Four-Manifolds,

Einstein Metrics, &

Differential Topology

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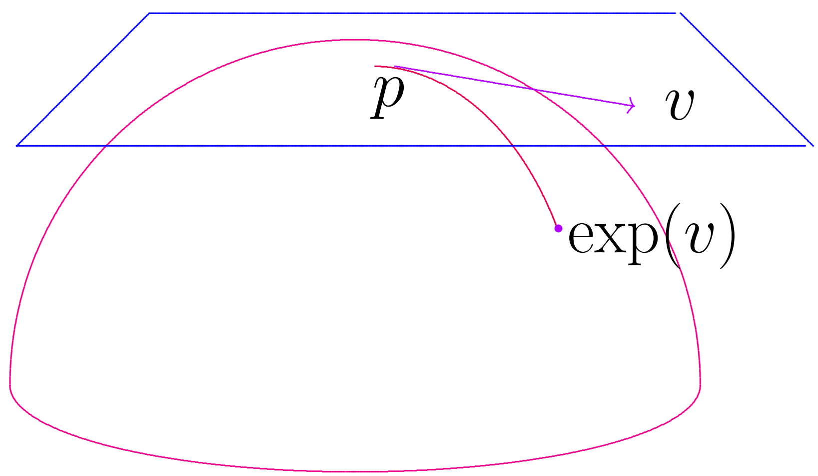

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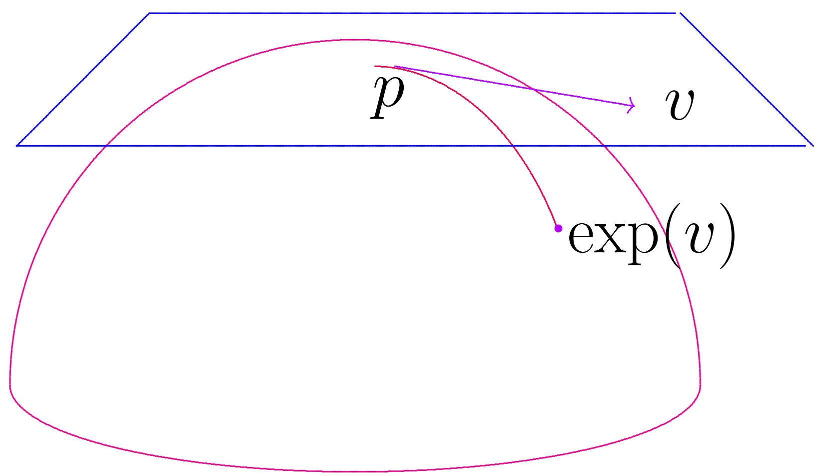

Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

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"...the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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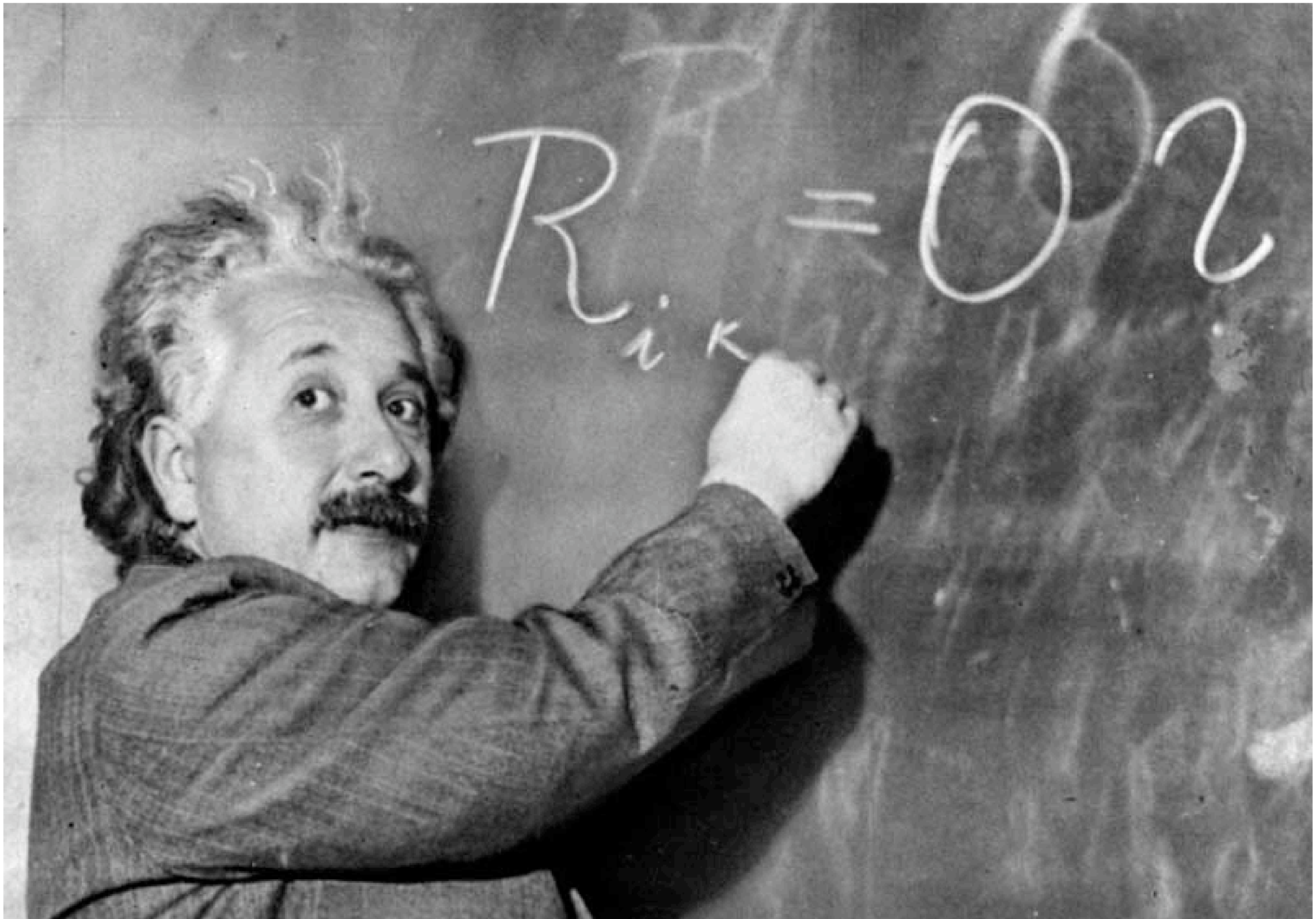

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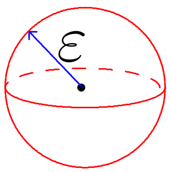

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$$\Delta x^j = 0 \Longrightarrow r_{jk} = \frac{1}{2}\Delta g_{jk} + \ell ots.$$

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- When $n \geq 6$, wide open. Maybe???

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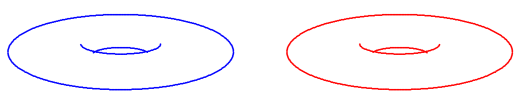

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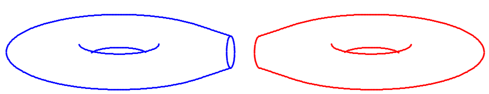

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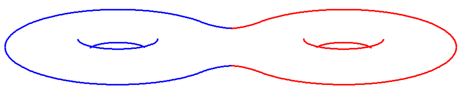

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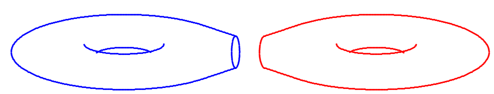

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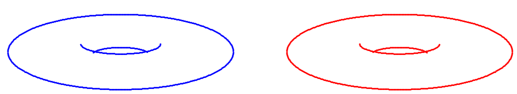

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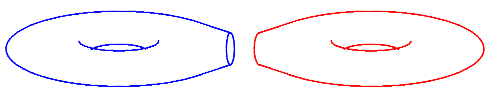

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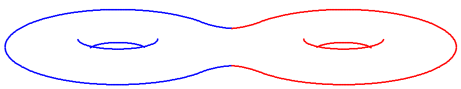

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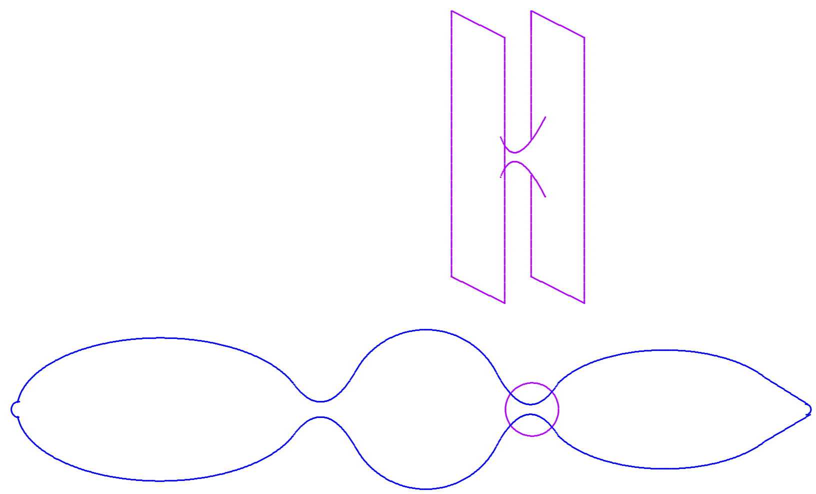

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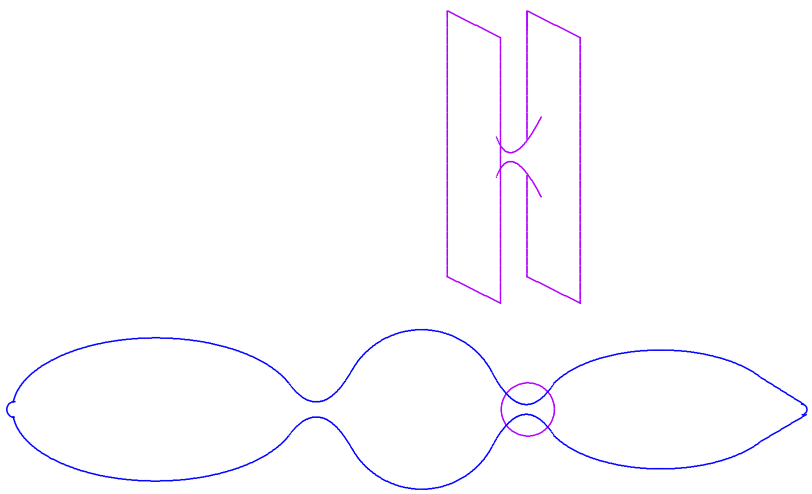

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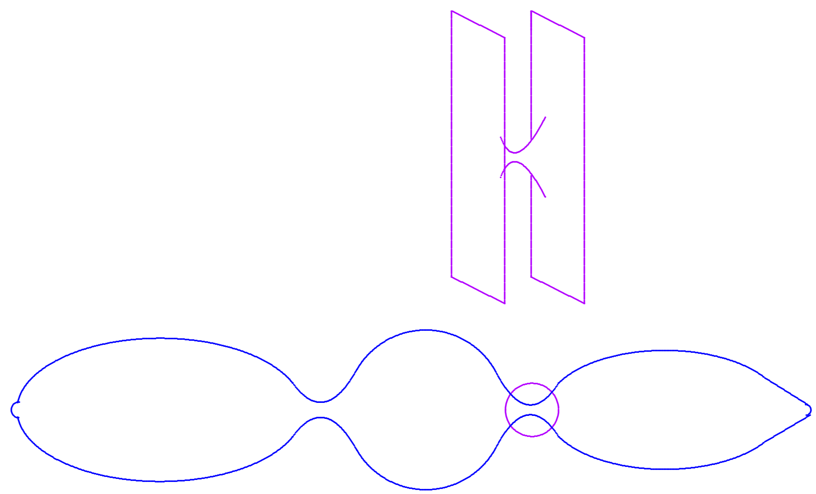

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Similar results for most simply connected spin 5-manifolds. (Boyer, Galicki, Kollár, et al.)

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By contrast, high-dimensional Einstein metrics too common; have little to do with geometrization.

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$$\star^2 = 1.$$

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Λ^-	$\mathring{r}$	$W_- + \frac{s}{12}$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature

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Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

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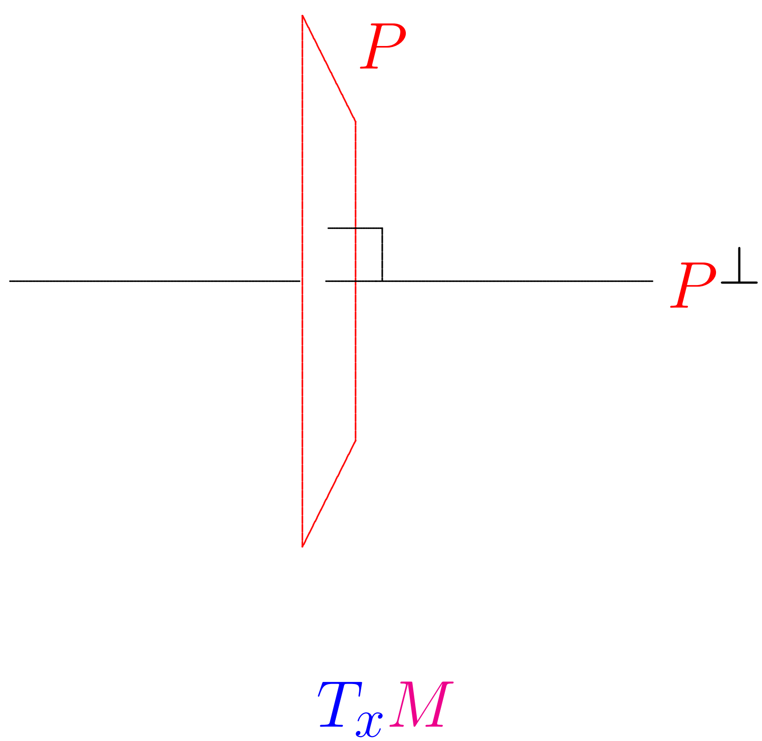

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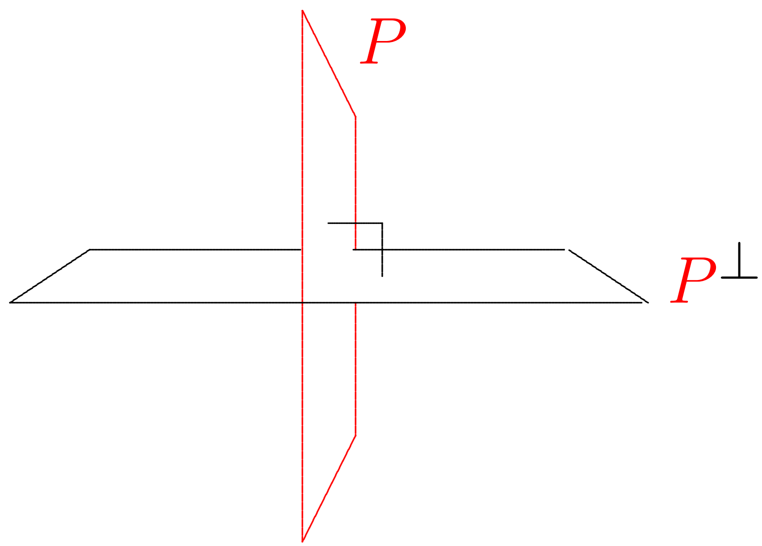

T_xM

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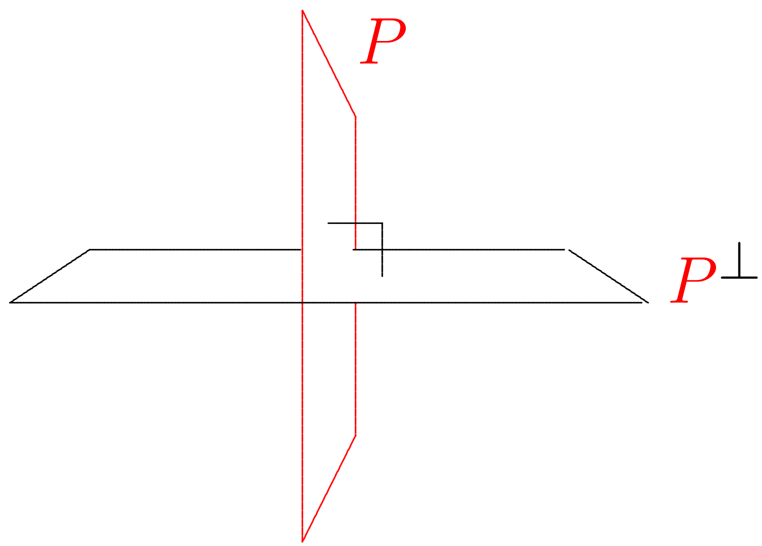

$$K(P) = K(P^{\perp})$$

(M, g) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + \qquad \qquad \qquad \qquad \right) d\mu$$

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for Euler-characteristic $\chi(M) = \sum_j (-1)^j b_j(M)$.

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For (M^4, g) compact oriented Riemannian,

Euler characteristic

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Typically, one homeotype $\longleftrightarrow$ ∞ many diffeotypes.

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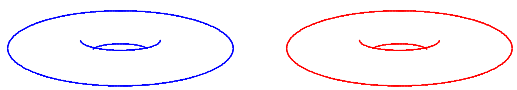

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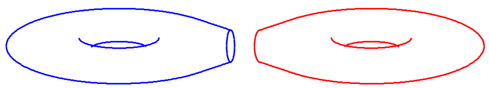

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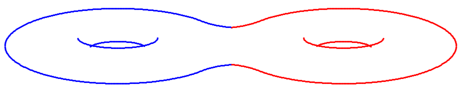

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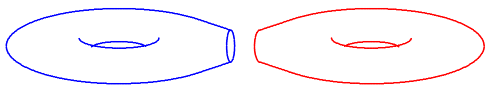

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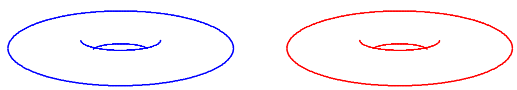

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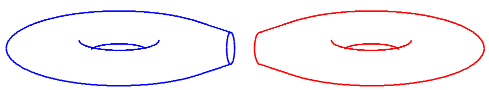

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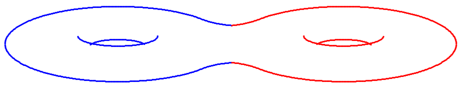

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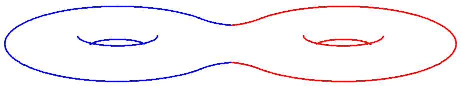

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Spin, $\chi = 24$, $\tau = -16$.

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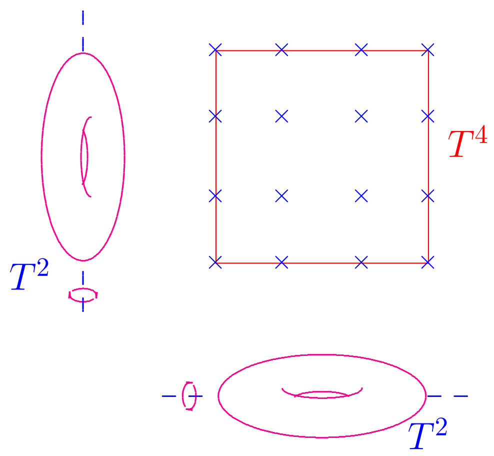

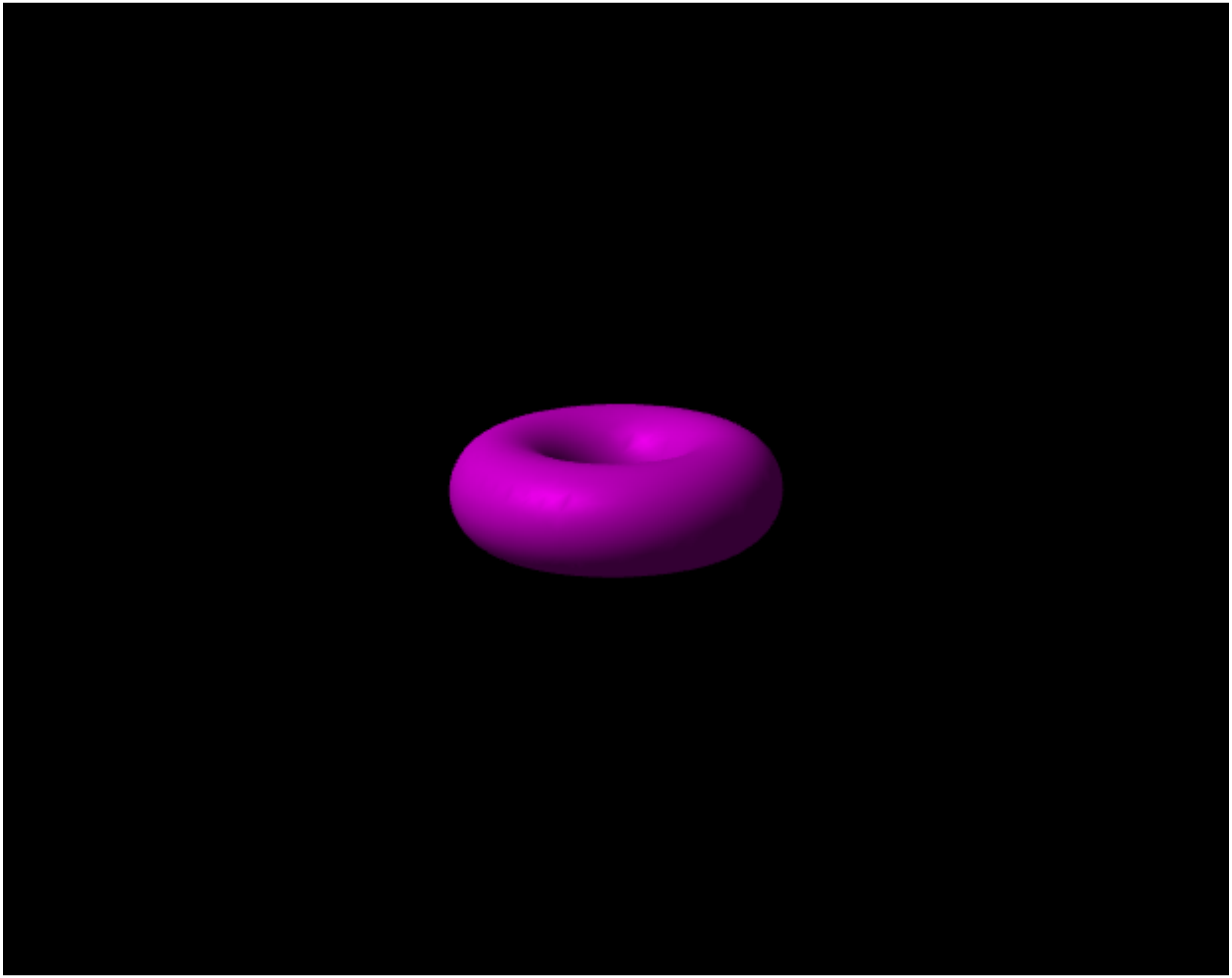

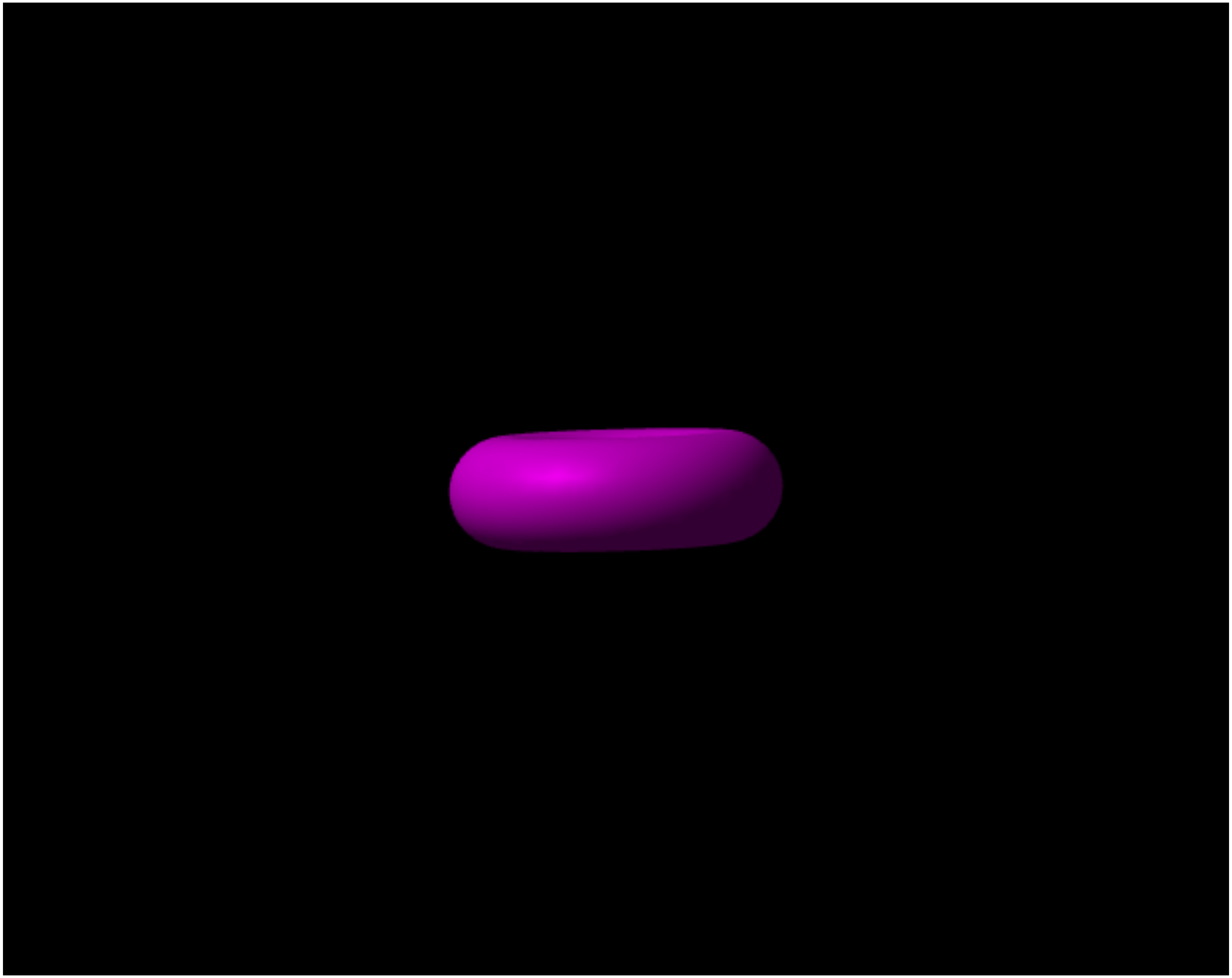

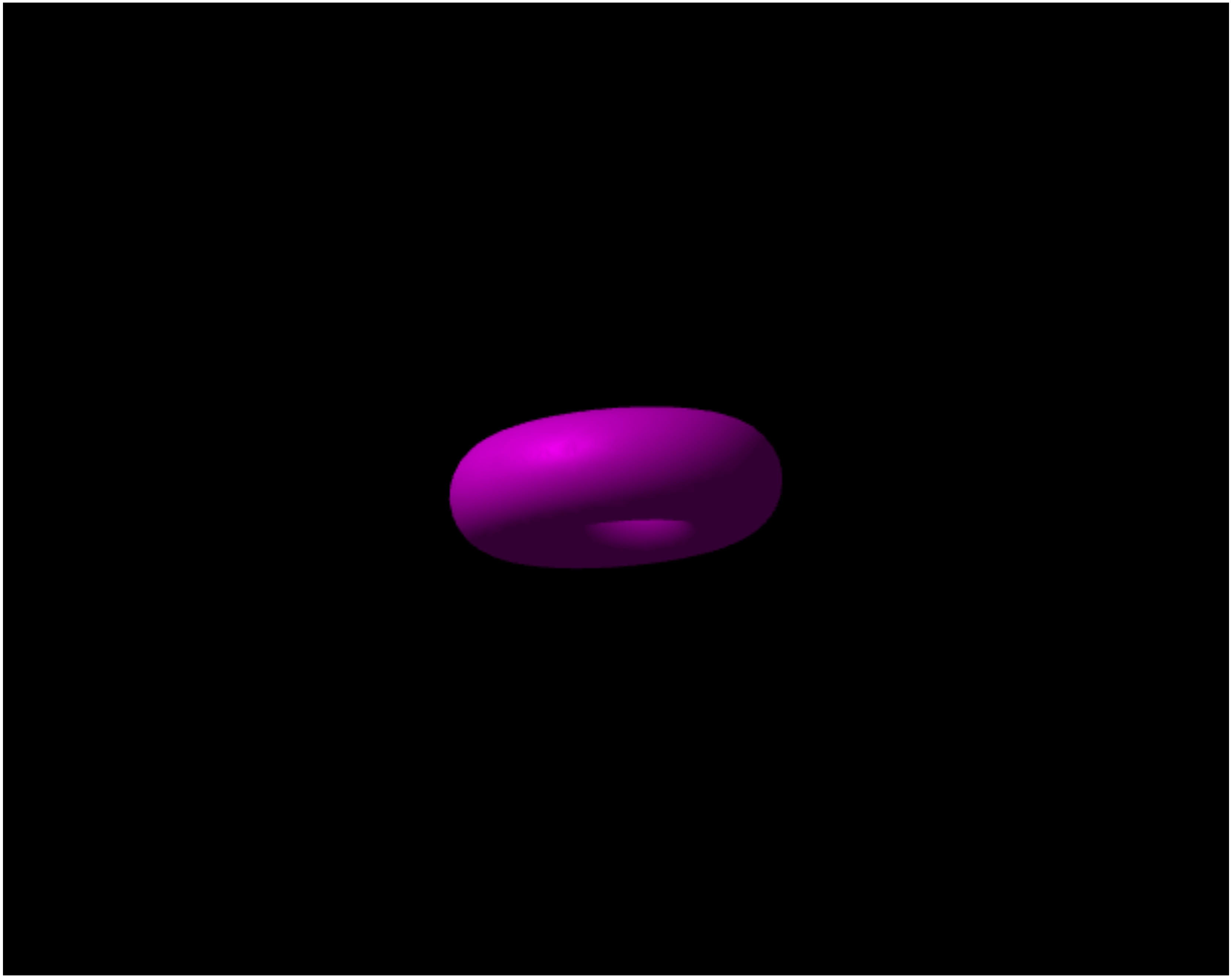

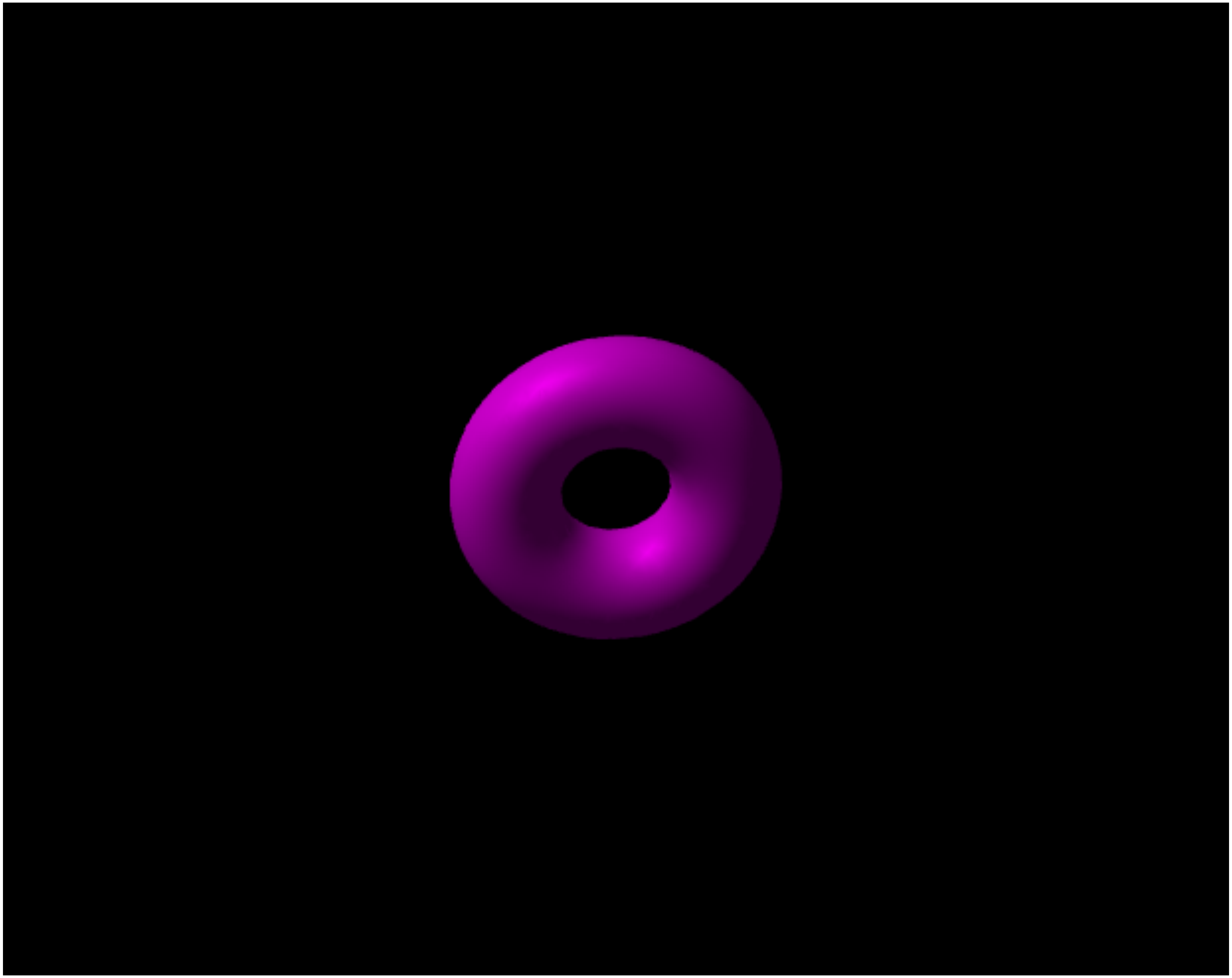

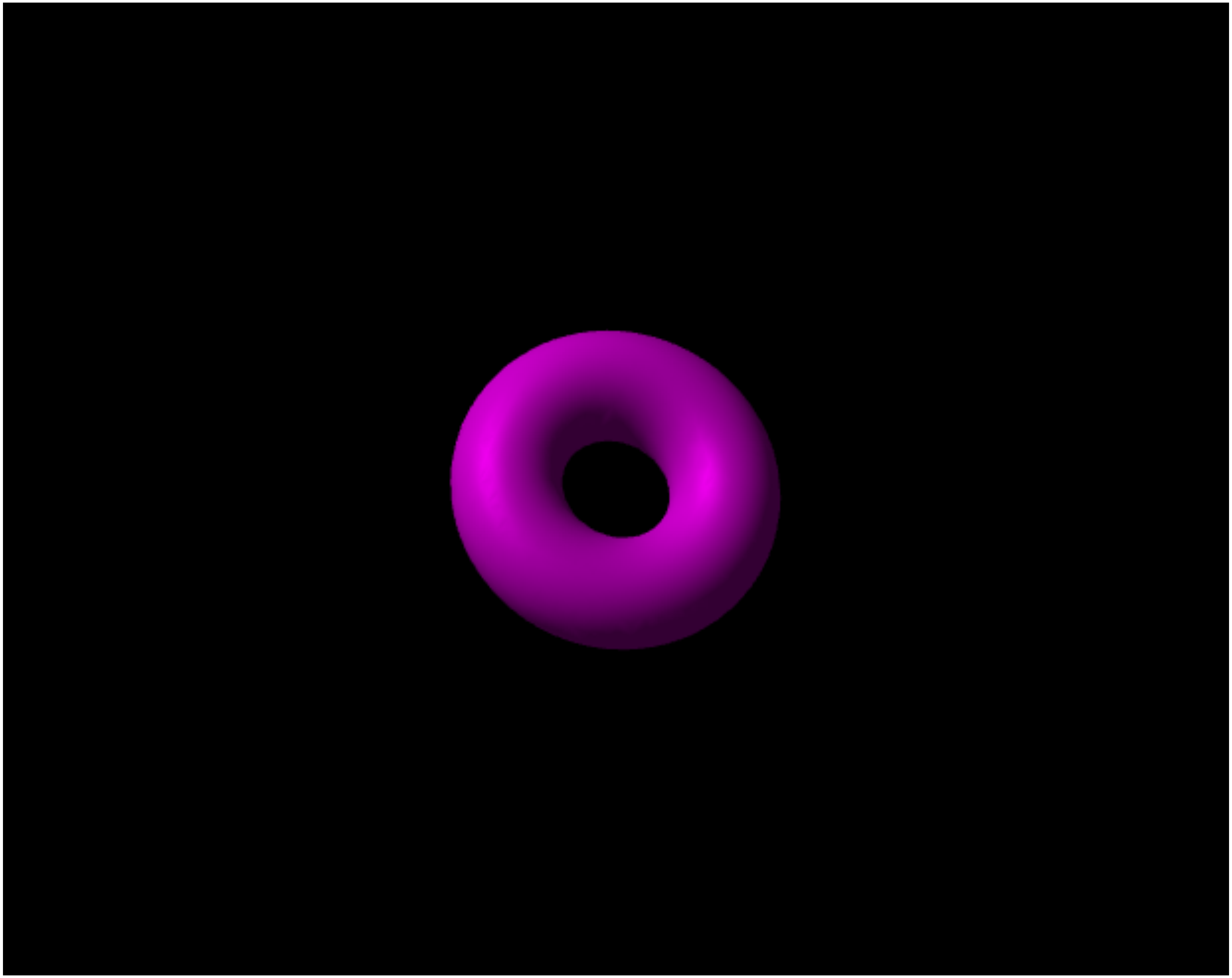

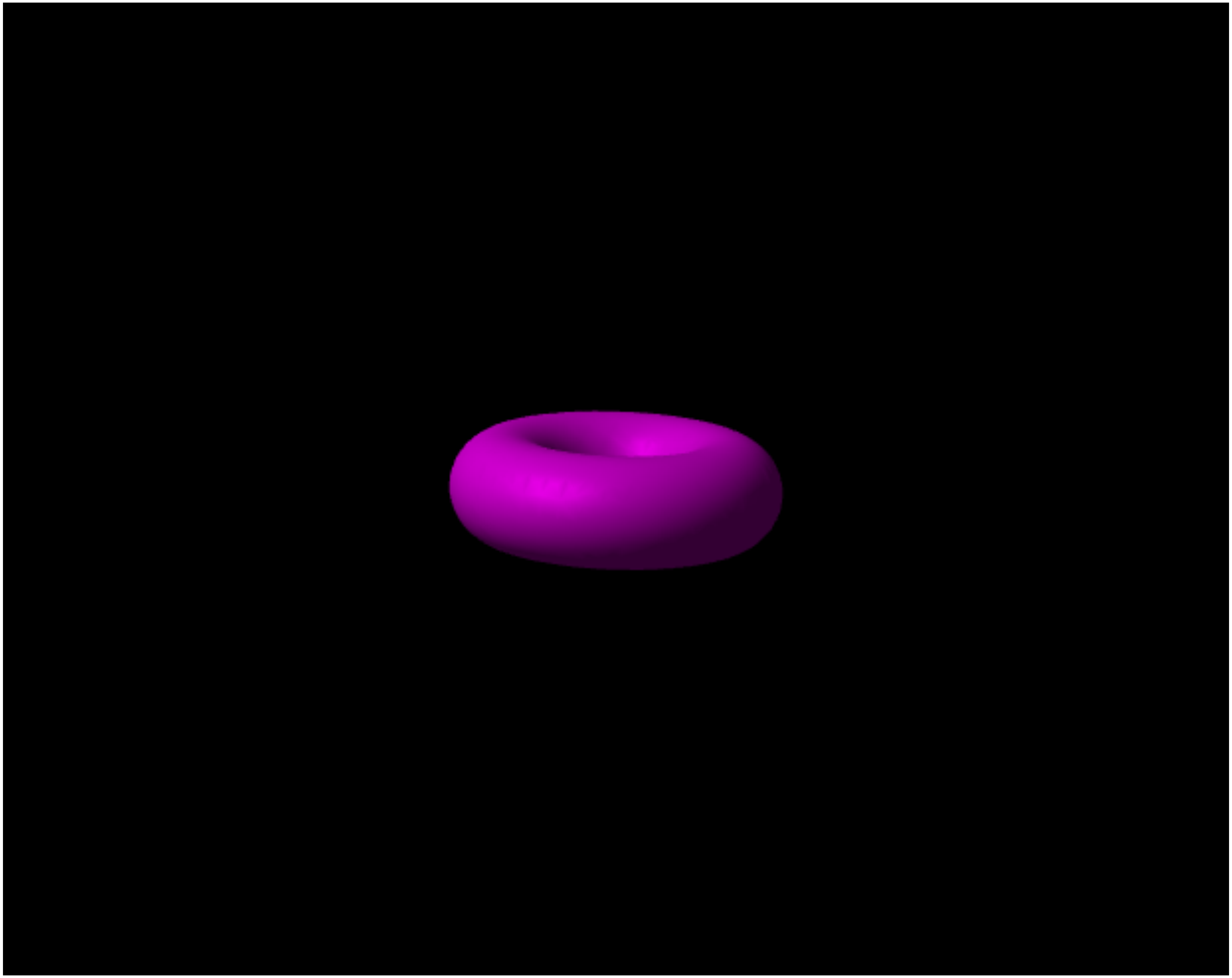

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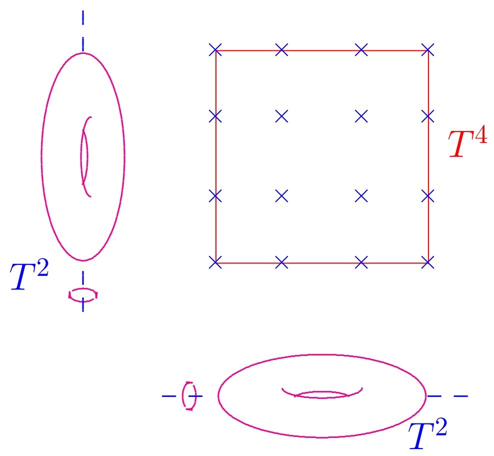

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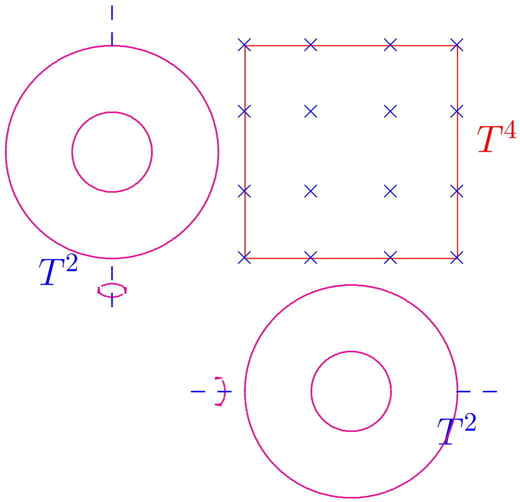

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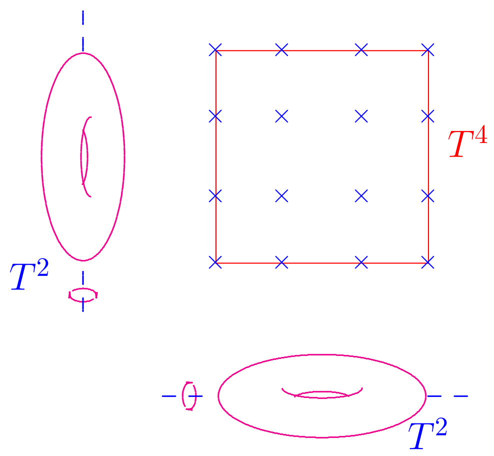

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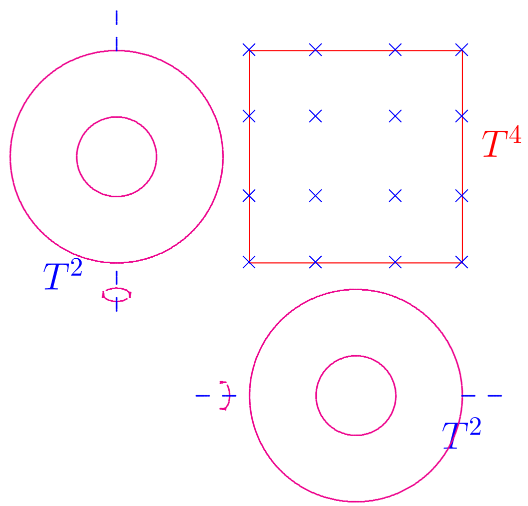

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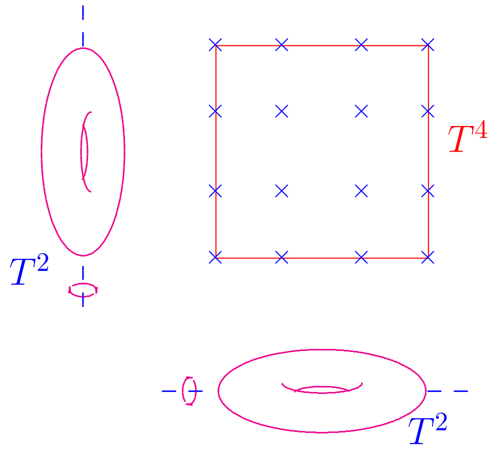

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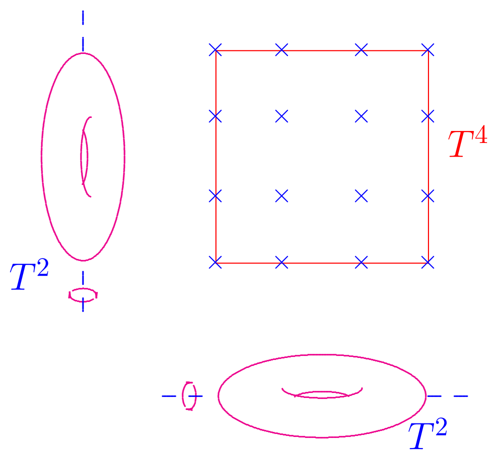

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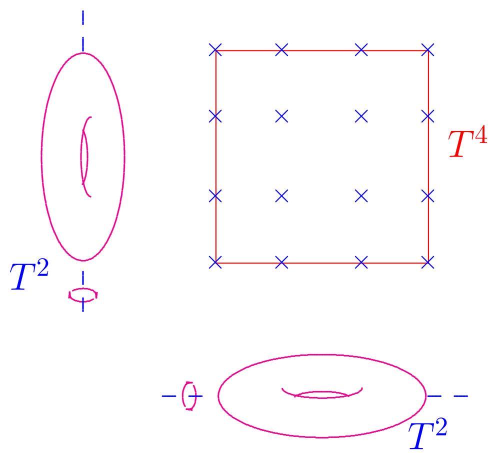

Replace $\mathbb{R}^4/\mathbb{Z}_2$ neighborhood of each singular point with copy of T^*S^2.

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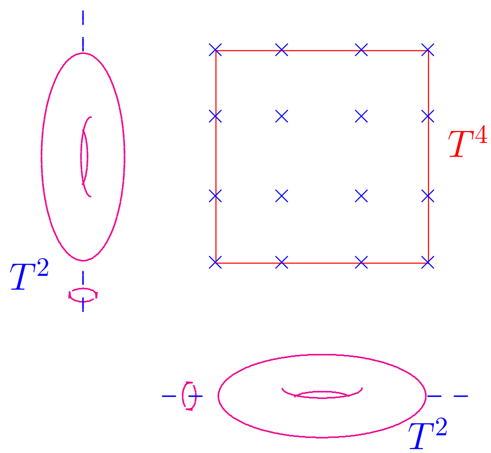

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Result is a $K3$ surface.

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T^4 = Picard torus of curve of genus 2.

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Conjecture (11/8 Conjecture)**.** *Any smooth compact* simply connected *spin* 4*-manifold* M *is (unorientedly) homeomorphic to either* S^4 *or a connected sum* $jK3\#k(S^2 \times S^2)$.

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Certainly true of all examples in this lecture!

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Even Narrower Question. *If (M^4, J) is a compact complex surface, when does M^4 admit an Einstein metric g (unrelated to J) with Einstein constant $\lambda \geq 0$?*

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Similarly when M symplectic instead of complex.

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No others: Hitchin-Thorpe, Seiberg-Witten, . . .

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Definitive list . . .

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Ricci-flat Kähler:

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"Calabi-Yau metrics."

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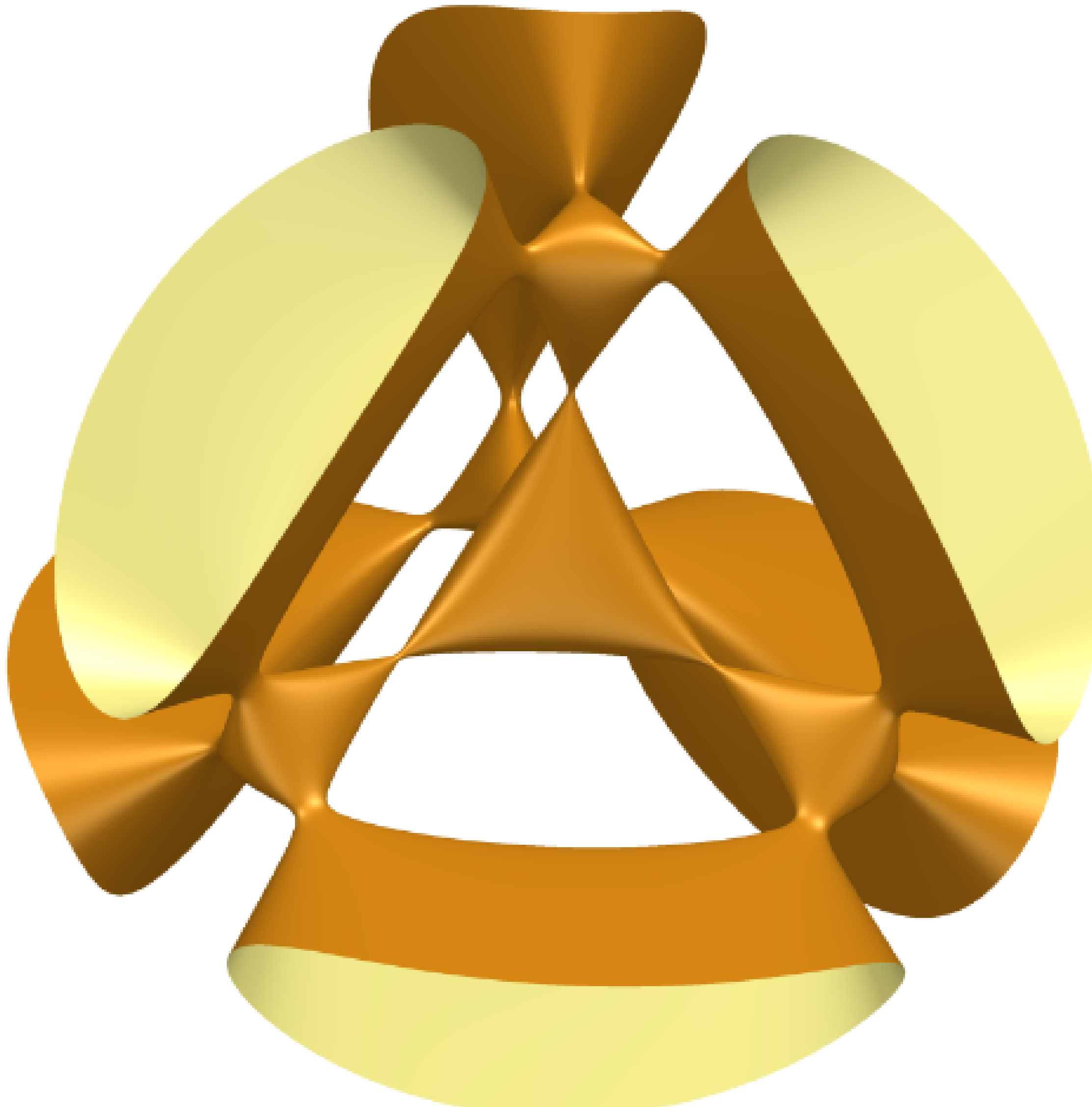

Corollary. $\exists$ $\lambda = 0$ *Einstein metrics on $K3$.*

Indeed, $\exists$ sequences of these $\longrightarrow$ flat orbifold $T^4/\mathbb{Z}_2$.

Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

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Theorem (Hitchin-Thorpe Inequality)**.** *If smooth compact oriented M^4 admits Einstein g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality only if (M, g) finitely covered by flat T^4 or Calabi-Yau $K3$.

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Moduli space $\mathscr{E}(M)$ completely understood.

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Moduli space $\mathscr{E}(M)$ connected!

Above the line:

$$\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

$$K3,$$
$$K3/\mathbb{Z}_2,$$
$$T^4,$$
$$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$$
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Know an Einstein metric on each manifold.

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Moduli space $\mathscr{E}(M) \neq \varnothing$.

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\hline
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Derdziński '83: breakthrough paper on this subject.

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everywhere on M. This scalar condition is a conformally invariant analog of the more familiar condition $s > 0$.

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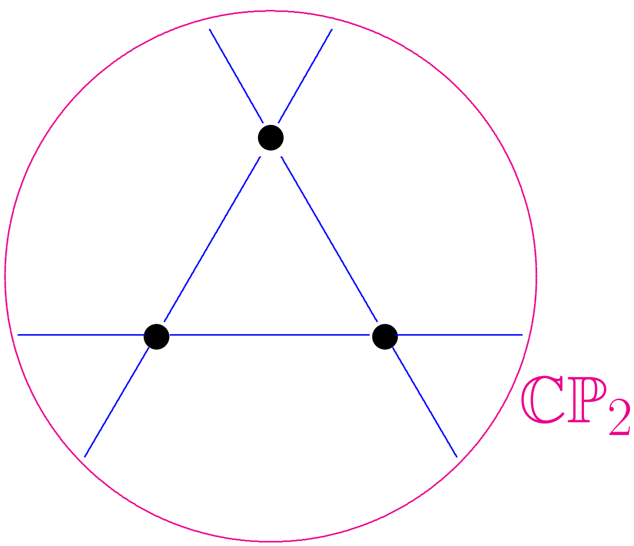

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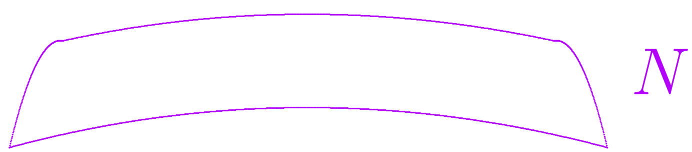

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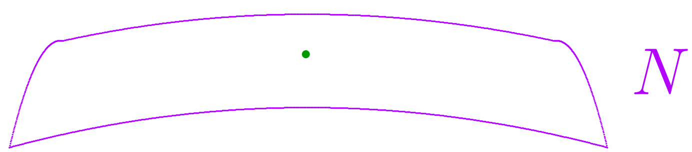

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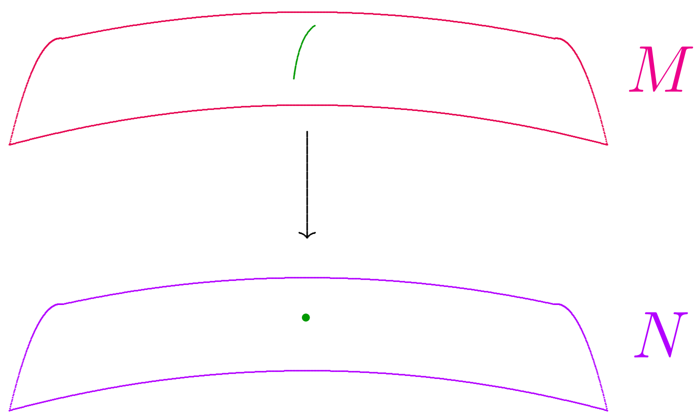

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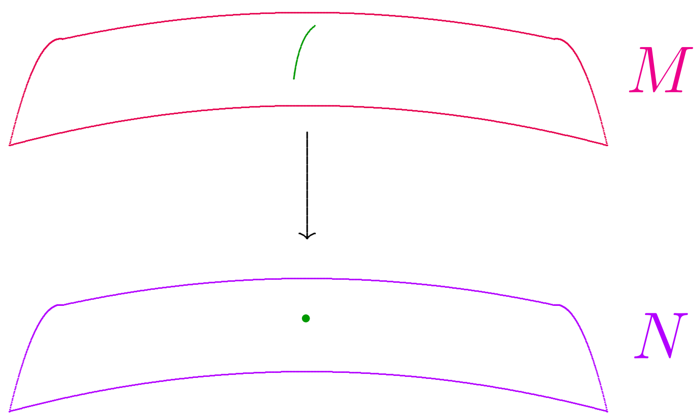
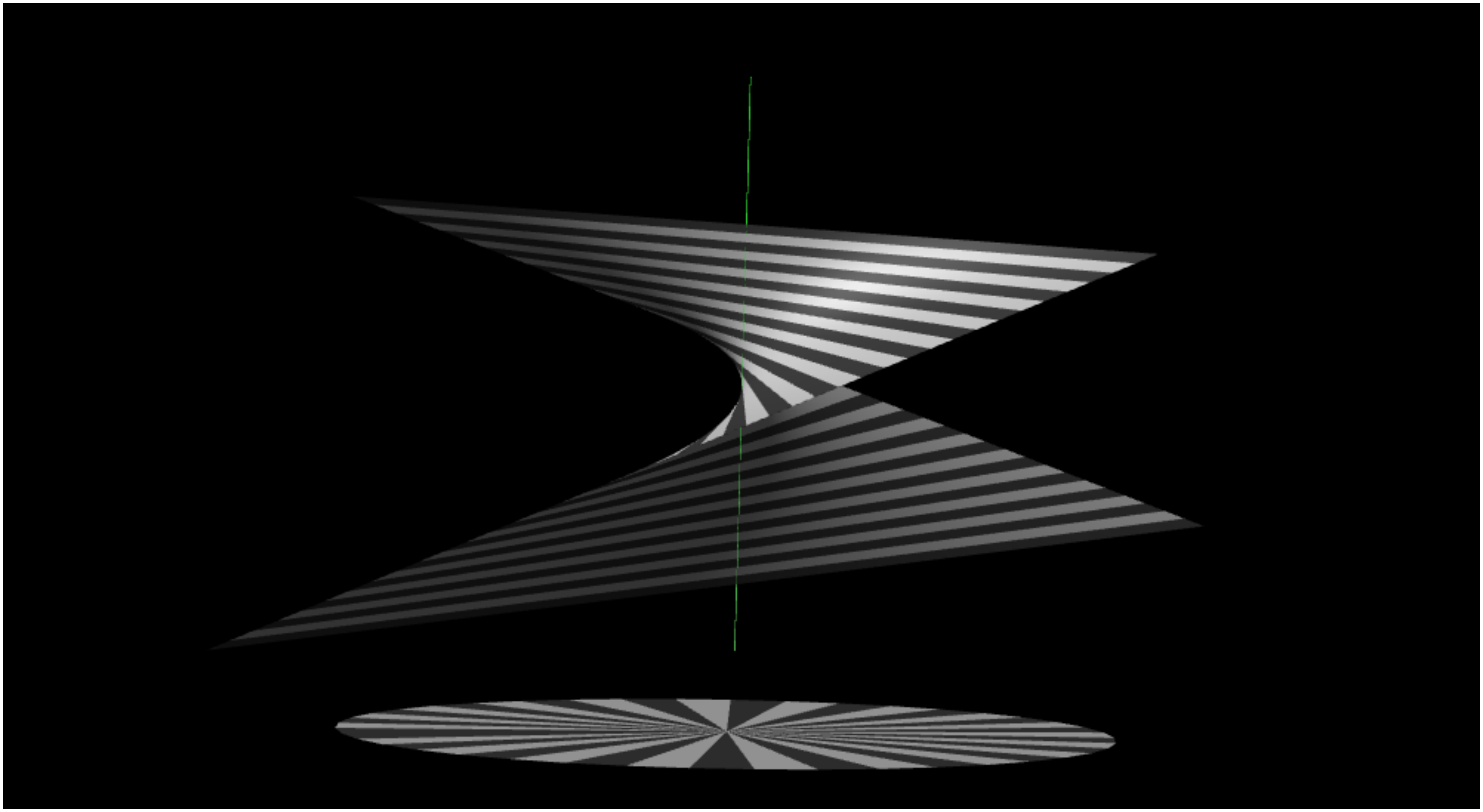
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.





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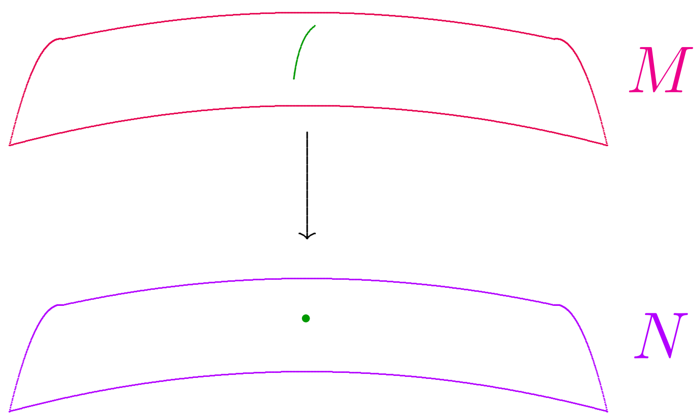

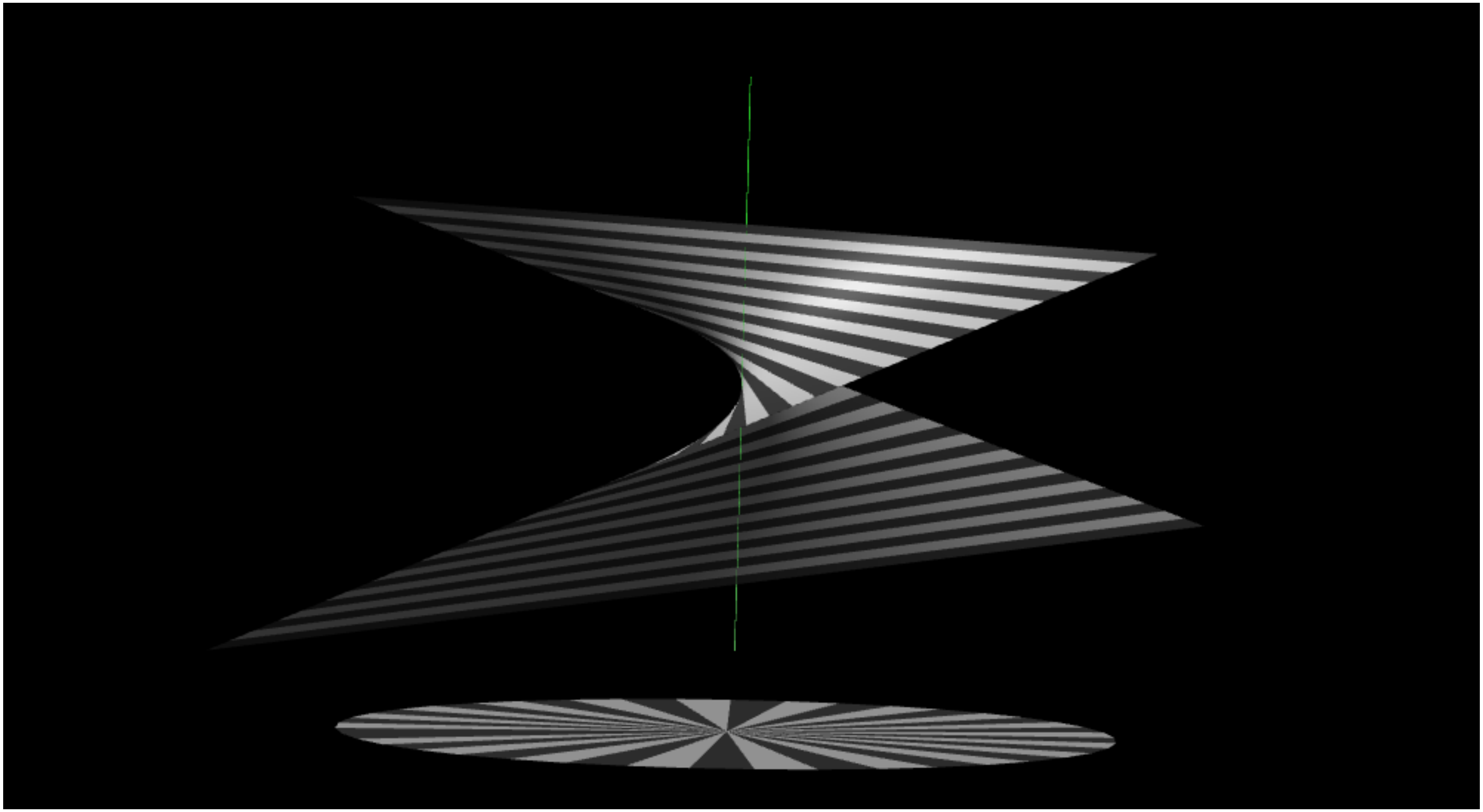

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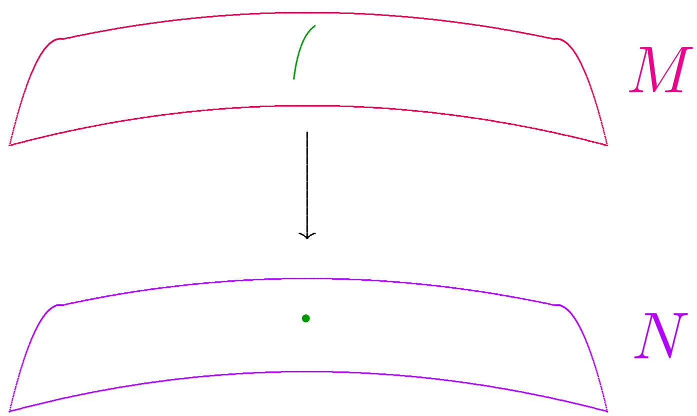

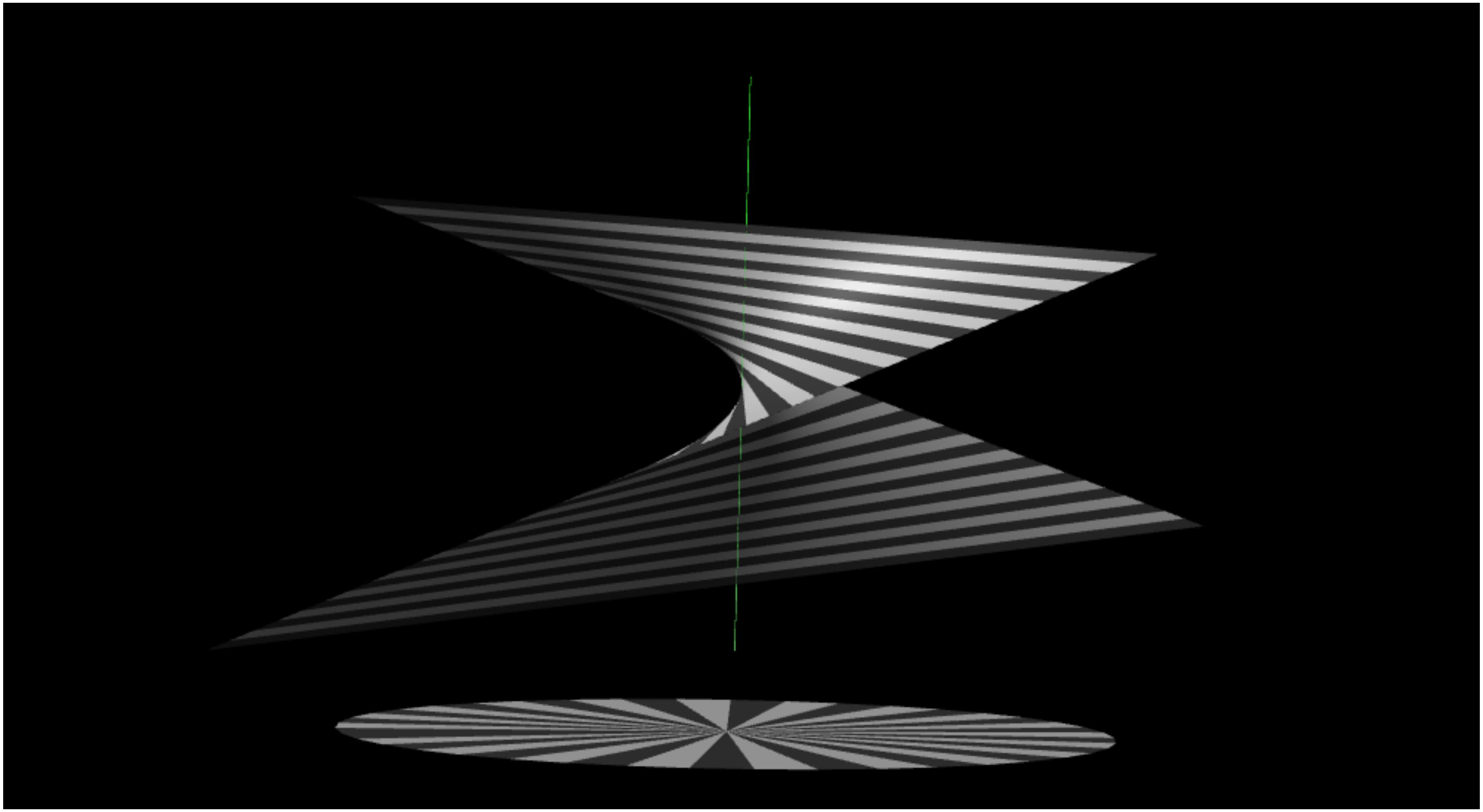

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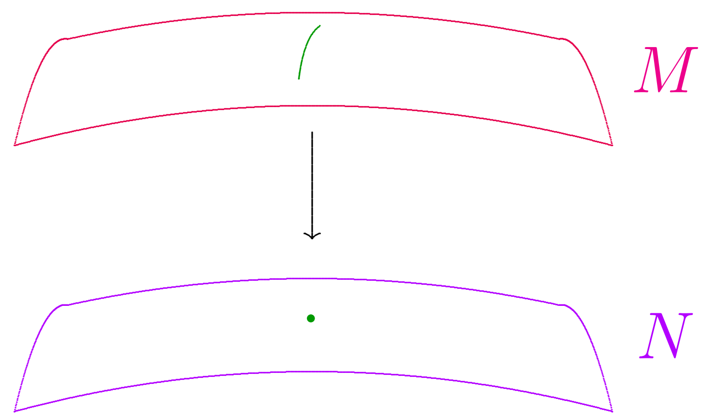

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

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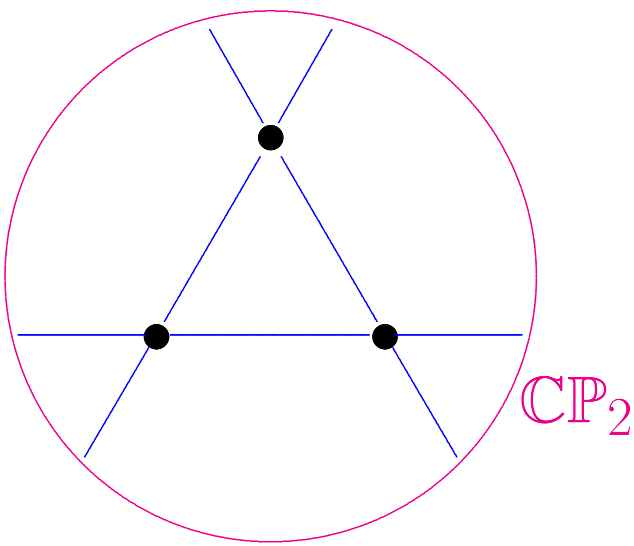

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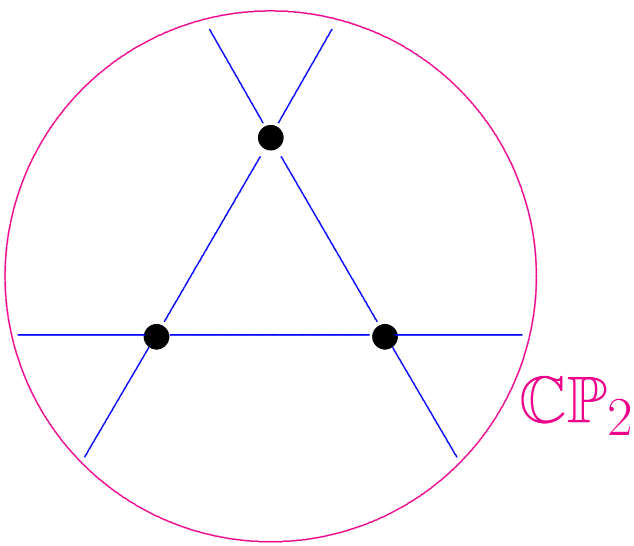

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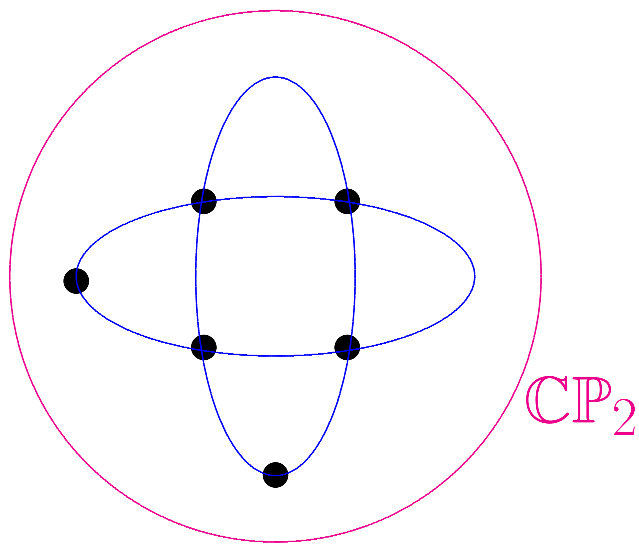

No 3 on a line, no 6 on conic,

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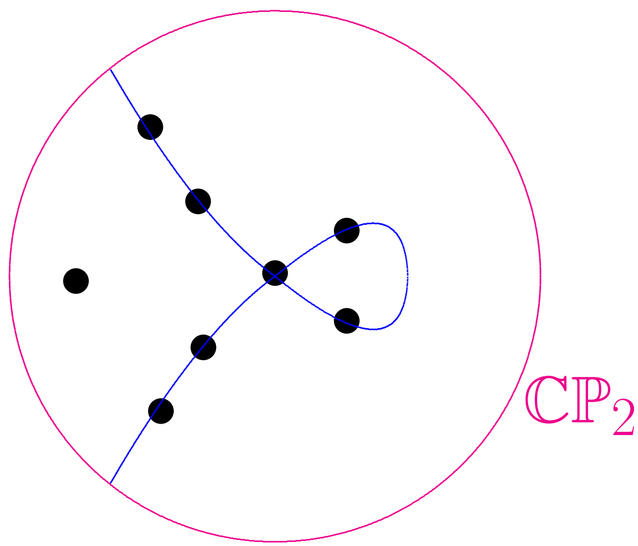

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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Theorem. *Each Del Pezzo (M^4, J) admits a compatible conformally Kähler Einstein metric, and this metric is unique up to automorphisms and rescaling.*

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Existence: Tian, Odaka-Spotti-Sun, Chen-L-Weber...

Uniqueness: Bando-Mabuchi, L '12...

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Just a point if $b_2(M) \leq 5$.

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Theorem (L '14).

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Theorem (L '14)**.** $\mathscr{E}^+_\omega(M)$ *is connected. Moreover, if* $b_2(M) \leq 5$, *then* $\mathscr{E}^+_\omega(M) = \{point\}$.

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Corollary.

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Corollary. *$\mathscr{E}^+_\omega(M)$ is exactly one connected component of $\mathscr{E}(M)$.*

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But that would be the subject of an an **entirely different** colloquium!