Bach-Flat

Kähler Surfaces

Claude LeBrun
Stony Brook University

PRIMA 2017: Oaxaca, Mexico
August 17, 2017

Bach-Flat

Kähler Surfaces

Bach-Flat

Kähler Surfaces

arXiv:1702.03840 [math.DG]

Bach-Flat

Kähler Surfaces

arXiv:1702.03840 [math.DG]

submitted to
J. Geom. Analysis

On Riemannian n-manifold (M, g),

On Riemannian n-manifold (M, g), $n \geq 3$,

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2} \mathring{r}^{[a}{}_{[c} \delta^{b]}{}_{d]} + \frac{2}{n(n-1)} \, s \, \delta^{a}_{[c} \delta^{b}_{d]}$$

On Riemannian n-manifold (M, g),

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2} \mathring{r}^{[a}_{[c} \delta^{b]}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c} \delta^{b}_{d]}$$

where

$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W &= \text{Weyl curvature} \end{aligned}$$

On Riemannian n-manifold (M, g),

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2} \mathring{r}^{[a}_{[c} \delta^{b]}_{d]} + \frac{2}{n(n-1)} \, s \, \delta^{a}_{[c} \delta^{b}_{d]}$$

where

$$s = \text{scalar curvature}$$

$$\mathring{r} = \text{trace-free Ricci curvature}$$

W = Weyl curvature *(conformally invariant)*

On Riemannian n-manifold (M, g),

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}{}_{[c}\delta^{b]}{}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c}\delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature *(conformally invariant)*

$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

On Riemannian n-manifold (M, g),

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2} \mathring{r}^{[a}_{[c}\delta^{b]}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c}\delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature *(conformally invariant)*

Proposition. *Assume* $n \geq 4$*. Then* (M^n, g) *locally conformally flat* $\Longleftrightarrow W \equiv 0$.

On Riemannian n-manifold (M, g),

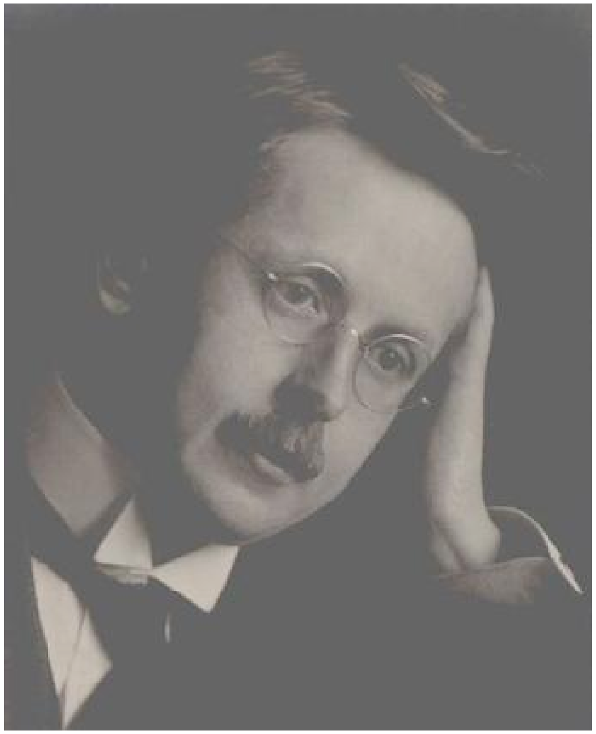
$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}{}_{[c}\delta^{b]}{}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c}\delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature *(conformally invariant)*



For metrics on fixed M^n,

For metrics on fixed M^n, Weyl functional

For metrics on fixed M^n, Weyl functional

$$\mathscr{W} : \mathcal{G}_M \longrightarrow \mathbb{R}$$

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}(g) = \int_M |W_g|^{n/2} d\mu_g$$

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}(g) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

$$\mathscr{W} : \mathcal{G}_M / (C^\infty)^+ \longrightarrow \mathbb{R}$$

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

Measures deviation $[g]$ from conformal flatness.

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

Measures deviation $[g]$ from conformal flatness.

Basic problems: For given smooth compact M,

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

Measures deviation $[g]$ from conformal flatness.

Basic problems: For given smooth compact M,

- Are there any critical points?

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

Measures deviation $[g]$ from conformal flatness.

Basic problems: For given smooth compact M,

- Are there any critical points?
- Can we classify them?

Dimension Four is Exceptional

Dimension Four is Exceptional

For M^4,

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2}\mathring{r}^{cd})W_{acbd}$$

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}$$

called Bach tensor.

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}$$

called Bach tensor.

Solutions called Bach-flat metrics.

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}$$

called Bach tensor.

Solutions called Bach-flat metrics.

Bianchi $\Longrightarrow$ Any Einstein (M^4, g) is Bach-flat.

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}$$

called Bach tensor.

Solutions called Bach-flat metrics.

Bianchi $\Longrightarrow$ Any Einstein (M^4, g) is Bach-flat.

Of course, conformally Einstein good enough!

By contrast:

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

has degenerate Euler-Lagrange equation

$$|W_g|^{(n-4)/2}(\nabla\nabla\cdot W + \cdots) = 0$$

when $n > 4$.

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

has degenerate Euler-Lagrange equation

$$|W_g|^{(n-4)/2}(\nabla\nabla\cdot W + \cdots) = 0$$

when $n > 4$.

Einstein metrics are usually not critical points.

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

has degenerate Euler-Lagrange equation

$$|W_g|^{(n-4)/2}(\nabla\nabla\cdot W + \cdots) = 0$$

when $n > 4$.

Einstein metrics are usually not critical points.

Calabi-Yau $\times$ flat on $K3 \times T^\ell$ never critical

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

has degenerate Euler-Lagrange equation

$$|W_g|^{(n-4)/2}(\nabla\nabla\cdot W + \cdots) = 0$$

when $n > 4$.

Einstein metrics are usually not critical points.

Calabi-Yau × flat on $K3 \times T^\ell$ never critical

when $\ell > 0$,

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

has degenerate Euler-Lagrange equation

$$|W_g|^{(n-4)/2}(\nabla\nabla\cdot W + \cdots) = 0$$

when $n > 4$.

Einstein metrics are usually not critical points.

Calabi-Yau × flat on $K3 \times T^\ell$ never critical

when $\ell > 0$, because $\mathscr{W} \propto \mathrm{Vol}(T^\ell)$!

Henceforth, assume M compact, real dimension 4.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No!

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds are also Bach-flat.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

$W_+ := \frac{1}{2}(W + \star W)$ called self-dual Weyl tensor.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $\Leftrightarrow W = -\star W$.

$W_+ := \frac{1}{2}(W + \star W)$ called self-dual Weyl tensor.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $\Leftrightarrow W = W_-$.

$W_- := \frac{1}{2}(W - \star W)$ is anti-self-dual Weyl tensor.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Thom-Hirzebruch signature formula:

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Thom-Hirzebruch signature formula:

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

Hence

$$\mathscr{W}([g]) = -12\pi^2 \tau(M) + 2 \int_M |W_+|^2 d\mu_g$$

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Thom-Hirzebruch signature formula:

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

Hence

$$\mathscr{W}([g]) = -12\pi^2 \tau(M) + 2 \int_M |W_+|^2 d\mu_g$$

ASD metrics minimize $\mathscr{W}$, and so are Bach-flat.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Thom-Hirzebruch signature formula:

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

Hence

$$\mathscr{W}([g]) = -12\pi^2 \tau(M) + 2 \int_M |W_+|^2 d\mu_g$$

ASD metrics minimize $\mathscr{W}$, and so are Bach-flat.

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}$$

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Thom-Hirzebruch signature formula:

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

Hence

$$\mathscr{W}([g]) = -12\pi^2 \tau(M) + 2 \int_M |W_+|^2 d\mu_g$$

ASD metrics minimize $\mathscr{W}$, and so are Bach-flat.

$$B_{ab} := 2(\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd})(W_+)_{acbd}$$

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Examples: Any scalar-flat Kähler surface (M, g, J).

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Examples: Any scalar-flat Kähler surface (M, g, J).

Kähler surfaces:

$$|W_+|^2 = \frac{s^2}{24}$$

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Examples: Any scalar-flat Kähler surface (M, g, J).

Kähler surfaces:

$$W_+ = 0 \quad \Leftrightarrow \quad s = 0$$

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Examples: Any scalar-flat Kähler surface (M, g, J).

Special constant-scalar-curvature Kähler manifolds.

Henceforth, assume M compact, real dimension 4.

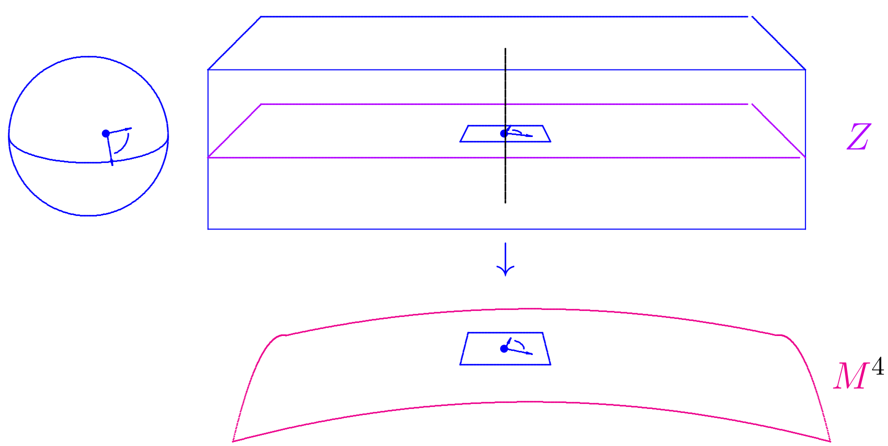
If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Examples: Any scalar-flat Kähler surface (M, g, J).

Special constant-scalar-curvature Kähler manifolds.

Twistor-related ideas $\rightsquigarrow$ understood long before cscK.



Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Examples: Any scalar-flat Kähler surface (M, g, J).

Special constant-scalar-curvature Kähler manifolds.

Twistor-related ideas $\rightsquigarrow$ understood long before cscK.

If not Ricci-flat Kähler, then

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Examples: Any scalar-flat Kähler surface (M, g, J).

Special constant-scalar-curvature Kähler manifolds.

Twistor-related ideas $\rightsquigarrow$ understood long before cscK.

If not Ricci-flat Kähler, then

rational or ruled, with ${c_1}^2 < 0$.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Examples: Any scalar-flat Kähler surface (M, g, J).

Special constant-scalar-curvature Kähler manifolds.

Twistor-related ideas $\rightsquigarrow$ understood long before cscK.

If not Ricci-flat Kähler, then

rational or ruled, with $(2\chi + 3\tau)(M) < 0$.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Examples: Any scalar-flat Kähler surface (M, g, J).

Special constant-scalar-curvature Kähler manifolds.

Twistor-related ideas $\rightsquigarrow$ understood long before cscK.

If not Ricci-flat Kähler, then

rational or ruled, with $(2\chi + 3\tau)(M) < 0$.

Violate Hitchin-Thorpe, so $\nexists$ Einstein on such M.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Examples: Any scalar-flat Kähler surface (M, g, J).

Special constant-scalar-curvature Kähler manifolds.

Twistor-related ideas $\rightsquigarrow$ understood long before cscK.

If not Ricci-flat Kähler, then

rational or ruled, with $(2\chi + 3\tau)(M) < 0$.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Examples: Any scalar-flat Kähler surface (M, g, J).

Special constant-scalar-curvature Kähler manifolds.

Twistor-related ideas $\rightsquigarrow$ understood long before cscK.

If not Ricci-flat Kähler, then

rational or ruled, with $(2\chi + 3\tau)(M) < 0$.

L-Singer '93, Kim-L-Pontecorovo '97 Any rational/ruled (M, J) has blow-ups admitting SFK.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Another possibility:

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Another possibility: Double Poincaré-Einstein.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Another possibility: Double Poincaré-Einstein.

$h = f^{-2}g$ Einstein, when $f \neq 0$,

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Another possibility: Double Poincaré-Einstein.

$h = f^{-2}g$ Einstein, $\lambda < 0$, when $f \neq 0$,

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Another possibility: Double Poincaré-Einstein.

$h = f^{-2} g$ Einstein, $\lambda < 0$, when $f \neq 0$,

$f : M \to \mathbb{R}$ with $df \neq 0$ along $f^{-1}(0) \neq \varnothing$.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Another possibility: Double Poincaré-Einstein.

$h = f^{-2}g$ Einstein, when $f \neq 0$,

$f : M \to \mathbb{R}$

$$0 = f\mathring{r} + 2\mathrm{Hess}_0 f$$

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Another possibility: Double Poincaré-Einstein.

Prototype:

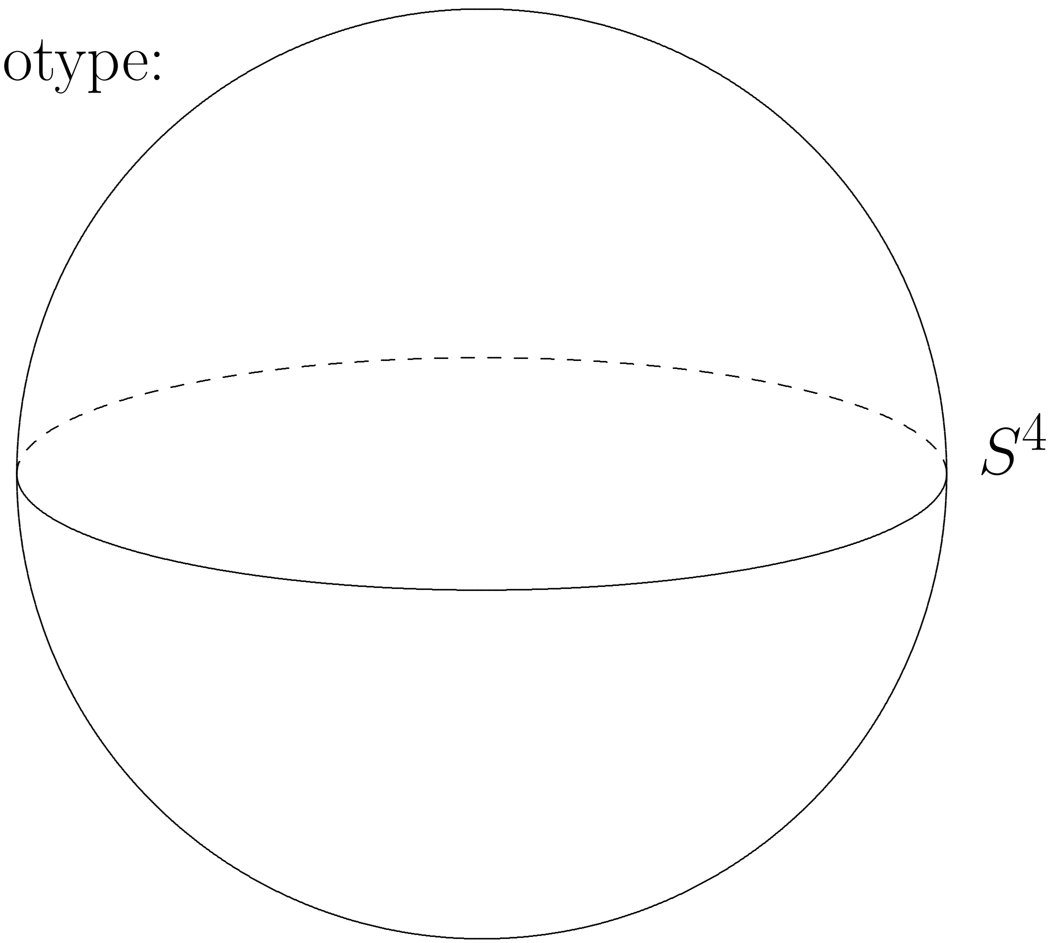
Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Another possibility: Double Poincaré-Einstein.

Prototype:



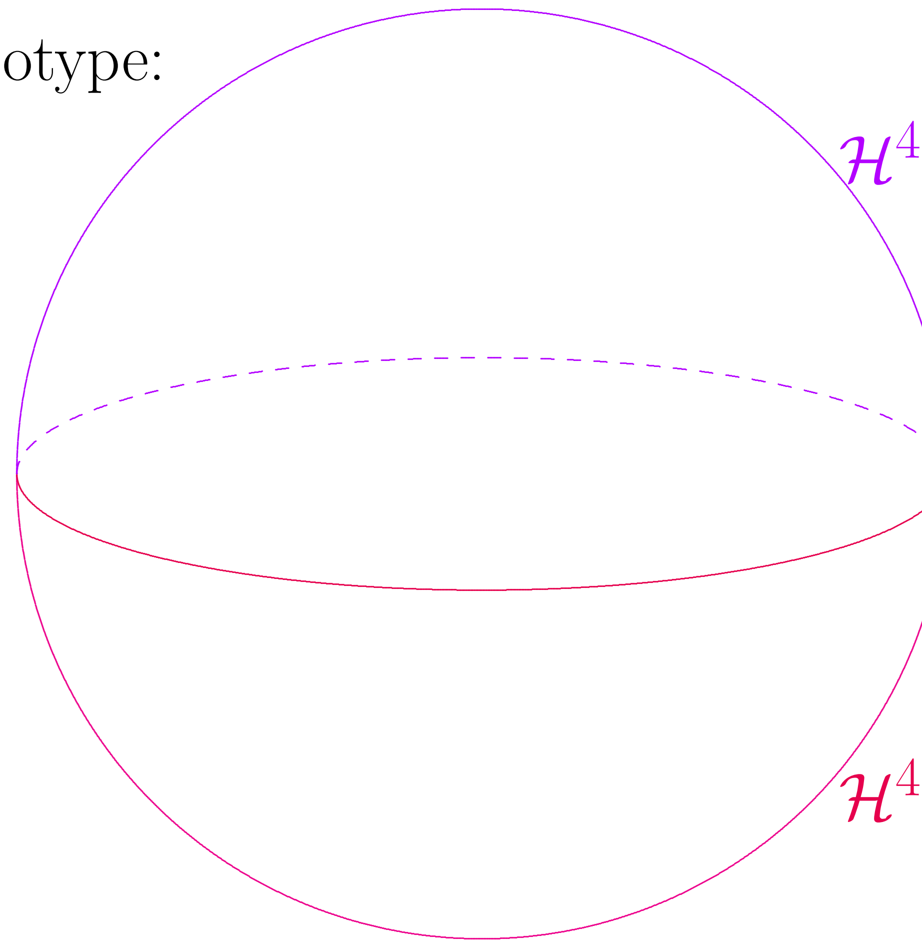
Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Another possibility: Double Poincaré-Einstein.

Prototype:



Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Another possibility: Double Poincaré-Einstein.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Another possibility: Double Poincaré-Einstein.

This prototype is rather degenerate:

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Another possibility: Double Poincaré-Einstein.

This prototype is rather degenerate:

S^4 is also Einstein, ASD.

Henceforth, assume M compact, real dimension 4.

If $(M^4, [g])$ is Bach-flat, is it conformally Einstein?

No! Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

Another possibility: Double Poincaré-Einstein.

This prototype is rather degenerate.

But $\exists$ genuine examples that aren't.

Henceforth, assume M compact, real dimension 4.

Henceforth, assume M compact, real dimension 4.

Open Problem:

Henceforth, assume M compact, real dimension 4.

Open Problem:

Every Bach-flat 4-manifold one of these three types?

Henceforth, assume M compact, real dimension 4.

Open Problem:

Every Bach-flat 4-manifold one of these three types?

Einstein, $\pm$ASD, Double Poincaré-Einstein?

Henceforth, assume M compact, real dimension 4.

Open Problem:

Every Bach-flat 4-manifold one of these three types?

Einstein, $\pm$ASD, Double Poincaré-Einstein?

Locally this is wildly false!

Henceforth, assume M compact, real dimension 4.

Open Problem:

Every Bach-flat 4-manifold one of these three types?

Einstein, $\pm$ASD, Double Poincaré-Einstein?

Locally this is wildly false!

But no compact counter-examples are known!

Henceforth, assume M compact, real dimension 4.

Today:

Henceforth, assume M compact, real dimension 4.

Today:

Bach-flat Kähler

Henceforth, assume M compact, real dimension 4.

Today:

Bach-flat Kähler $\Longrightarrow$ one of these three types.

Henceforth, assume M compact, real dimension 4.

Today:

Bach-flat Kähler $\Longrightarrow$ one of these three types.

Builds on earlier local results of Andrzej Derdziński.

Henceforth, assume M compact, real dimension 4.

Today:

Bach-flat Kähler $\Longrightarrow$ one of these three types.

Builds on earlier local results of Andrzej Derdziński.

Scalar curvature s plays the starring role.

Henceforth, assume M compact, real dimension 4.

Today:

Bach-flat Kähler $\Longrightarrow$ one of these three types.

Builds on earlier local results of Andrzej Derdziński.

Scalar curvature s plays the starring role.

Kähler surfaces:

$$|W_+|^2 = \frac{s^2}{24}$$

Henceforth, assume M compact, real dimension 4.

Today:

Bach-flat Kähler $\Longrightarrow$ one of these three types.

Builds on earlier local results of Andrzej Derdziński.

Scalar curvature s plays the starring role.

Kähler surfaces:

$$|W_+|^2 = \frac{s^2}{24}$$

$$\int_M |W_+|^2 d\mu = \frac{1}{24} \int_M s^2 d\mu$$

Henceforth, assume M compact, real dimension 4.

Today:

Bach-flat Kähler $\Longrightarrow$ one of these three types.

Builds on earlier local results of Andrzej Derdziński.

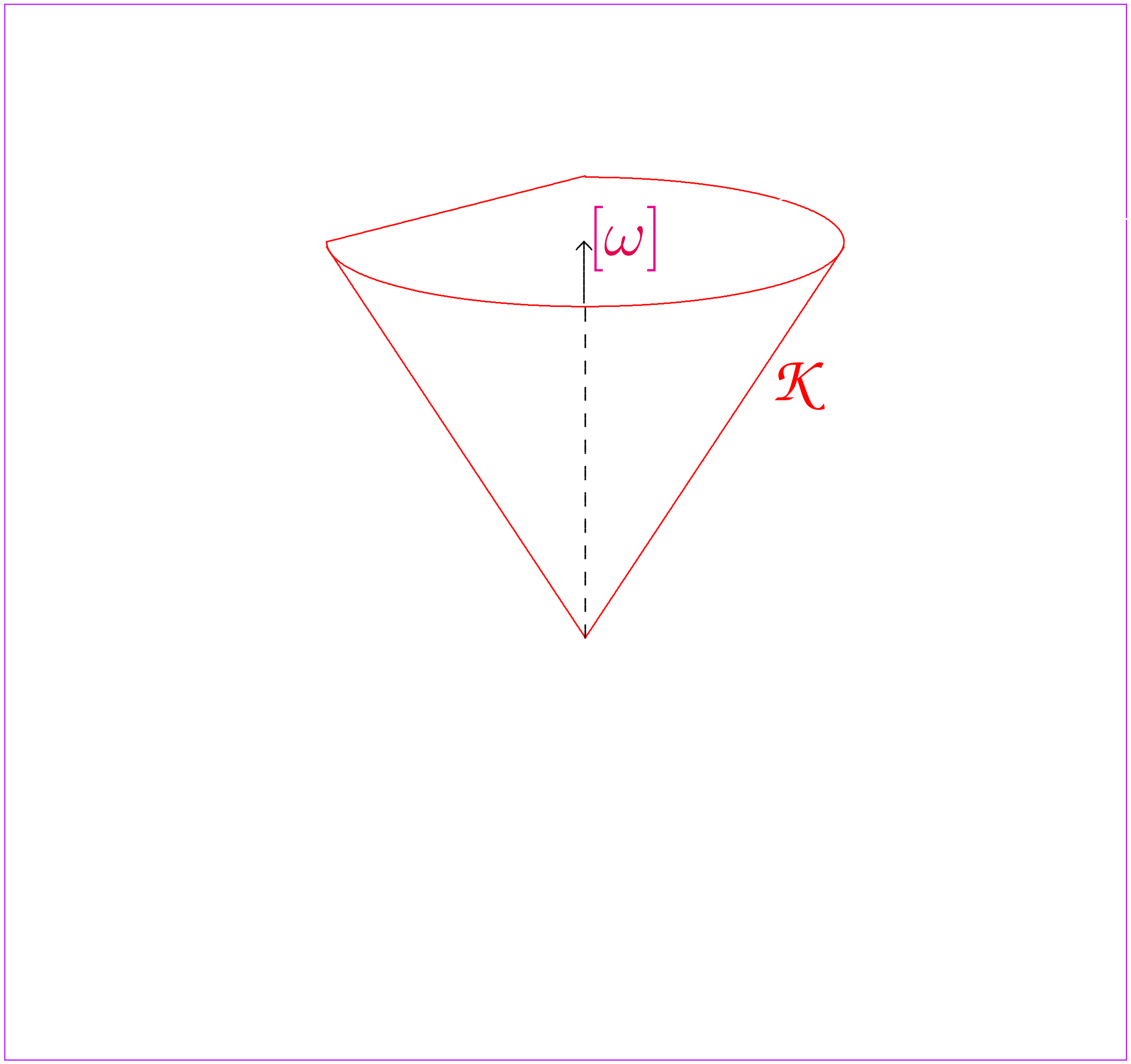
Scalar curvature s plays the starring role.

Kähler surfaces:

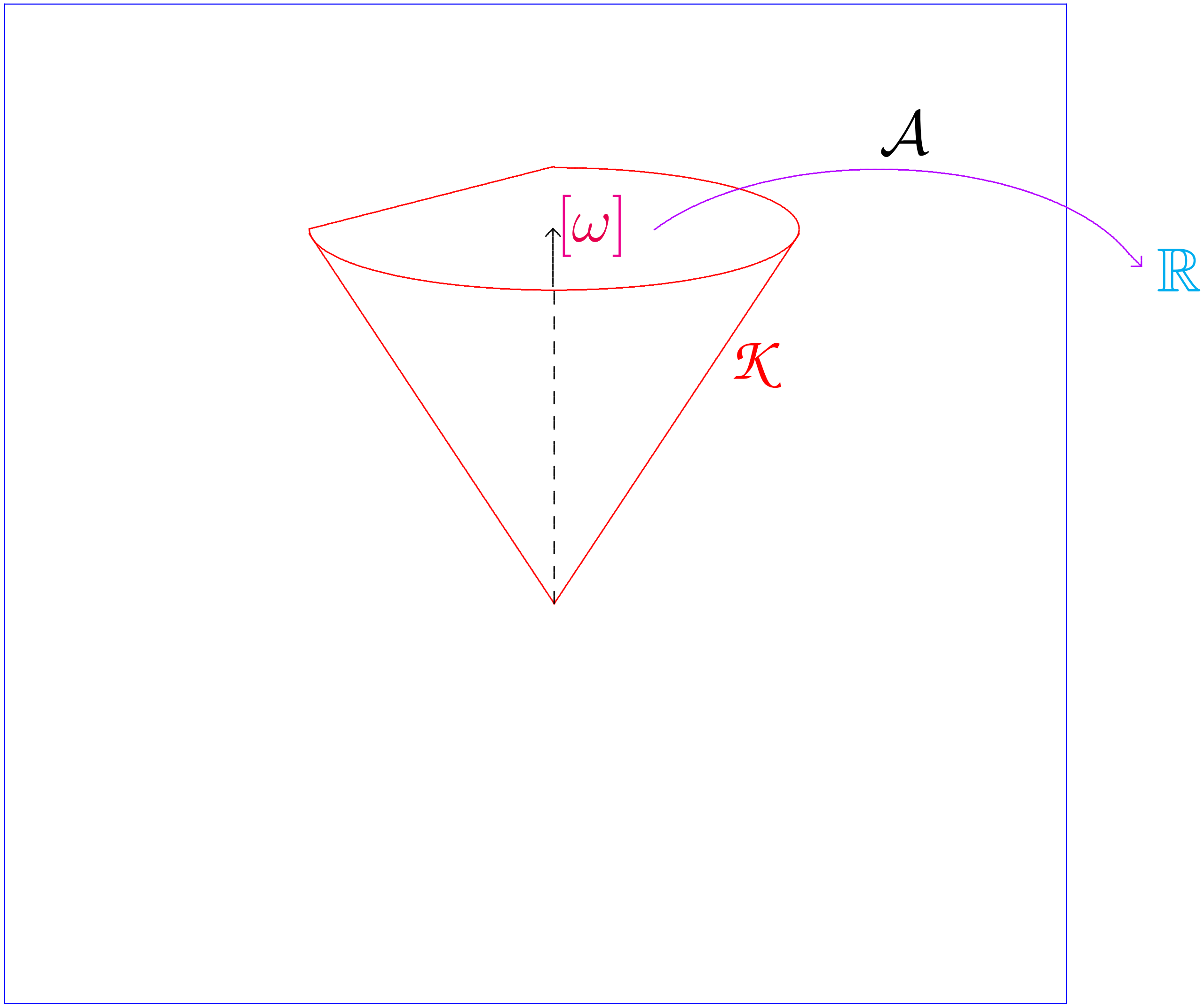
$$|W_+|^2 = \frac{s^2}{24}$$

$$\int_M |W_+|^2 d\mu = \frac{1}{24} \int_M s^2 d\mu$$

Bach-flat Kähler $\Longrightarrow$ extremal Kähler



$$\mathcal{K} \subset H^{1,1}(M, \mathbb{R}) \subset H^2(M, \mathbb{R})$$



$$\mathcal{K} \subset H^{1,1}(M,\mathbb{R}) \subset H^2(M,\mathbb{R})$$

Action Function on Kähler Cone

For any extremal Kähler (M^4, g, J),

$$\frac{1}{32\pi^2}\int s^2 d\mu_g = \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2}\|\mathcal{F}_{[\omega]}\|^2$$
$$=: \mathcal{A}([\omega])$$

where $\mathcal{F}$ is Futaki invariant.

$\mathcal{A}$ is function on Kähler cone $\mathcal{K} \subset H^2(M, \mathbb{R})$.

Action Function on Kähler Cone

For any extremal Kähler (M^4, g, J),

$$\frac{1}{32\pi^2}\int s^2 d\mu_g = \frac{(c_1\cdot[\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2}\|\mathcal{F}_{[\omega]}\|^2$$
$$=: \mathcal{A}([\omega])$$

where $\mathcal{F}$ is Futaki invariant.

$\mathcal{A}$ is function on Kähler cone $\mathcal{K} \subset H^2(M, \mathbb{R})$.

Proposition. *If g is a Kähler metric on a compact complex surface (M^4, J), with Kähler class $[\omega]$,*

Action Function on Kähler Cone

For any extremal Kähler (M^4, g, J),

$$\frac{1}{32\pi^2}\int s^2 d\mu_g = \frac{(c_1\cdot[\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2}\|\mathcal{F}_{[\omega]}\|^2$$
$$=: \mathcal{A}([\omega])$$

where $\mathcal{F}$ is Futaki invariant.

$\mathcal{A}$ is function on Kähler cone $\mathcal{K} \subset H^2(M, \mathbb{R})$.

Proposition. *If g is a Kähler metric on a compact complex surface (M^4, J), with Kähler class $[\omega]$, then g satisfies $B = 0$ $\Longleftrightarrow$*

Action Function on Kähler Cone

For any extremal Kähler (M^4, g, J),

$$\frac{1}{32\pi^2}\int s^2 d\mu_g = \frac{(c_1\cdot[\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2}\|\mathcal{F}_{[\omega]}\|^2$$
$$=: \mathcal{A}([\omega])$$

where $\mathcal{F}$ is Futaki invariant.

$\mathcal{A}$ is function on Kähler cone $\mathcal{K} \subset H^2(M, \mathbb{R})$.

Proposition. *If g is a Kähler metric on a compact complex surface (M^4, J), with Kähler class $[\omega]$, then g satisfies $B = 0 \iff$*

- *g is an extremal Kähler metric; and*

Action Function on Kähler Cone

For any extremal Kähler (M^4, g, J),

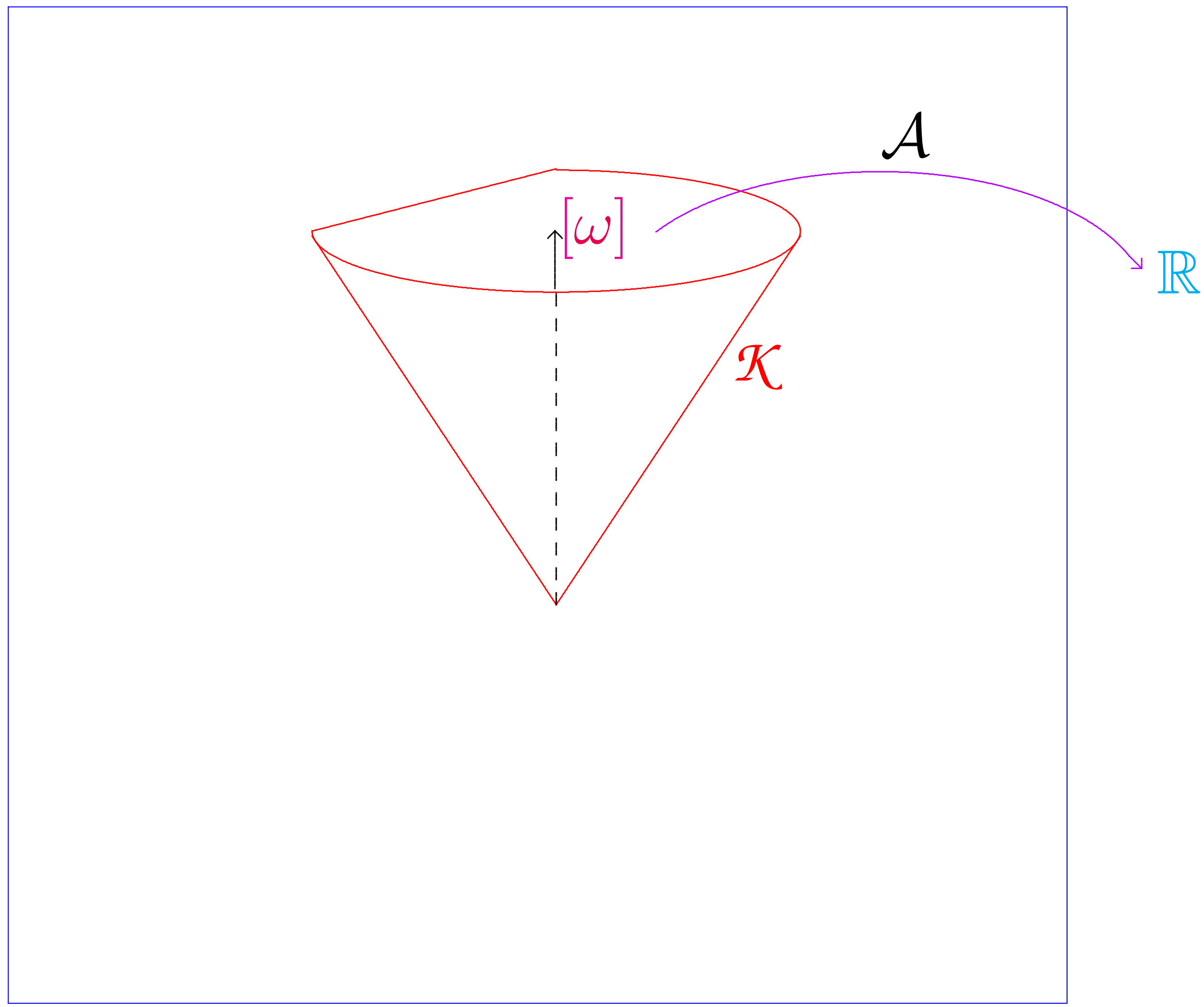
$$\frac{1}{32\pi^2}\int s^2 d\mu_g = \frac{(c_1\cdot[\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2}\|\mathcal{F}_{[\omega]}\|^2$$
$$=: \mathcal{A}([\omega])$$

where $\mathcal{F}$ is Futaki invariant.

$\mathcal{A}$ is function on Kähler cone $\mathcal{K} \subset H^2(M, \mathbb{R})$.

Proposition. *If g is a Kähler metric on a compact complex surface (M^4, J), with Kähler class $[\omega]$, then g satisfies $B = 0 \Longleftrightarrow$*

- *g is an extremal Kähler metric; and*
- *$[\omega]$ is a critical point of $\mathcal{A} : \mathcal{K} \to \mathbb{R}$.*



$$\mathcal{K} \subset H^{1,1}(M, \mathbb{R}) \subset H^2(M, \mathbb{R})$$

Henceforth, assume M compact, real dimension 4.

Today:

Bach-flat Kähler $\Longrightarrow$ one of these three types.

Builds on earlier local results of Andrzej Derdziński.

Scalar curvature s plays the starring role.

Kähler surfaces:

$$|W_+|^2 = \frac{s^2}{24}$$

$$\int_M |W_+|^2 d\mu = \frac{1}{24}\int_M s^2 d\mu$$

Bach-flat Kähler $\quad\Longrightarrow\quad$ extremal Kähler

Today:

Bach-flat Kähler $\Longrightarrow$ one of these three types.

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface.*

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

I. $\min s > 0$. *Then*

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein,* $\lambda > 0$, $\mathit{Hol} = \mathbf{SO}(4)$.

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$*; or else*

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein,* $\lambda > 0$*,* $Hol = \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$; *or else*

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein,* $\lambda > 0$*,* *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda < 0$*; or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein.*

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$*; or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$*,*

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein,* $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda < 0$*; or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$, *and* $M - \mathcal{Z}$ *has exactly two components.*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$; *or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$, *and* $M - \mathcal{Z}$ *has exactly two components.*

I. $s > 0$ *everywhere. Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein,* $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $s < 0$ *somewhere. Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda < 0$*; or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$*, and* $M - \mathcal{Z}$ *has exactly two components.*

If not Kähler-Einstein:

I. s *is positive. Then*

$(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. s *is zero. Then*

(M, g, J) *SFK, but not Ricci-flat.*

III. s *changes sign. Then*

$(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$*, and* $M - \mathcal{Z}$ *has exactly two components.*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$*; or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathscr{Z}^3$*, and* $M - \mathscr{Z}$ *has exactly two components.*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

This happens $\iff$ $c_1 > 0$.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

This happens $\iff$ $c_1 > 0$.

$\iff$ (M^4, J) is a Del Pezzo surface.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

This happens $\iff$ $c_1 > 0$.

$\iff$ (M^4, J) is a Del Pezzo surface.

Existence: Chen-L-Weber '08, et. al:

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

This happens $\iff$ $c_1 > 0$.

$\iff$ (M^4, J) is a Del Pezzo surface.

Existence: Chen-L-Weber '08, et. al.

Page, Siu, Yau, Tian, Odaka-Spotti-Sun, ...

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

This happens $\iff$ $c_1 > 0$.

$\iff$ (M^4, J) is a Del Pezzo surface.

L '12: Bach-flat Käher g uniquely determined by J up to complex automorphisms and homothety.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

This happens $\iff$ $c_1 > 0$.

$\iff$ (M^4, J) is a Del Pezzo surface.

L '12: Bach-flat Käher g uniquely determined by J up to complex automorphisms and homothety.

Inspired by numerical experiments of Gideon Maschler.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $\mathit{Hol} = \mathbf{SO}(4)$.

This happens $\iff$ $c_1 > 0$.

$\iff$ (M^4, J) is a Del Pezzo surface.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

This happens $\iff$ $c_1 > 0$.

$\iff$ (M^4, J) is a Del Pezzo surface.

(a) when $\mathrm{Aut}_0(M, J)$ reductive.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

This happens $\iff$ $c_1 > 0$.

$\iff$ (M^4, J) is a Del Pezzo surface.

(a) when $\mathrm{Aut}_0(M, J)$ reductive.

(b) when $\mathrm{Aut}_0(M, J)$ non-reductive.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

This happens $\iff$ $c_1 > 0$.

$\iff$ (M^4, J) is a Del Pezzo surface.

(a) when $\mathrm{Aut}_0(M, J)$ reductive.

(b) when $M = \mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ or $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein,* $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

Previously discussed this case: $W_+ = 0$.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

Previously discussed this case: $W_+ = 0$.

Main point: if $\min s = 0$,

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein,* $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

Previously discussed this case: $W_+ = 0$.

Main point: if $\min s = 0$, then $s \equiv 0$.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

Previously discussed this case: $W_+ = 0$.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

Previously discussed this case: $W_+ = 0$.

(a) $\Longrightarrow$ $\text{Kod}\,(M, J) = 0$.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

Previously discussed this case: $W_+ = 0$.

(a) $\Longrightarrow$ Kod $(M, J) = 0$.

(b) $\Longrightarrow$ Kod $(M, J) = -\infty$.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

(b) (M, g, J) *anti-self-dual*, *but not Einstein*.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$; *or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$, *and* $M - \mathcal{Z}$ *has exactly two components.*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein,* $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda < 0$*; or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$*, and* $M - \mathcal{Z}$ *has exactly two components.*

If $\min s < 0$,

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$*; or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$*, and* $M - \mathcal{Z}$ *has exactly two components.*

If $\min s < 0$, then s either constant,

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *$Hol = \mathbf{SO}(4)$.*

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$*; or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,*
$s = 0$ *defines smooth connected* $\mathcal{Z}^3$*, and*
$M - \mathcal{Z}$ *has exactly two components.*

If $\min s < 0$, then s either constant, or changes sign.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$; *or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$, *and* $M - \mathcal{Z}$ *has exactly two components.*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$; *or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$, *and* $M - \mathcal{Z}$ *has exactly two components.*

(a) $\Longrightarrow \text{Kod}\,(M, J) = 2$.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$; *or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$, *and* $M - \mathcal{Z}$ *has exactly two components.*

(a) $\Longrightarrow$ Kod $(M, J) = 2$. (b) $\Longrightarrow$ Kod $(M, J) = -\infty$.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein,* $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda < 0$*; or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathscr{Z}^3$, *and* $M - \mathscr{Z}$ *has exactly two components.*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$*; or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$, *and* $M - \mathcal{Z}$ *has exactly two components.*

Examples of (b): Hwang-Simanca, Tønnesen-Friedman

A few words about the proof. . .

For Kähler metrics g,

For Kähler metrics g,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\Big]$$

For Kähler metrics g,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\Big]$$

so Bach-flat $\Longrightarrow$ g extremal

For Kähler metrics g,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\Big]$$

so Bach-flat $\Longrightarrow$ g extremal

Lemma. *Suppose* (M^4, g, J) *Bach-flat Kähler, with* s *non-constant.*

For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\right]$$

so Bach-flat $\Longrightarrow$ g extremal

Lemma. *Suppose (M^4, g, J) Bach-flat Kähler, with s non-constant. Then $s : M \to \mathbb{R}$ is a Morse-Bott function, with critical submanifolds either complex curves, or isolated points.*

For Kähler metrics g,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \text{Hess}_0(s) + 3J^*\text{Hess}_0(s)\Big]$$

so Bach-flat $\Longrightarrow$ g extremal

Lemma. *Suppose* (M^4, g, J) *Bach-flat Kähler, with* s *non-constant. Then* $s : M \to \mathbb{R}$ *is a Morse-Bott function, with critical submanifolds either complex curves, or isolated points.*

Reason: $J\nabla s$ is Killing field.

For Kähler metrics g,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\Big]$$

so Bach-flat $\Longrightarrow$ g extremal

Lemma. *Suppose* (M^4, g, J) *Bach-flat Kähler, with* s *non-constant. Then* $s : M \to \mathbb{R}$ *is a Morse-Bott function, with critical submanifolds either complex curves, or isolated points.*

Reason: $J\nabla s$ is Killing field. In particular,

$$(\nabla s)\Big|_p = 0 \quad \Longrightarrow \quad (\nabla\nabla s)\Big|_p \neq 0,$$

For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\right]$$

so Bach-flat $\Longrightarrow$ g extremal

Lemma. *Suppose (M^4, g, J) Bach-flat Kähler, with s non-constant. Then $s : M \to \mathbb{R}$ is a Morse-Bott function, with critical submanifolds either complex curves, or isolated points.*

Reason: $J\nabla s$ is Killing field. In particular,

$$(\nabla s)\Big|_p = 0 \quad \Longrightarrow \quad (\nabla\nabla s)\Big|_p \neq 0,$$

$$\Delta s \neq 0 \quad \text{at } \min s \text{ and } \max s.$$

For Kähler metrics g,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\Big]$$

so Bach-flat $\Longrightarrow$ g extremal

Lemma. *Suppose* (M^4, g, J) *Bach-flat Kähler, with* s *non-constant. Then* $s : M \to \mathbb{R}$ *is a Morse-Bott function, with critical submanifolds either complex curves, or isolated points.*

Reason: $J\nabla s$ is Killing field. In particular,

$$(ds)\Big|_p = 0 \quad \Longrightarrow \quad \mathrm{Hess}(s)\Big|_p \neq 0,$$

For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\right]$$

so Bach-flat $\Longrightarrow$ g extremal

For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\right]$$

so Bach-flat $\Longrightarrow$ g extremal and

For Kähler metrics g,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\Big]$$

so Bach-flat $\Longrightarrow$ g extremal and

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

For Kähler metrics g,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\Big]$$

so Bach-flat $\Longrightarrow$ g extremal and

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\right]$$

so Bach-flat $\Longrightarrow$ g extremal and

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3,$$

For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\right]$$

so Bach-flat $\Longrightarrow$ g extremal and

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3,$$

where $\Delta = -\nabla^a\nabla_a$.

For Kähler metrics g,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\Big]$$

so Bach-flat $\Longrightarrow$ g extremal and

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

For Kähler metrics g,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\Big]$$

so Bach-flat $\Longrightarrow$ g extremal and

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

Lemma. *The function κ is constant,*

For Kähler metrics g,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\Big]$$

so Bach-flat $\Longrightarrow$ g extremal and

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

Lemma. *The function κ is constant, and has the same sign $(+, -, 0)$ as* $\min s$*.*

For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\right]$$

so Bach-flat $\Longrightarrow$ g extremal and

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

Lemma. *The function κ is constant, and has the same sign $(+, -, 0)$ as* $\min s$*.*

Obvious if s constant.

For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \text{Hess}_0(s) + 3J^*\text{Hess}_0(s)\right]$$

so Bach-flat $\Longrightarrow$ g extremal and

$$0 = s\mathring{r} + 2\text{Hess}_0(s).$$

On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

Lemma. *The function κ is constant, and has the same sign $(+, -, 0)$ as* $\min s$.

Otherwise $\kappa = (s^2 - 6\Delta s)s$ at $\min s$.

For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \text{Hess}_0(s) + 3J^*\text{Hess}_0(s)\right]$$

so Bach-flat $\Longrightarrow$ g extremal and

$$0 = s\mathring{r} + 2\text{Hess}_0(s).$$

On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

Lemma. *The function κ is constant, and has the same sign $(+, -, 0)$ as* $\min s$.

Otherwise $\kappa = (\qquad + \qquad)s$ at $\min s$.

For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\right]$$

so Bach-flat $\Longrightarrow$ g extremal and

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

Lemma. *The function κ is constant, and has the same sign $(+, -, 0)$ as* $\min s$.

Otherwise $\kappa = (s^2 - 6\Delta s)s$ at $\min s$.

For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \text{Hess}_0(s) + 3J^*\text{Hess}_0(s)\right]$$

so Bach-flat $\Longrightarrow$ g extremal and

$$0 = s\mathring{r} + 2\text{Hess}_0(s).$$

On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

Lemma. *The function κ is constant, and has the same sign $(+, -, 0)$ as* $\min s$.

Otherwise $\kappa = (\qquad + \qquad)s$ at $\min s$.

For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\right]$$

so Bach-flat $\Longrightarrow$ g extremal and

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

Lemma. *The function κ is constant, and has the same sign $(+, -, 0)$ as* $\min s$.

For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\right]$$

so Bach-flat $\Longrightarrow$ g extremal and

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

Lemma. *The function κ is constant, and has the same sign $(+, -, 0)$ as $\min s$. On set where $s \neq 0$, the constant κ = scalar curvature of $s^{-2}g$.*

Let's just see what happens when $\min s = 0$.

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set:

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set:

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set:

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

$$s(p) = 0 \implies \mathrm{Hess}_0(s)\Big|_p = 0.$$

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set:

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

$$s(p) = 0 \implies (\nabla\nabla s)\Big|_p = ag.$$

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set:

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

$$s(p) = 0 \quad \Longrightarrow \quad (\nabla\nabla s)\Big|_p = ag.$$

$$(\nabla s)\Big|_p = 0 \quad \Longrightarrow \quad (\nabla\nabla s)\Big|_p \neq 0,$$

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set:

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

$$s(p) = 0 \implies (\nabla\nabla s)\Big|_p = ag.$$

$$a > 0$$

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set.

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set.

$W_+ \neq 0$ everywhere else.

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

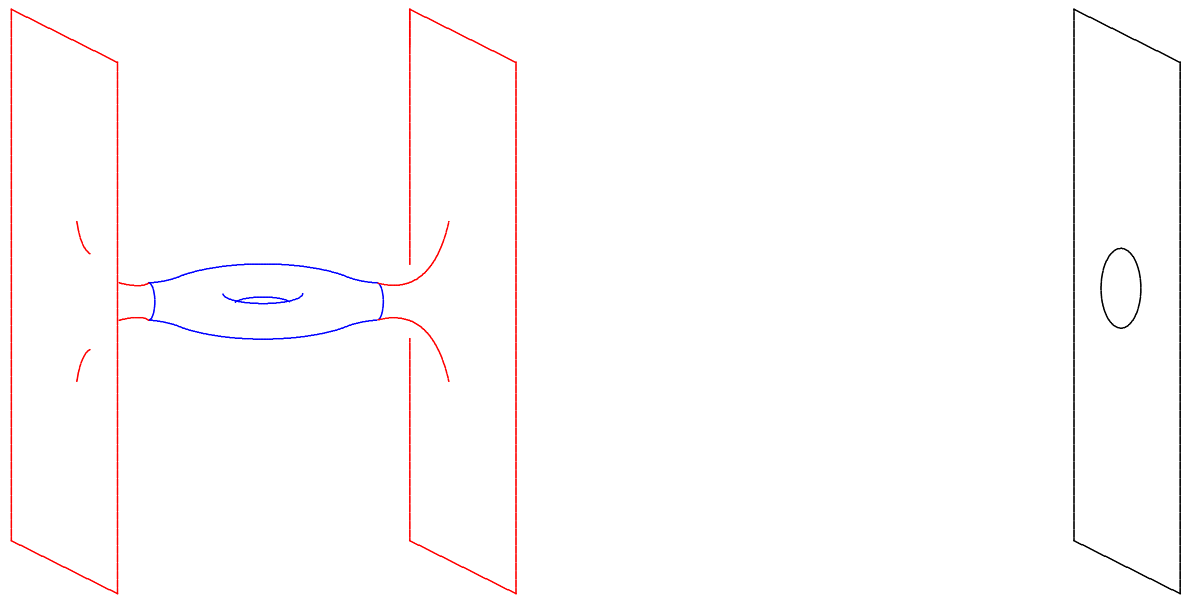
Want to show that s is constant.

If not, $s = 0$ only at finite set.

$W_+ \neq 0$ everywhere else.

$h = s^{-2}g$ is Ricci-flat, asymptotically Euclidean.

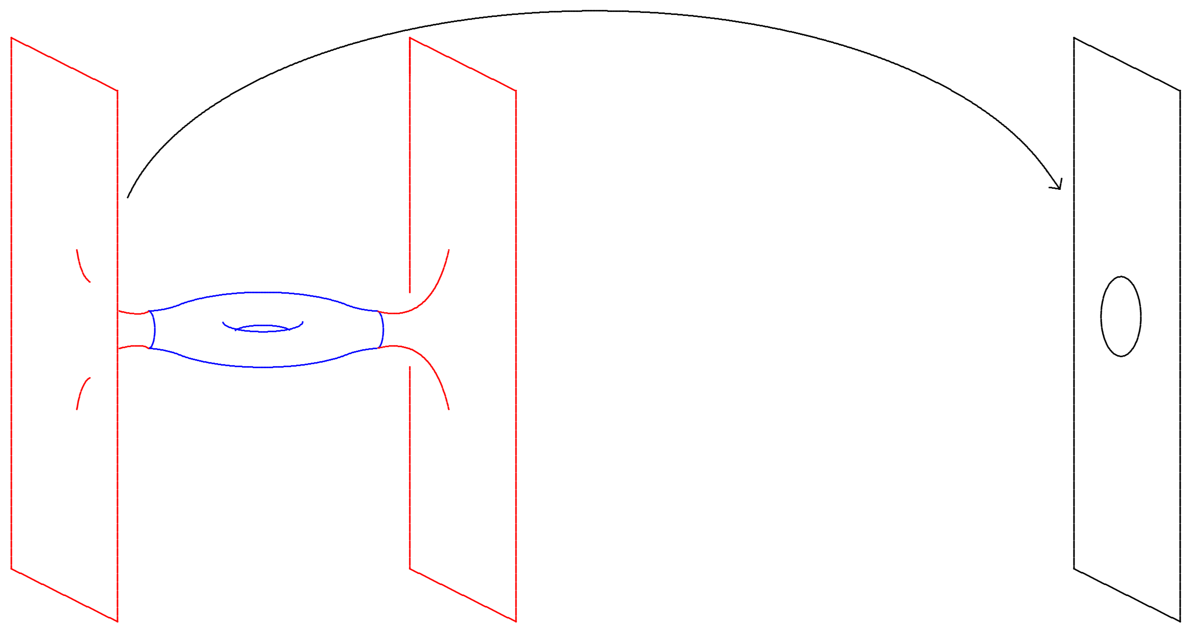
Definition. *A complete, non-compact Riemannian n-manifold (M^n, g) is called* asymptotically Euclidean *(AE) if there is a compact set $K \subset M$ such that each component of $M - K$ is diffeomorphic to $\mathbb{R}^n - D^n$ in such a manner that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

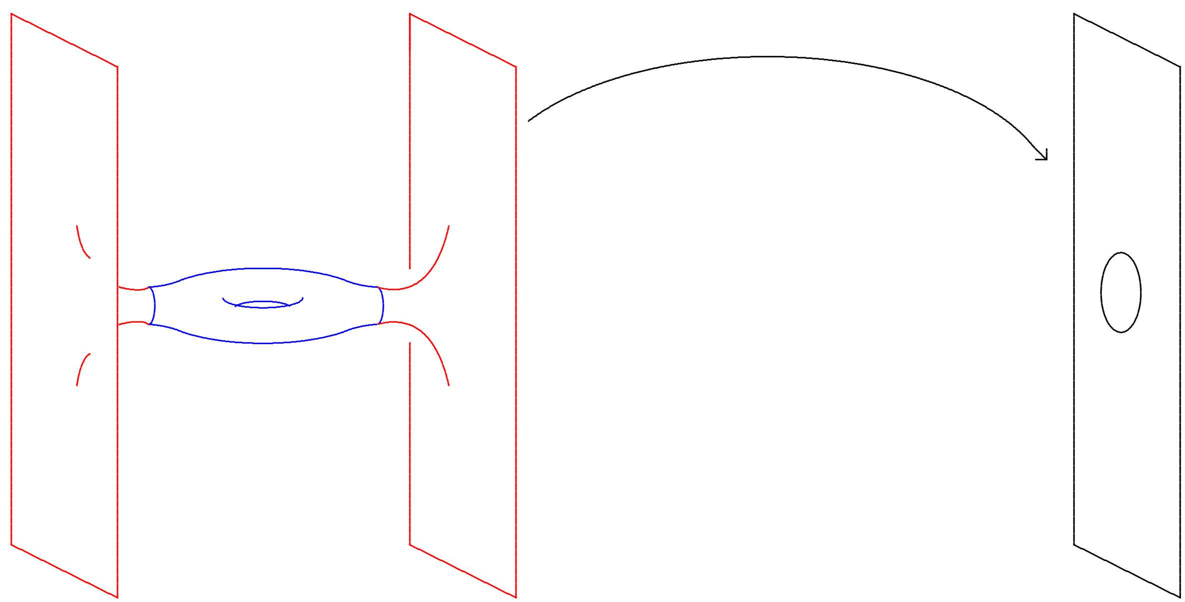
Definition. *A complete, non-compact Riemannian n-manifold (M^n, g) is called* asymptotically Euclidean *(AE) if there is a compact set $K \subset M$ such that each component of $M - K$ is diffeomorphic to $\mathbb{R}^n - D^n$ in such a manner that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

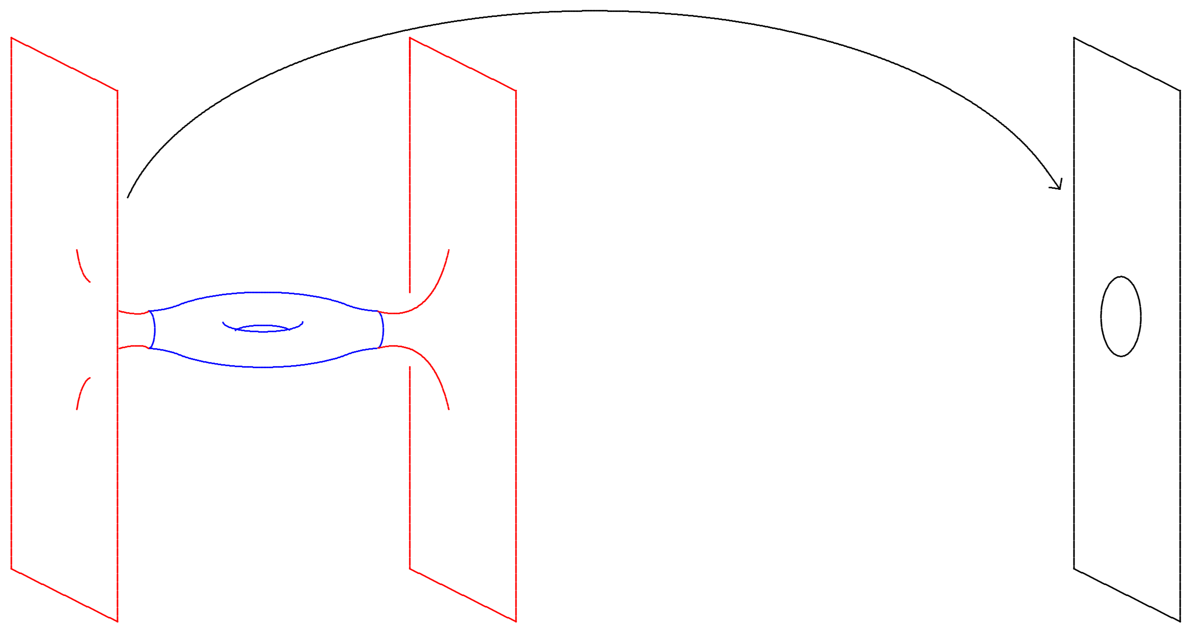
Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g) *is called* asymptotically Euclidean *(AE) if there is a compact set* $K \subset M$ *such that each component of* $M - K$ *is diffeomorphic to* $\mathbb{R}^n - D^n$ *in such a manner that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

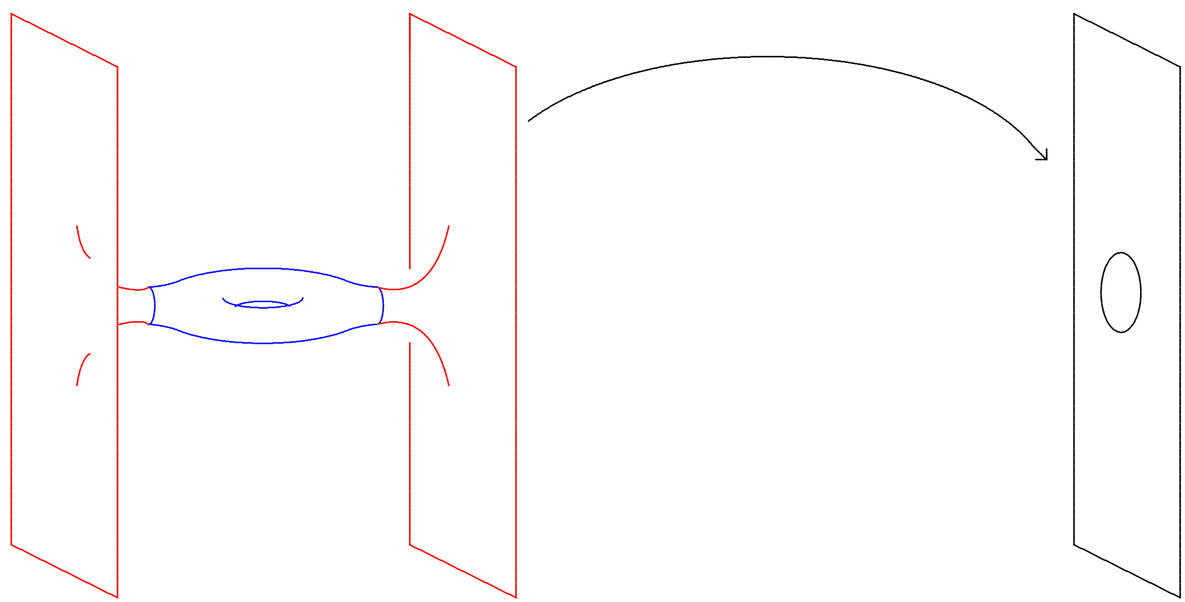
Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g) *is called* asymptotically Euclidean *(AE) if there is a compact set* $K \subset M$ *such that each component of* $M - K$ *is diffeomorphic to* $\mathbb{R}^n - D^n$ *in such a manner that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

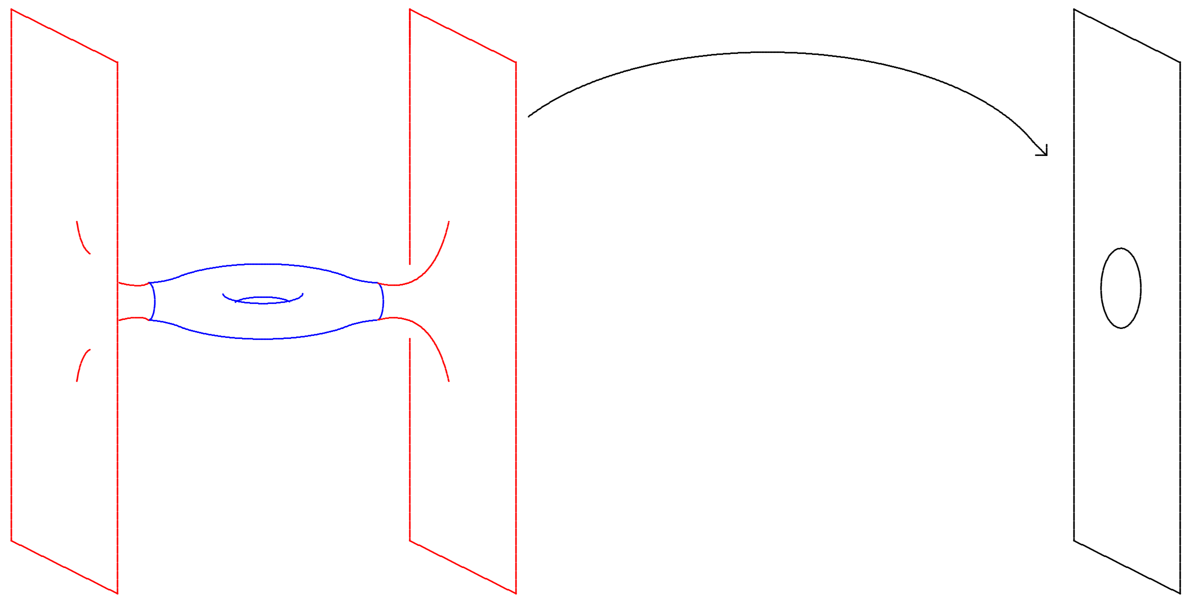
Definition. *A complete, non-compact Riemannian n-manifold (M^n, g) is called* asymptotically Euclidean *(AE) if there is a compact set $K \subset M$ such that each component of $M - K$ is diffeomorphic to $\mathbb{R}^n - D^n$ in such a manner that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

Definition. *A complete, non-compact Riemannian n-manifold (M^n, g) is called* asymptotically Euclidean *(AE) if there is a compact set $K \subset M$ such that each* "*end*" *is diffeomorphic to $\mathbb{R}^n - D^n$ in such a manner that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set.

$W_+ \neq 0$ everywhere else.

$h = s^{-2}g$ is Ricci-flat, asymptotically Euclidean.

Let's just see what happens when min $s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set.

$W_+ \neq 0$ everywhere else.

$h = s^{-2}g$ is Ricci-flat, asymptotically Euclidean.

Positive mass theorem

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set.

$W_+ \neq 0$ everywhere else.

$h = s^{-2}g$ is Ricci-flat, asymptotically Euclidean.

Positive mass theorem (or Bishop-Gromov):

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set.

$W_+ \neq 0$ everywhere else.

$h = s^{-2}g$ is Ricci-flat, asymptotically Euclidean.

Positive mass theorem (or Bishop-Gromov):

Ricci-flat h must be flat!

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set.

$W_+ \neq 0$ everywhere else.

$h = s^{-2}g$ is Ricci-flat, asymptotically Euclidean.

Positive mass theorem (or Bishop-Gromov):

Ricci-flat h must be flat!

So $W_+ \equiv 0$.

Let's just see what happens when $\min s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set.

$W_+ \neq 0$ everywhere else.

$h = s^{-2} g$ is Ricci-flat, asymptotically Euclidean.

Positive mass theorem (or Bishop-Gromov):

Ricci-flat h must be flat!

So $W_+ \equiv 0$.

Contradiction!

Let's just see what happens when min $s = 0$.

Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set.

$W_+ \neq 0$ everywhere else.

$h = s^{-2}g$ is Ricci-flat, asymptotically Euclidean.

Positive mass theorem (or Bishop-Gromov):

Ricci-flat h must be flat!

So $W_+ \equiv 0$.

Contradiction! So $s \equiv 0$.

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$*; or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$*, and* $M - \mathcal{Z}$ *has exactly two components.*

If not Kähler-Einstein,

$$\min s < 0 \Longrightarrow$$

$(M, s^{-2}g)$ double Poincaré-Einstein. Here, $s = 0$ defines smooth connected $\mathcal{Z}^3$, and $M - \mathcal{Z}$ has exactly two components.

Bochner forbids having $s < 0$ everywhere, because otherwise $(M, s^{-2}g)$ would be $\lambda < 0$ Einstein with non-trivial Killing field $J\nabla s$.

Bochner forbids having $s < 0$ everywhere, because otherwise $(M, s^{-2}g)$ would be $\lambda < 0$ Einstein with non-trivial Killing field $J\nabla s$.

Thus $\max s \geq 0$.

Bochner forbids having $s < 0$ everywhere, because otherwise $(M, s^{-2}g)$ would be $\lambda < 0$ Einstein with non-trivial Killing field $J\nabla s$.

Thus $\max s \geq 0$. But

$$0 > \kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

Bochner forbids having $s < 0$ everywhere, because otherwise $(M, s^{-2}g)$ would be $\lambda < 0$ Einstein with non-trivial Killing field $J\nabla s$.

Thus $\max s \geq 0$. But

$$0 > \kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

So $\nabla s \neq 0$ when $s = 0$.

Bochner forbids having $s < 0$ everywhere, because otherwise $(M, s^{-2}g)$ would be $\lambda < 0$ Einstein with non-trivial Killing field $J\nabla s$.

Thus $\max s \geq 0$. But

$$0 > \kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

So $\nabla s \neq 0$ when $s = 0$.

Hence $\max s > 0$,

Bochner forbids having $s < 0$ everywhere, because otherwise $(M, s^{-2}g)$ would be $\lambda < 0$ Einstein with non-trivial Killing field $J\nabla s$.

Thus $\max s \geq 0$. But

$$0 > \kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

So $\nabla s \neq 0$ when $s = 0$.

Hence $\max s > 0$, and $s = 0$ smooth hypersurface.

Bochner forbids having $s < 0$ everywhere, because otherwise $(M, s^{-2}g)$ would be $\lambda < 0$ Einstein with non-trivial Killing field $J\nabla s$.

Thus $\max s \geq 0$. But

$$0 > \kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

So $\nabla s \neq 0$ when $s = 0$.

Hence $\max s > 0$, and $s = 0$ smooth hypersurface.

$$s : M \to \mathbb{R}$$

Morse-Bott without critical manifolds of odd index

Bochner forbids having $s < 0$ everywhere, because otherwise $(M, s^{-2}g)$ would be $\lambda < 0$ Einstein with non-trivial Killing field $J\nabla s$.

Thus $\max s \geq 0$. But

$$0 > \kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

So $\nabla s \neq 0$ when $s = 0$.

Hence $\max s > 0$, and $s = 0$ smooth hypersurface.

$$s : M \to \mathbb{R}$$

Morse-Bott without critical manifolds of odd index

$\Longrightarrow$ regions $s < 0$ and $s > 0$ are both connected.

Bochner forbids having $s < 0$ everywhere, because otherwise $(M, s^{-2}g)$ would be $\lambda < 0$ Einstein with non-trivial Killing field $J\nabla s$.

Thus $\max s \geq 0$. But

$$0 > \kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

So $\nabla s \neq 0$ when $s = 0$.

Hence $\max s > 0$, and $s = 0$ smooth hypersurface.

$$s : M \to \mathbb{R}$$

Morse-Bott without critical manifolds of odd index

$\Longrightarrow$ regions $s < 0$ and $s > 0$ are both connected.

Similarly, hypersurface $s = 0$ connected, too.

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$*; or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$, *and* $M - \mathcal{Z}$ *has exactly two components.*