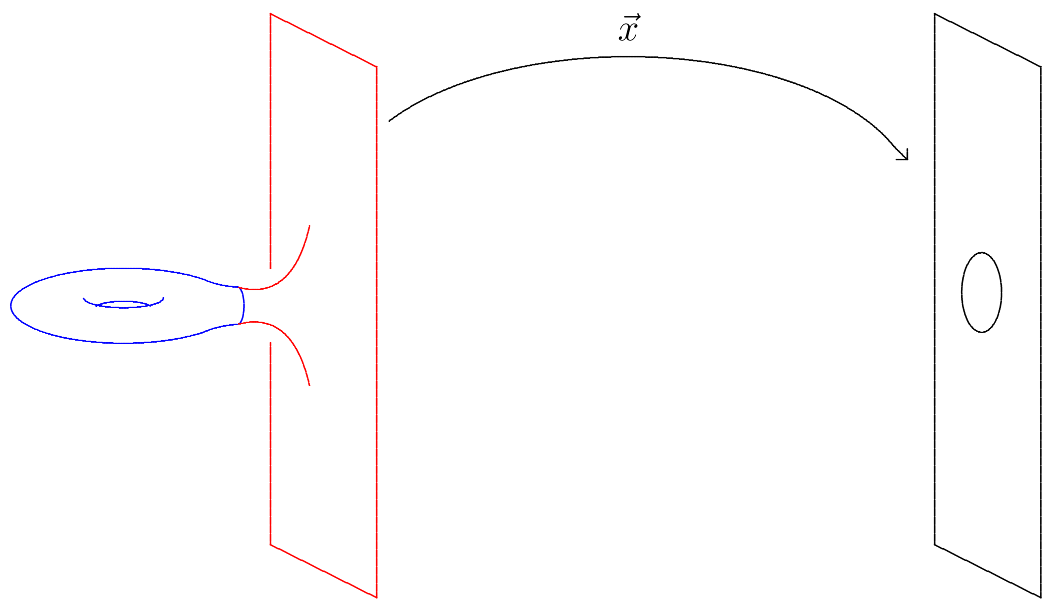

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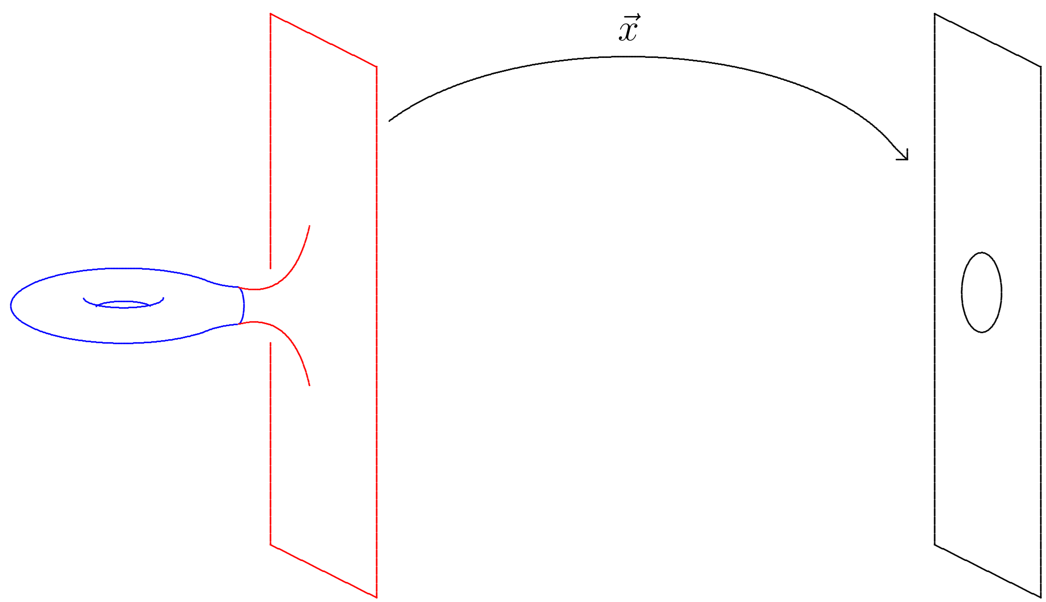

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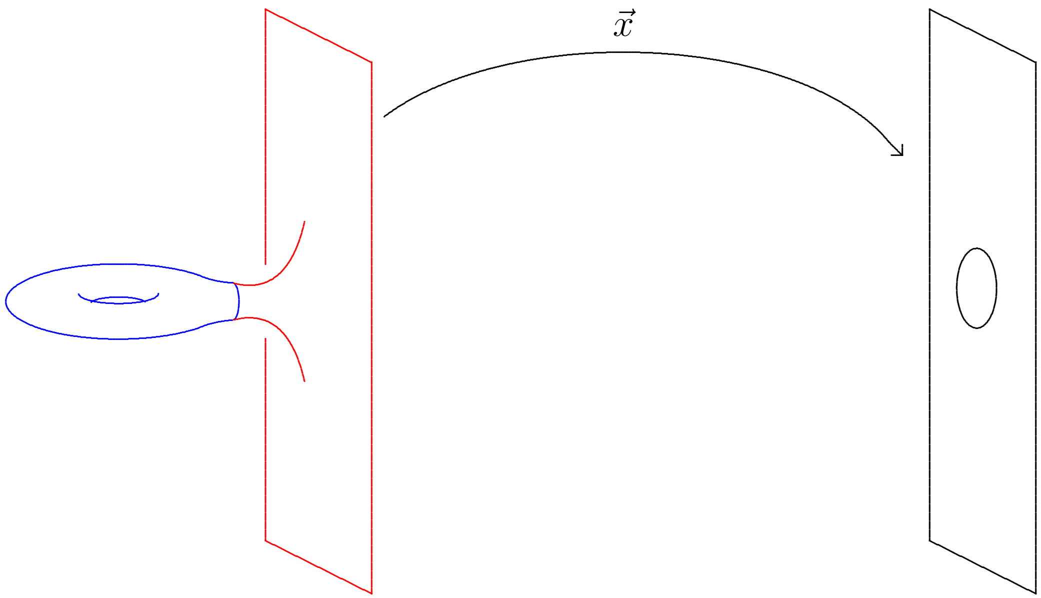

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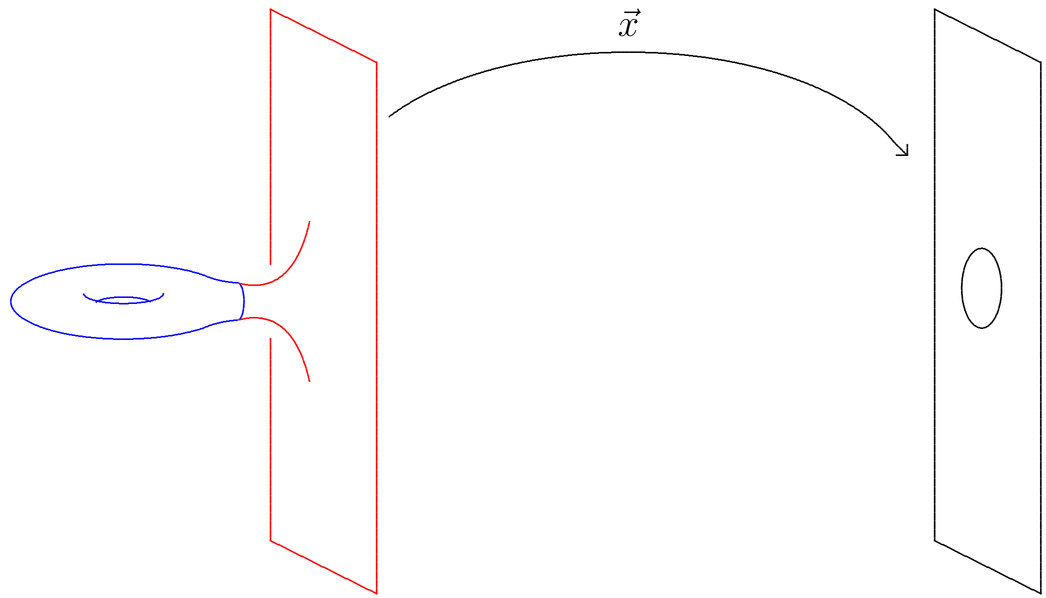

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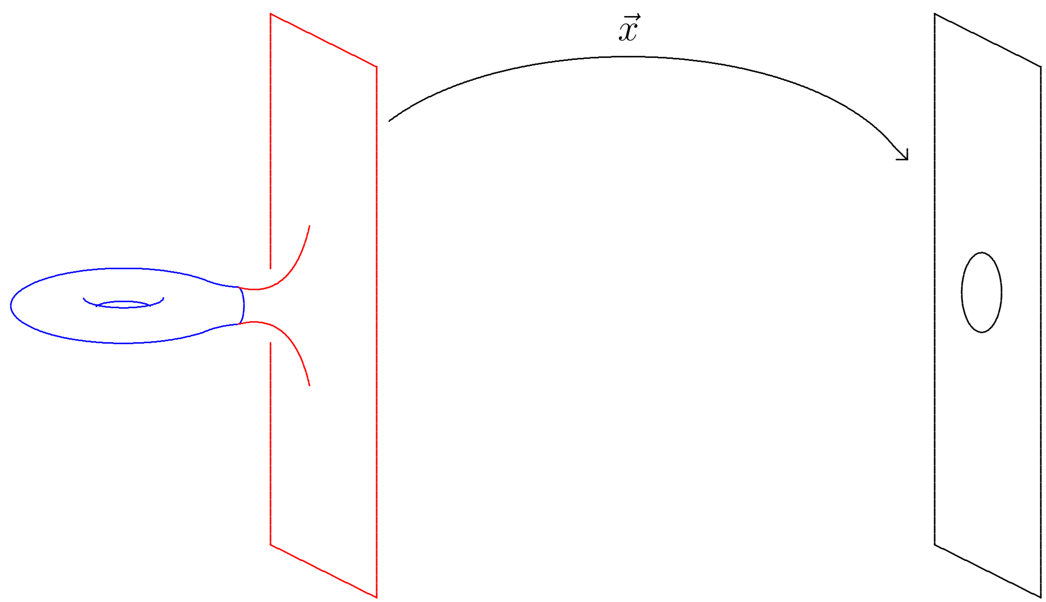

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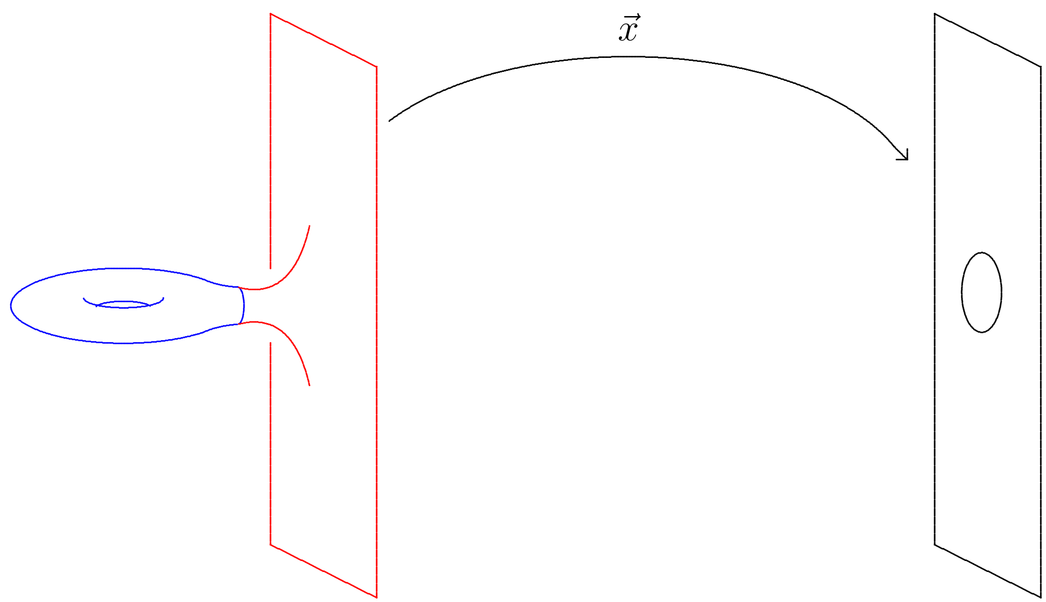

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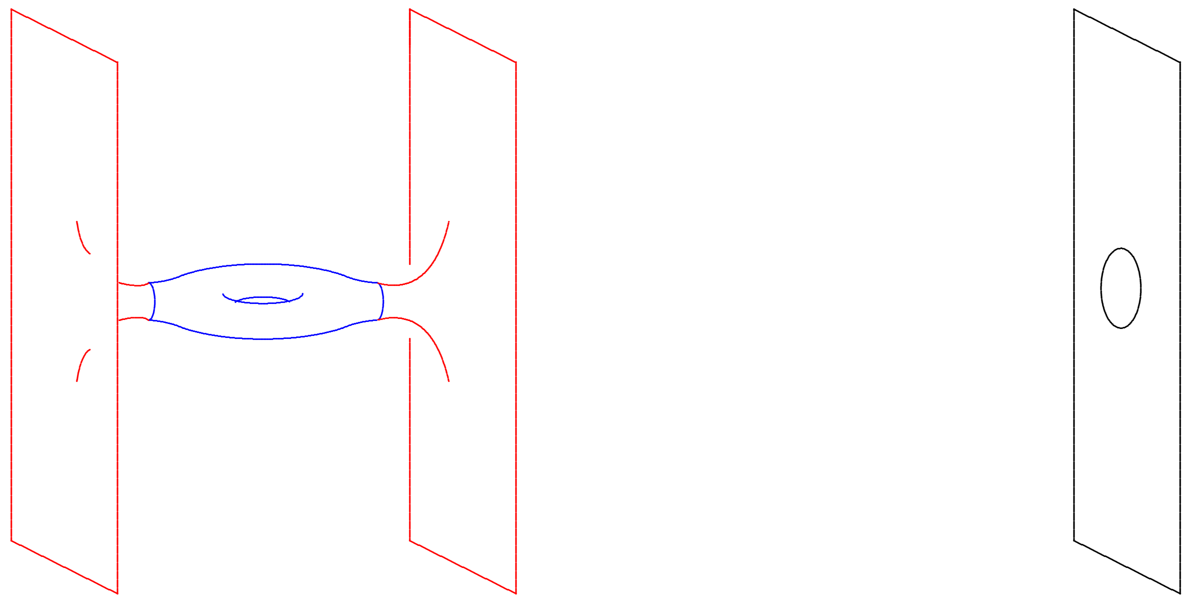

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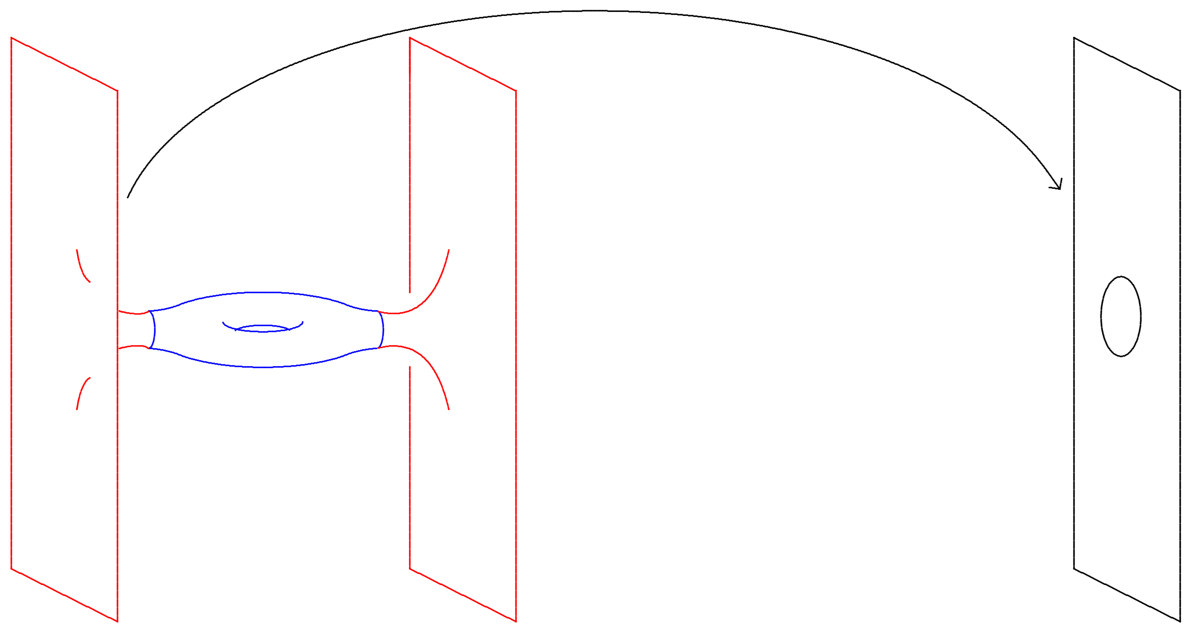

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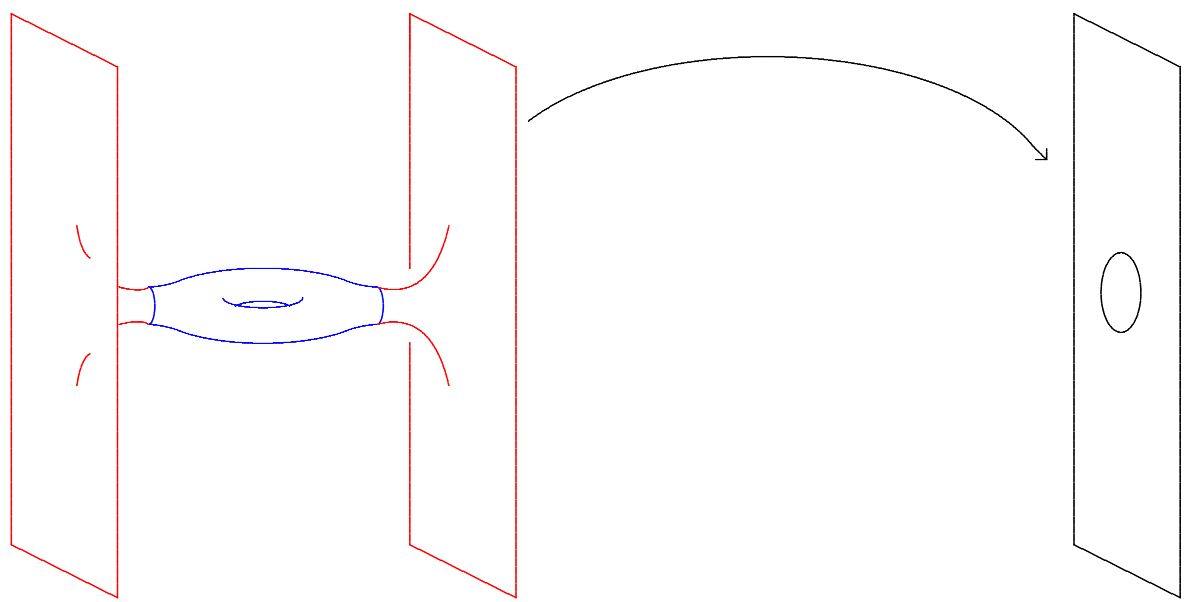

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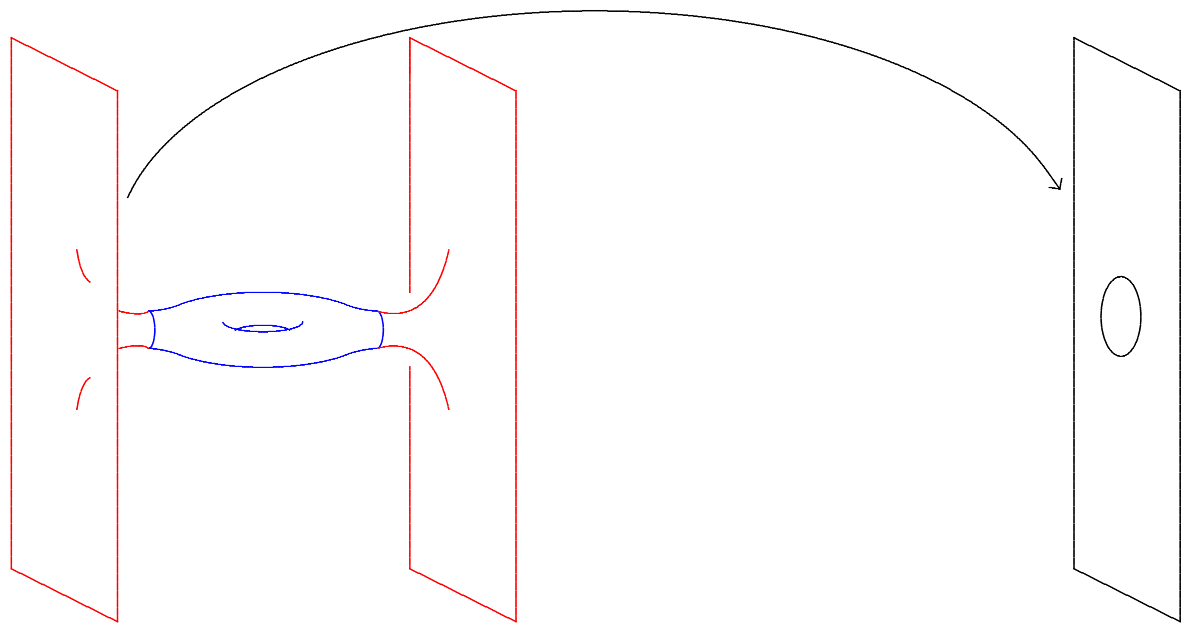

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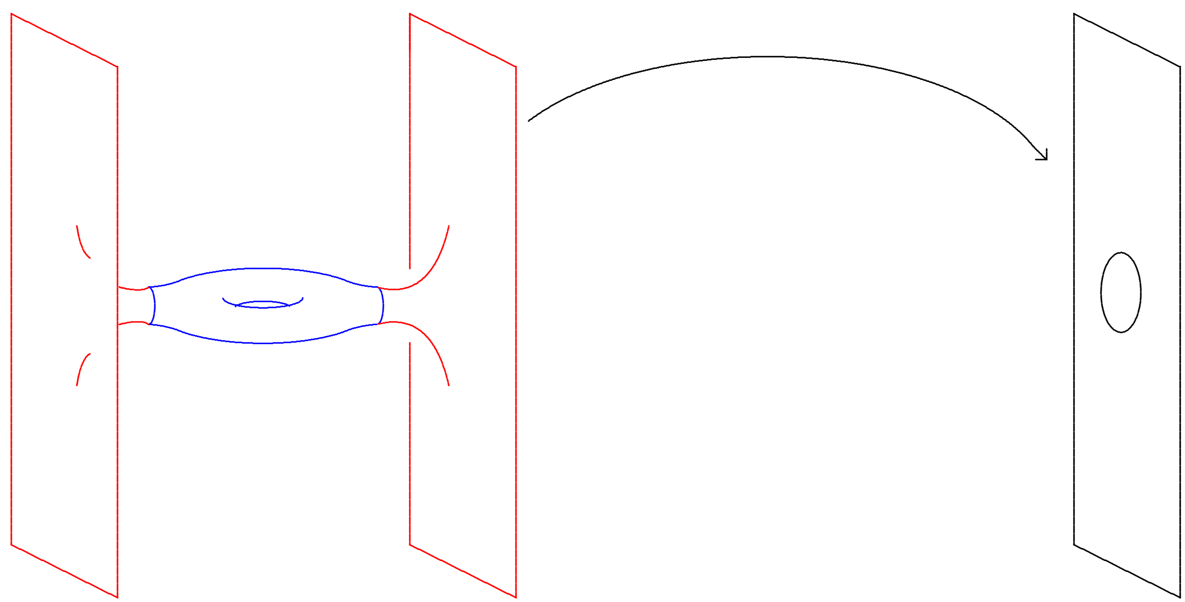

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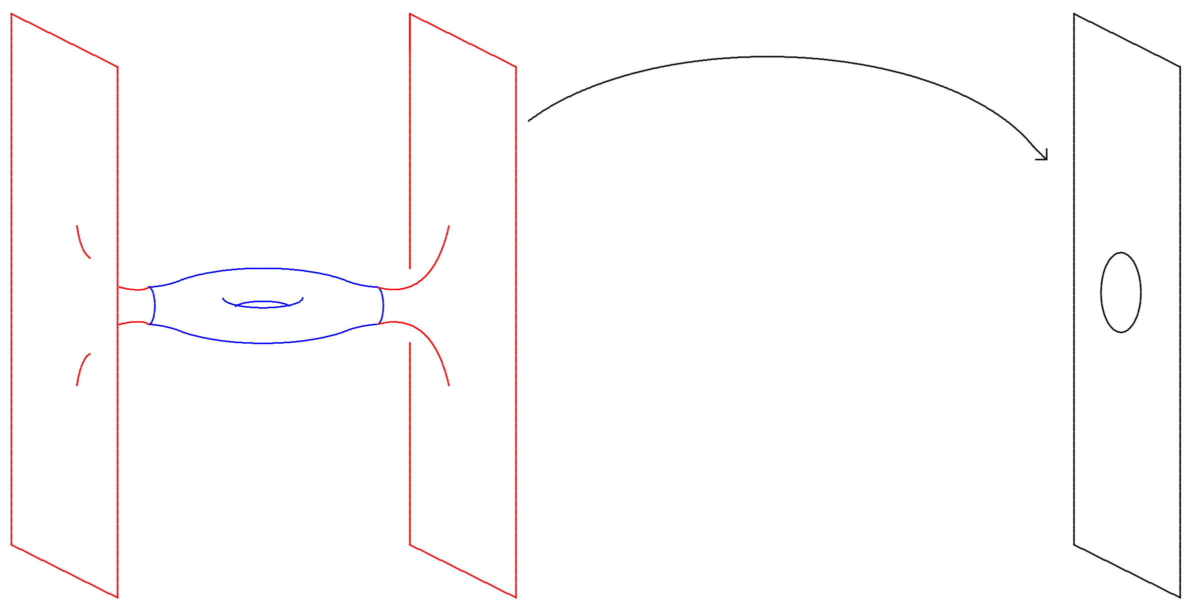

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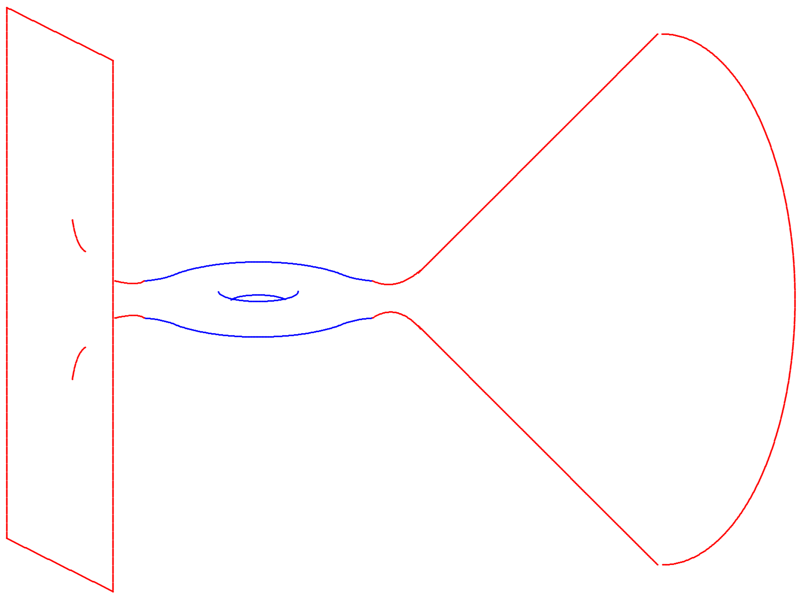

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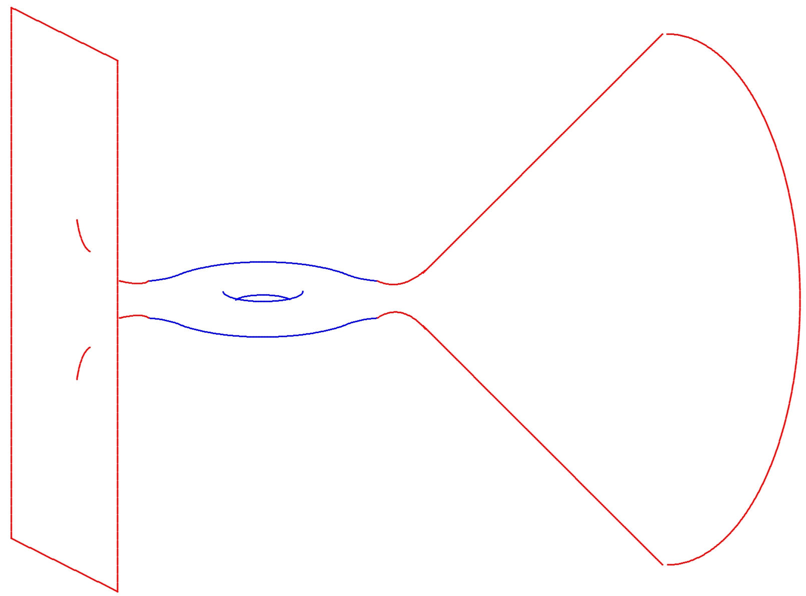

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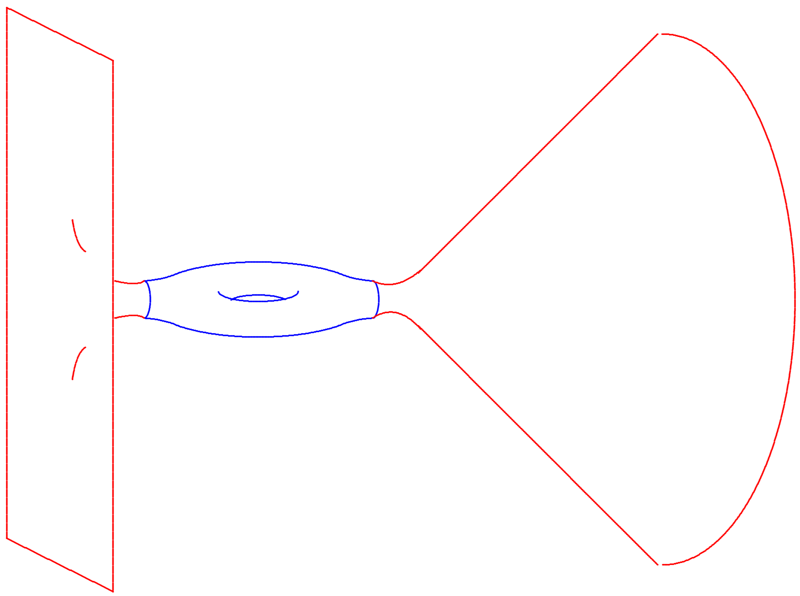

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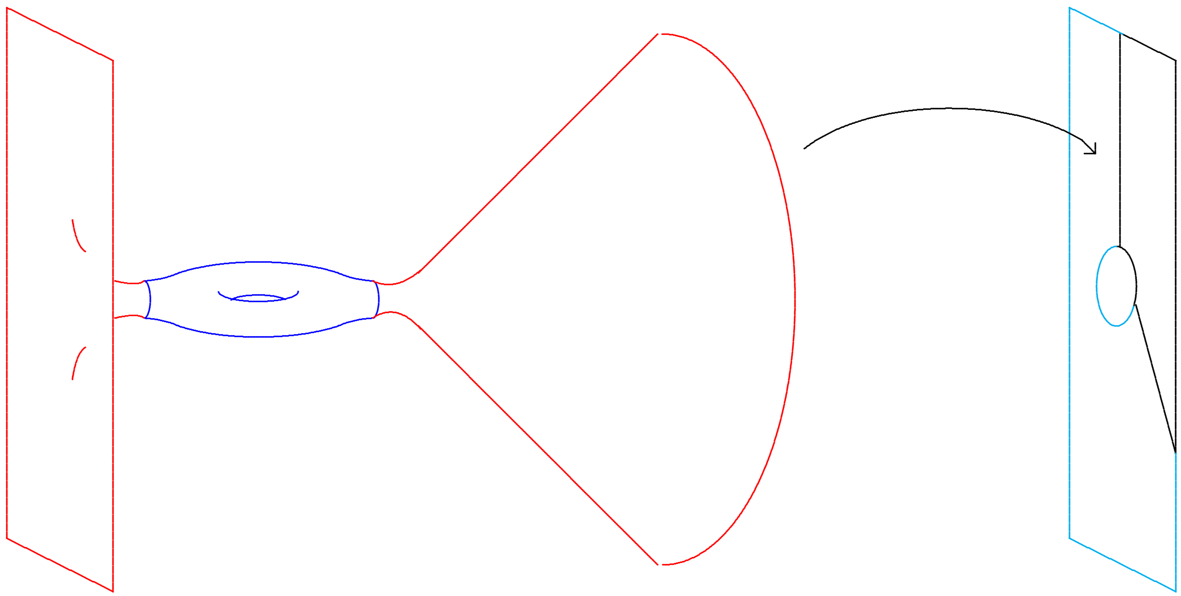

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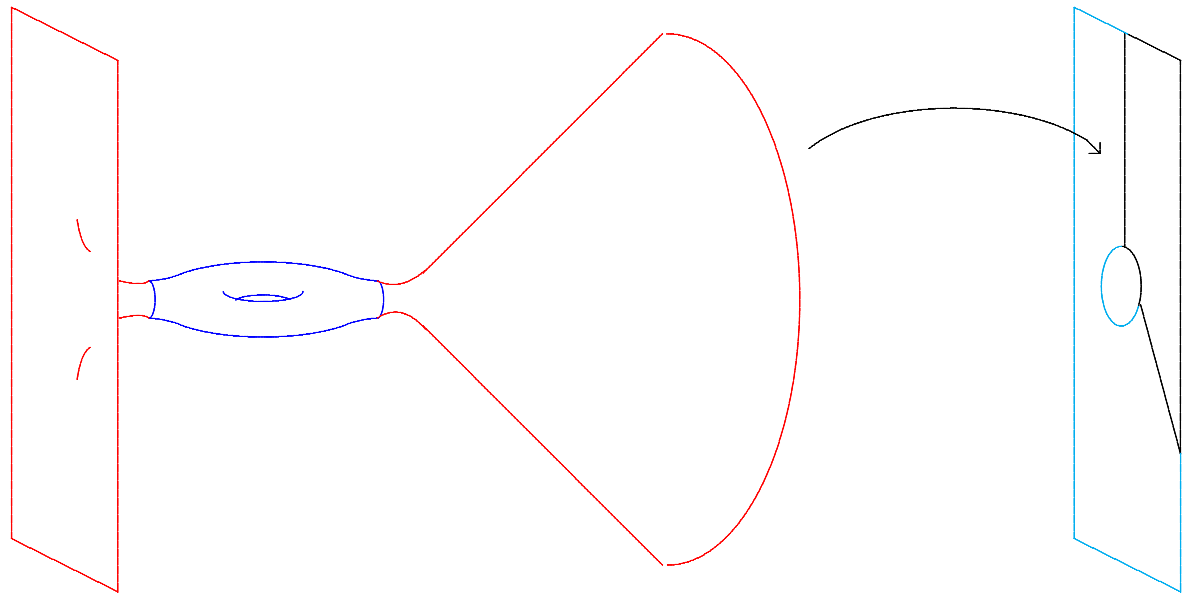

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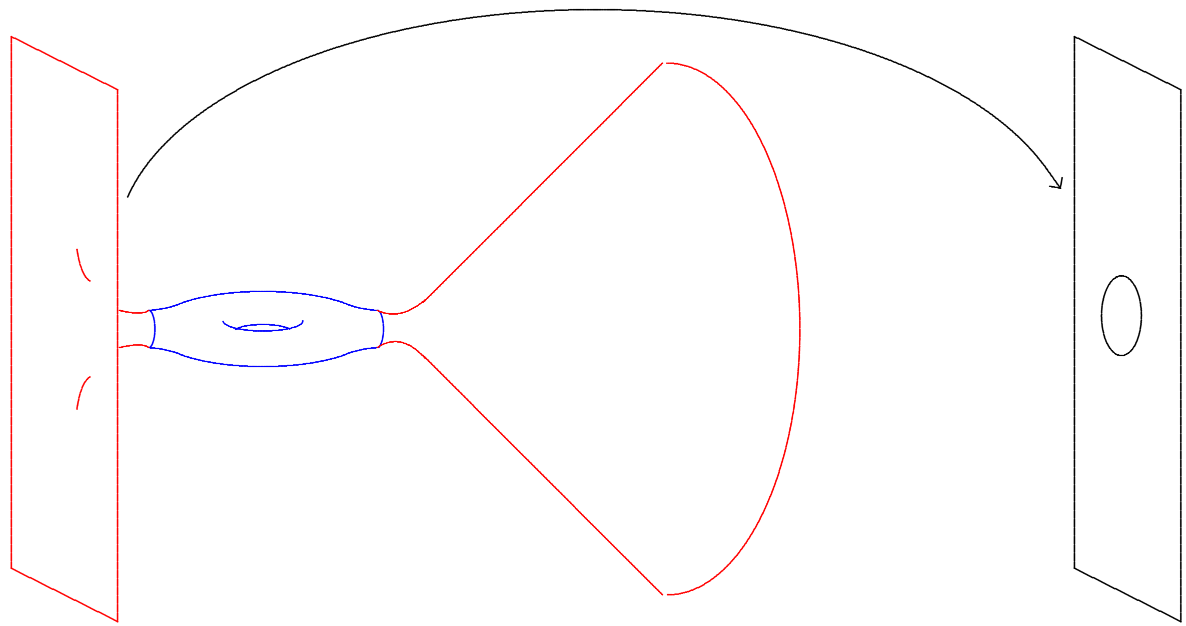

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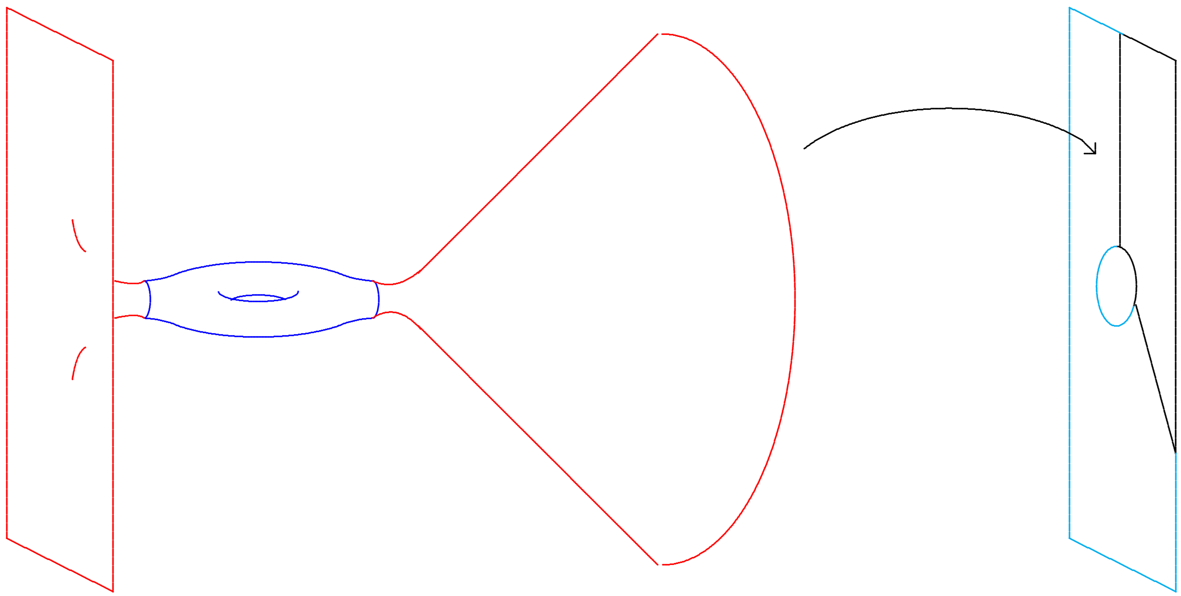

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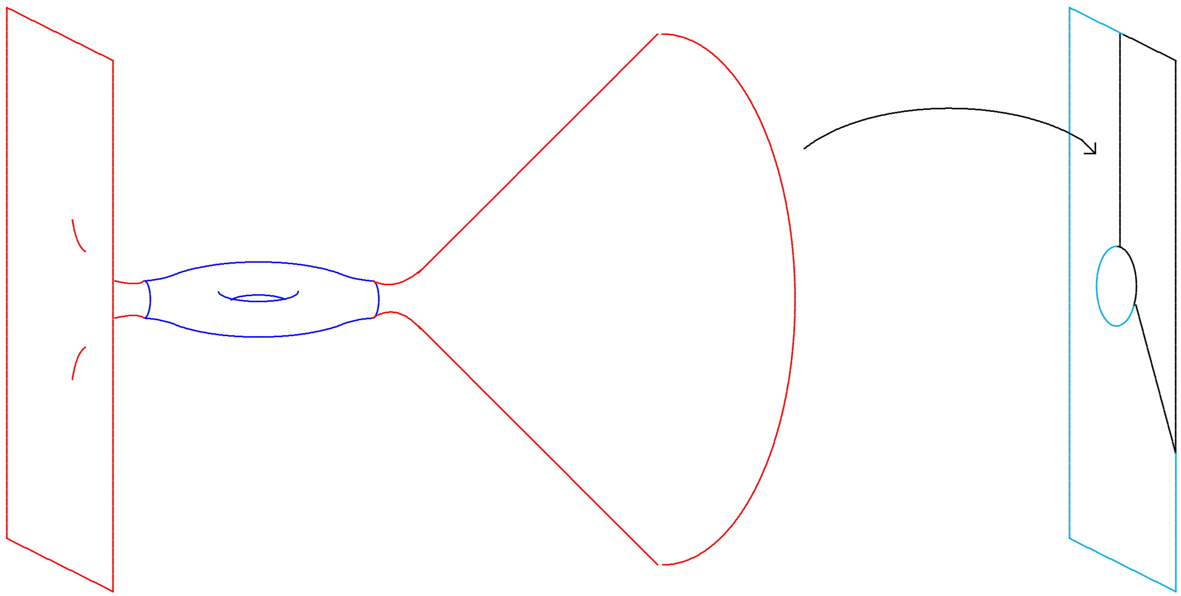

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Why consider ALE spaces?

Key examples:

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By contrast, any Ricci-flat AE manifold must be flat, by the Bishop-Gromov inequality...

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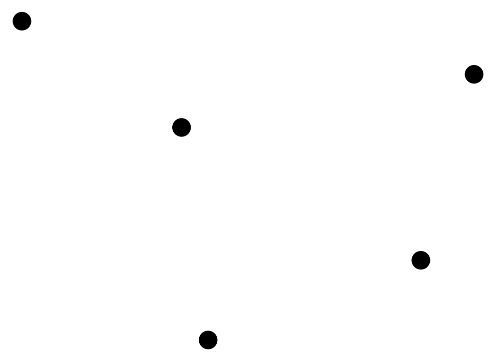

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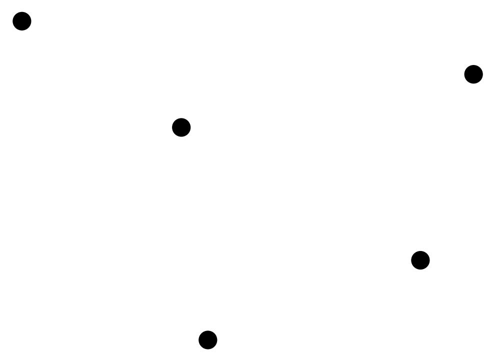
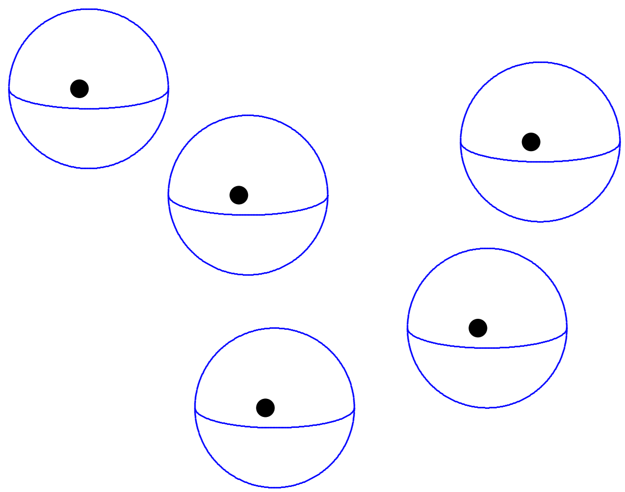


Data: ℓ points in $\mathbb{R}^3$. $\Longrightarrow$ V with $\Delta V = 0$

$$V = \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$



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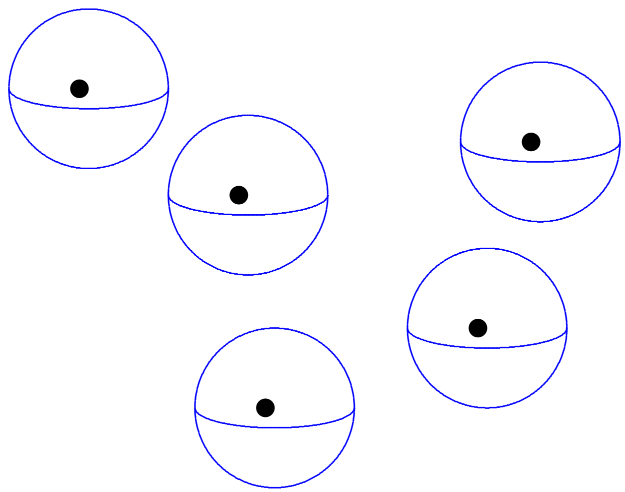

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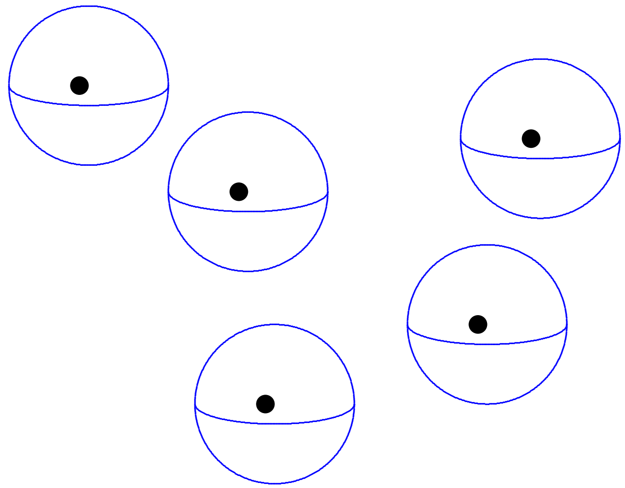

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on P. Then take M^4 = Riemannian completion.

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The G-H metrics are hyper-Kähler, and were soon rediscovered independently by Hitchin.

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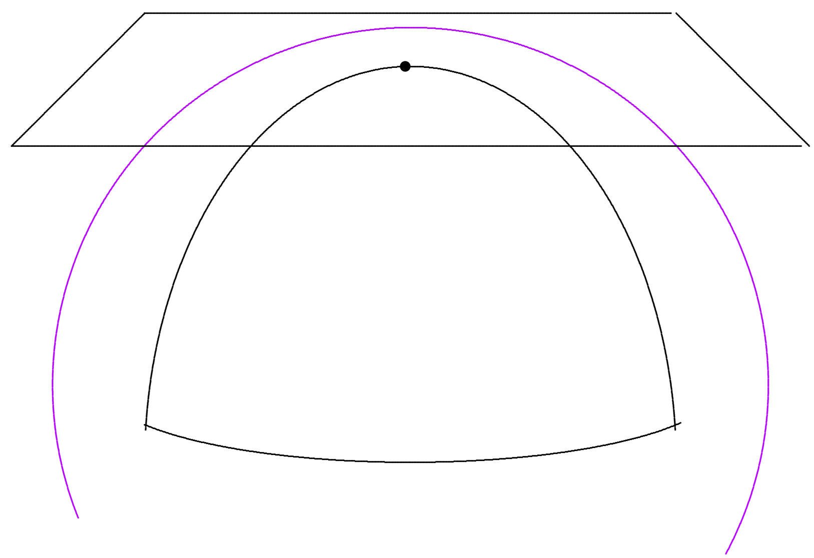

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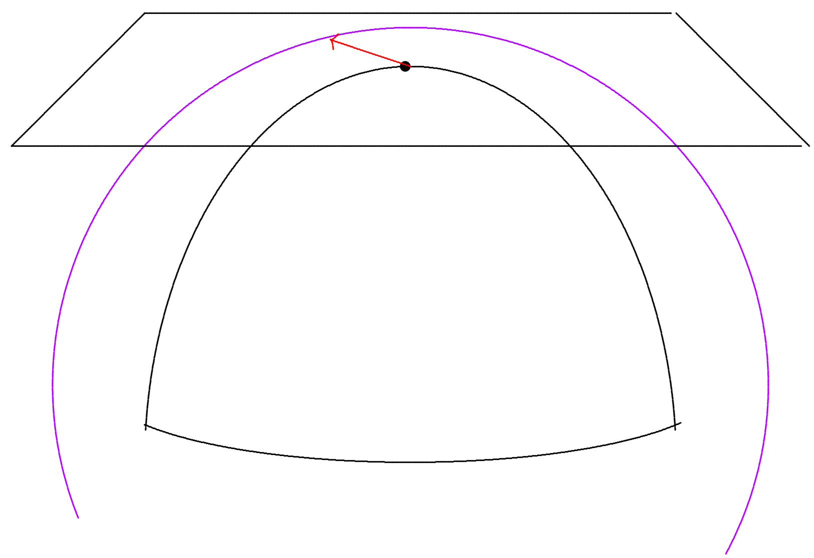

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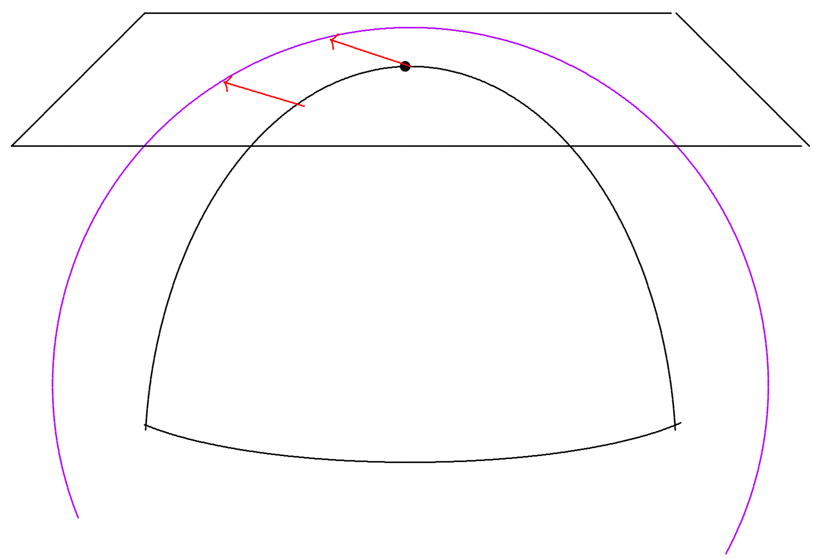

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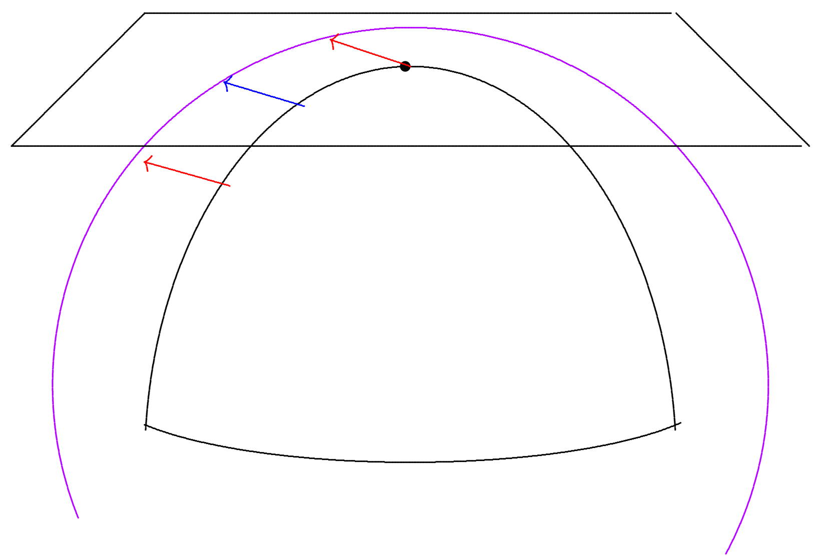

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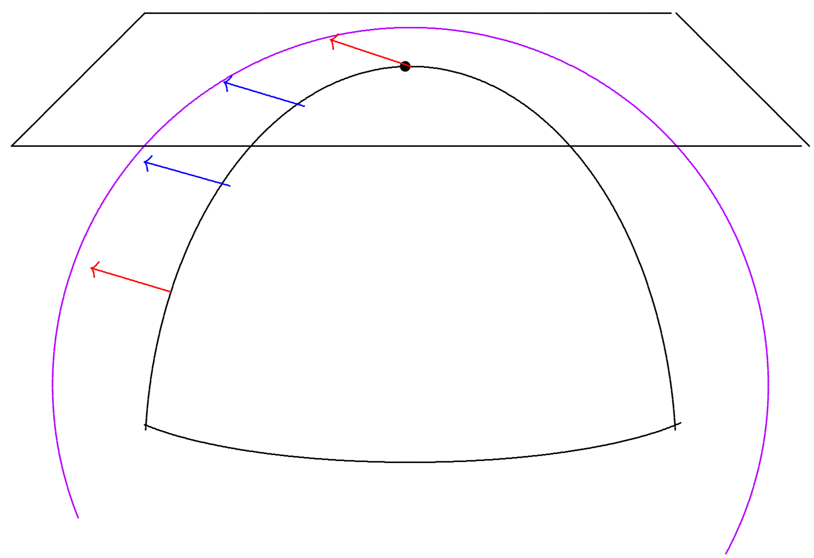

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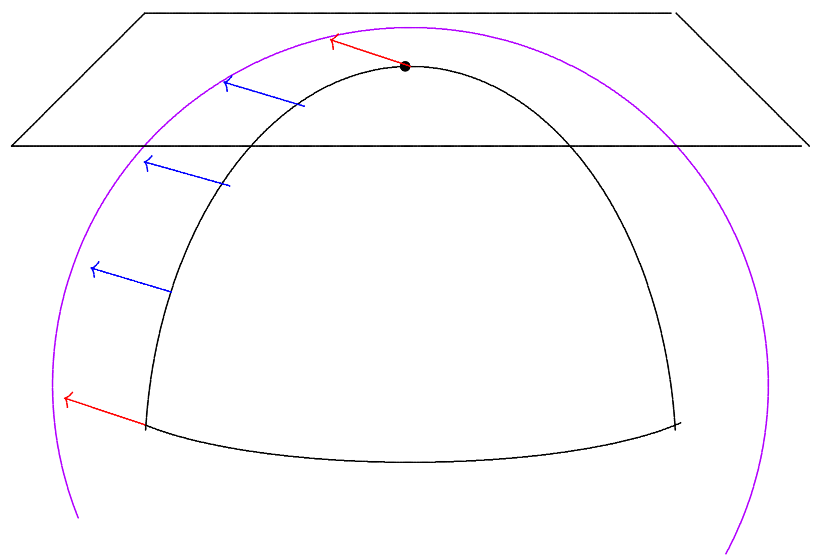

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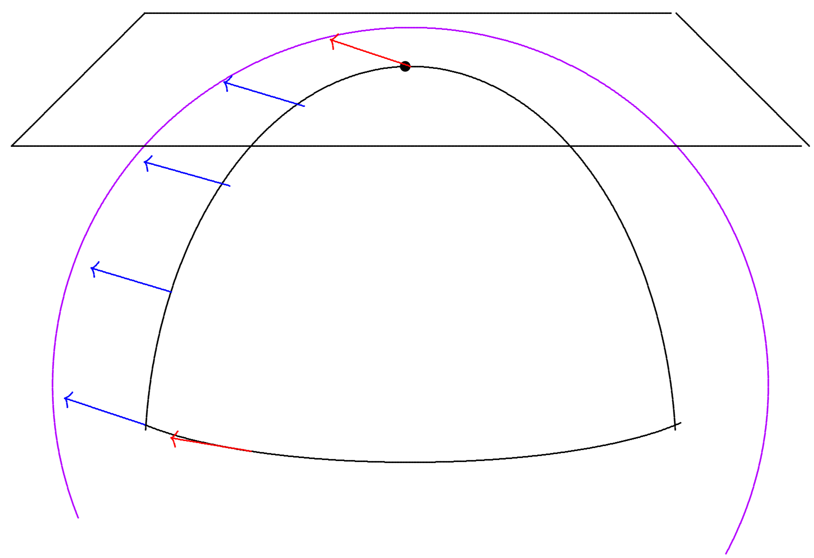

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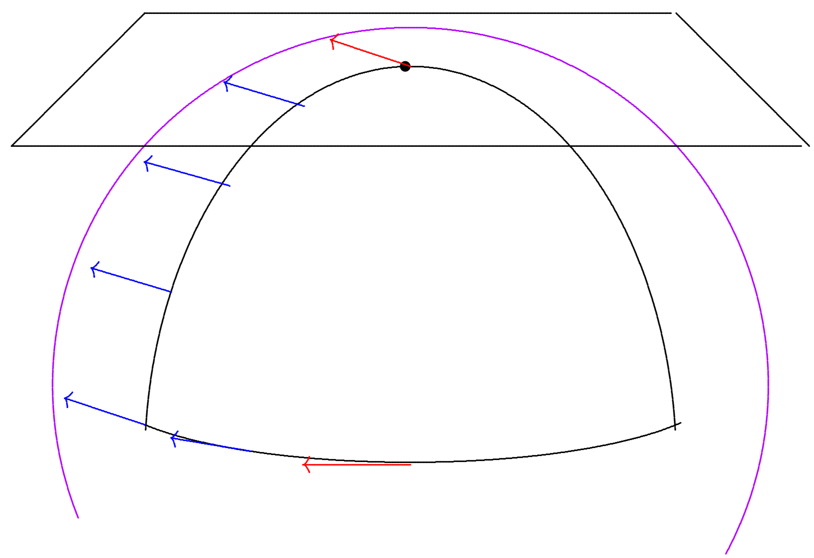

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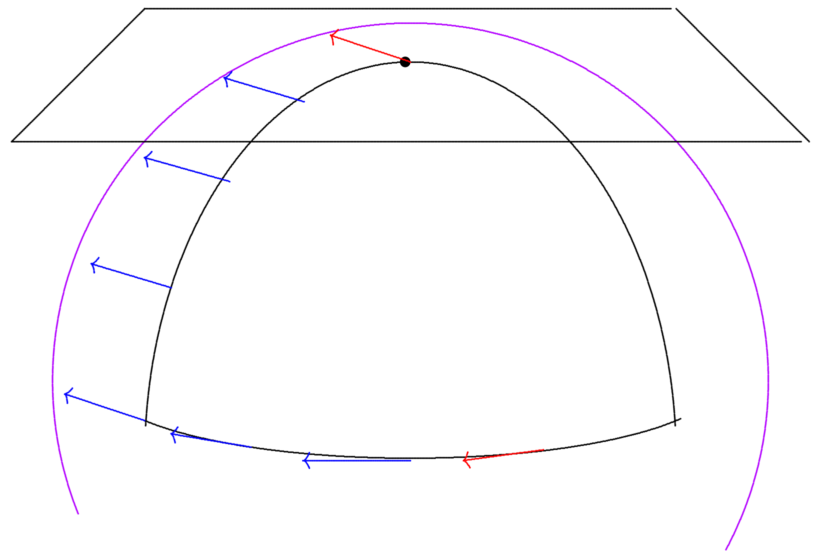

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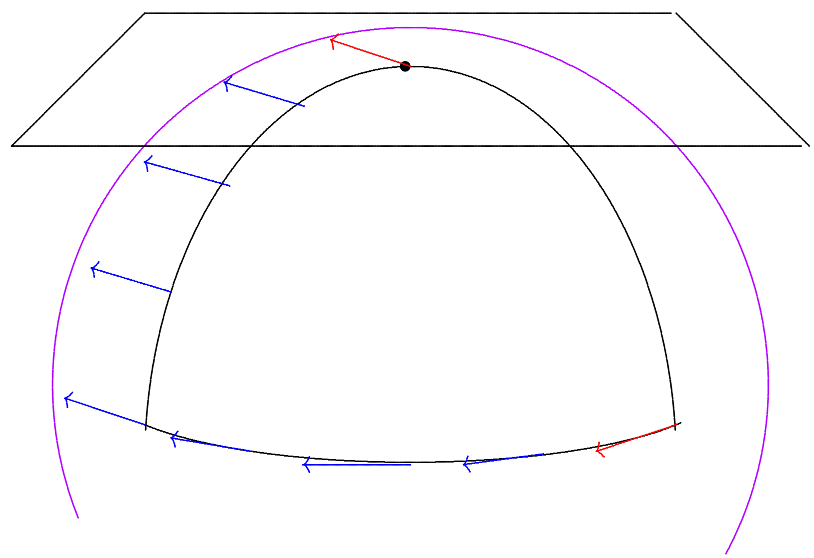

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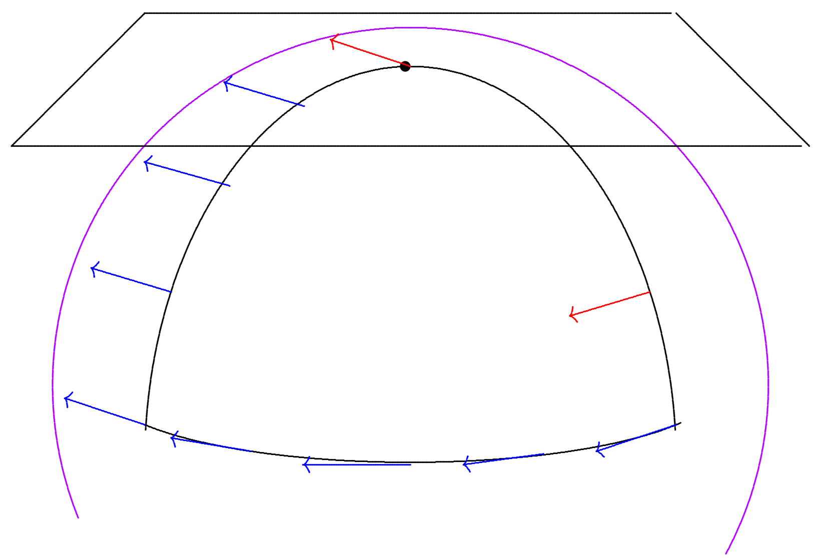

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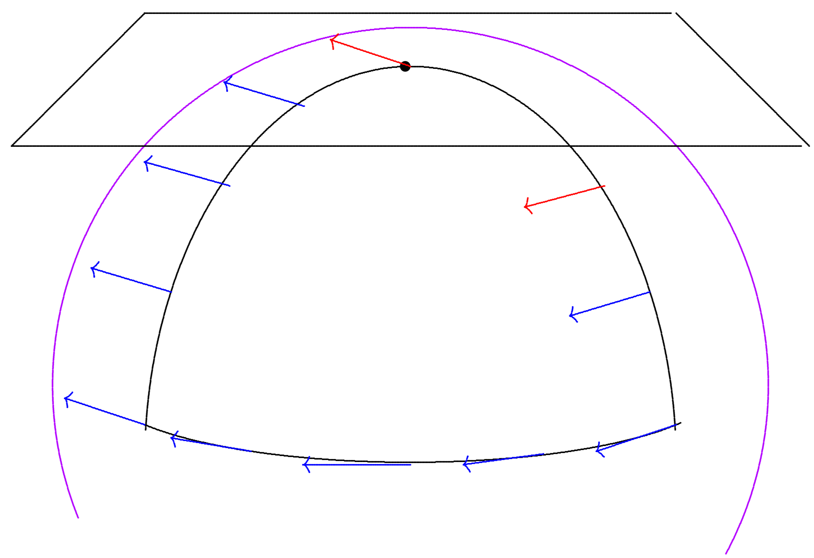

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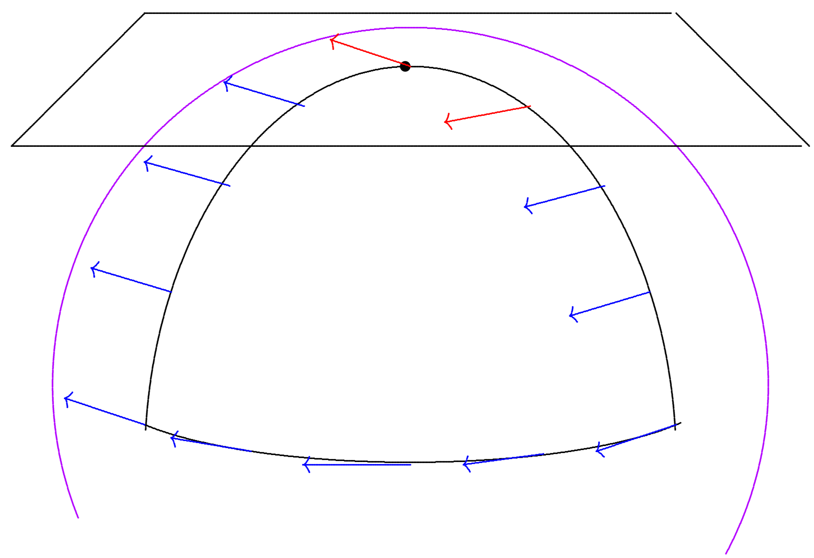

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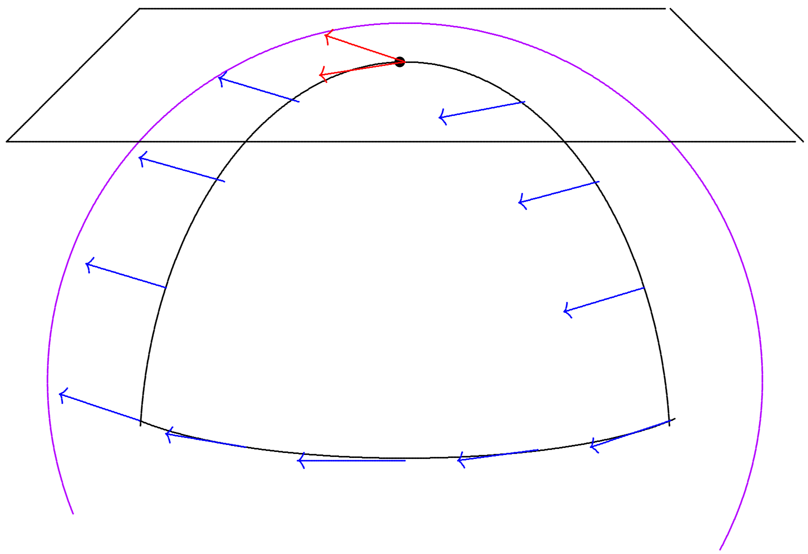

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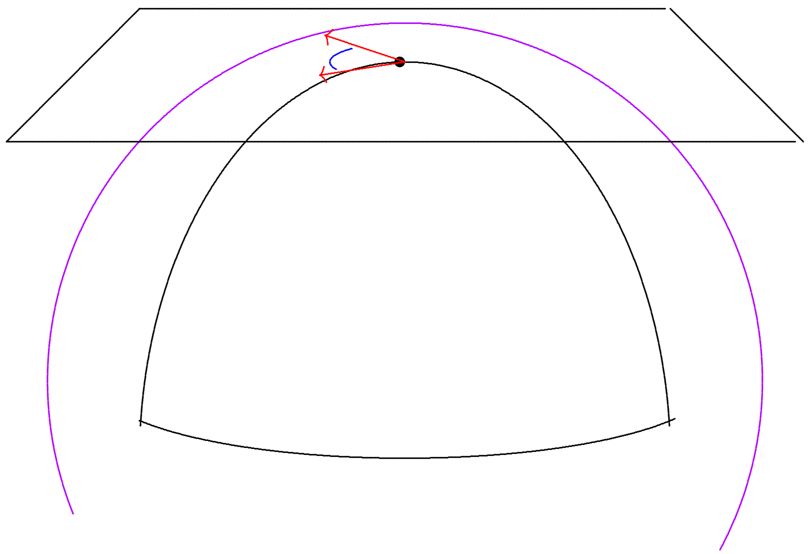
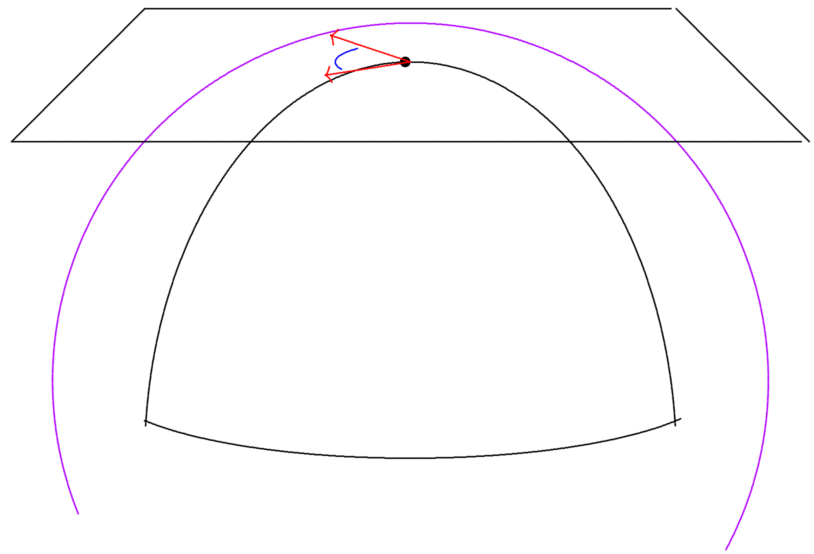


Hyper-Kähler metrics:

(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



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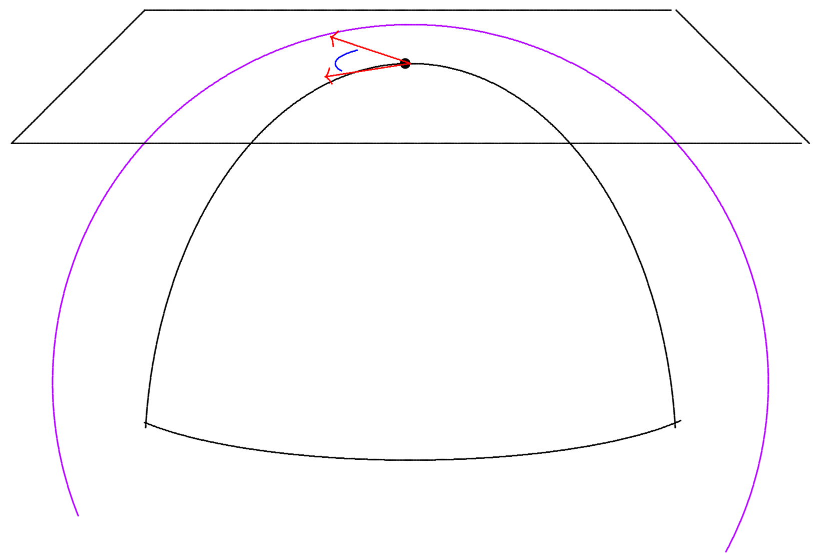

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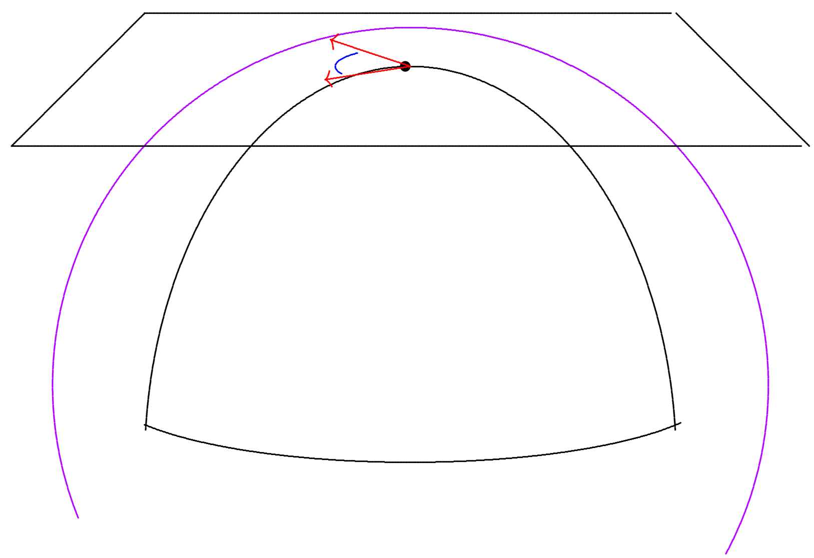

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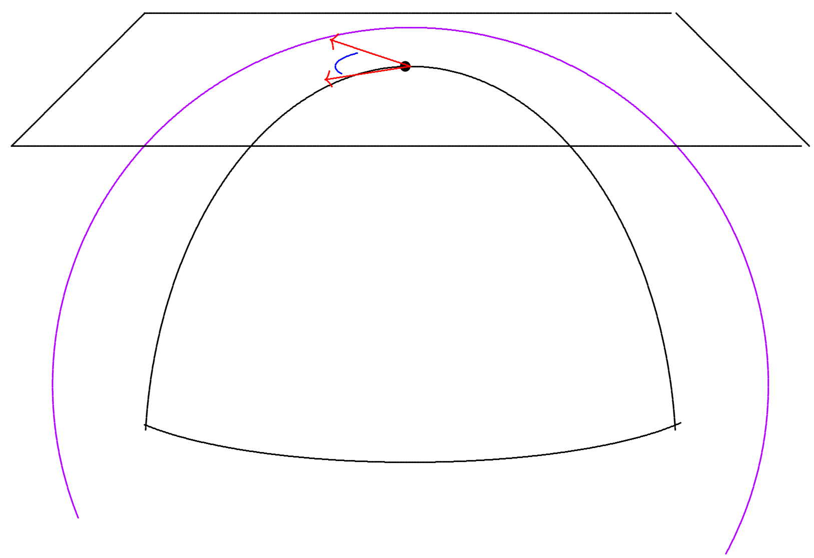

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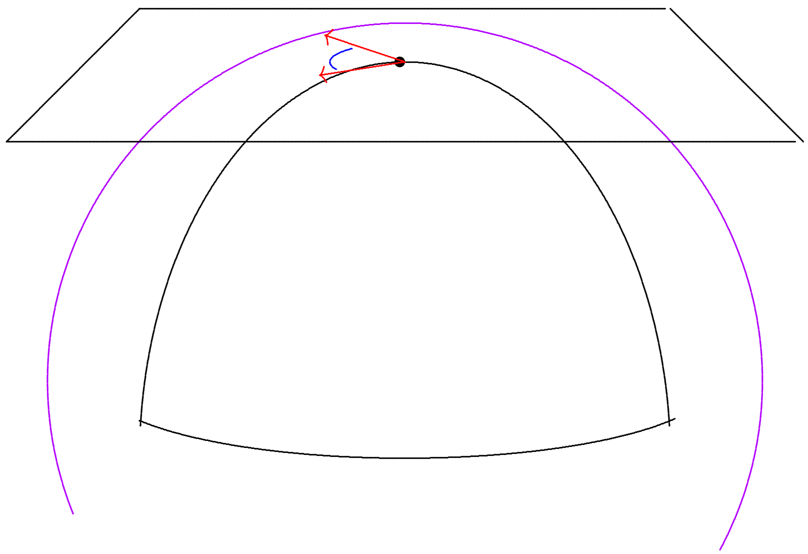

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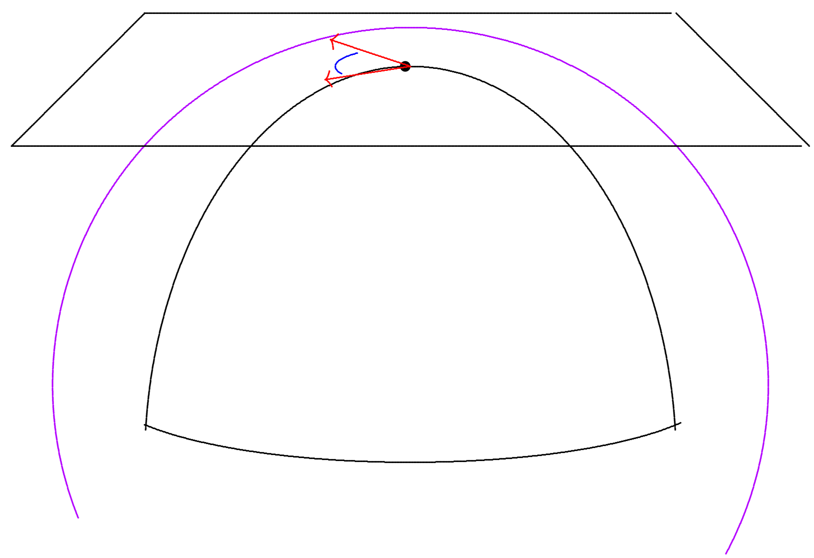

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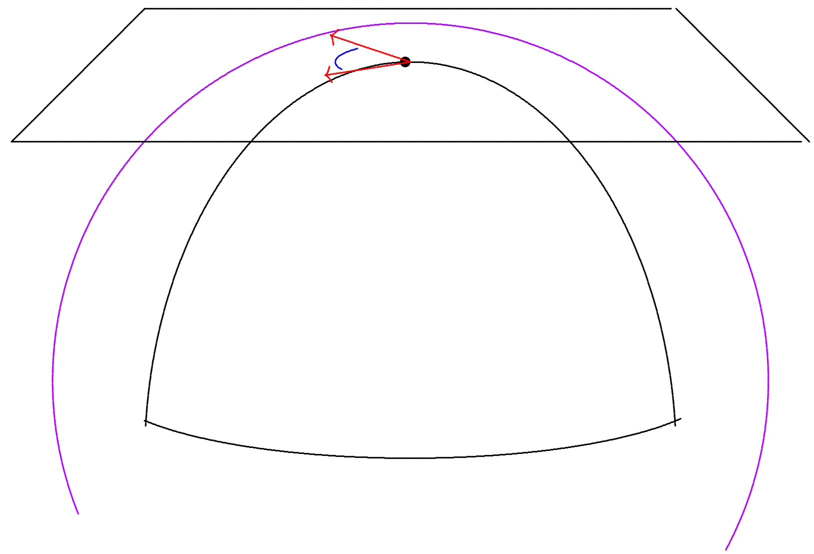

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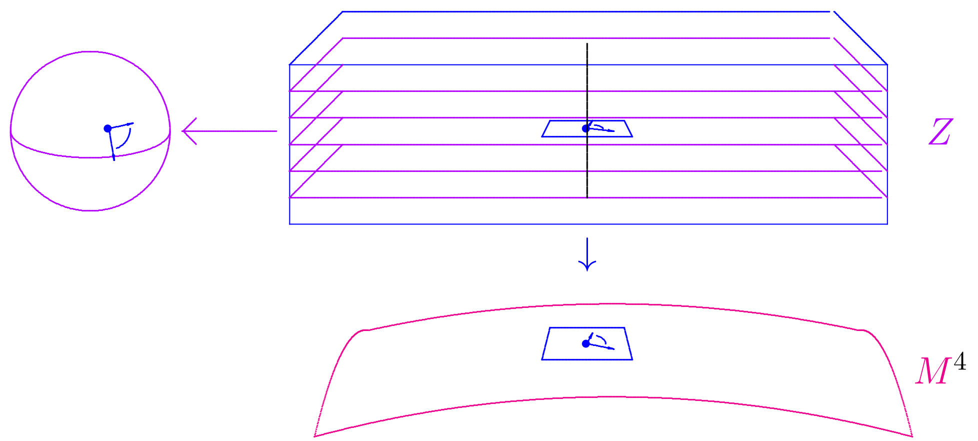

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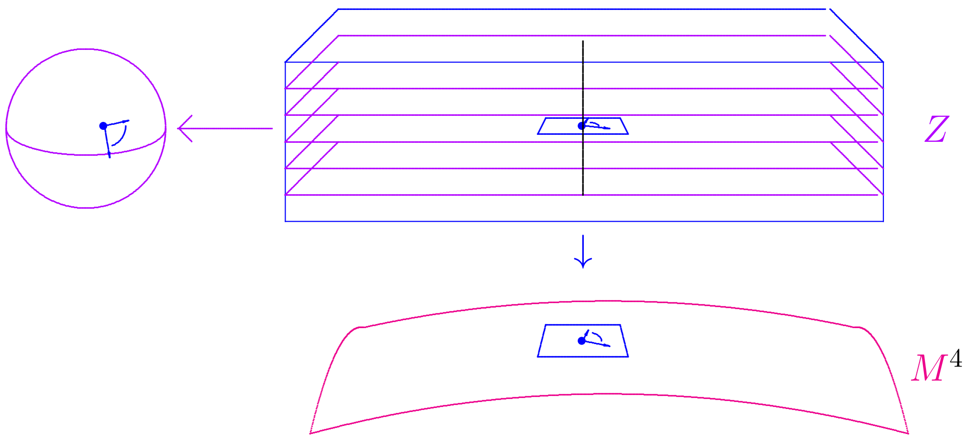

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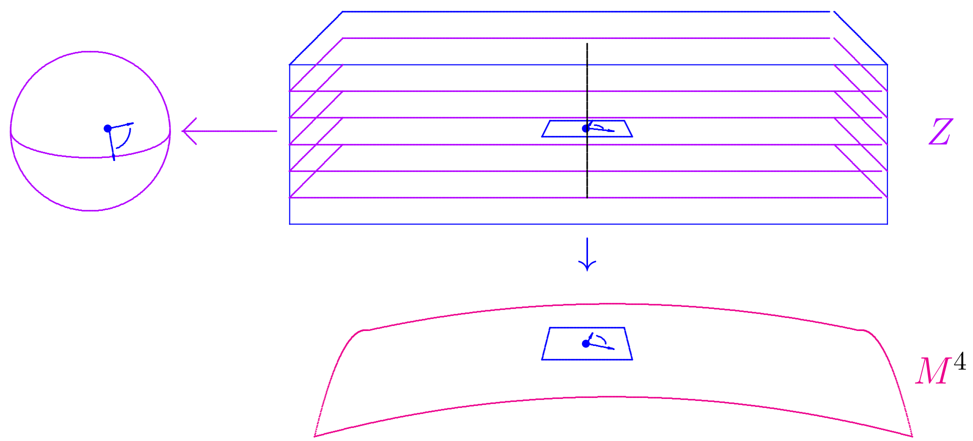

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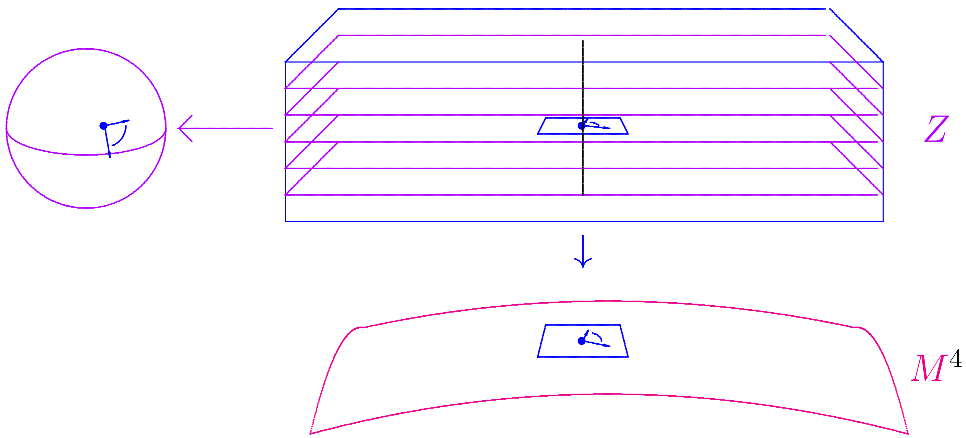

Riemannian non-linear graviton construction.

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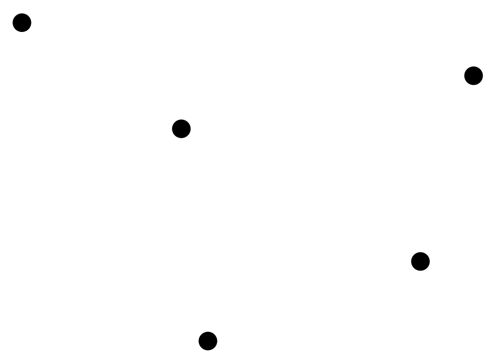

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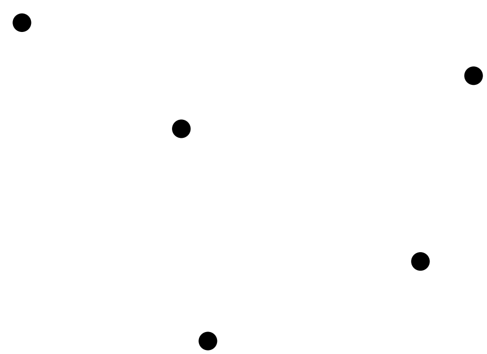

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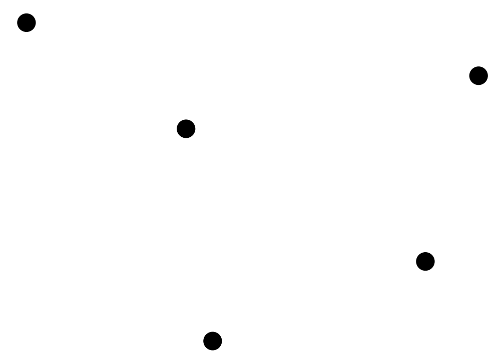

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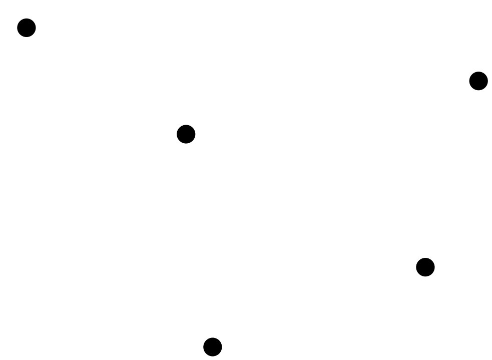

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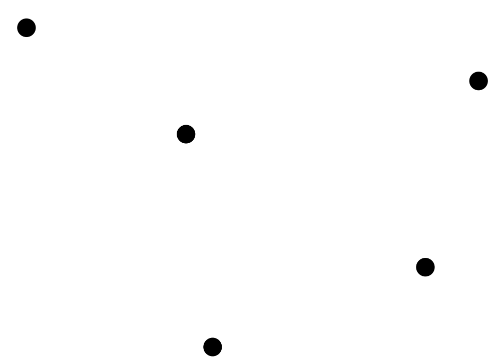

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is the twistor space of a Gibbons-Hawking metric.

Key examples:

Term ALE coined by Gibbons & Hawking, 1979.

They wrote down various explicit Ricci-flat ALE 4-manifolds they called gravitational instantons.

Their examples have just one end, with

$$\Gamma \cong \mathbb{Z}_\ell \subset \mathbf{SU}(2) \subset \mathbf{O}(4).$$

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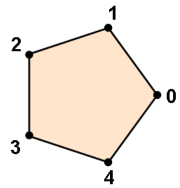
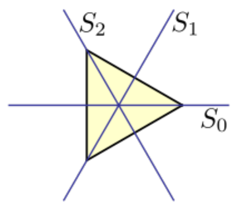
This conjecture was proved by Kronheimer, 1986.

Felix Klein, 1884: $\mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$

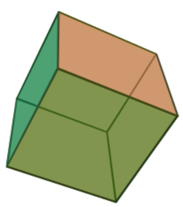


$\mathbb{Z}_{k+1} \longleftrightarrow xy + z^{k+1} = 0$

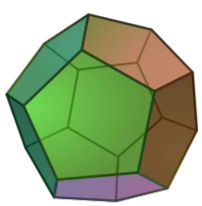


$\mathrm{Dih}^*_{k-2} \longleftrightarrow x^2 + z(y^2 + z^{k-2}) = 0$

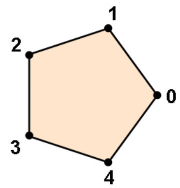
$T^* \longleftrightarrow x^2 + y^3 + z^4 = 0$



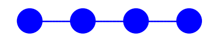
$O^* \longleftrightarrow x^2 + y^3 + yz^3 = 0$

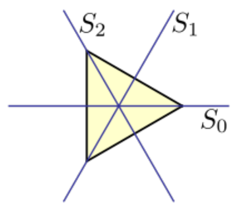


$I^* \longleftrightarrow x^2 + y^3 + z^5 = 0$



$$\mathbb{Z}_{k+1} \longleftrightarrow A_k$$





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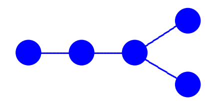
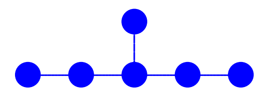
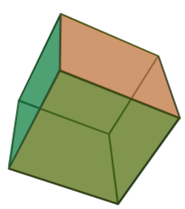
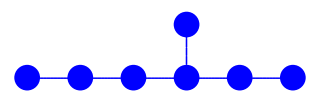


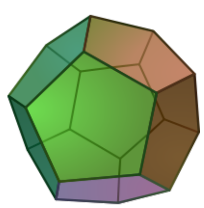
$$T^* \longleftrightarrow E_6$$





$$O^* \longleftrightarrow E_7$$





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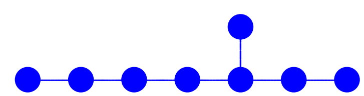
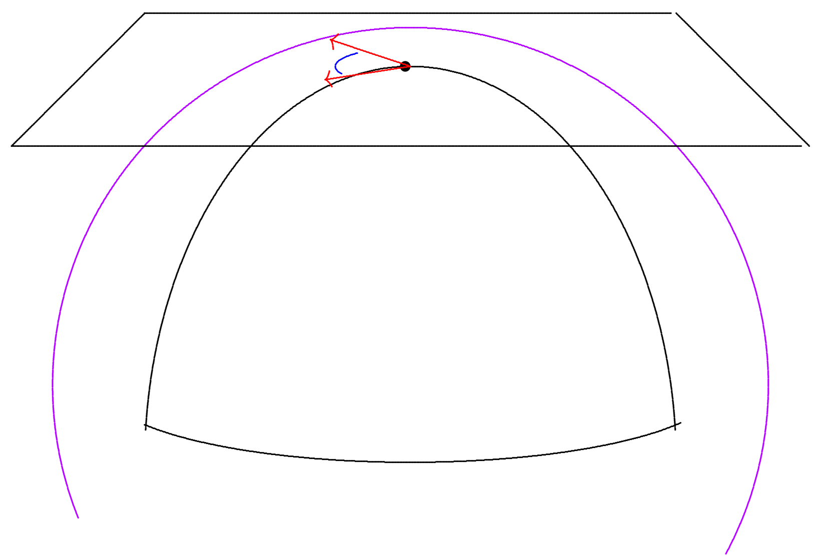


Kähler metrics:

(M^4, g) Kähler $\Longleftrightarrow$ holonomy $\subset$ $\mathbf{U}(2)$



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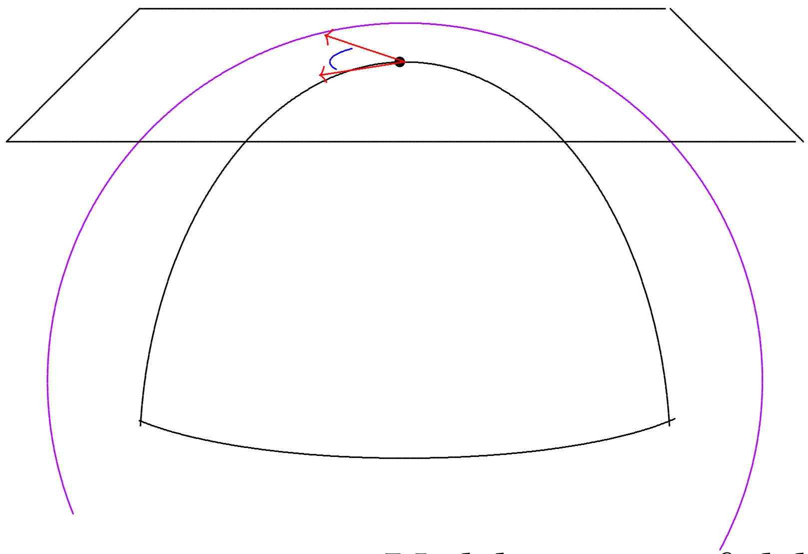

In real dimension 4, any Kähler manifold satisfies

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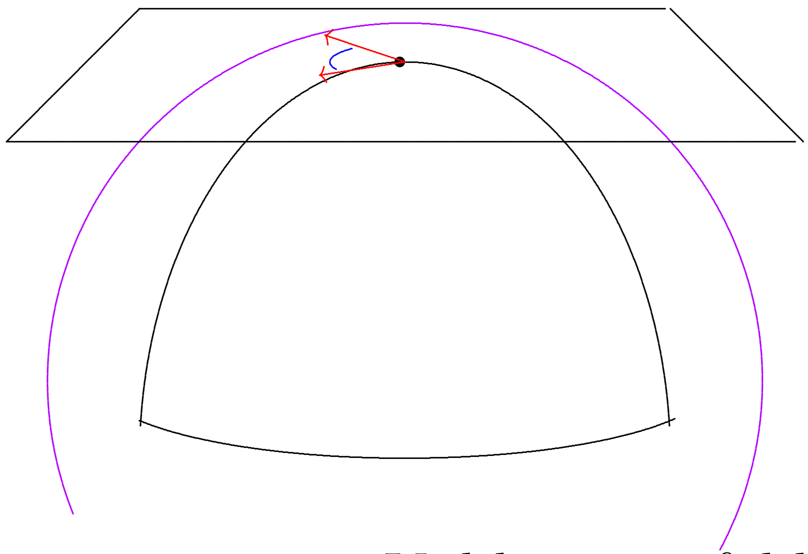

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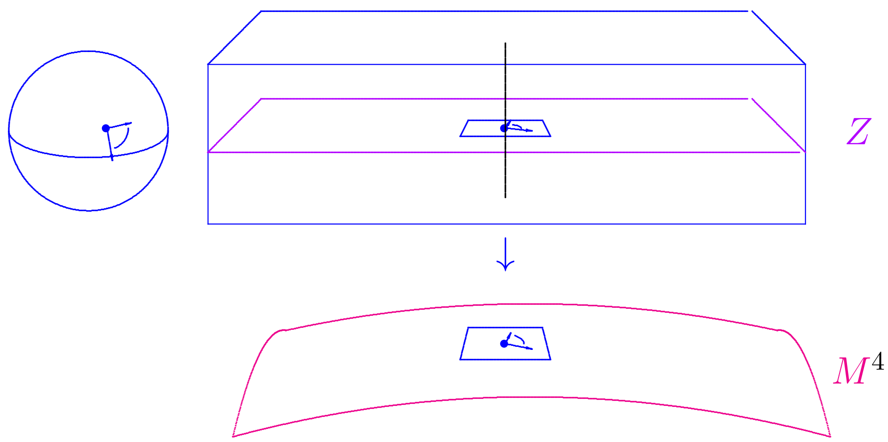
so that $W_+ = 0 \Longleftrightarrow s = 0$.

Any scalar-flat Kähler surface (M^4, g, J) has a

Penrose Twistor Space (Z, J),

which is once again a complex 3-manifold.



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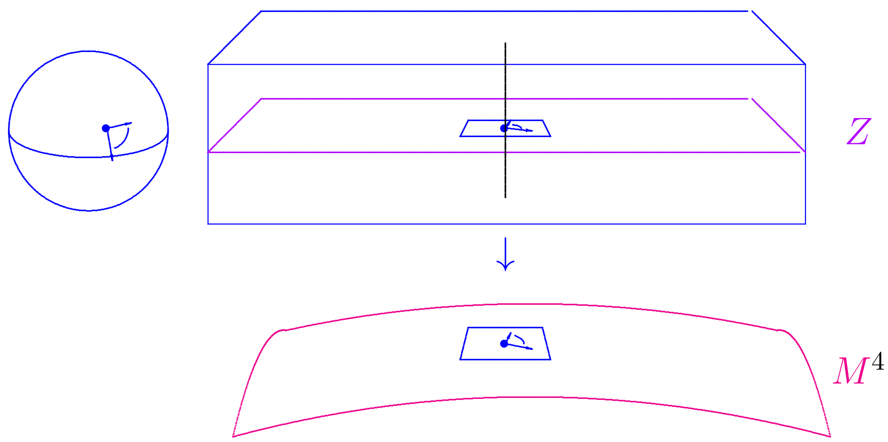

Kähler case: holomorphic section of $K_Z^{-1/2}$

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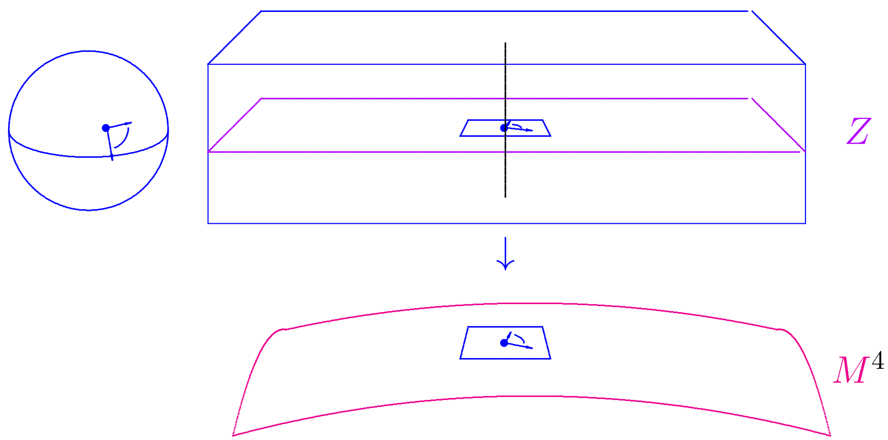

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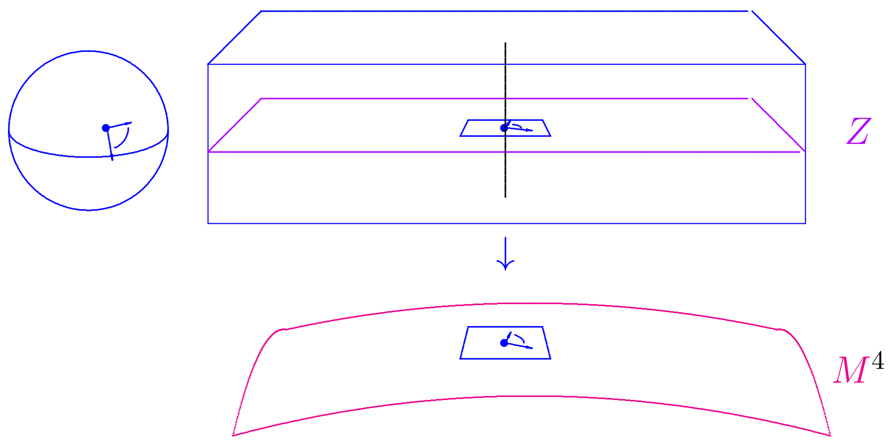

$$0 \to T^{1,0}M \to T^{1,0}Z|_M \to K_M{}^{-1} \to 0$$

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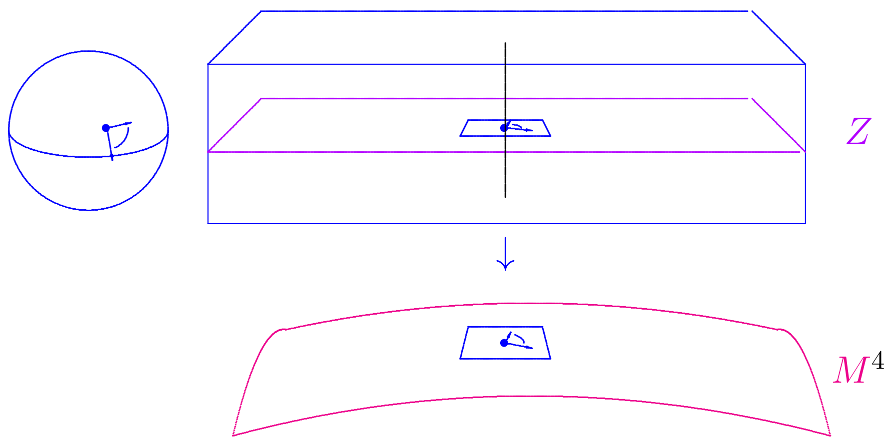

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Extension class: $\in H^1(M, \mathcal{O}(K_M \otimes T^{1,0}M))$.

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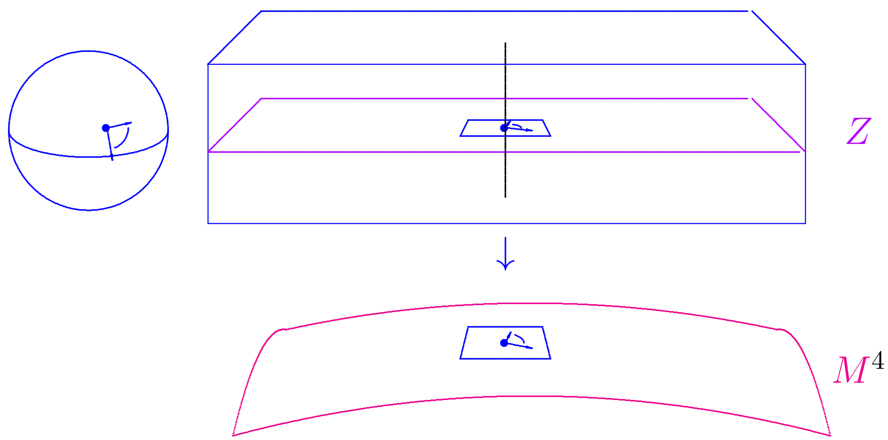

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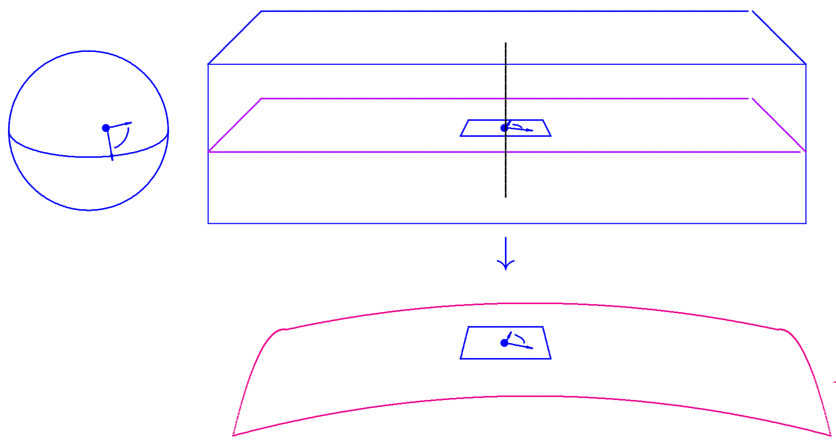

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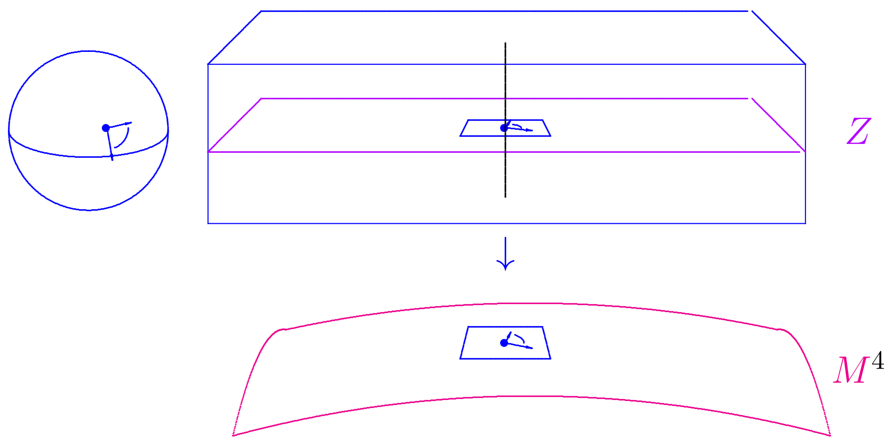

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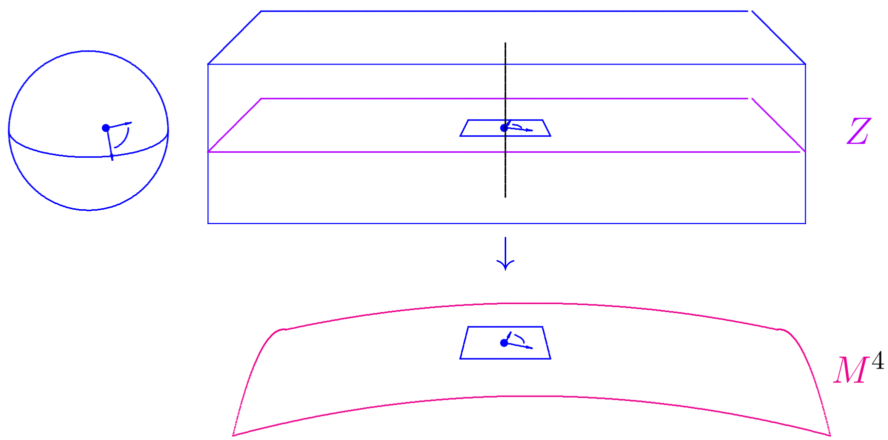

$$0 \to T^{1,0}M \to T^{1,0}Z|_M \to K_M{}^{-1} \to 0$$

Kähler form: $\omega = g(J\cdot, \cdot)$.

Some AE/ALE Scalar-Flat Kähler Surfaces:

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(L '91)

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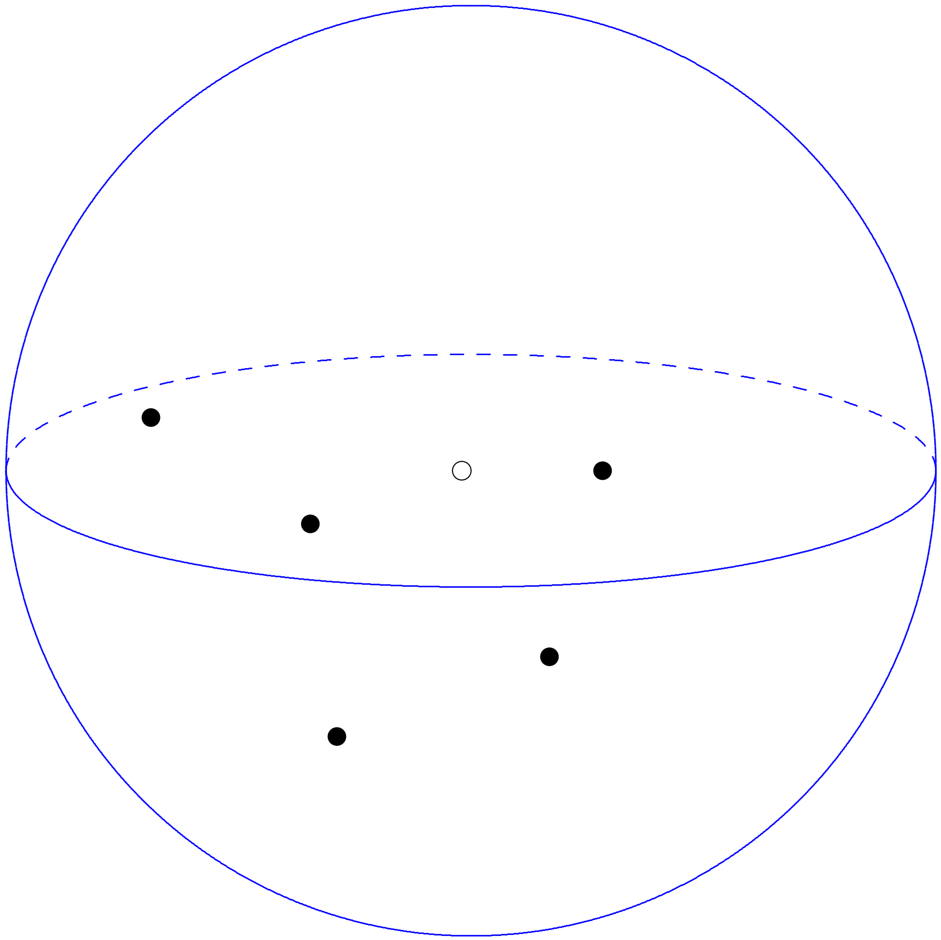

Data: $k+1$ points in $\mathcal{H}^3$.

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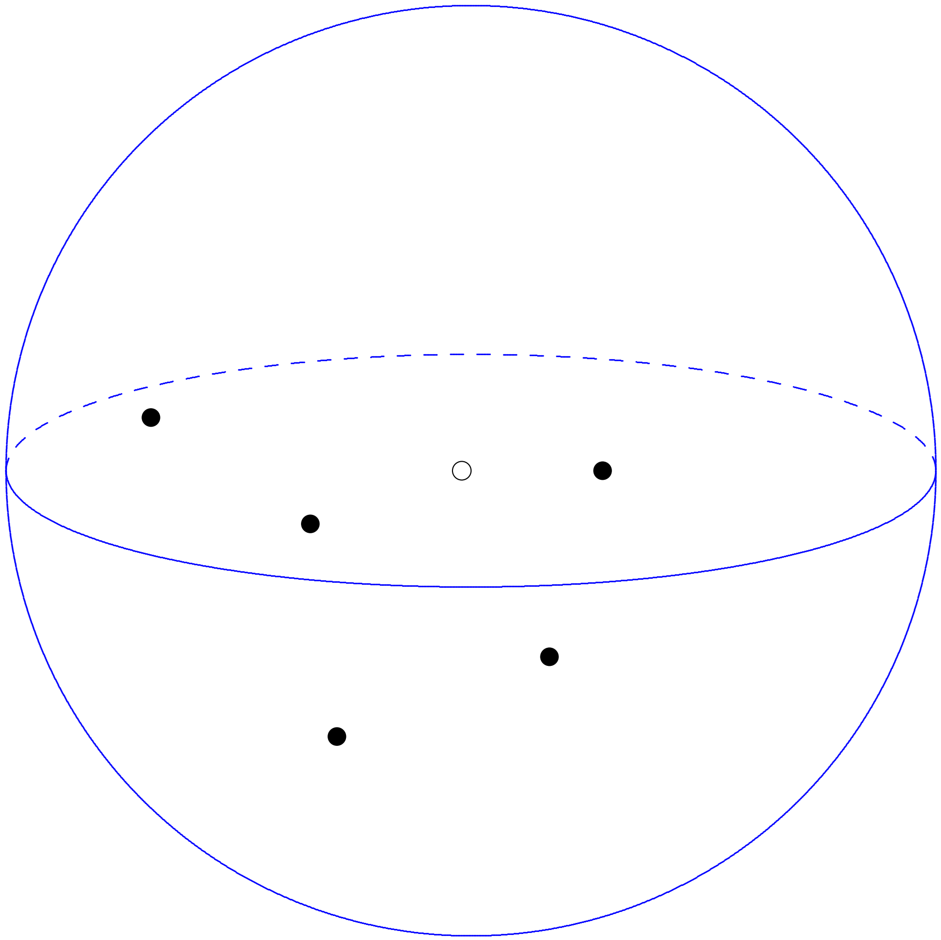

Data: $k+1$ points in $\mathcal{H}^3$. $\Longrightarrow$ V with $\Delta V = 0$

$$V = 1 + \frac{\ell}{e^{2\varrho_0} - 1} + \sum_{j=1}^{k} \frac{1}{e^{2\varrho_j} - 1}$$

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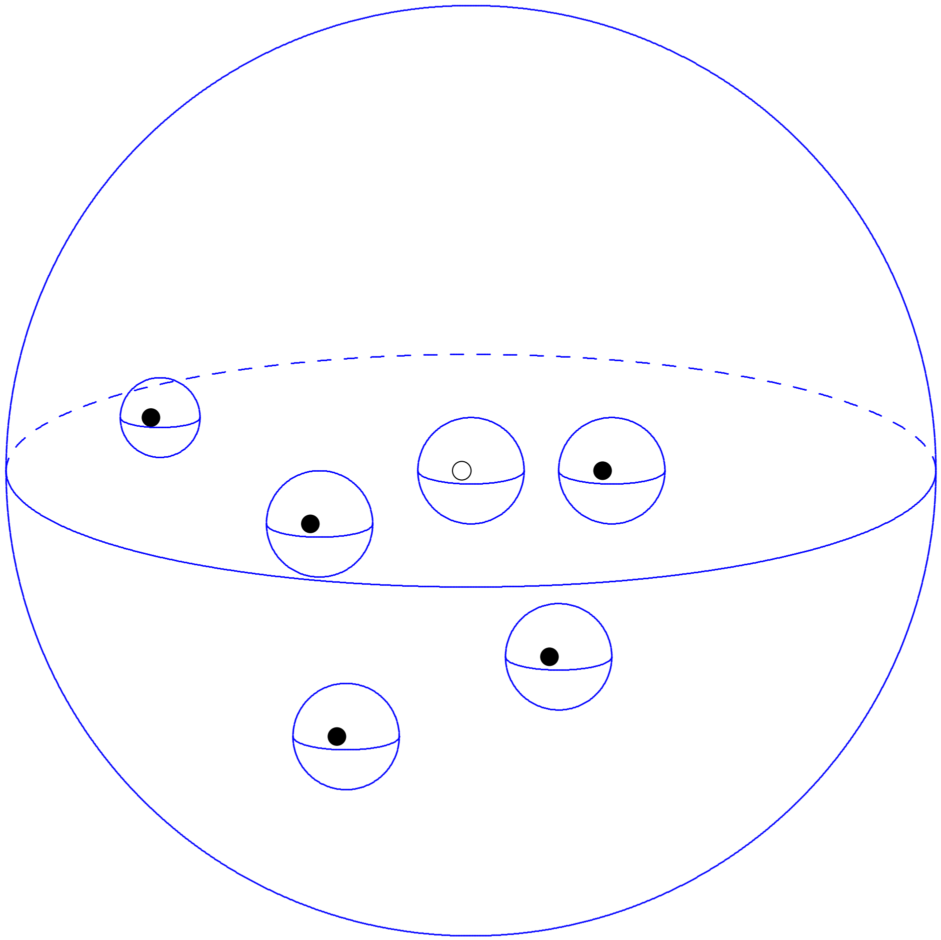
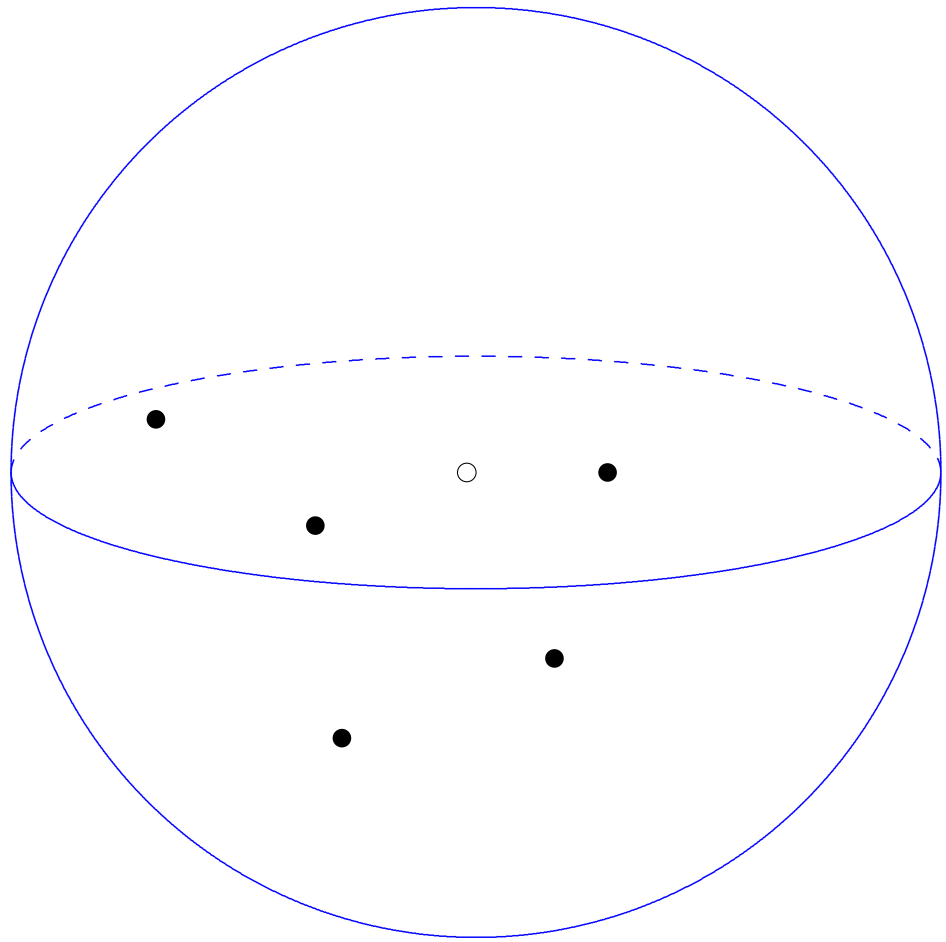


Data: $k+1$ points in $\mathcal{H}^3$. $\Longrightarrow V$ with $\Delta V = 0$

$F = \star dV$ curvature θ on $P \to \mathcal{H}^3 - \{\text{pts}\}$.

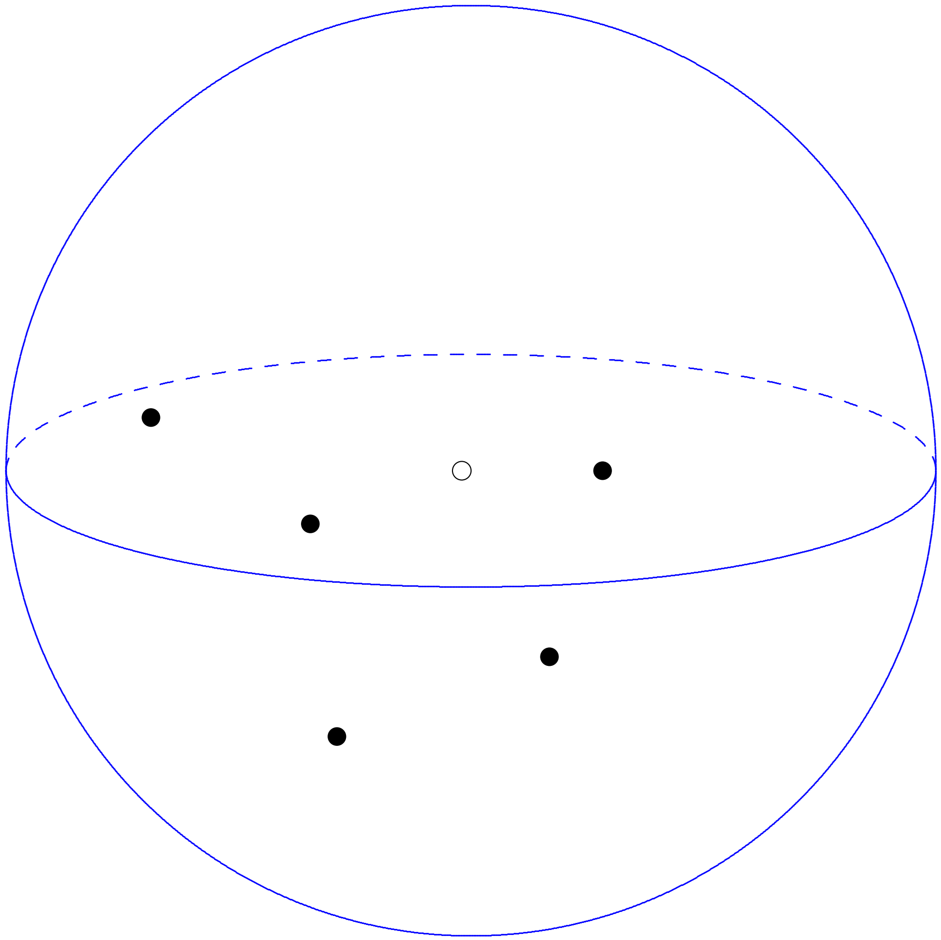
Some AE/ALE Scalar-Flat Kähler Surfaces:



Data: $k+1$ points in $\mathcal{H}^3$. $\Longrightarrow V$ with $\Delta V = 0$

$$g = \frac{1}{4\sinh^2 \varrho_0}\left(Vh + V^{-1}\theta^2\right)$$

Some AE/ALE Scalar-Flat Kähler Surfaces:



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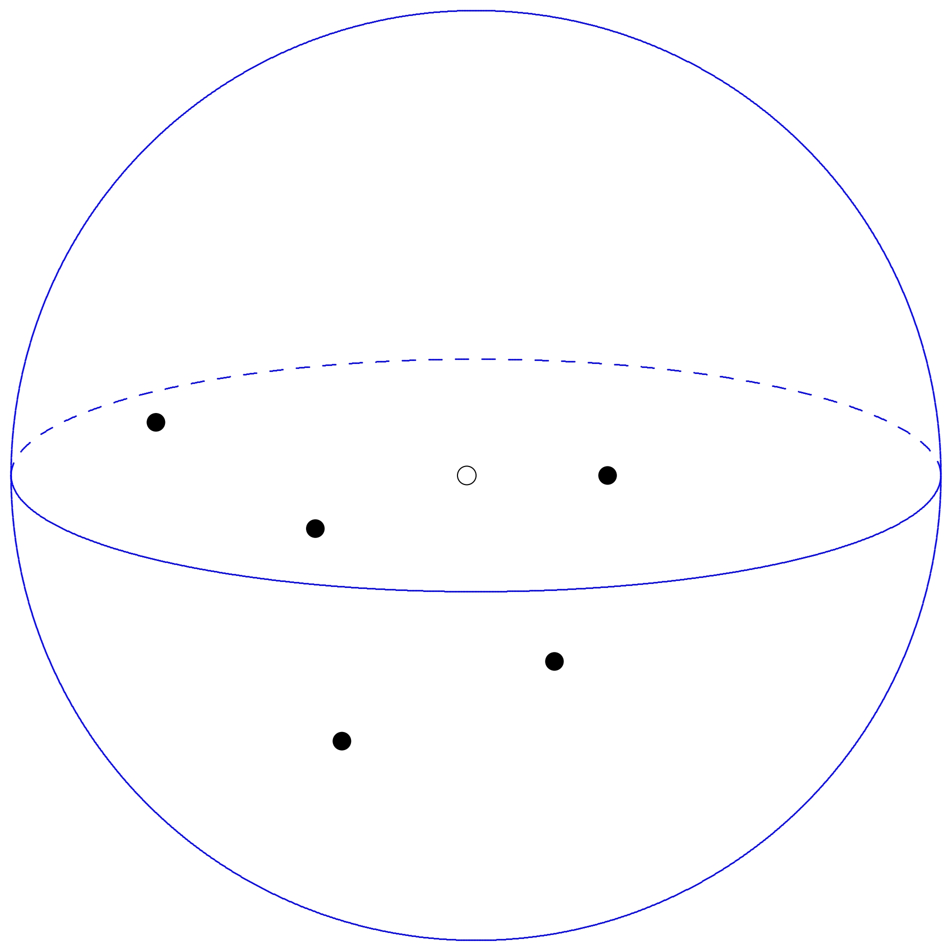

Riemannian completion is AE $\Longleftrightarrow$ $\ell = 1$:

$$V = 1 + \frac{\ell}{e^{2\varrho_0} - 1} + \sum_{j=1}^{k} \frac{1}{e^{2\varrho_j} - 1}$$

Twistor Spaces for These Metrics:

$$H^0(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(1,1)) = \mathbb{C}^4$$

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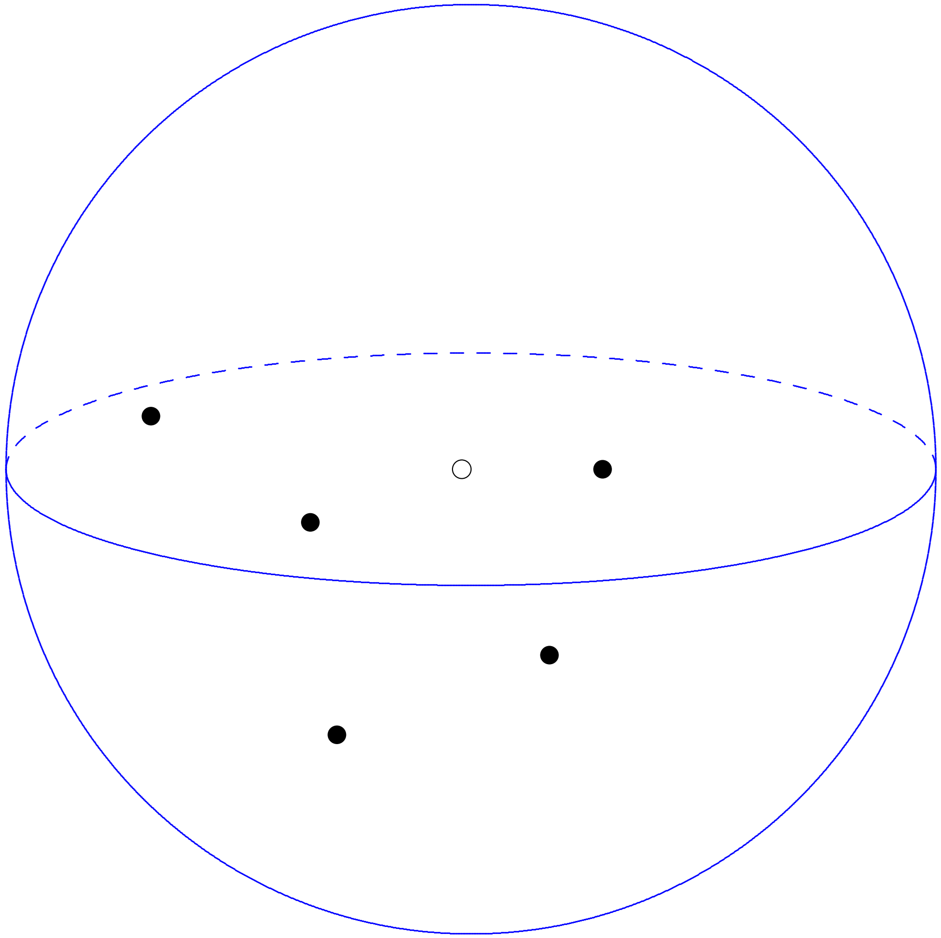

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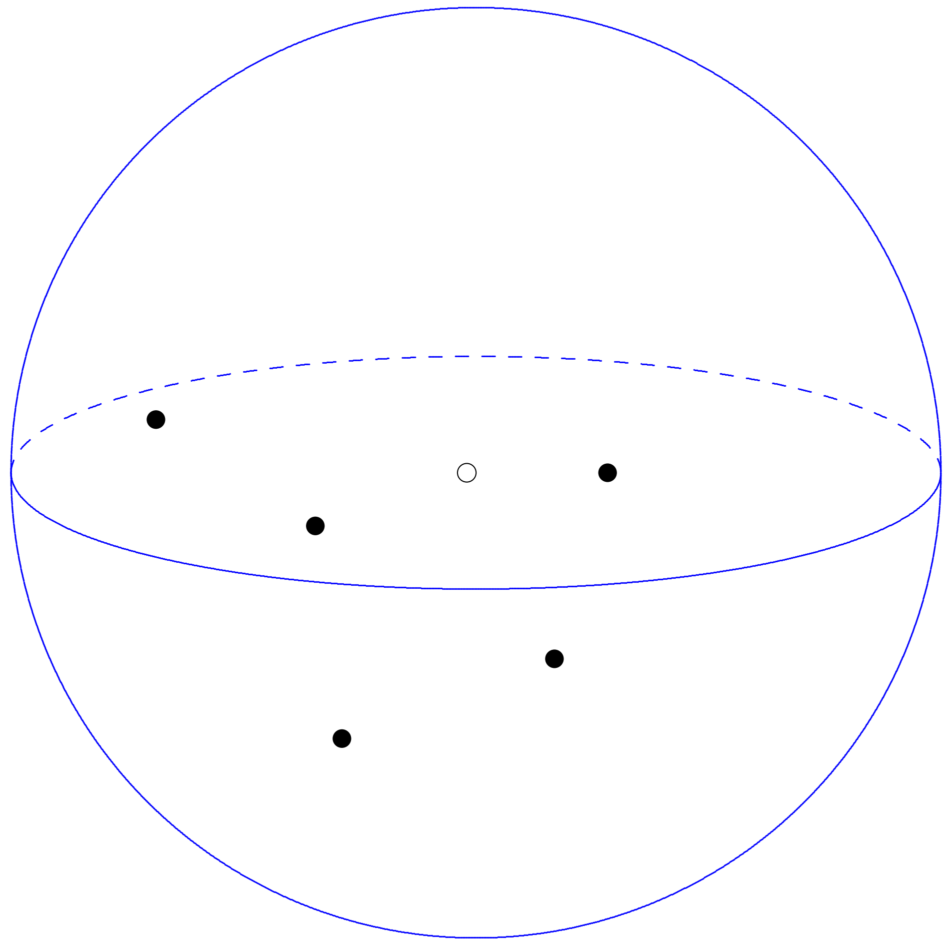

So $k+1$ points in $\mathcal{H}^3$ give rise to

$$P_0, P_1, \ldots, P_k \in H^0(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(1,1)).$$

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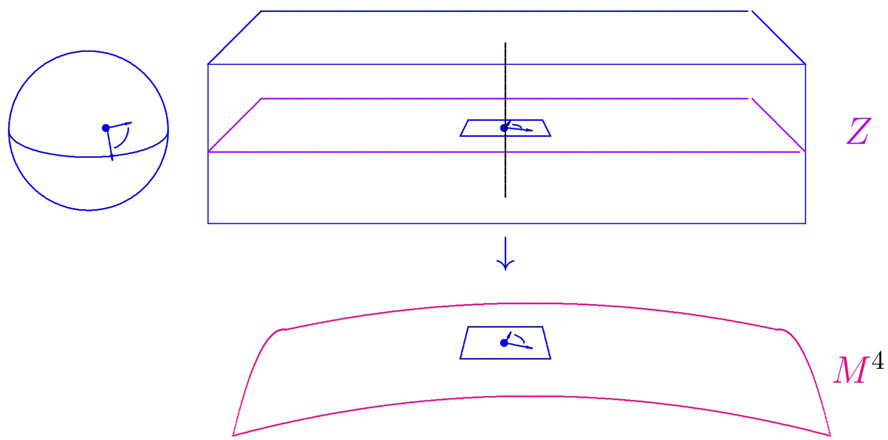
- making small resolutions of isolated singularities.

Any scalar-flat Kähler surface (M^4, g, J) has a

Penrose Twistor Space (Z, J),

which is once again a complex 3-manifold.



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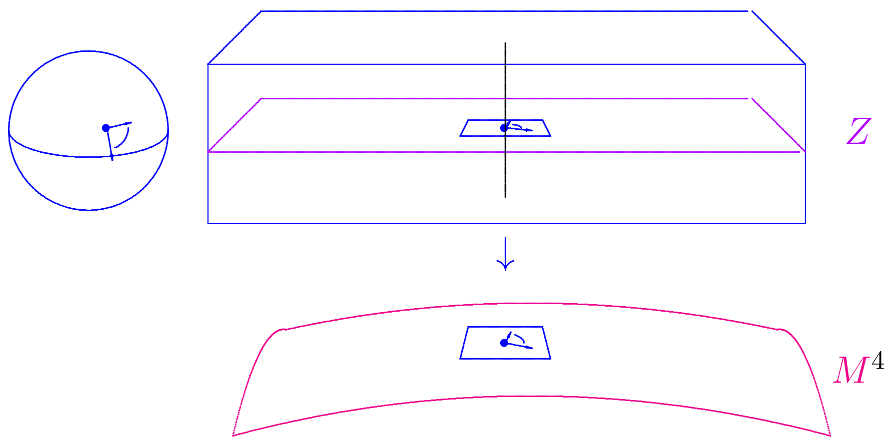

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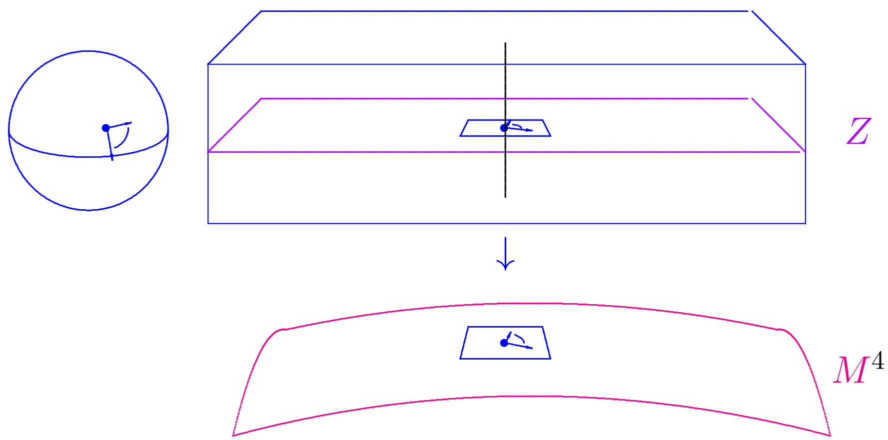

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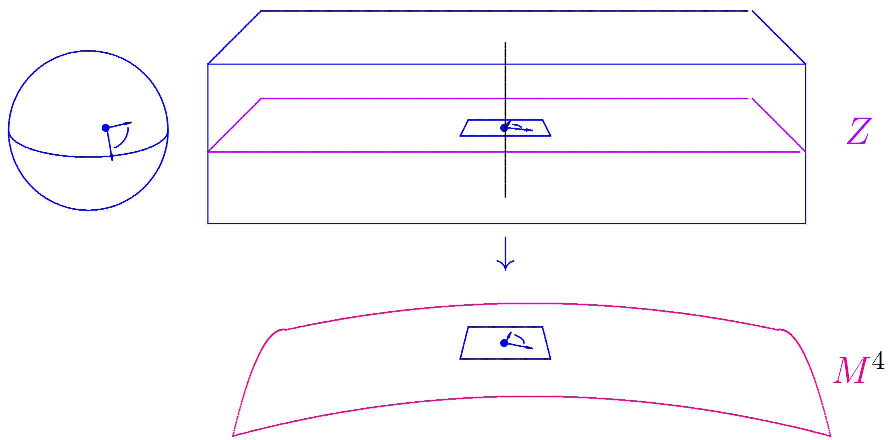

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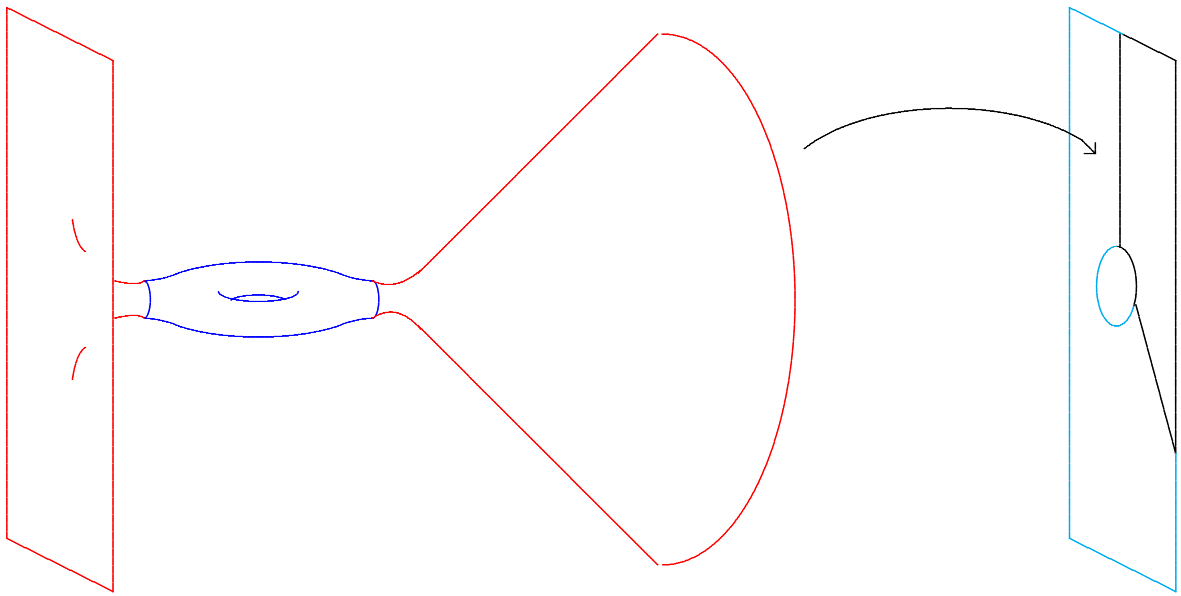
But full classification remains an open problem.

Definition. *Complete, non-compact* n*-manifold* (M^n, g) *is* asymptotically locally Euclidean *(ALE) if* $\exists$ *compact set* $K \subset M$ *such that* $M - K \approx \coprod_i (\mathbb{R}^n - D^n)/\Gamma_i$, *where* $\Gamma_i \subset \mathbf{O(n)}$, *such that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

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$$\mathfrak{m}(M, g) := \qquad \left[g_{ij,i} - g_{ii,j}\right]$$

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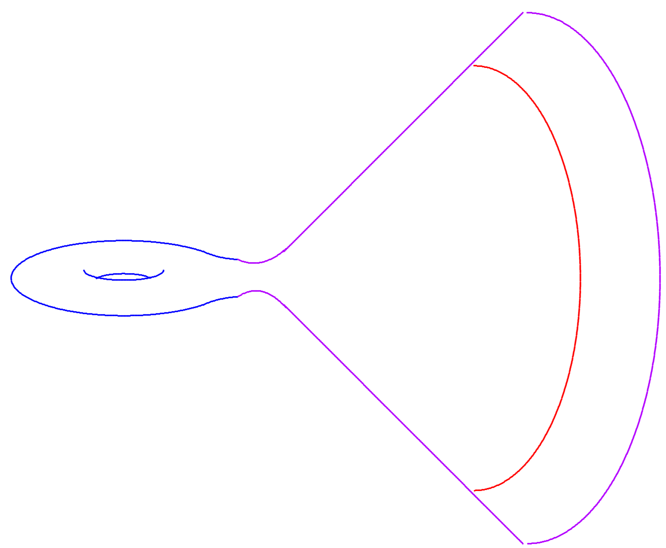

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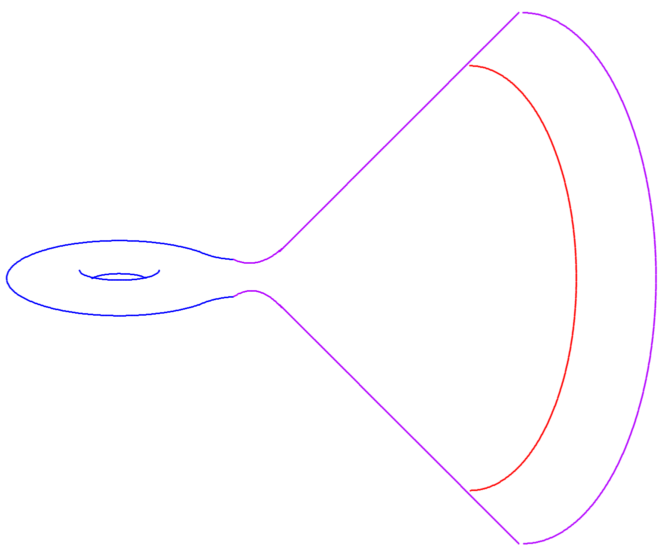

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Seems to depend on choice of coordinates!

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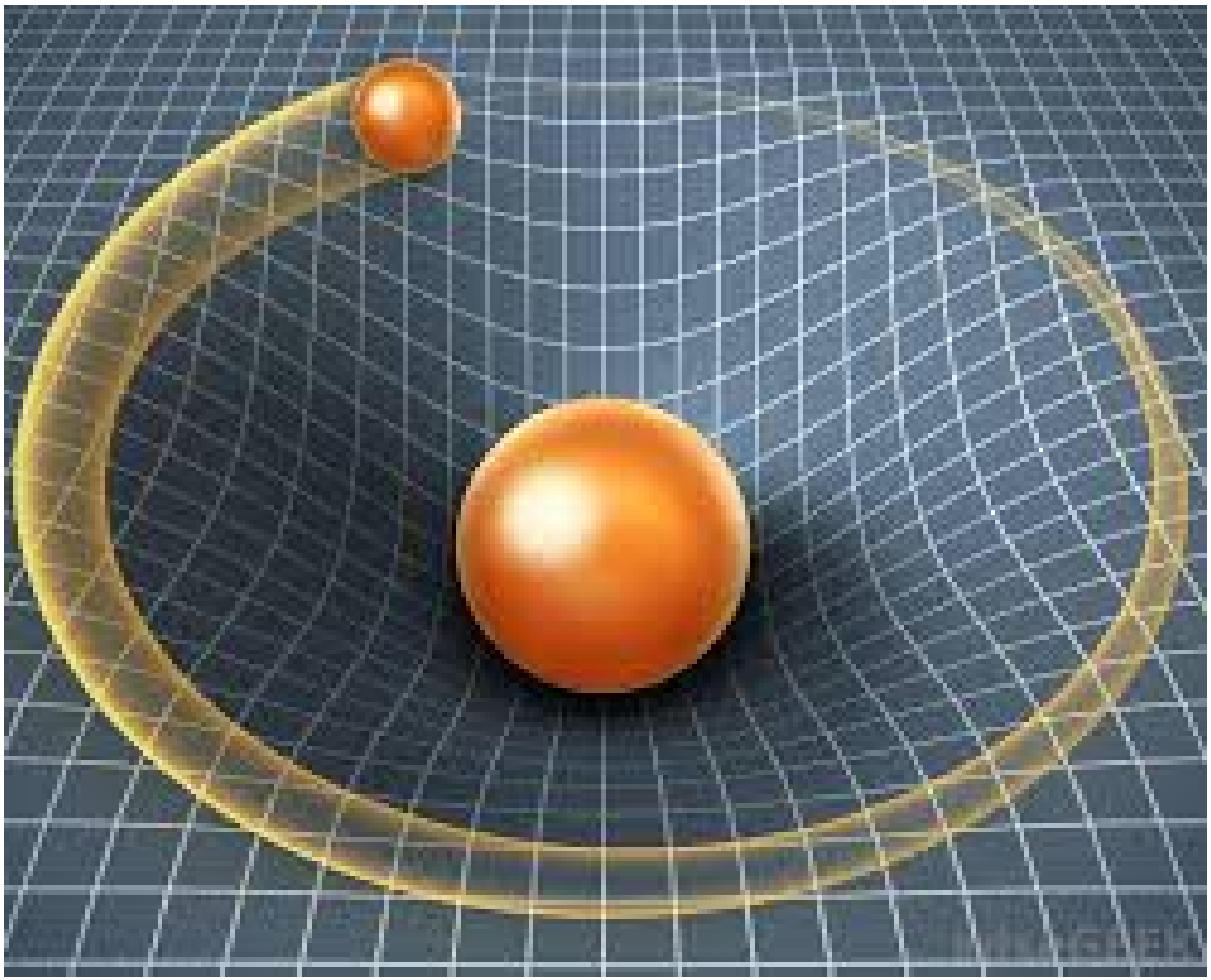

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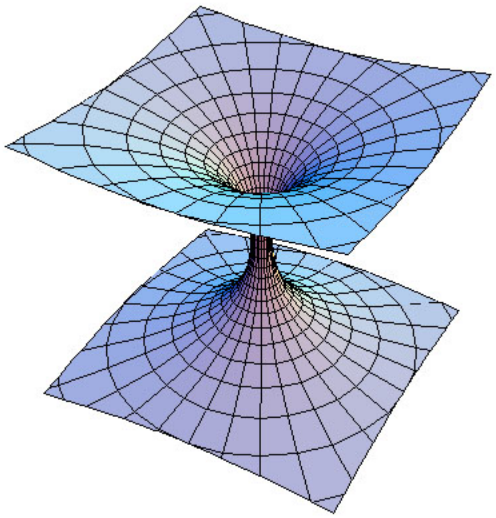

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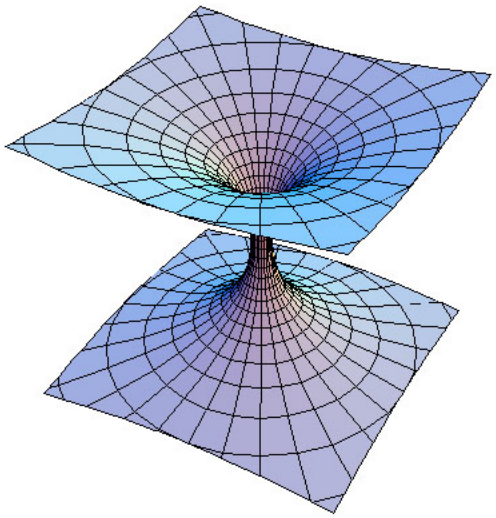

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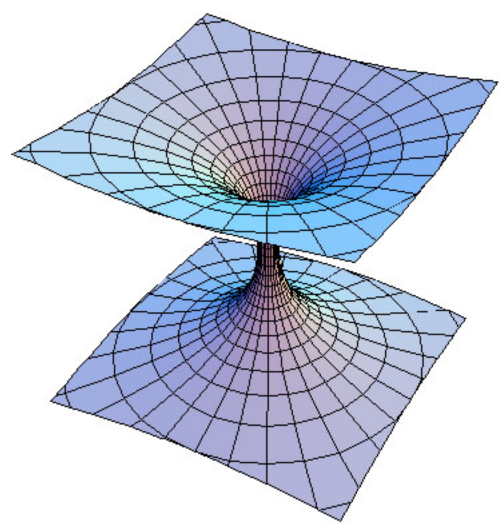

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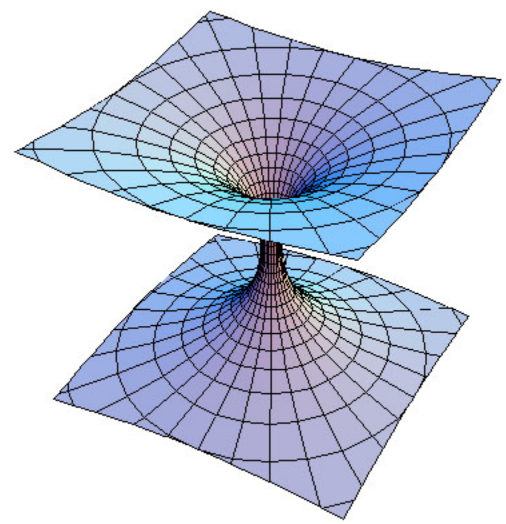

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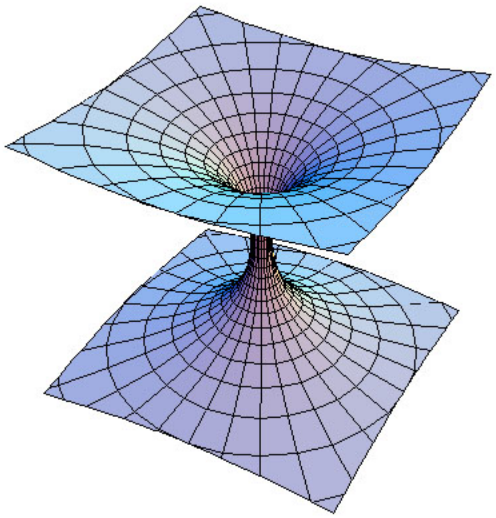

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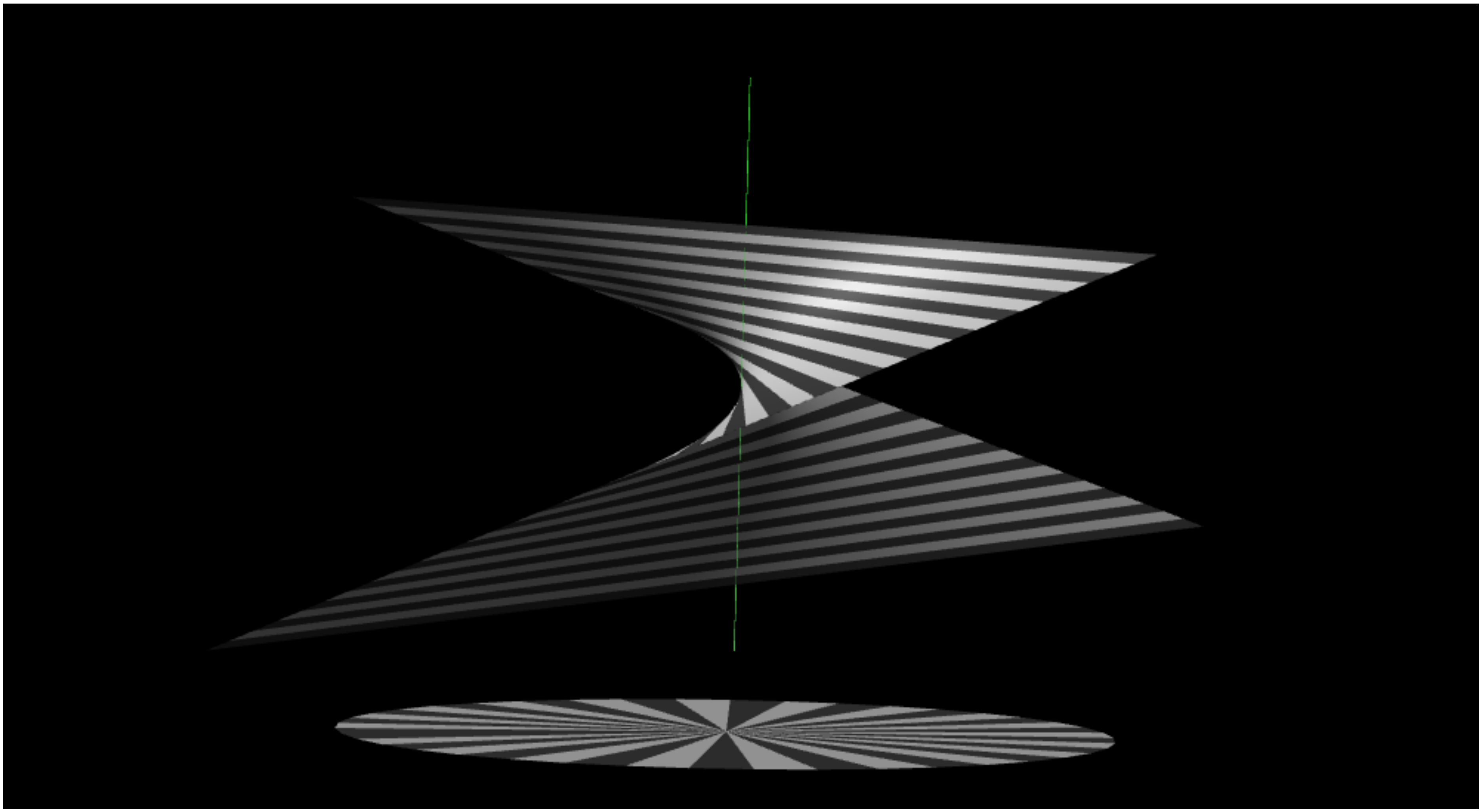

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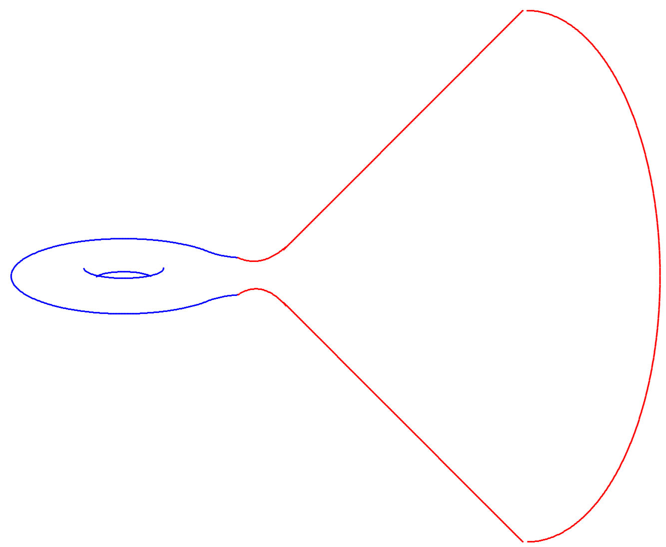

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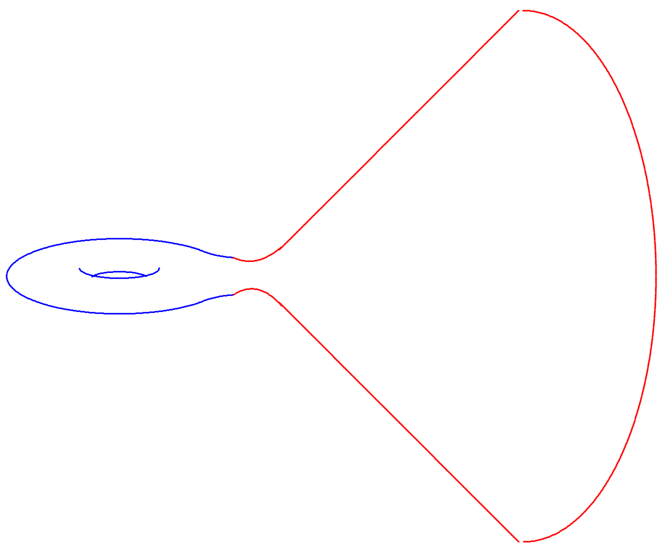

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

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Non-minimal resolutions typically admit families of such metrics for which the mass can be continuously deformed from negative to positive.

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So the mass is a "boundary correction" to the topological formula for the total scalar curvature.

Theorem C. *Any ALE Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

$$\mathfrak{m}(M,g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g d\mu_g$$

Corollary. *Any ALE scalar-flat Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}}.$$

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So **Theorem A** is an immediate consequence!

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$$g^{jk}\left(g_{j\ell,k} - g_{jk,\ell}\right)\nu^\ell \alpha_E = -\star d \log\left(\sqrt{\det g}\right) + O(\varrho^{-3-\varepsilon}).$$

$$\mathfrak{m}(M, g) = -\lim_{\varrho \to \infty} \frac{1}{12\pi^2} \int_{S_\varrho/\Gamma} \star\, d\left(\log \sqrt{\det g}\right)$$

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However, since $s = 0$,

$$d(\theta \wedge \omega) = \rho \wedge \omega = \frac{s}{4}\omega^2 = 0.$$

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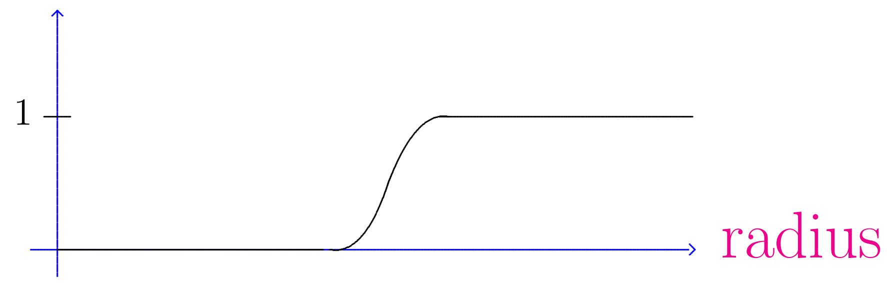

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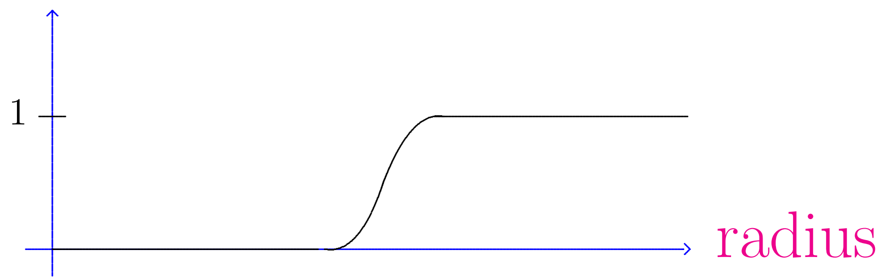

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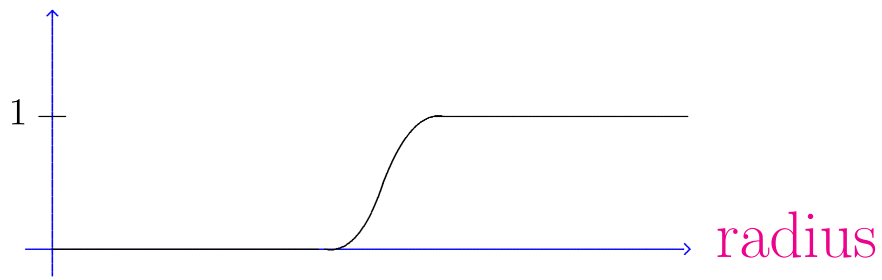

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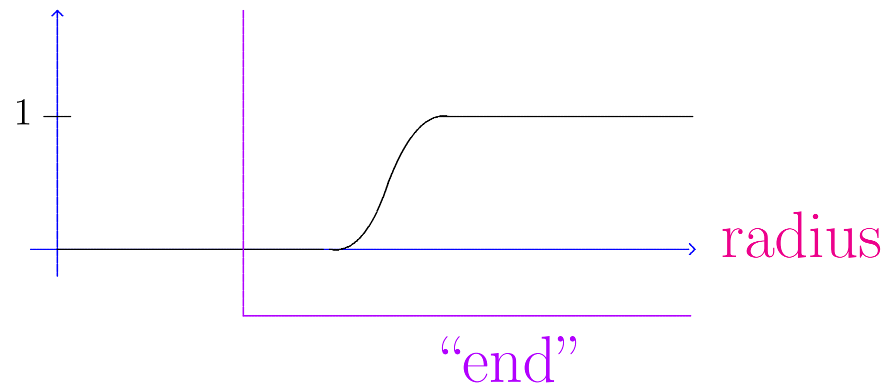

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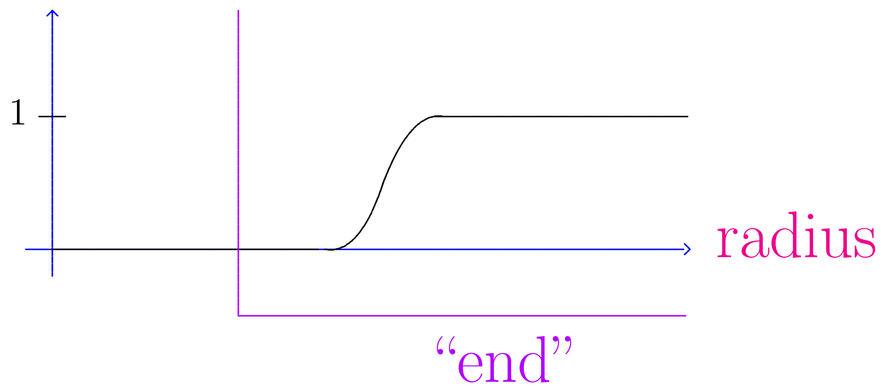

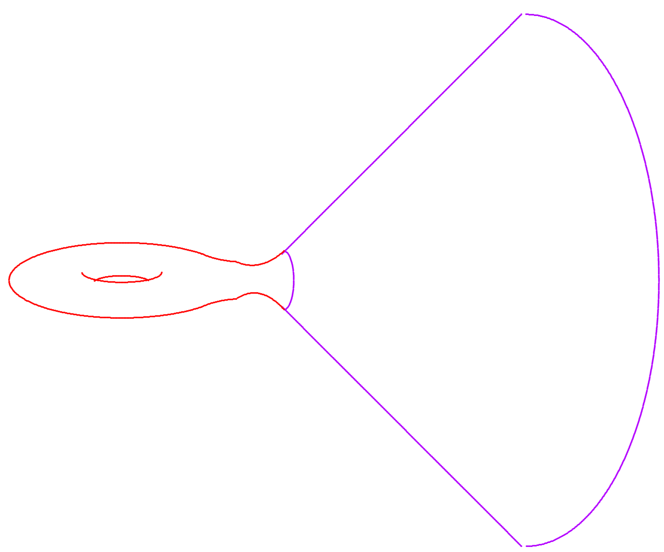

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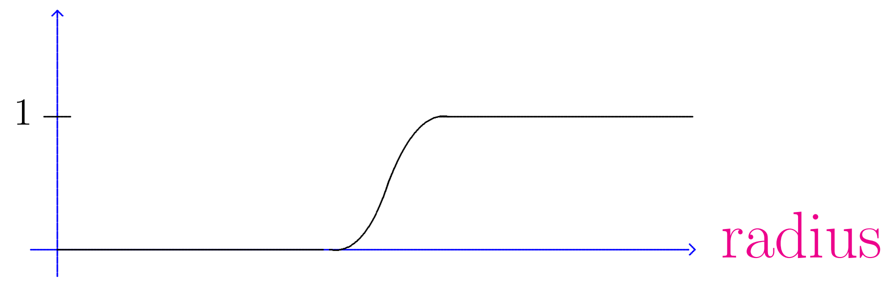

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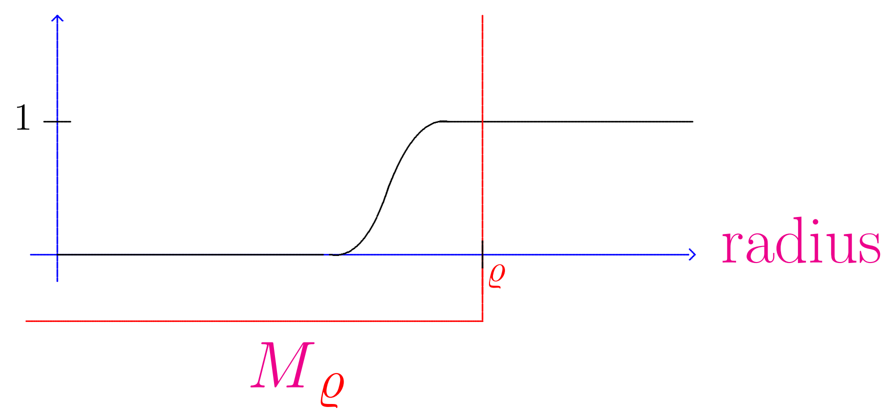

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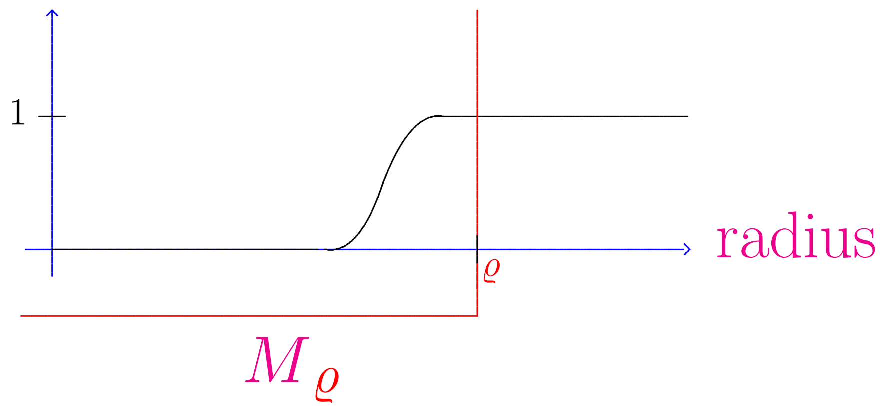

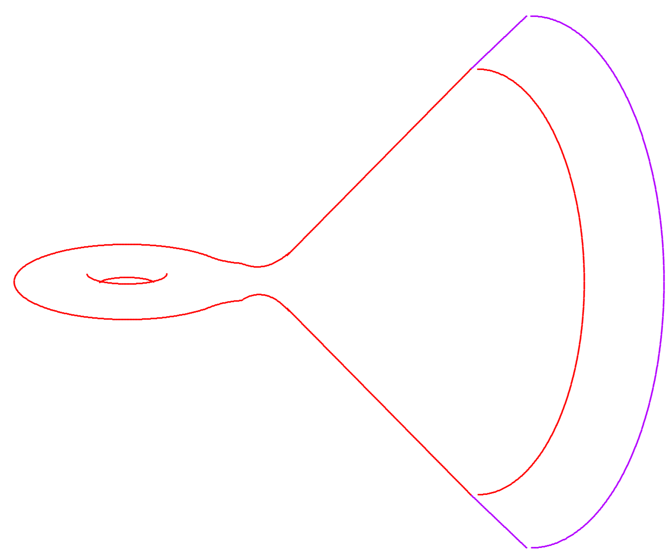

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Compactly supported, because $d\theta = \rho$ near infinity.

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where M_ϱ defined by radius $\leq \varrho$.

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The last point is serious.

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Seen in "gravitational instantons"

and other explicit examples.

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One argument proceeds by osculation:

$$J = J_0 + O(\varrho^{-3}), \qquad \nabla J = O(\varrho^{-4})$$

in suitable asymptotic coordinates adapted to g.

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Complete analytic family encodes info about J.

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This has some interesting consequences. . .

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Theorem D (Positive Mass Theorem). *Any AE Kähler manifold with non-negative scalar curvature has non-negative mass:*

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Proof actually shows something stronger!

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so the mass formula implies the claim.

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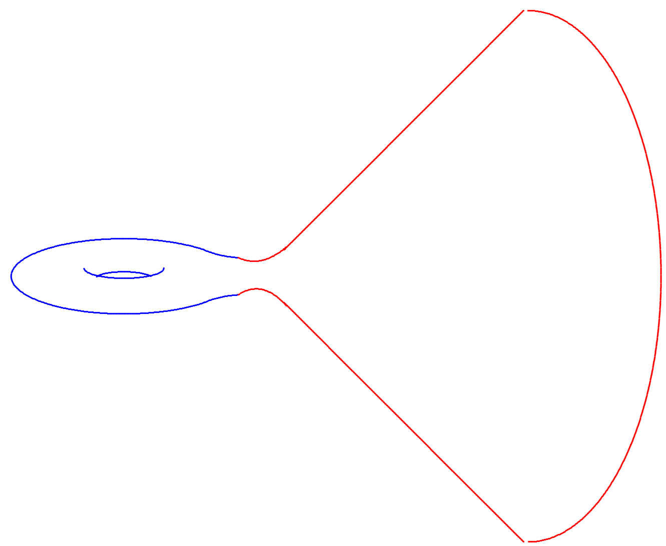

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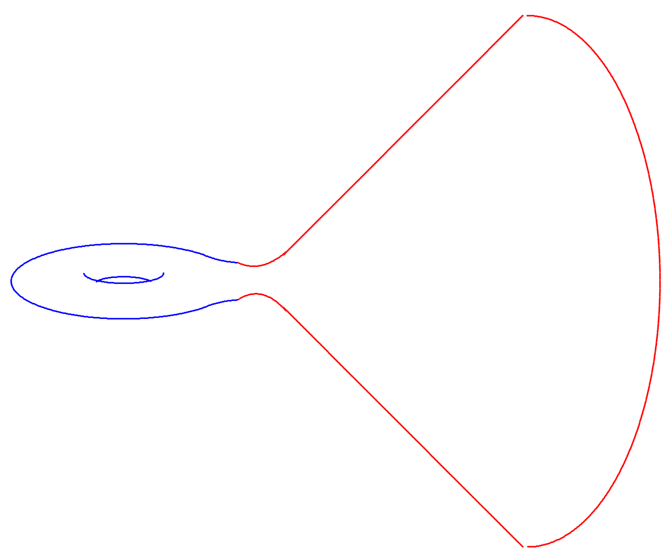

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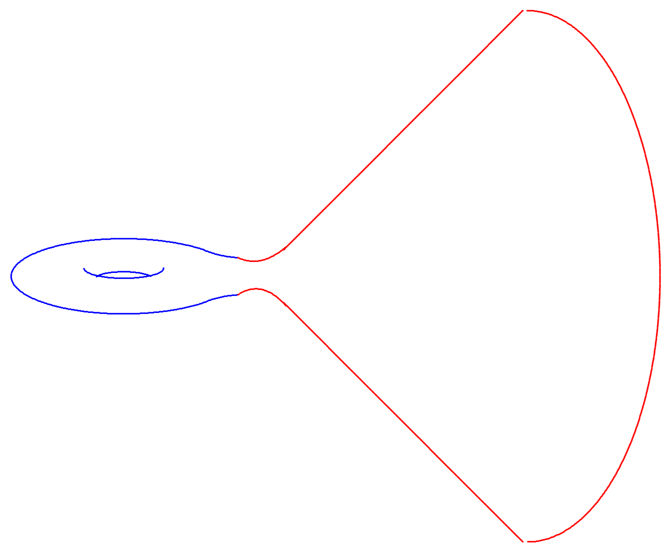
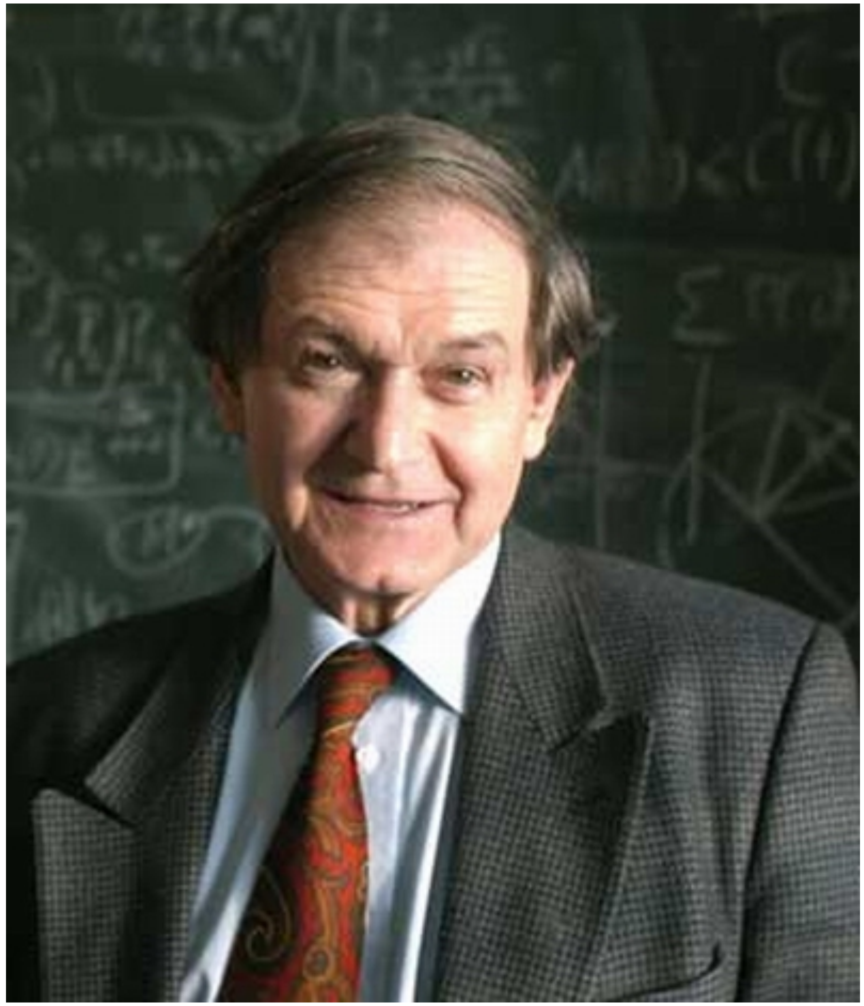
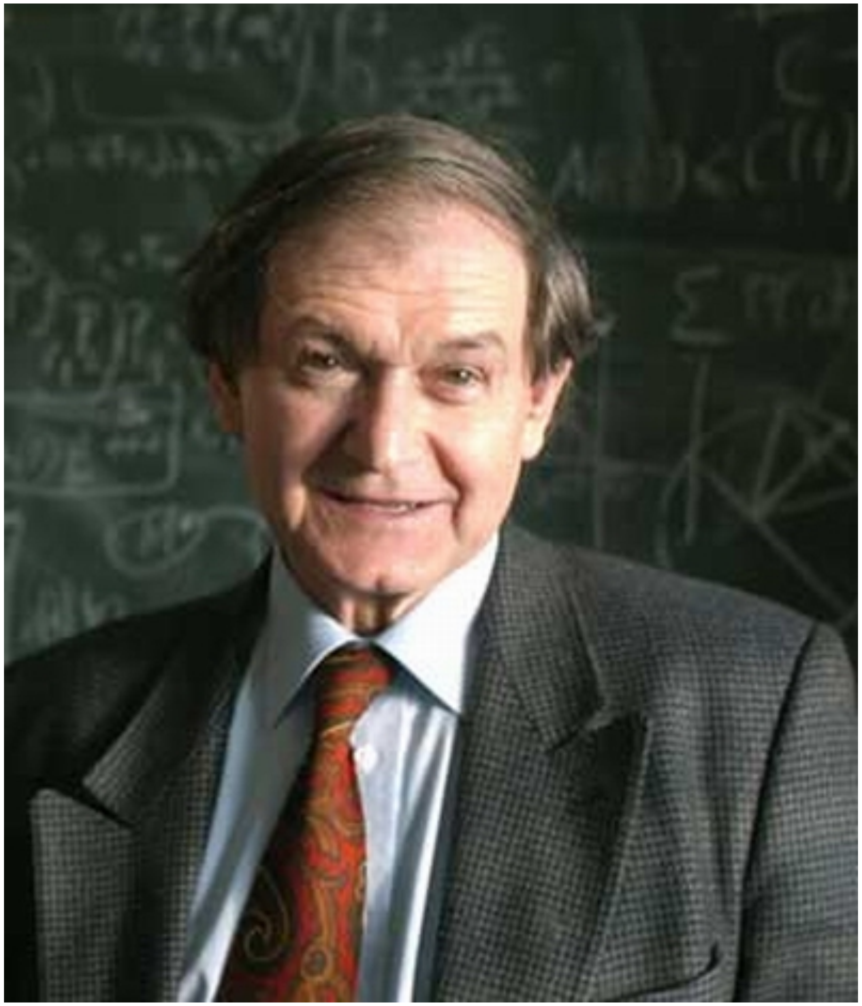
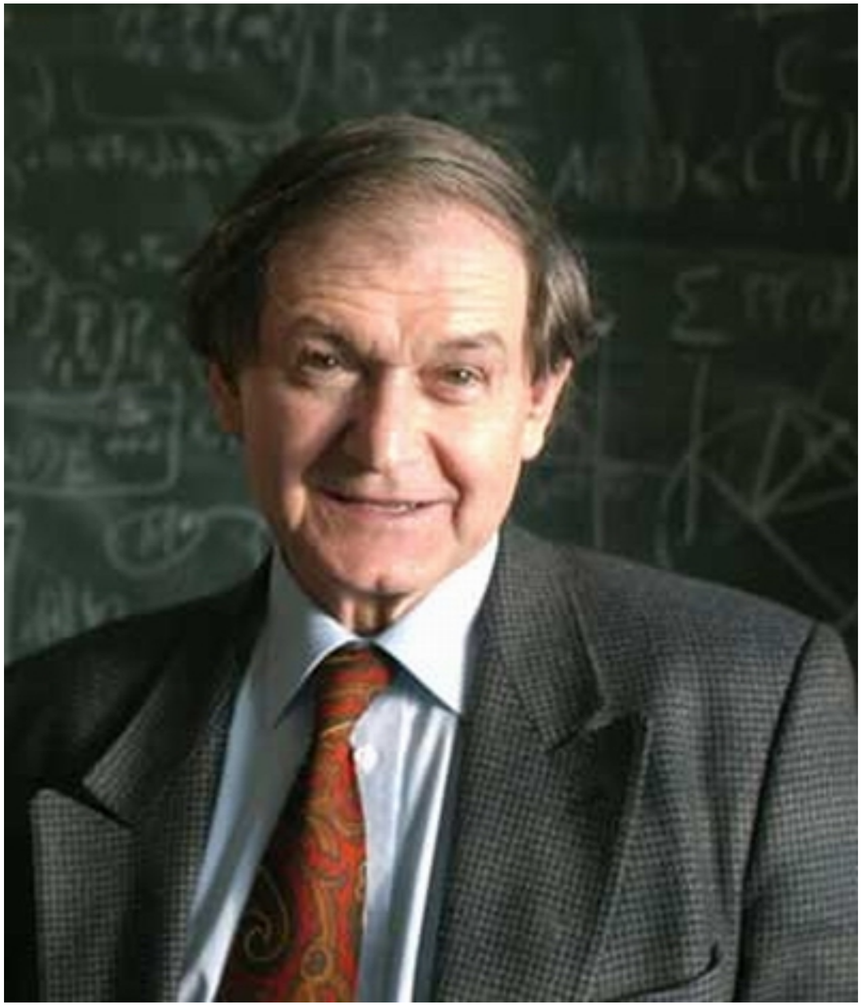






Happy Birthday, Roger!



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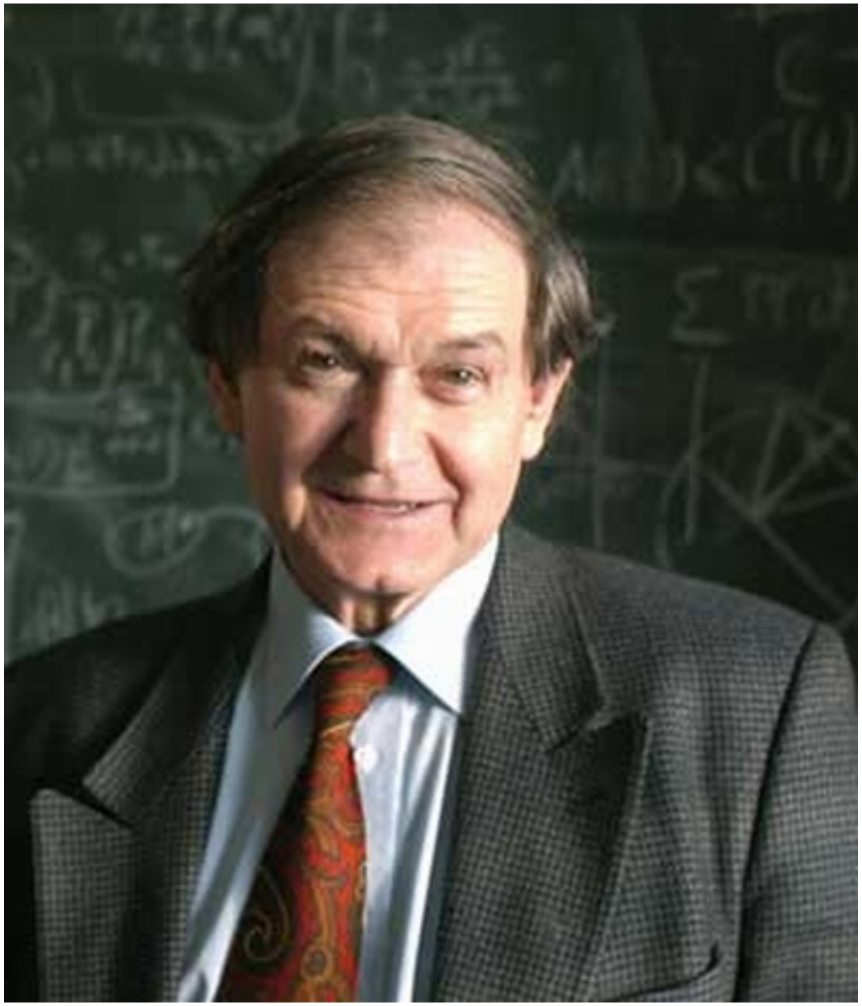

Happy Birthday, Roger!

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Happy Non-Retirement, Nick!