Einstein Metrics,

Harmonic Forms, &

Symplectic Four-Manifolds

Claude LeBrun
Stony Brook University

Oxford, July 22, 2015

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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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Has same sign as the *scalar curvature*

$$s = r_j^j = \mathcal{R}^{ij}{}_{ij}.$$

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Perhaps reasonable in other dimensions?

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When $n \geq 4$, situation is more encouraging...

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One key question:

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One key question:

Does enough rigidity really hold in dimension four to make this a genuine geometrization?

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Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric h (unrelated to ω)?*

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Some Suggestive Questions. *If* (M^4, ω) *is a symplectic* 4*-manifold, when does* M^4 *admit an Einstein metric* h *(unrelated to* ω*)? What if we also require* $\lambda \geq 0$*?*

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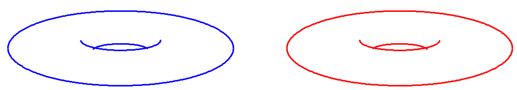

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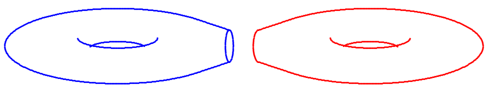

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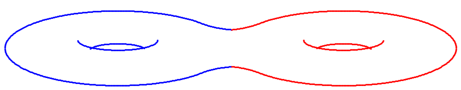

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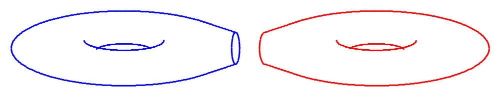

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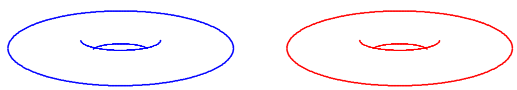

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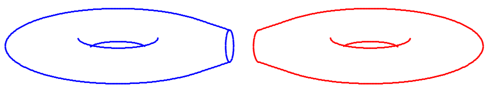

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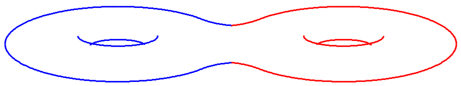

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Del Pezzo surfaces,
K3 surface, Enriques surface,
Abelian surface, Hyper-elliptic surfaces.

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Definitive list . . .

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\end{array}
$$

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

Above the line:

Know an Einstein metric on each manifold.

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- No other Einstein metrics belong to $\mathscr{U}$!

Formulation will depend on. . .

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Λ^+ self-dual 2-forms.
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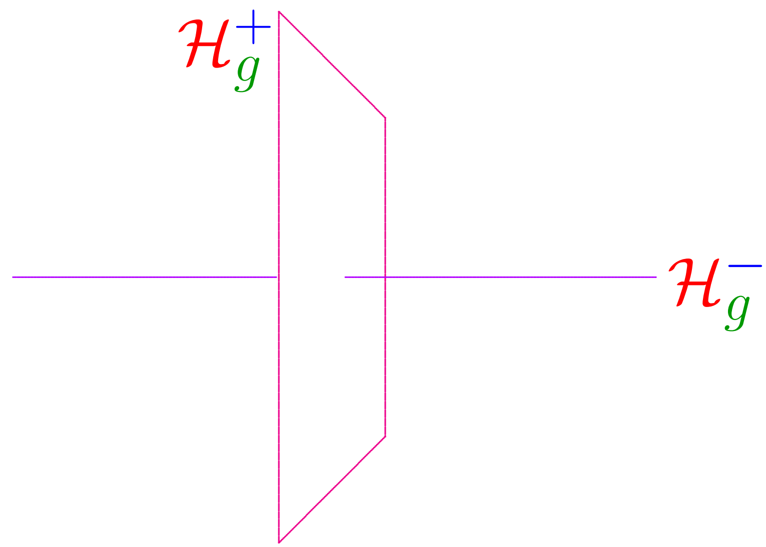

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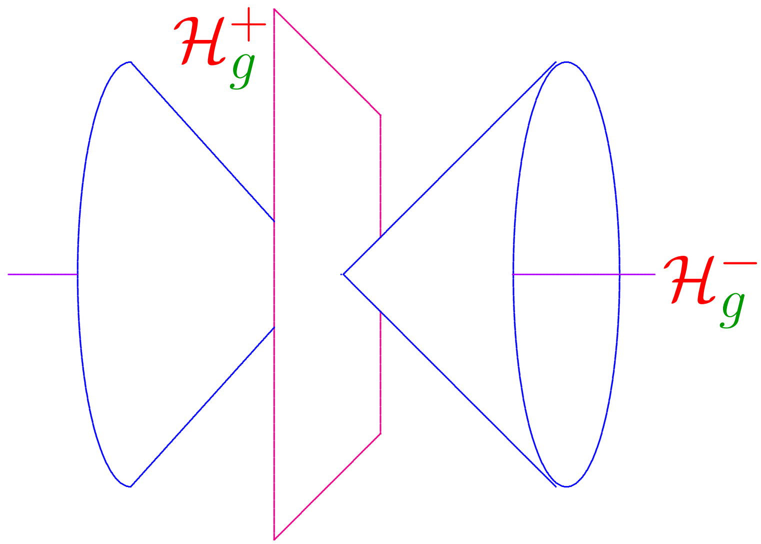

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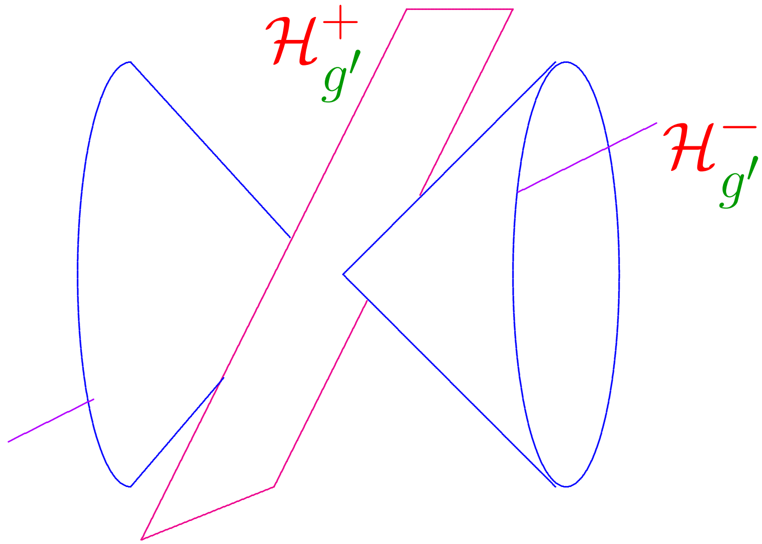

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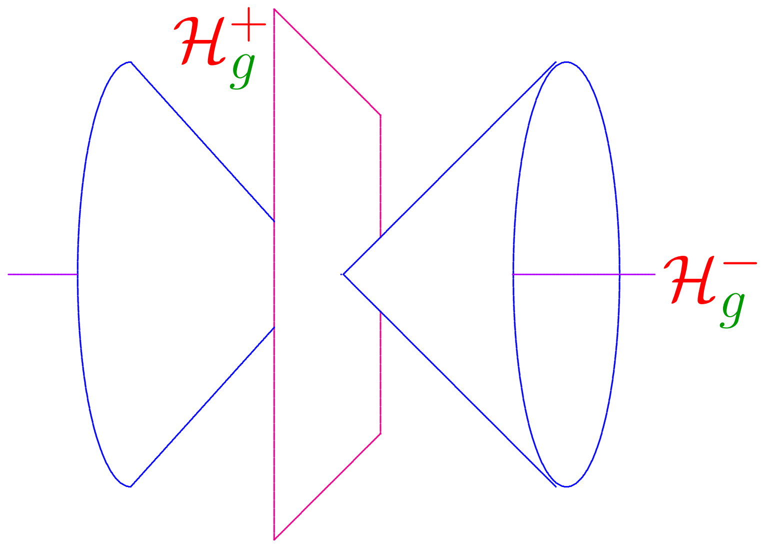

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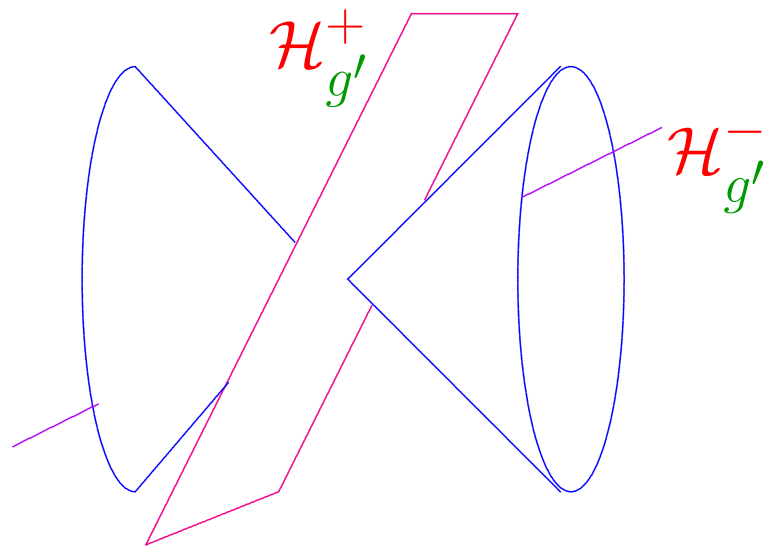

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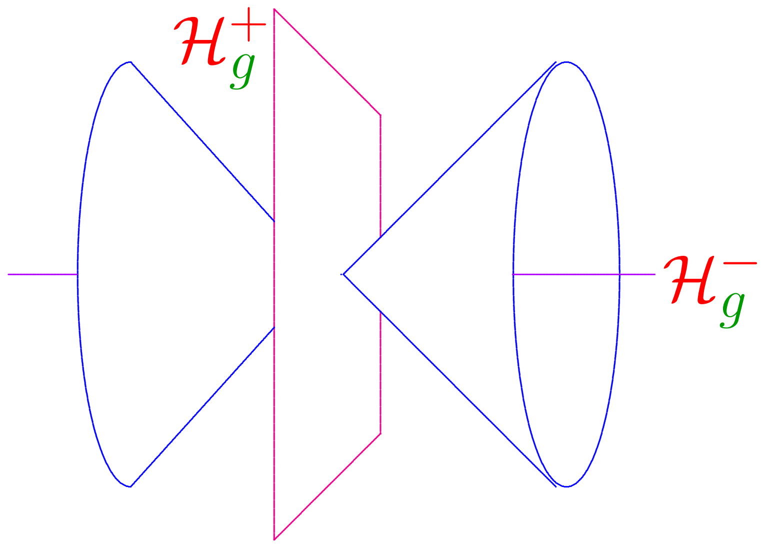

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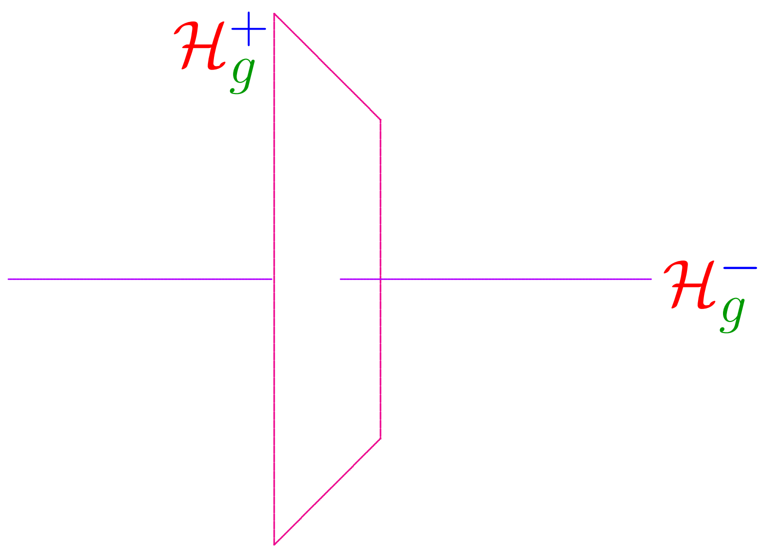

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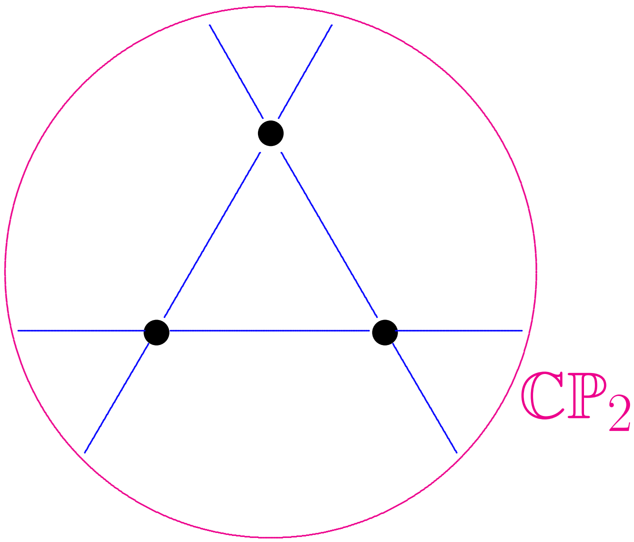

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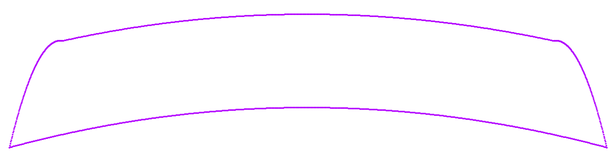

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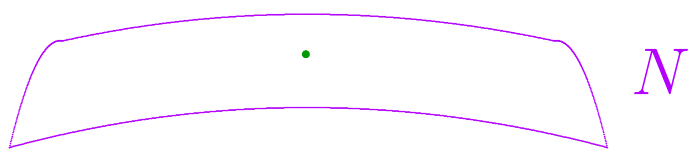

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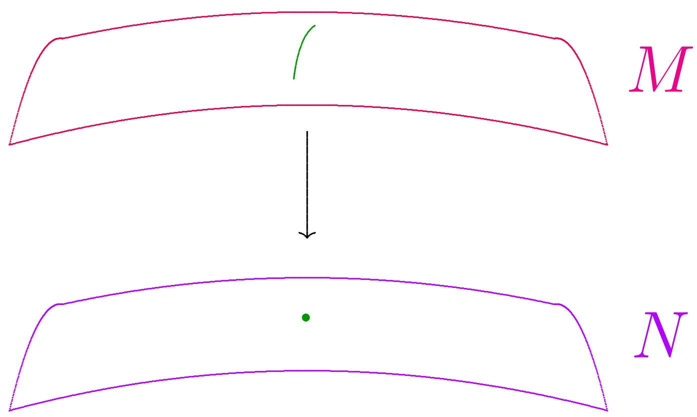

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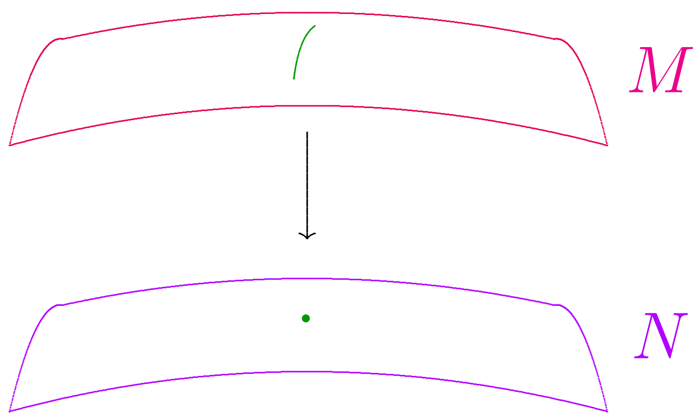
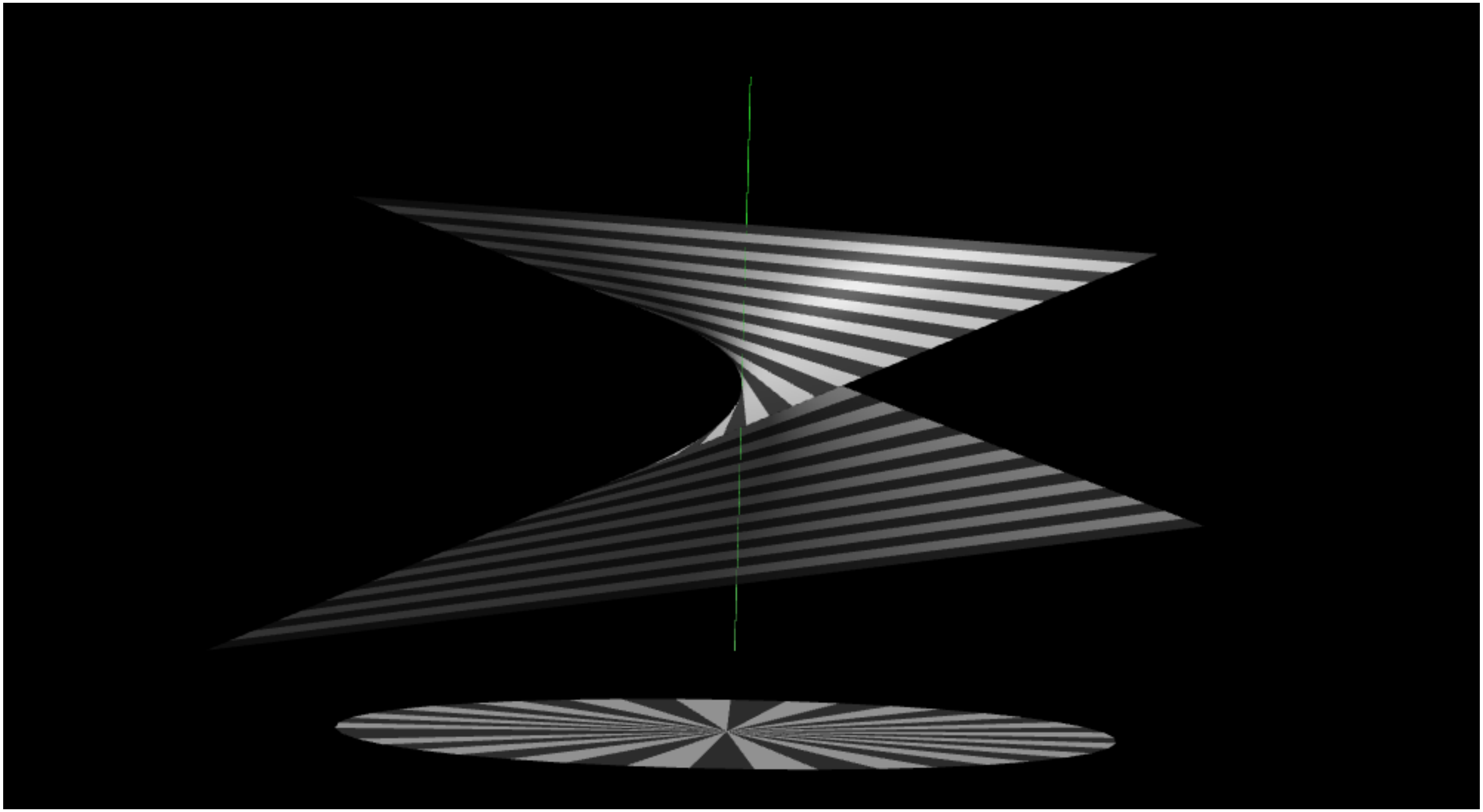
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.





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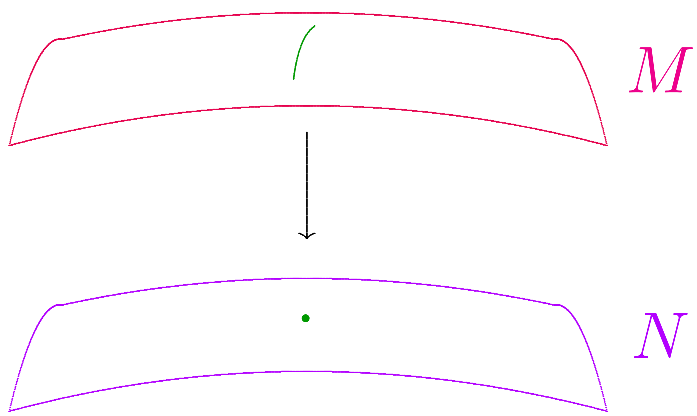

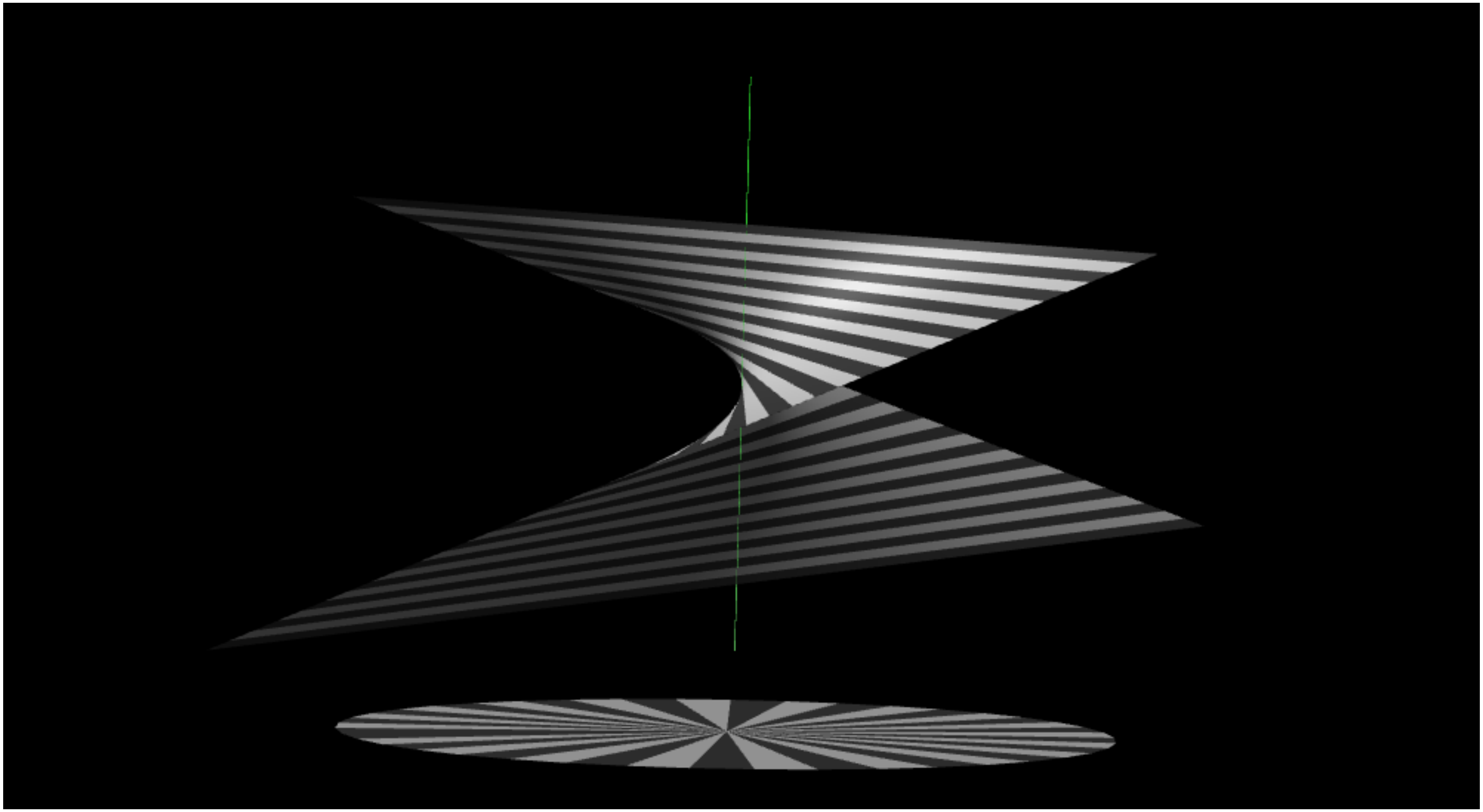

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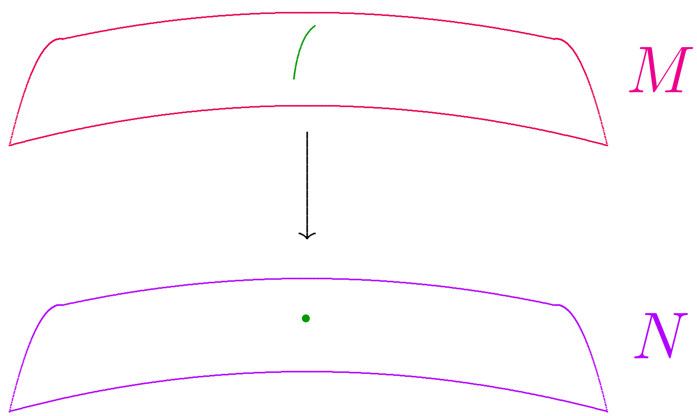

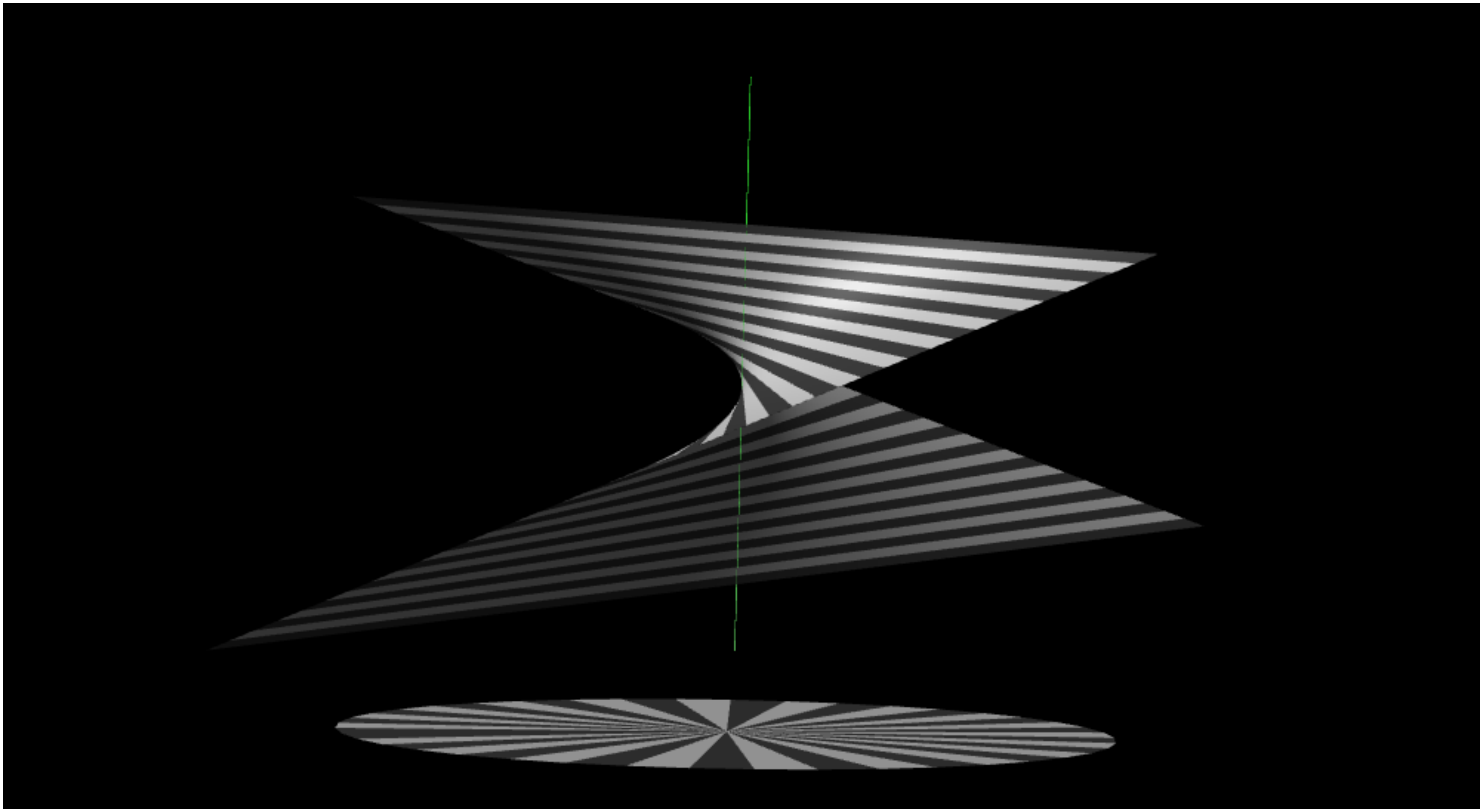

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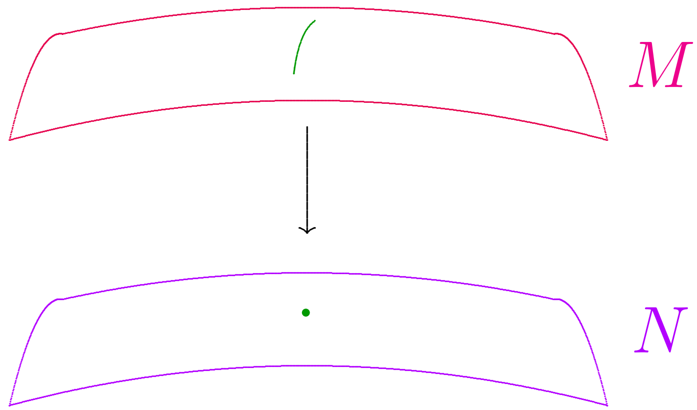
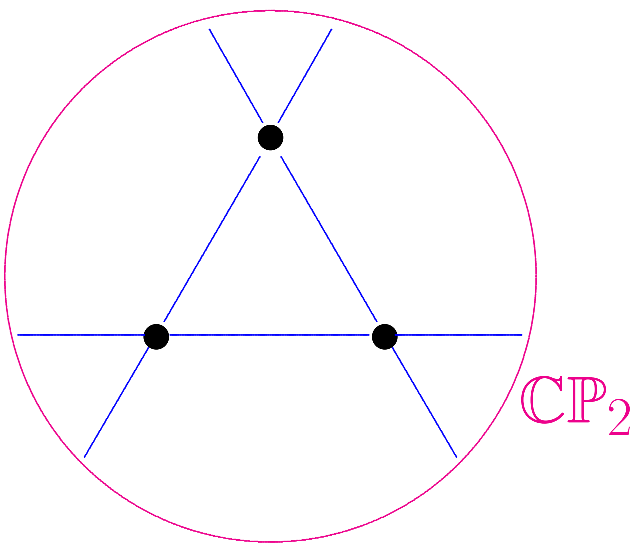


Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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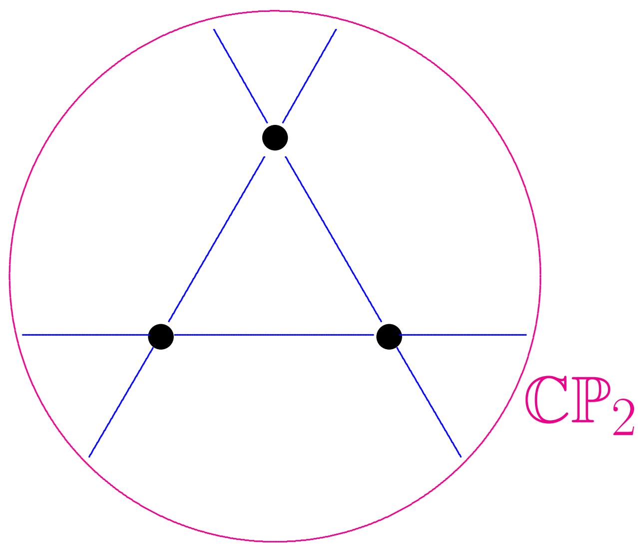

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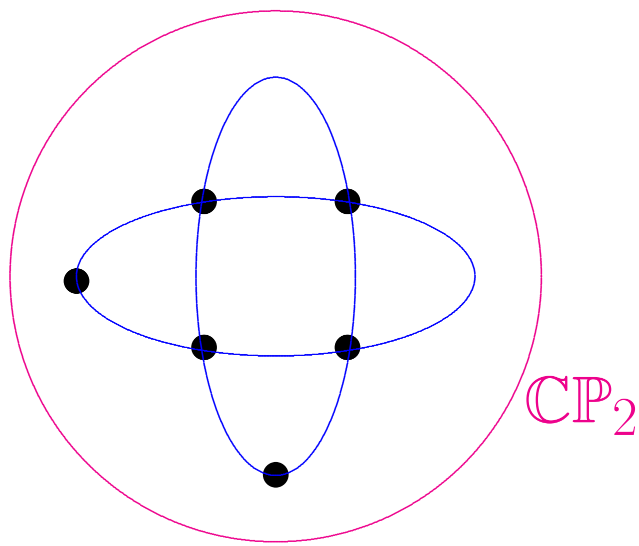

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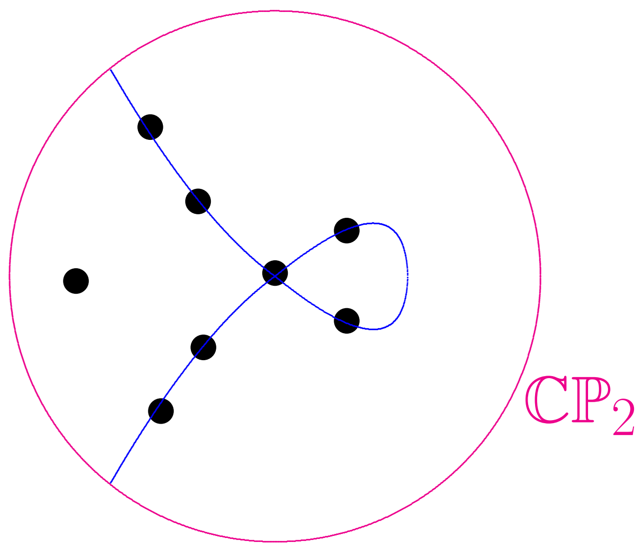

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Existence: Tian, Odaka-Spotti-Sun, Chen-L-Weber…

Uniqueness: Bando-Mabuchi, L 2012…

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Corollary. *$\mathscr{E}^+_\omega(M)$ is exactly one connected component of $\mathscr{E}(M)$.*

Method of Proof.

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for $fW^+ \in \mathrm{End}(\Lambda^+)$.

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Remark. If such metrics exist, $b_+(M) = 1$.

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Theorems A & B follow by restricting to the Einstein case and using previous results **(L 2012)**.