The Einstein-Weyl Equations,

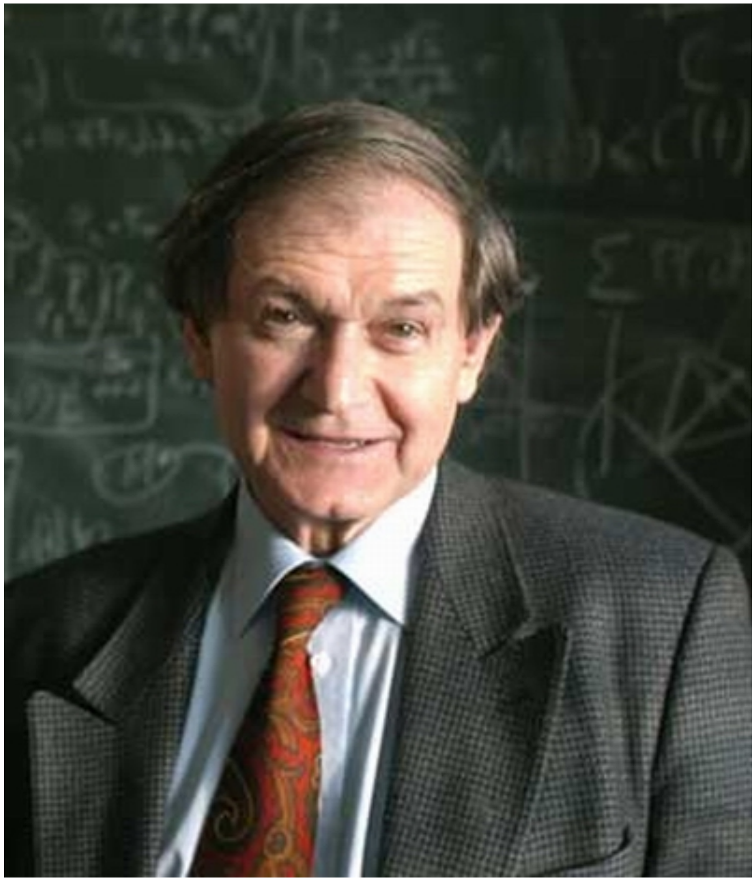
Scattering Maps, and

Holomorphic Disks

Claude LeBrun
Stony Brook University

Oxford, September 13, 2013

For Roger Penrose



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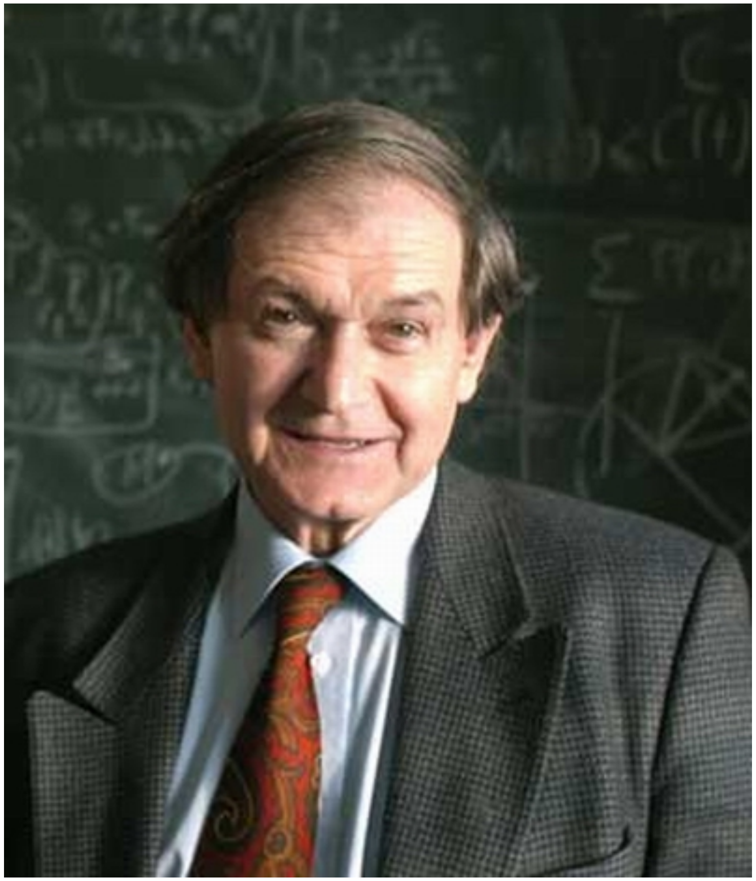

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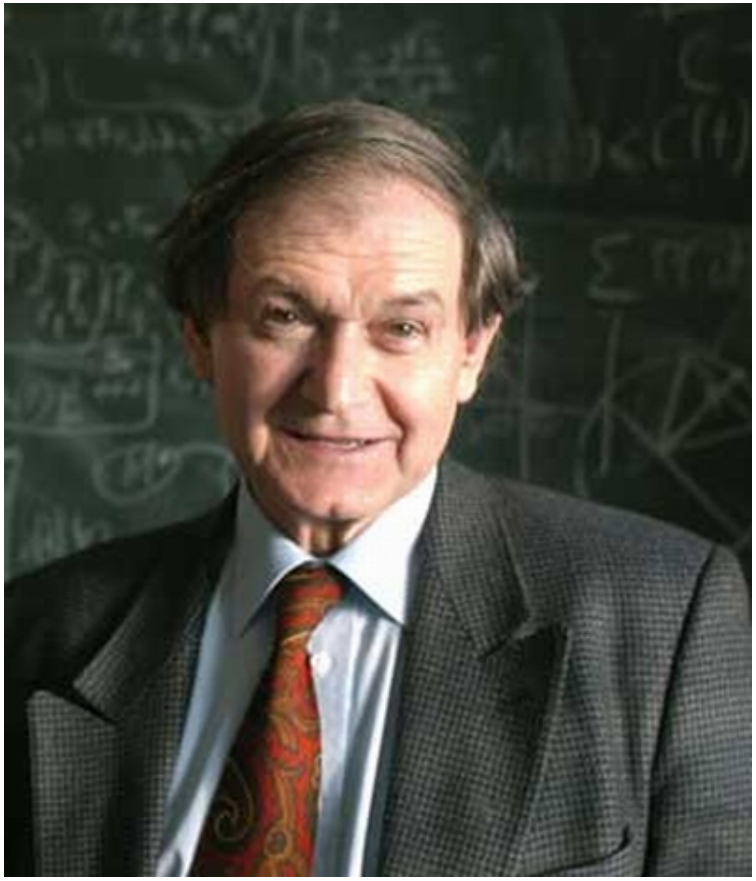

who discovered so many remarkable links between complex manifolds and space-time geometry;

and Paul Tod

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of the Einstein-Weyl equations.

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But — story takes place in $(2+1)$-dimensions!

Joint work with

Lionel Mason
University of Oxford

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Main references:

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- The Einstein-Weyl Equations, Scattering Maps, and Holomorphic Disks, Math. Res. Lett. 16 (2009) 291–301.

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$$\nabla_v g \propto g \quad \forall v$$

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where $\nu = d \log u$.

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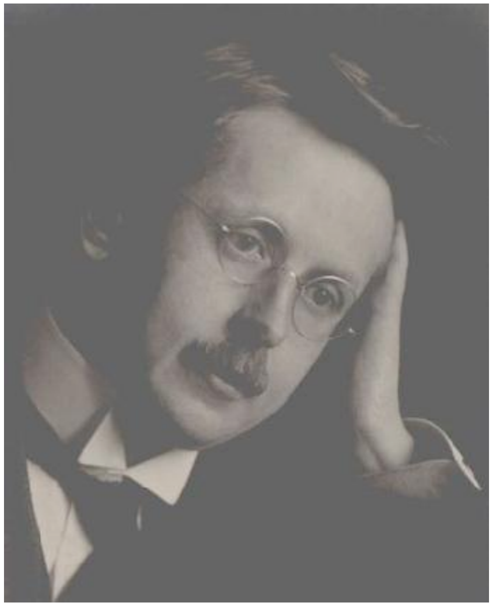
Hermann Weyl



$$\mathfrak{W} = (\mathfrak{G} + \alpha \mathfrak{l}) + \frac{\varepsilon^2}{4} \sqrt{g} \{ 1 - 3 (\varphi_i \varphi^i) \},$$

$$\Gamma^r_{ik} = \left\{ {ik \atop r} \right\} + \tfrac{1}{2} \varepsilon^2 (\delta^r_i \varphi_k + \delta^r_k \varphi_i - g_{ik} \varphi^r).$$

Unter Vernachlässigung der winzigen kosmologischen Terme erhalten wir hier also genau die klassische Maxwell-Einsteinsche Theorie der Elektrizität und Gravitation. Um Übereinstimmung mit den in § 34 verwendeten

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$$\boxed{r_{(jk)} = f g_{jk}}$$

for some function f.

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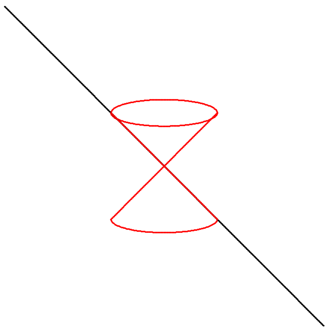

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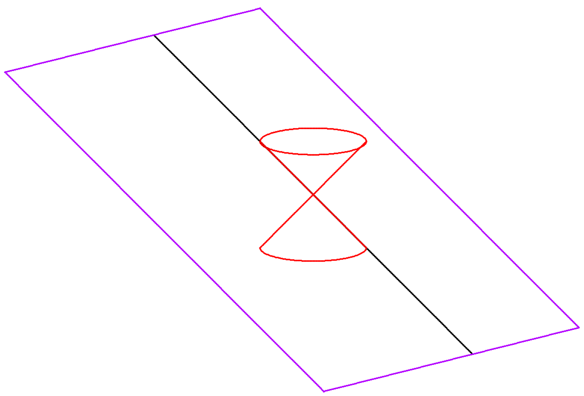

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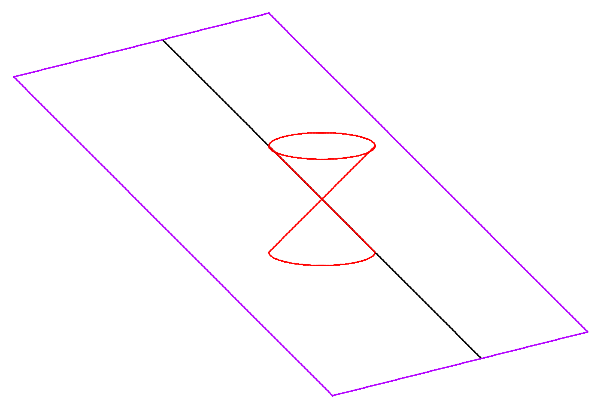

Equations $\Longleftrightarrow$ $\exists$ totally geodesic null surfaces.

Élie Cartan

$$\begin{cases} [\omega_1 \Omega_{23}] = [\omega_2 \Omega_{31}] = [\omega_3 \Omega_{12}], \\ [\omega_2 \Omega_{12}] + [\omega_3 \Omega_{31}] = 0, \\ [\omega_3 \Omega_{23}] + [\omega_1 \Omega_{12}] = 0, \\ [\omega_1 \Omega_{31}] + [\omega_2 \Omega_{23}] = 0. \end{cases}$$

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THÉORÈME. — *Les espaces de Weyl à trois dimensions qui admettent ∞^2 plans isotropes dépendent essentiellement de quatre fonctions arbitraires de deux arguments.*

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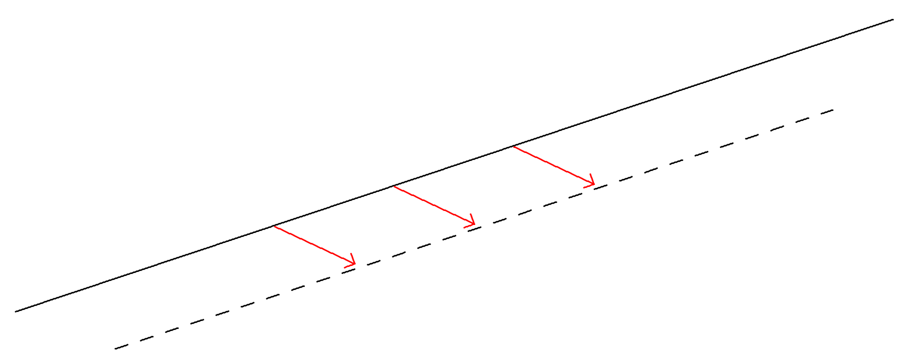

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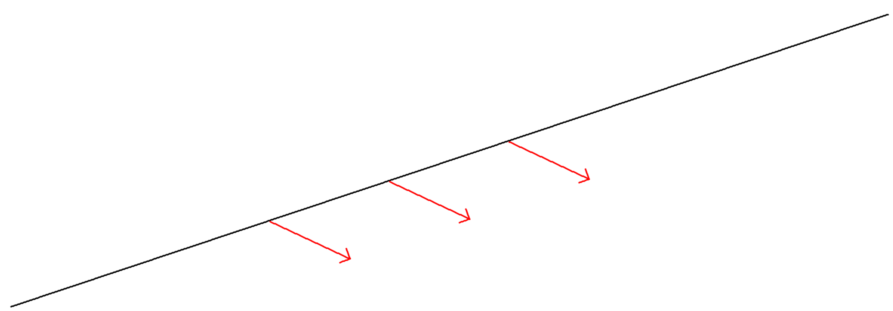

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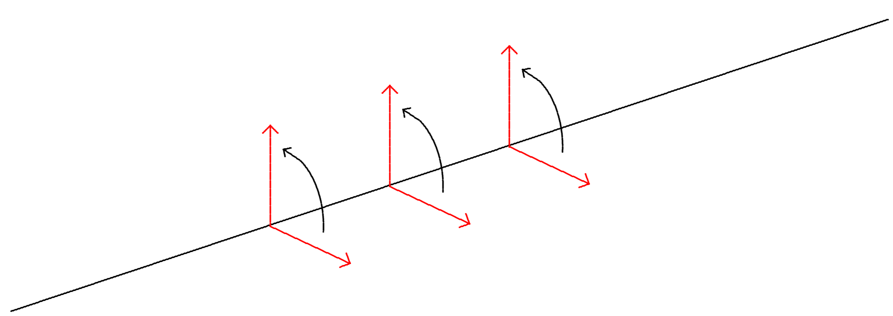

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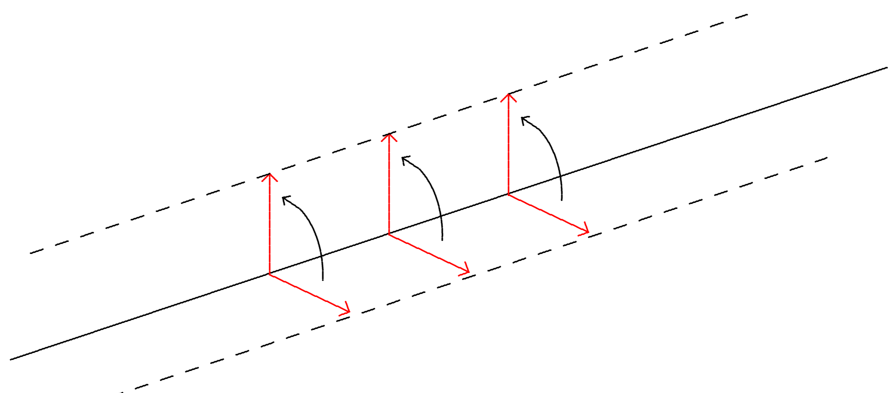

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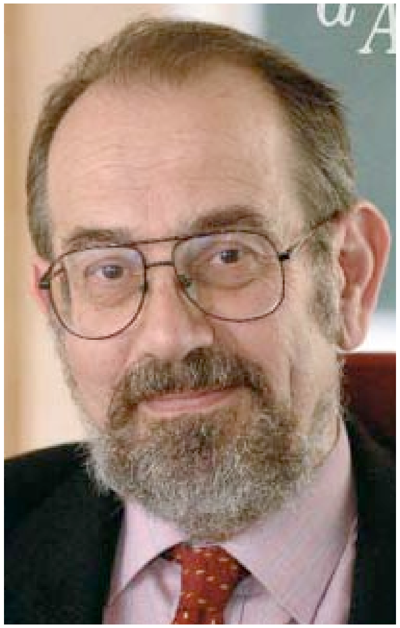

and so if

$$R(U \times V, U)U = U \times R(V, U)U, \tag{2.2}$$

then we can define a linear map

$$J(V) = U \times V \tag{2.3}$$

which satisfies

$$J^2(V) = U \times (U \times V) = (U, V)U - (U, U)V = -V$$

We thus have a real complex surface G with a family of real lines of self-intersection number 2. It can be shown that any such surface may be obtained by the above geodesic construction, but using a Weyl structure rather than a Riemannian structure. The integrability condition (2.2) is then the analogue of Einstein's equations ($R_{(ij)} = \Lambda g_{ij}$) for the Weyl structure (see [10]). This is the

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- *orientation-reversing diffeomorphisms*

$$\psi : \mathbb{CP}_1 \to \mathbb{CP}_1.$$

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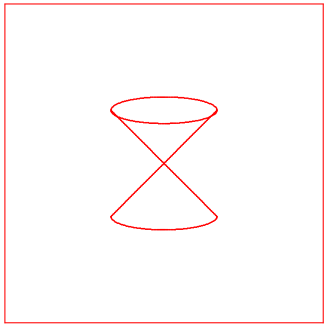

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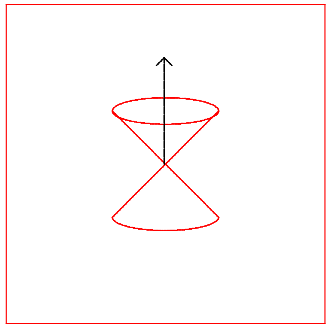

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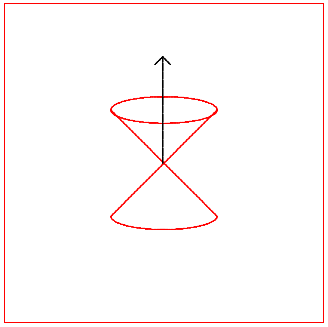

$\Longrightarrow$ M also oriented, in usual sense.

globally hyperbolic

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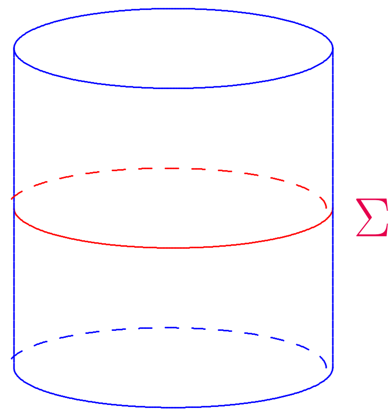

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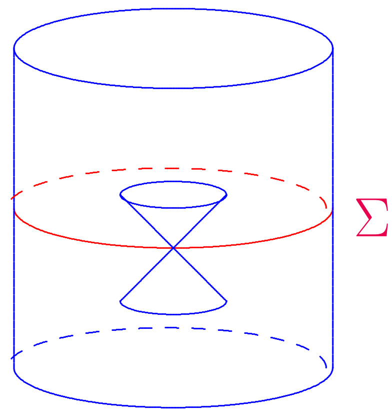

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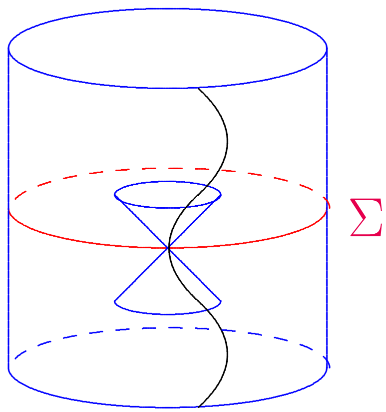

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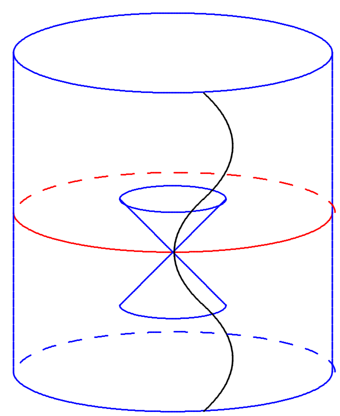

$\Longrightarrow M \approx \Sigma \times \mathbb{R}$

conformally compact

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β smooth 1-form with $\beta|_{\partial X} \equiv 0$.

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$(M, [g], \triangledown)$ conformally compact if

- $M = X - \partial X$ for compact man.-w.-boundary X
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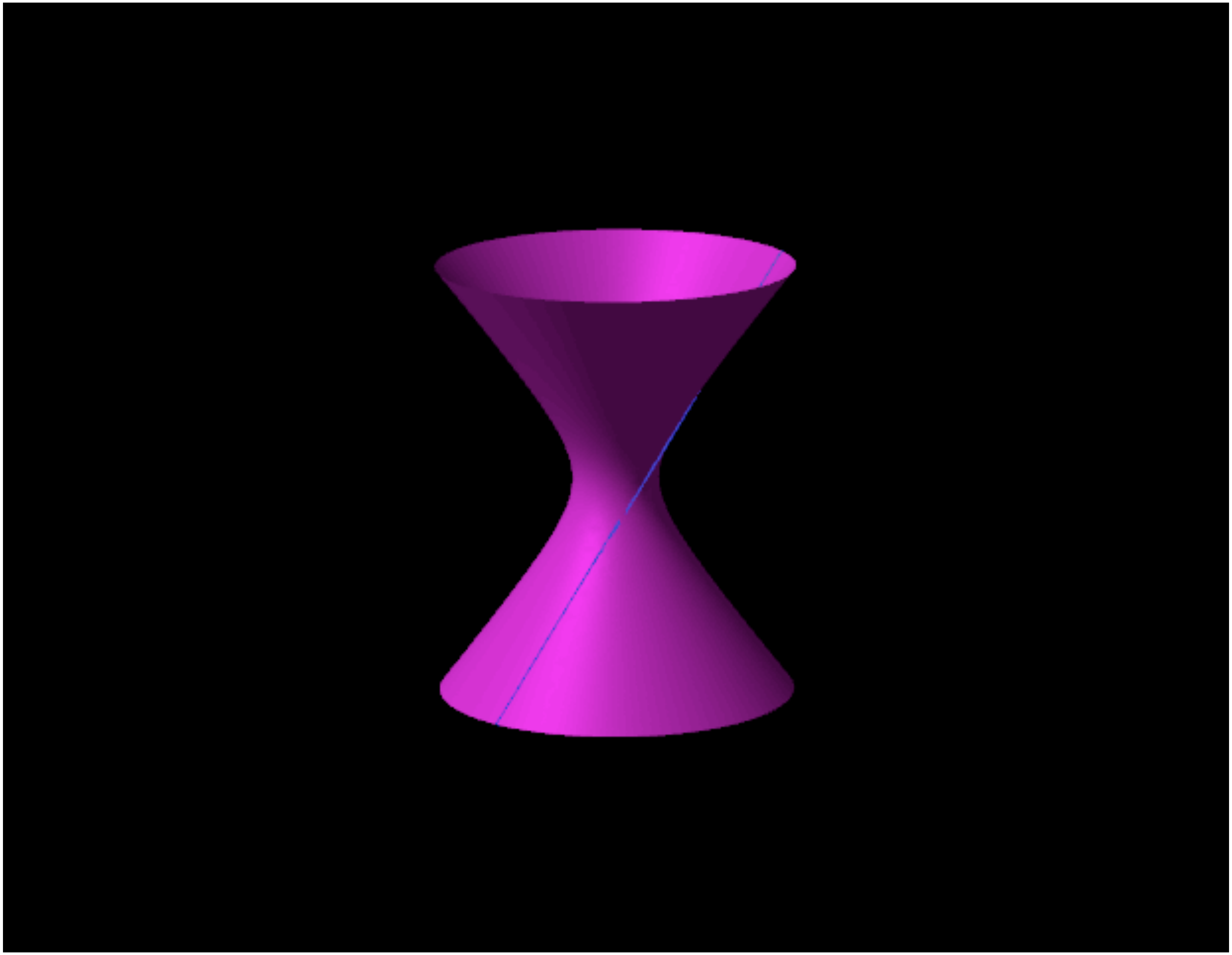

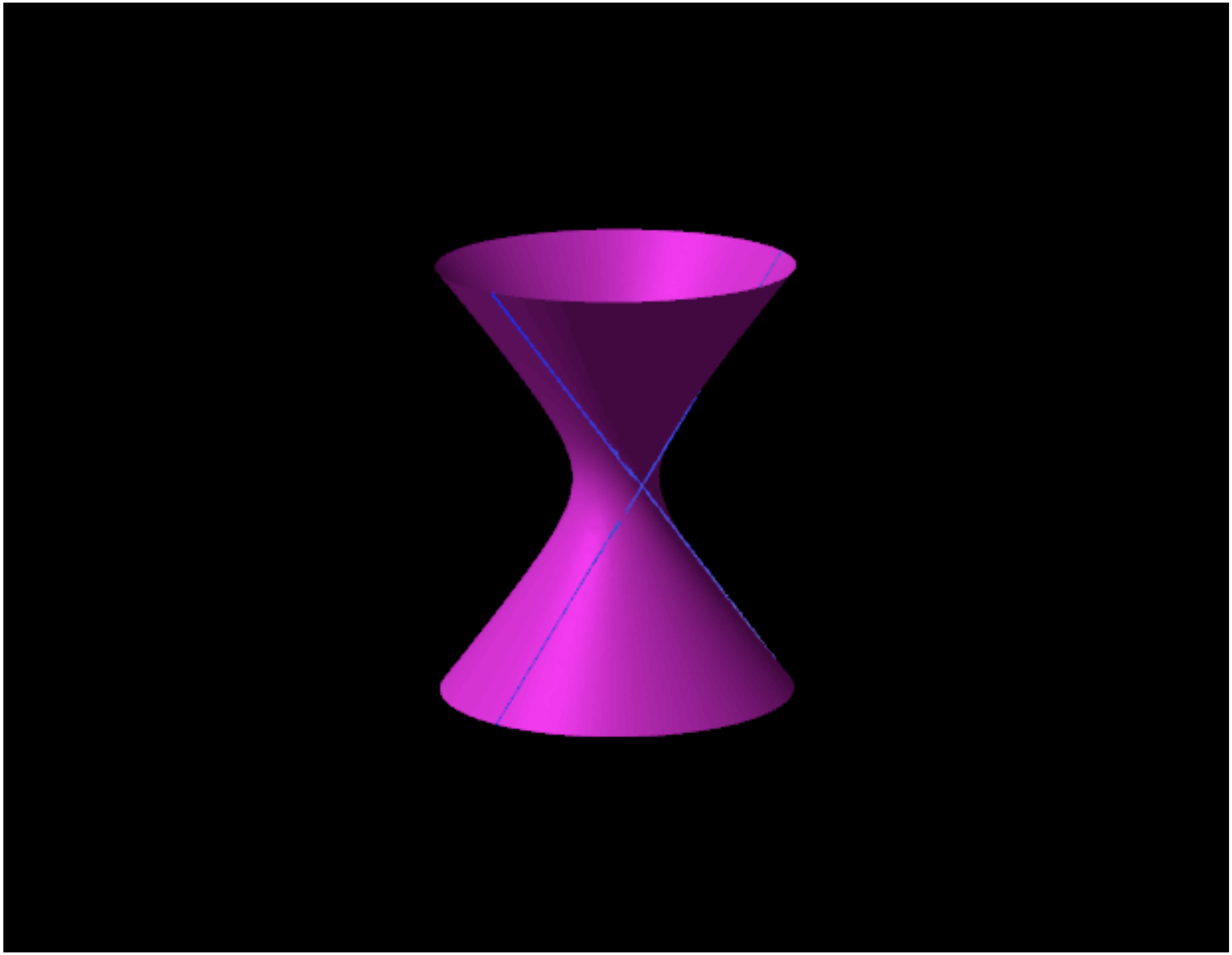

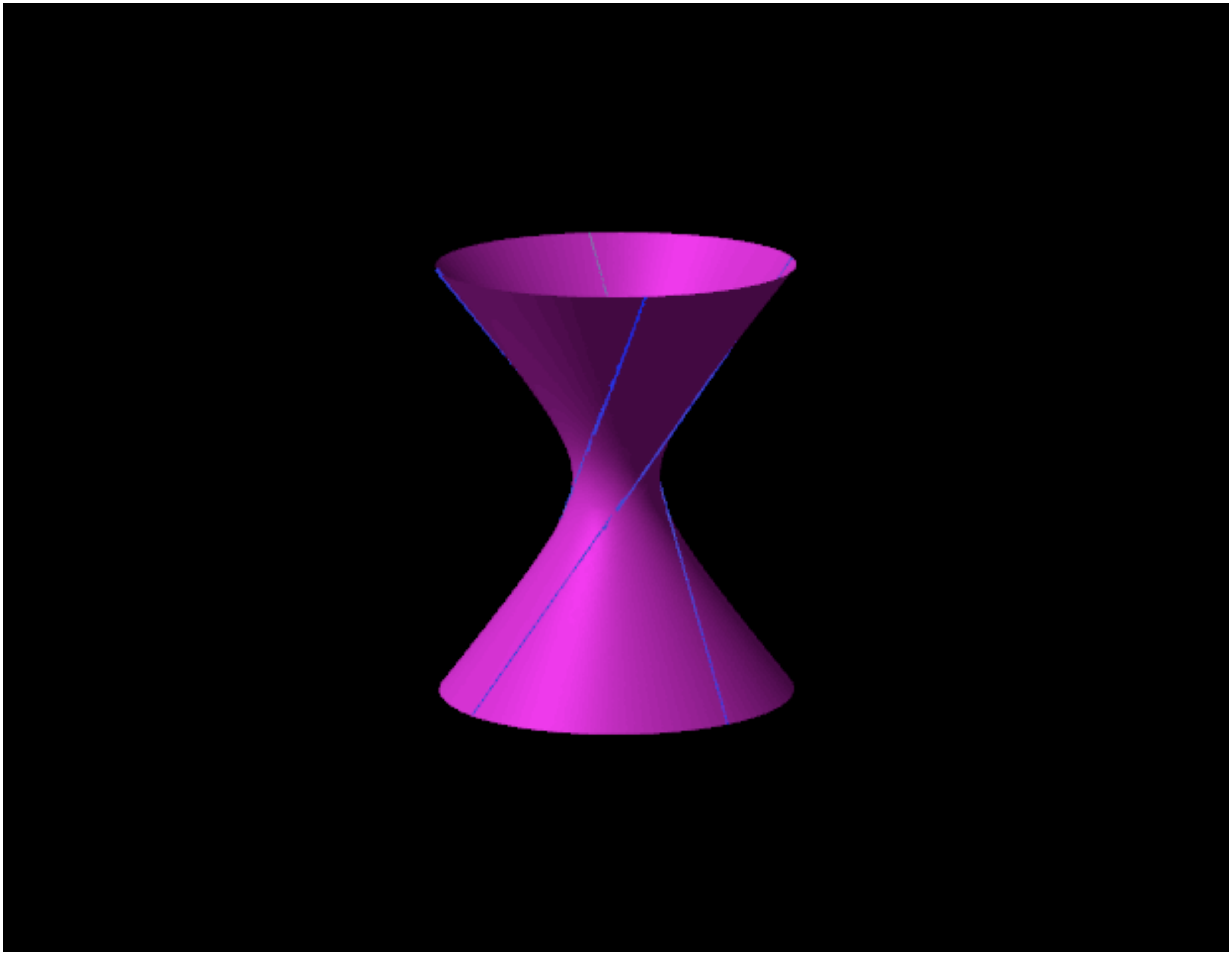

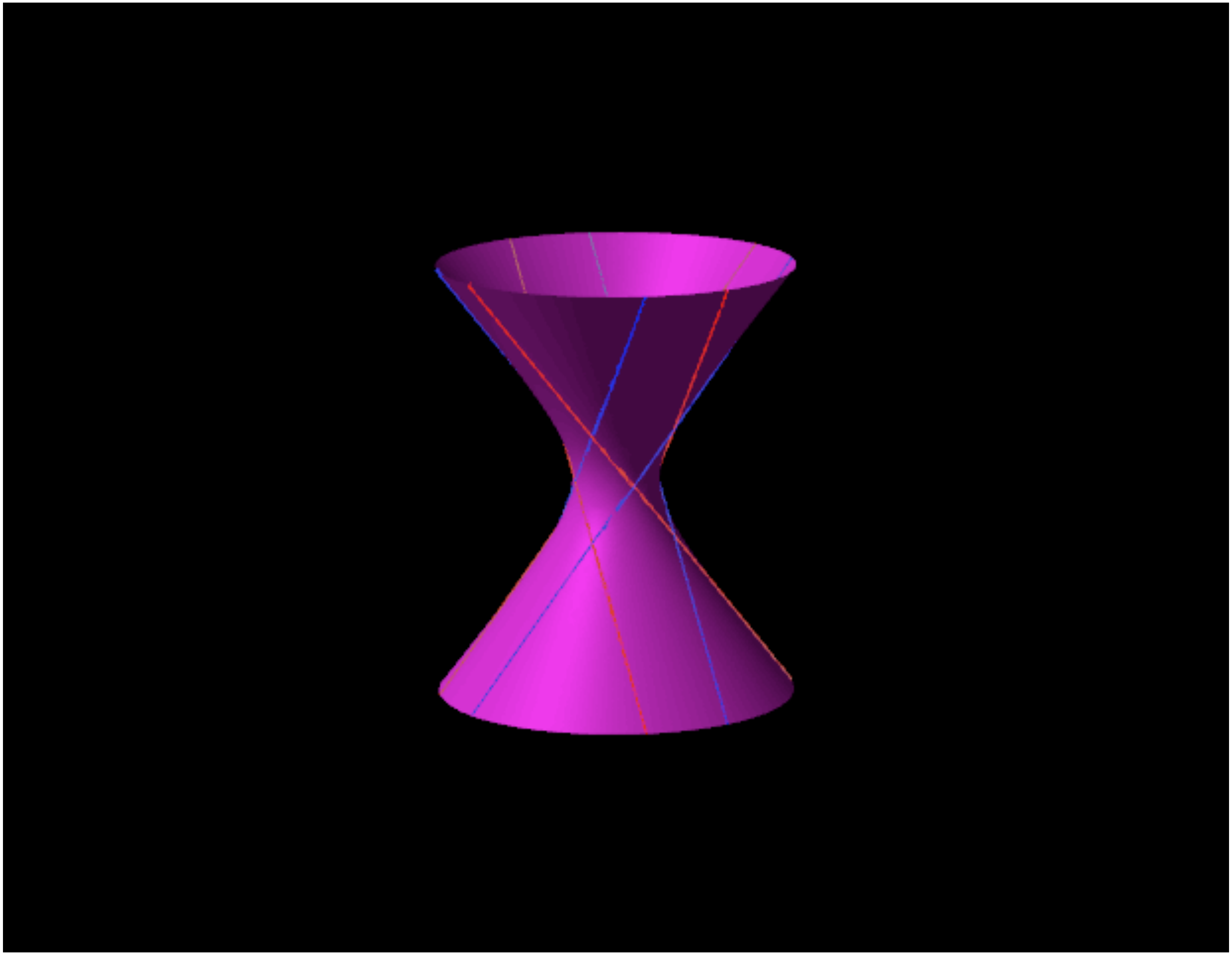

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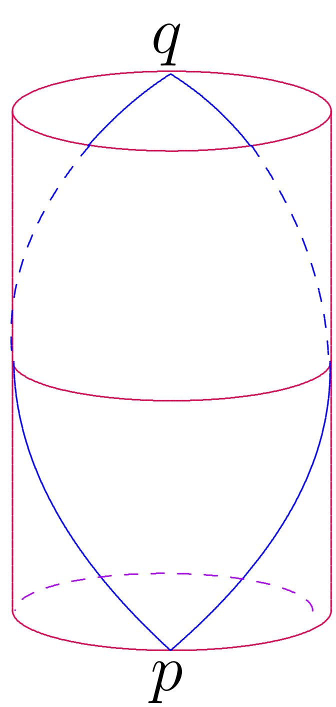

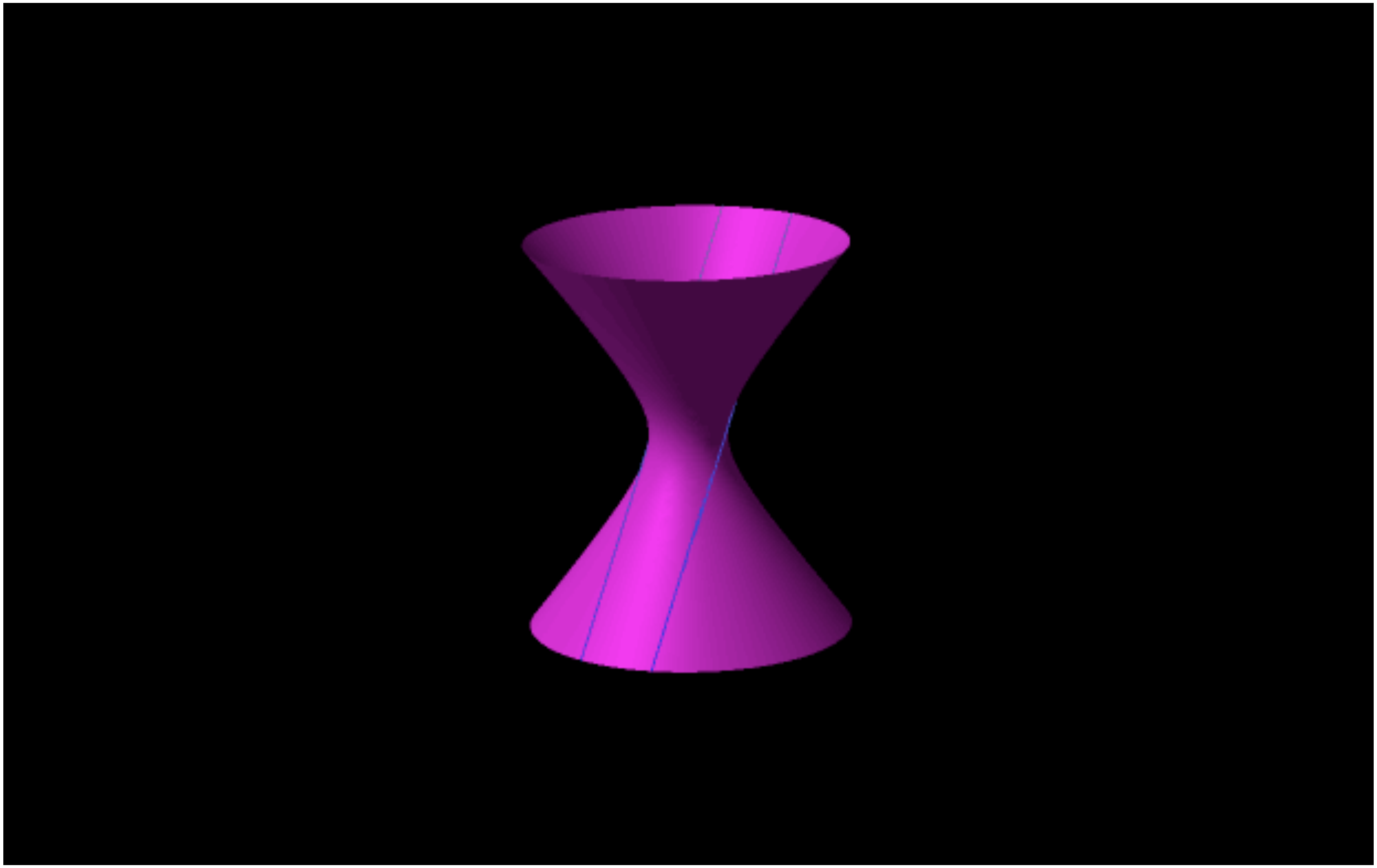

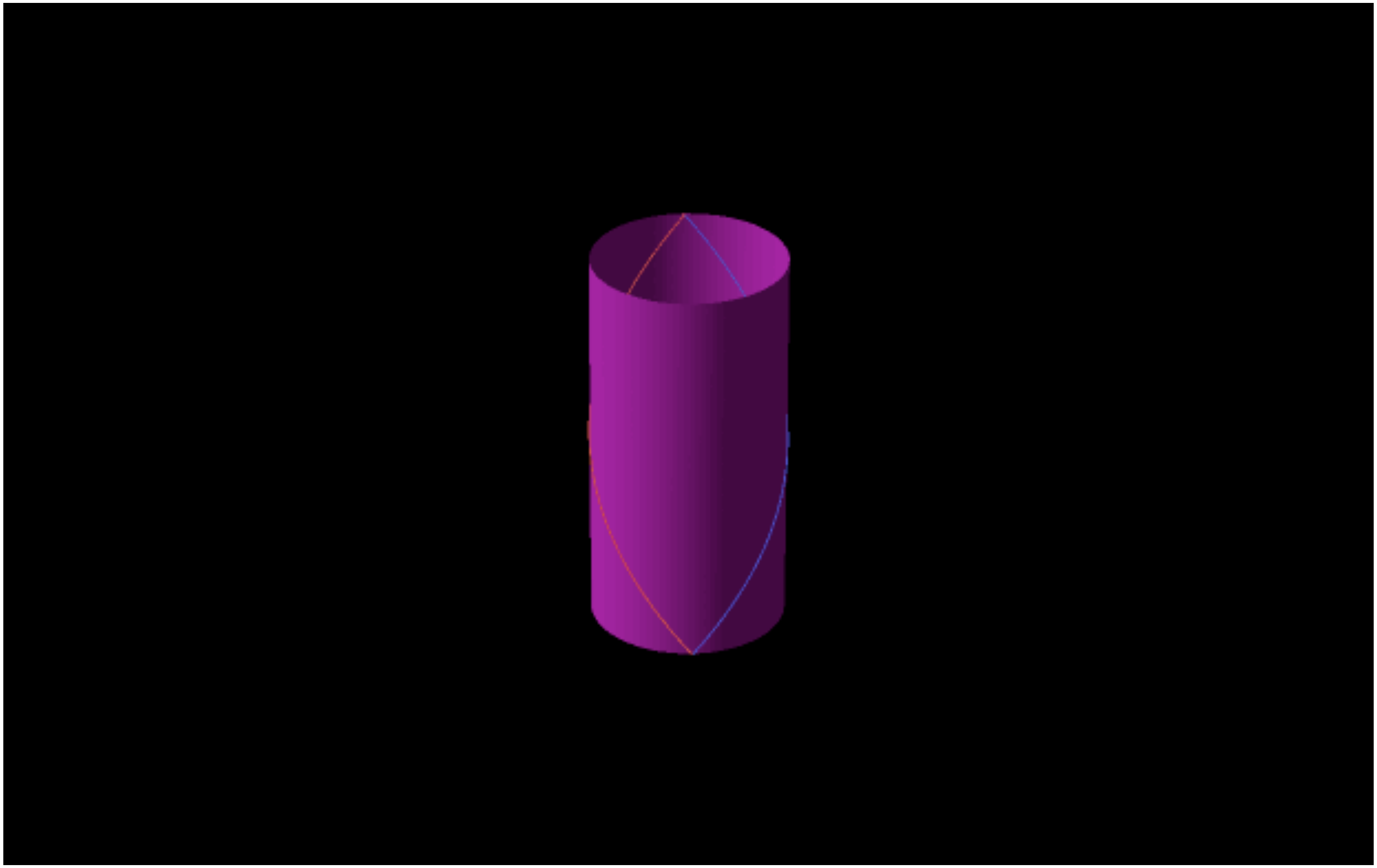

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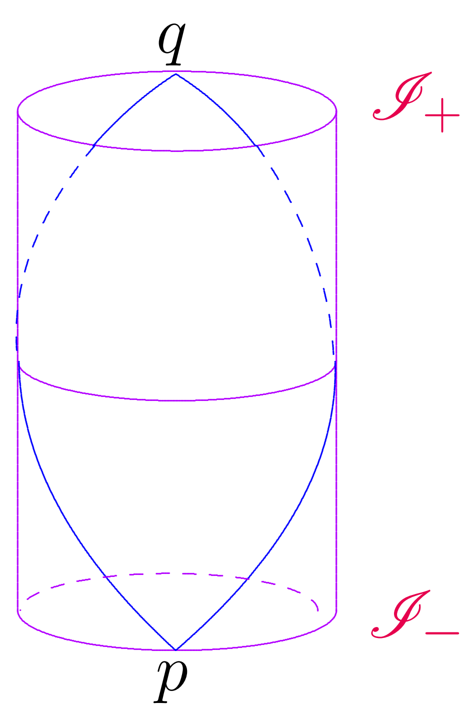

Theorem. *There is a natural one-to-one correspondence between*

- *smooth, space-time-oriented,* conformally compact*, globally hyperbolic Lorentzian Einstein-Weyl* 3*-manifolds* $(M, [g], \nabla)$*; and*

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$$\psi : \mathbb{CP}_1 \to \mathbb{CP}_1.$$

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$$\psi_1, \psi_2 : \mathbb{CP}_1 \to \mathbb{CP}_1$$

considered same iff

$$\psi_1 = \varphi \circ \psi_2 \circ \phi^{-1}$$

for Möbius transformations $\varphi, \phi \in PSL(2, \mathbb{C})$.

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Example: de Sitter $\longleftrightarrow$ antipodal map of $\mathbb{CP}_1$.

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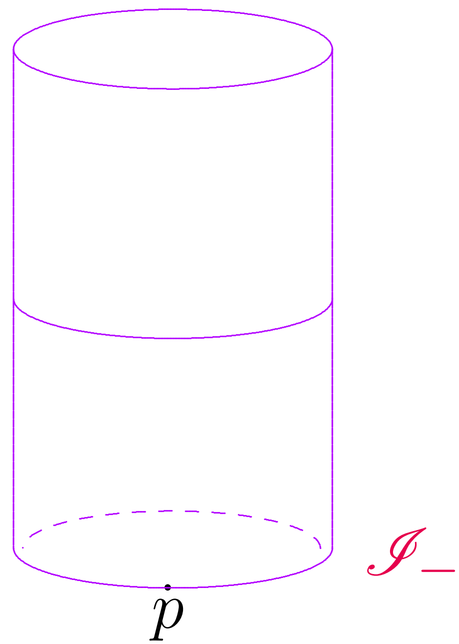
to Einstein-Weyl $(M^3, [g], \nabla)$ satisfying hypotheses.

Lemma. *For an Einstein-Weyl manifold as above, let* $p \in \mathscr{I}_-$ *be any point of past infinity.*



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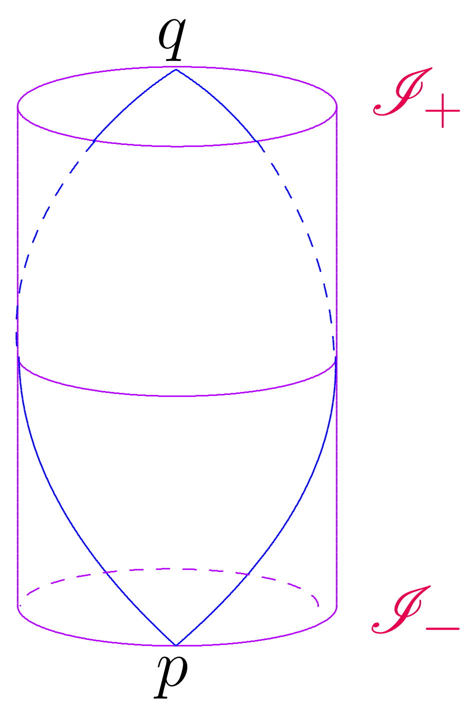

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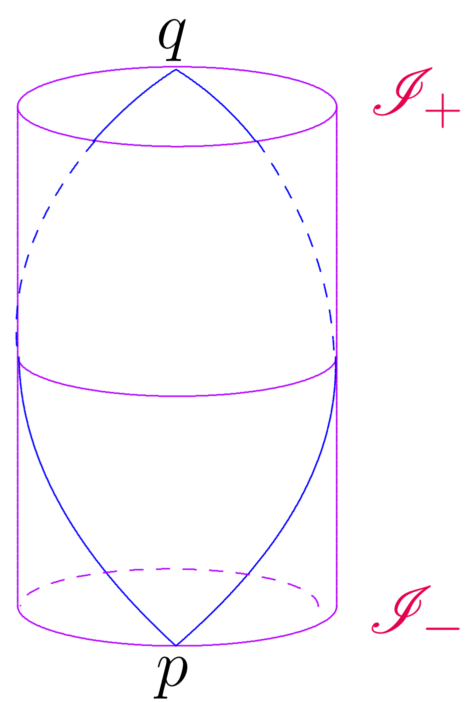

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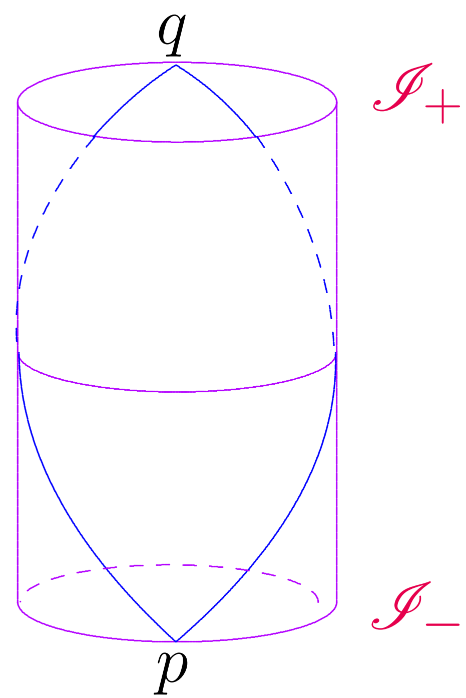

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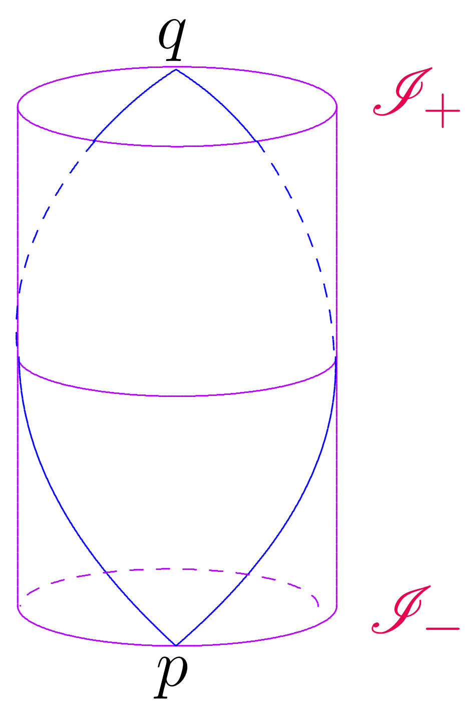
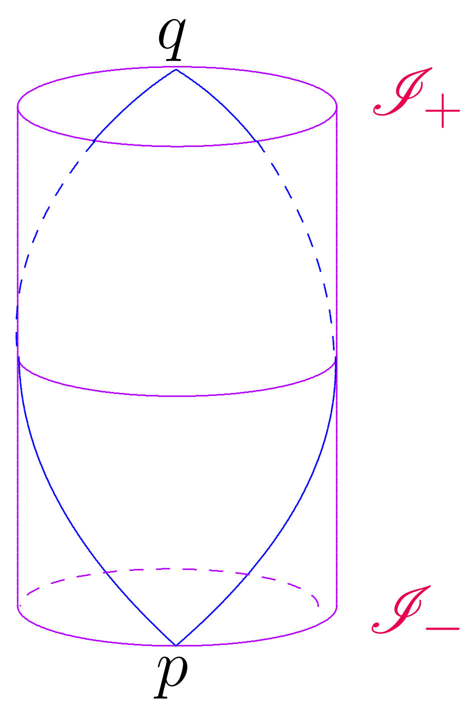


Thus define scattering map $\psi : \mathbb{CP}_1 \to \mathbb{CP}_1$ associated with $(M, [g], \nabla)$ to be function given by $\mathscr{I}_- \ni p \mapsto q \in \mathscr{I}_+$ after some choice of oriented conformal isomorphisms $\mathscr{I}_\pm \cong \mathbb{CP}_1$.



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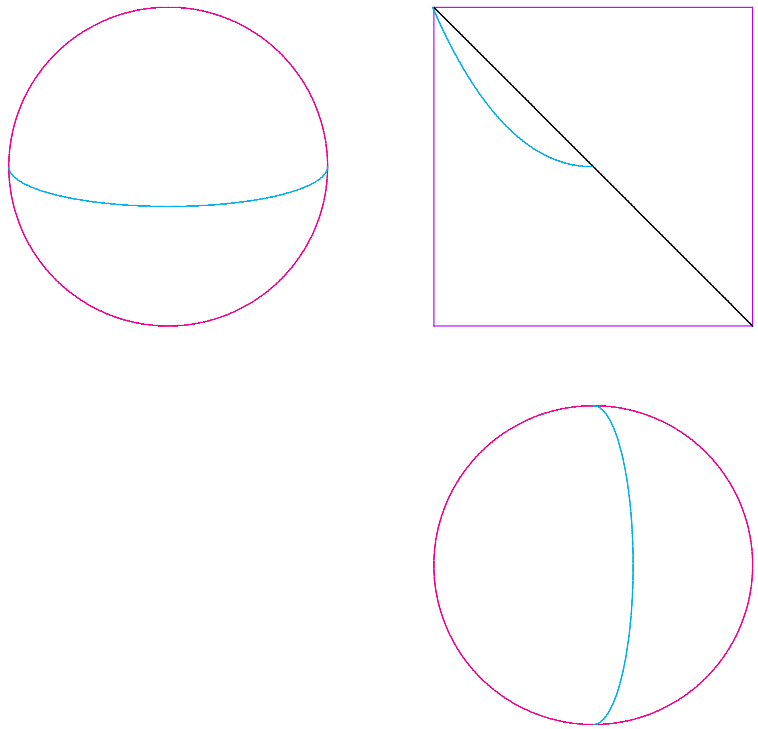
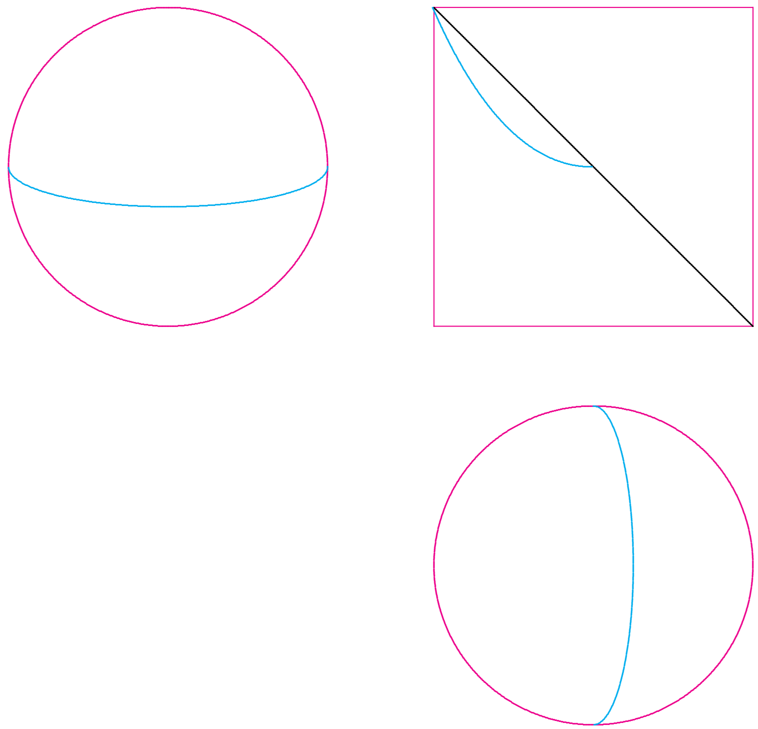
Strategy: construct 3-manifold $M = M_\psi$
as moduli space of holomorphic disks D
in $Z = \mathbb{CP}_1 \times \mathbb{CP}_1$ with ∂D on $P \subset Z$.



When ψ is the antipodal map,
disks are explicitly given by

$$\zeta \longmapsto ([a\zeta + b : c\zeta + d], [-\overline{d}\zeta - \overline{c} : \overline{b}\zeta + \overline{a}])$$

as ζ ranges over the unit disk $|\zeta| \leq 1$ in $\mathbb{C}$.



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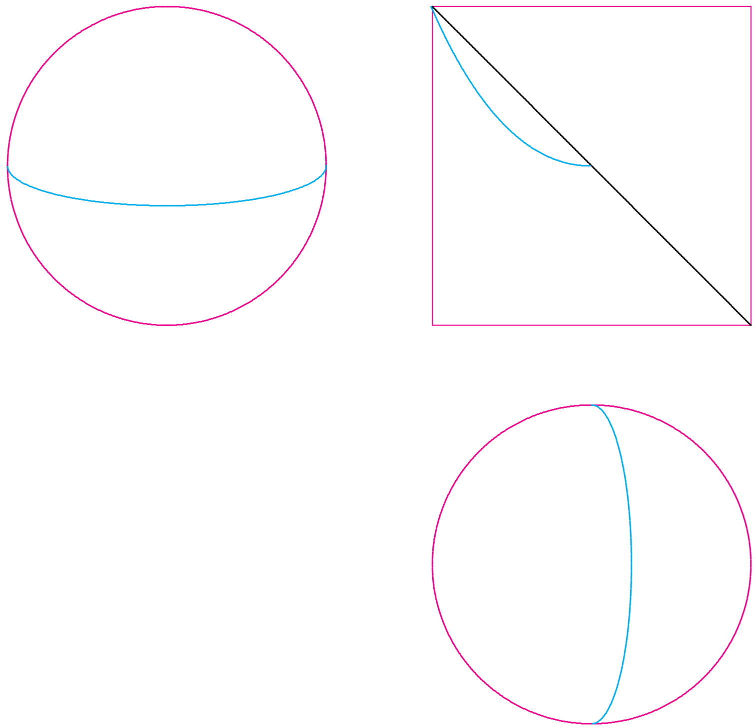
Moduli space M of disks mod reparameterization:
de Sitter space $SL(2, \mathbb{C})/SL(2, \mathbb{R})$.



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by replacing graph of anti-podal map with graph of orientation-reversing diffeomorphism

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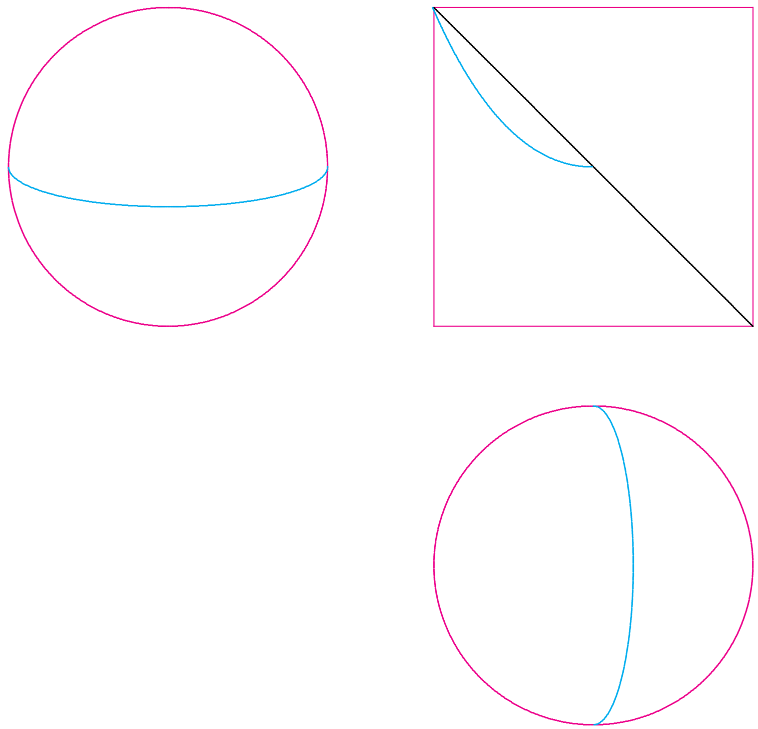

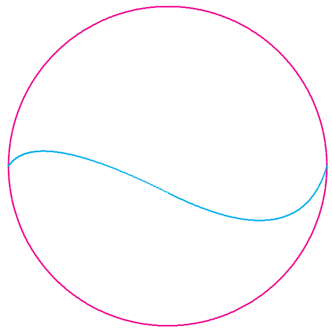

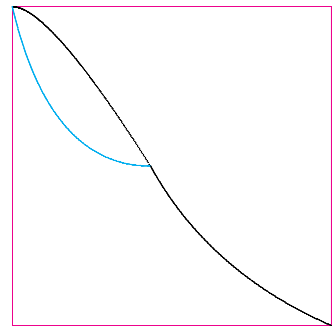

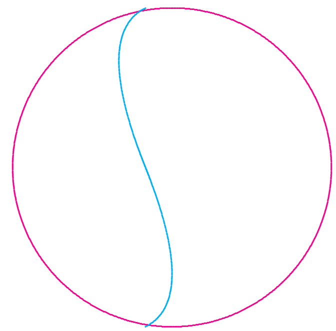

Lemma. *Let* $\psi : \mathbb{CP}_1 \to \mathbb{CP}_1$ *be any orientation-reversing diffeomorphism, and let*

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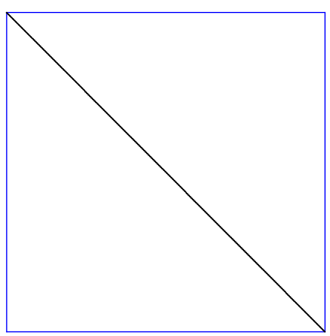

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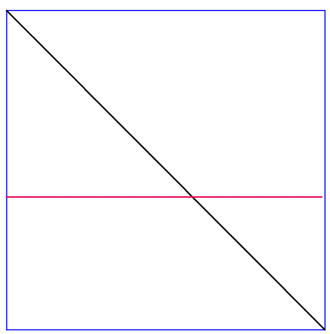

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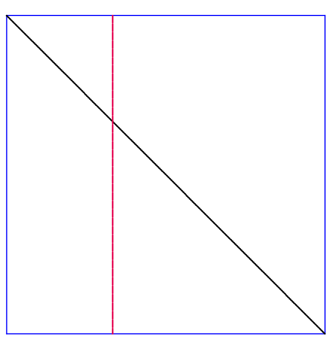

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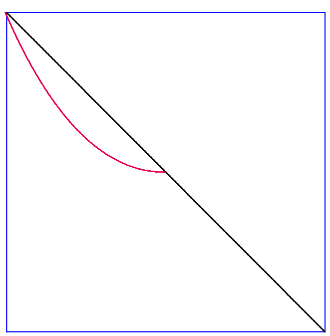

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If $(\Sigma, \partial\Sigma) \to (Z, P)$ *is any holomorphic curve with boundary representing* $\mathbf{a}$*, then* Σ *is either a holomorphic disk as above, or is a factor* $\mathbb{CP}_1$ *of* $Z = \mathbb{CP}_1 \times \mathbb{CP}_1$*.*

Proof.

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But need to show is that disk is actually a graph!

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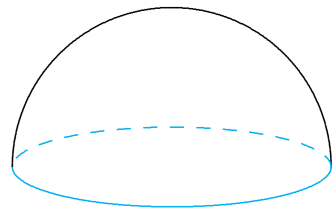

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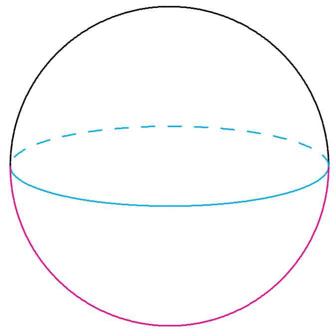

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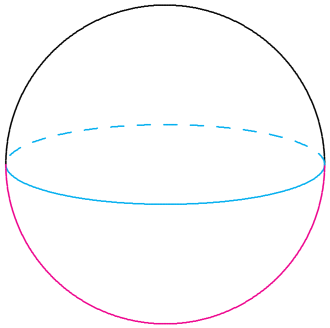

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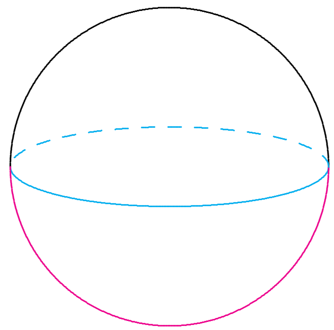

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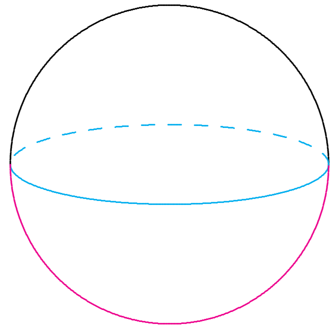

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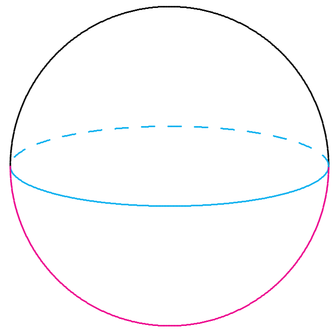

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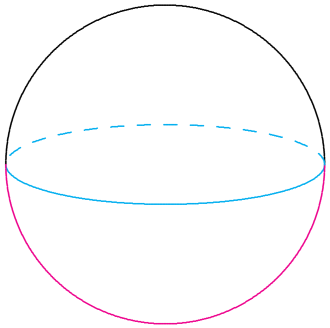

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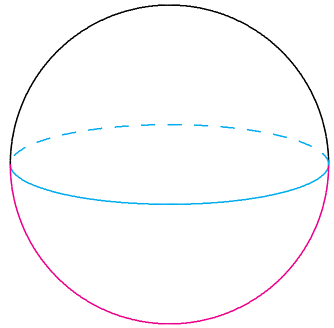

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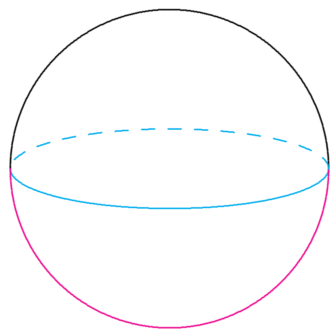

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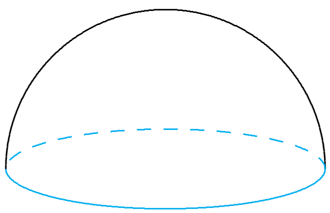

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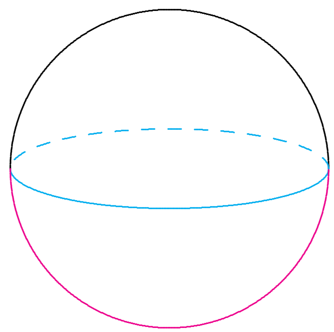

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Equals 2 in our case:

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$$h^1(\mathbb{CP}_1, \mathcal{O}(2)) = 0$$
$$h^0(\mathbb{CP}_1, \mathcal{O}(2)) = 3$$

cf. Kodaira's Theorem
on deformation of complex submanifolds

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Disks all have same ω-area. $\Longrightarrow$
any sequence has convergent subsequence...

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Tricky point: disks can degenerate to factor $\mathbb{CP}_1$.

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Since ψ is continuous deformation of antipodal,

Continuity method $\Rightarrow$ each level set non-empty!

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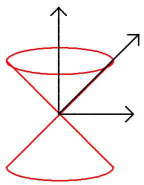
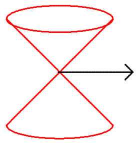
$\Rightarrow$ up to homothety $T_D M$ carries Lorentz metric, modelled on Killing form of $\mathfrak{sl}(2, \mathbb{R})$.

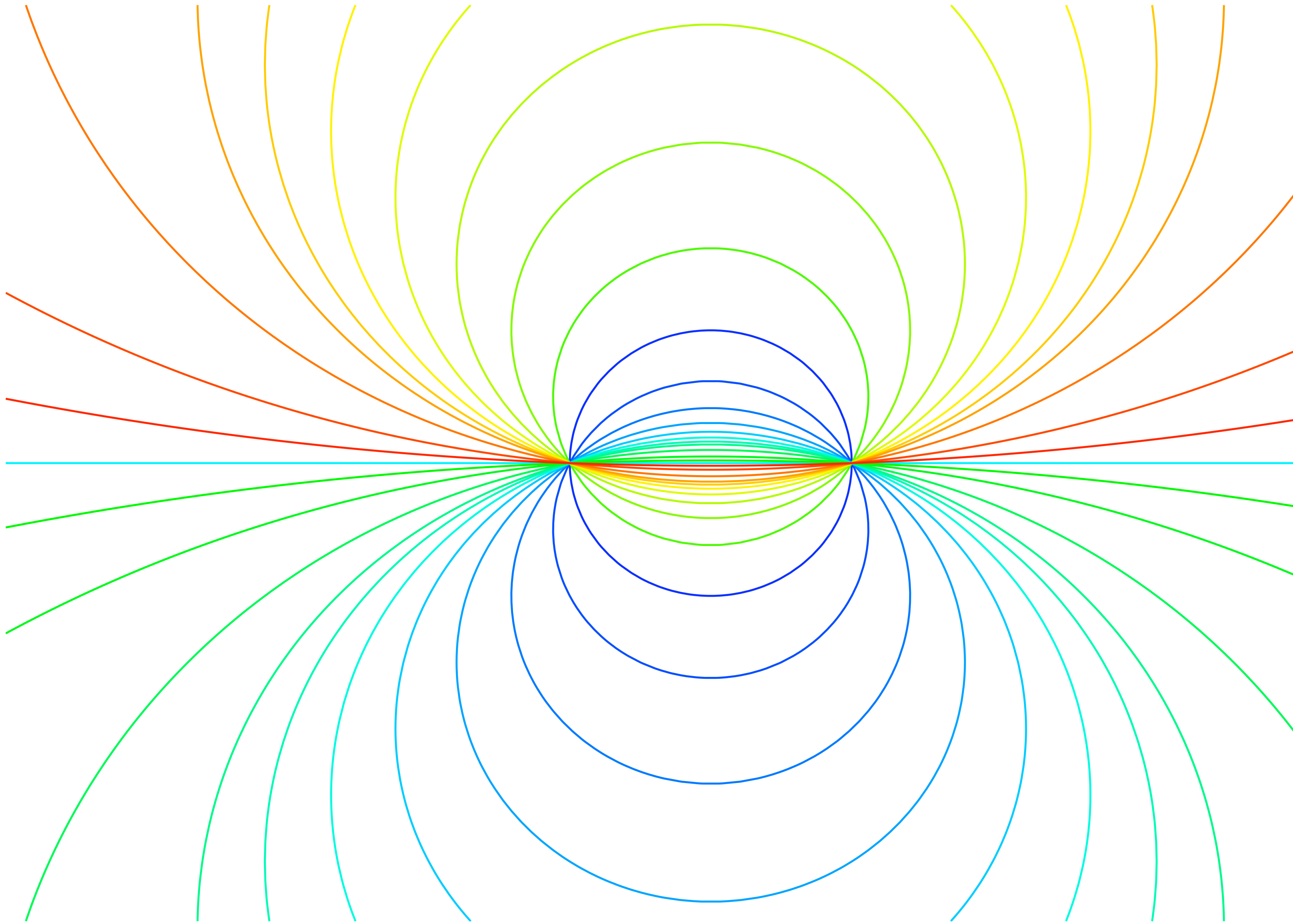
Trichotomy:

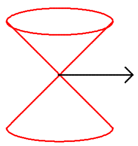
TM	$\mathfrak{sl}(2, \mathbb{R})$
space-like	hyperbolic
null	parabolic
time-like	elliptic





Space-like vector = infinitesimal variation with two distinct zeroes on ∂D.





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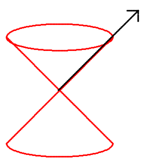

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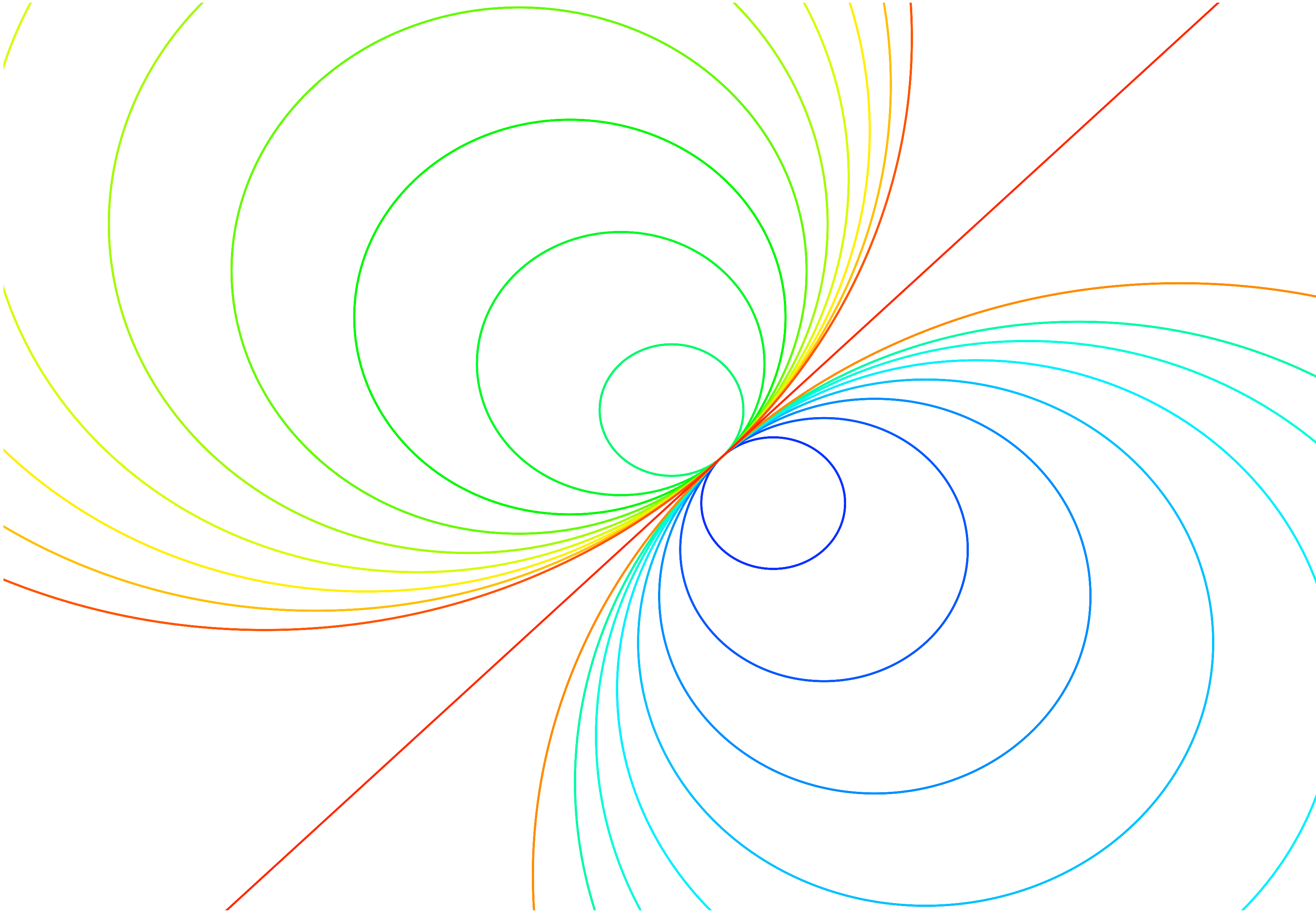

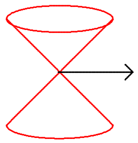

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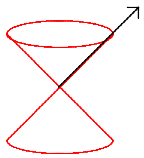

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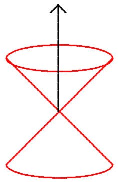
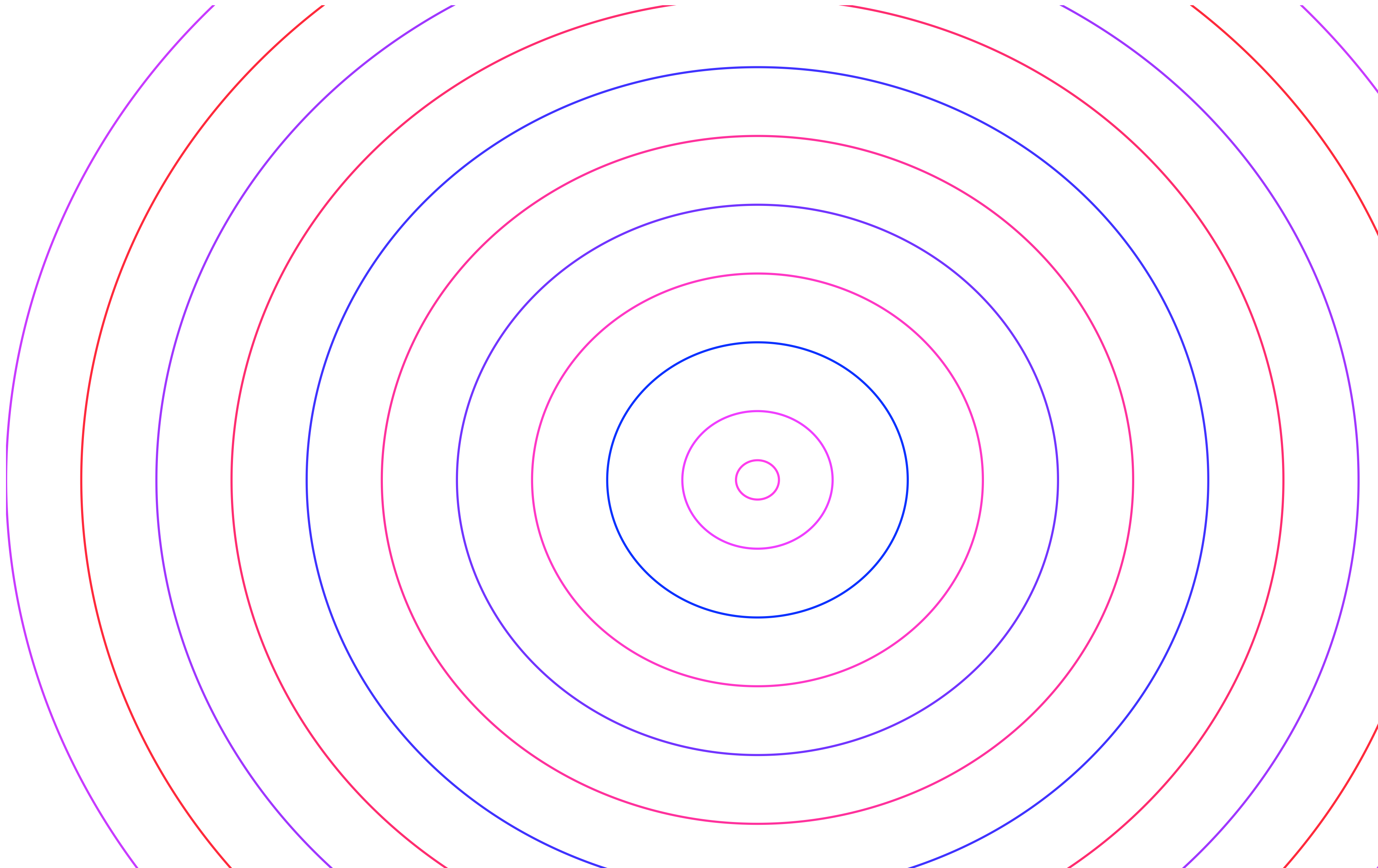


Time-like vector = infinitesimal variation with a single zero in interior of D: none along ∂D.



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By deformation: Cauchy surface topologically S^2.

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Space-like geodesic:
disks passing though a pair of distinct points
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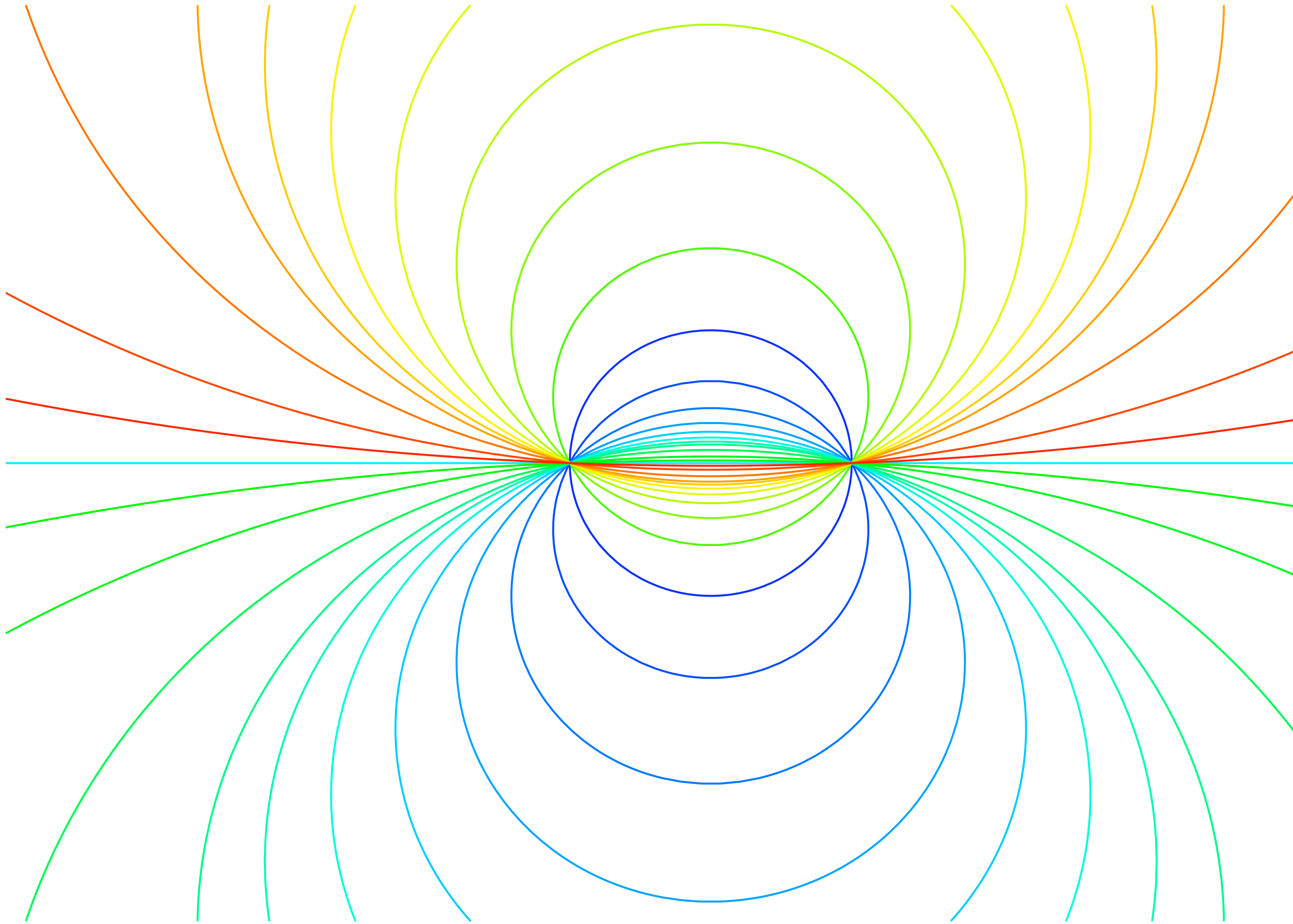

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Null geodesic:
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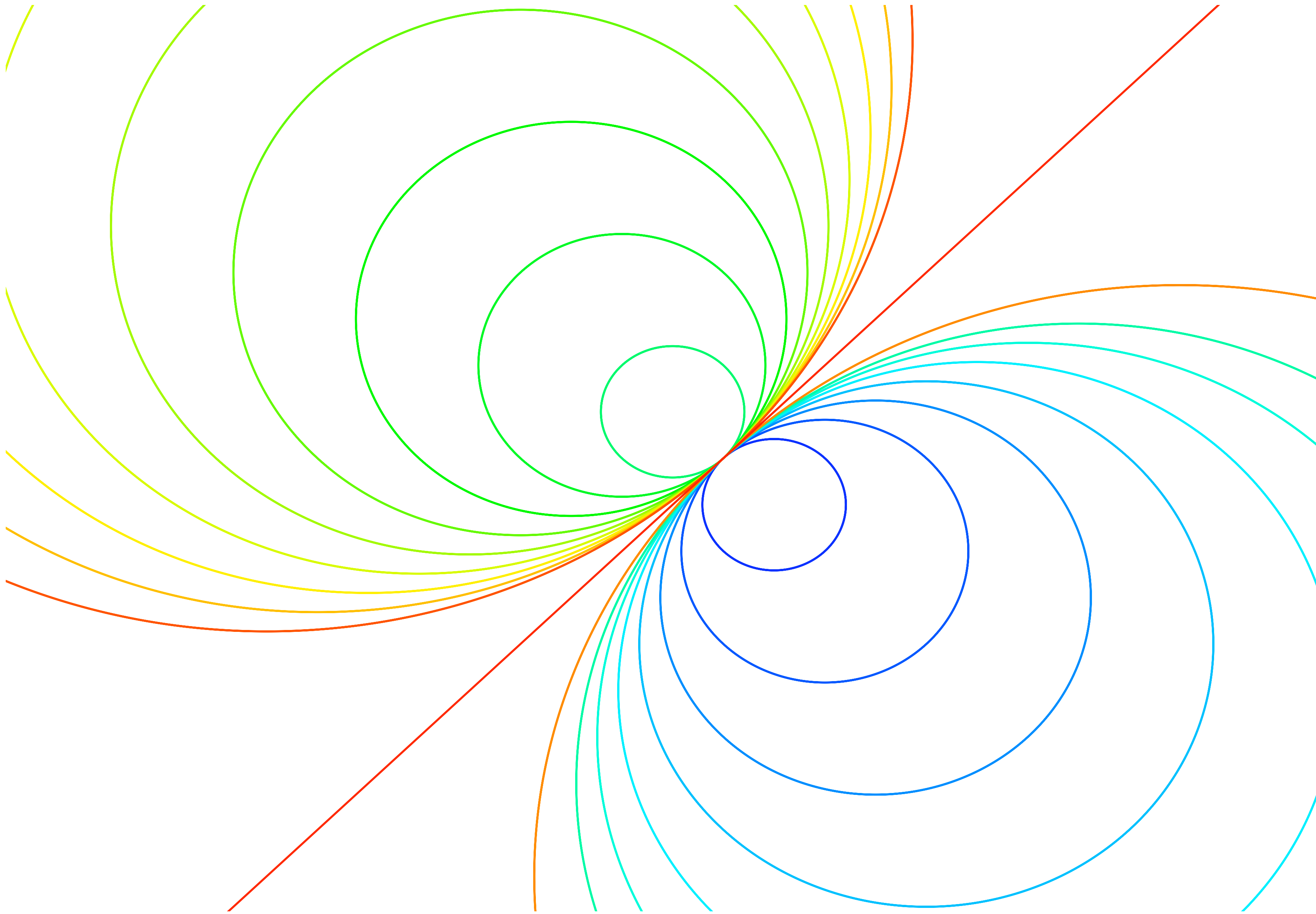

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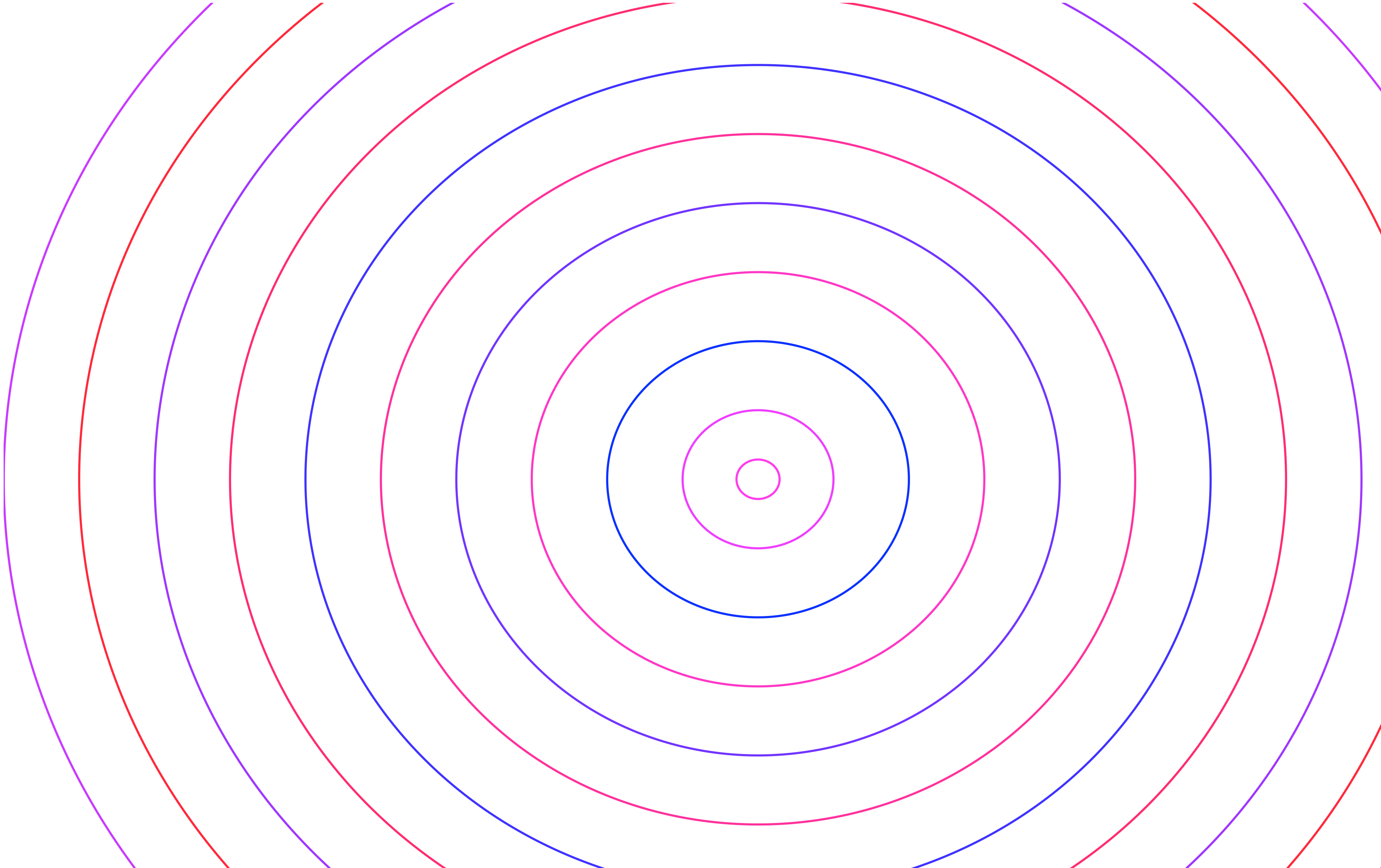

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- —— & Mason, Duke Math. J. 136 (2007) 205–273.

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Also gives direct proof of conformal compactness.

Theorem. *There is a natural one-to-one correspondence between*

- *smooth, space-time-oriented,* conformally compact, *globally hyperbolic Lorentzian Einstein-Weyl* 3*-manifolds* $(M, [g], \nabla)$*; and*

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