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Complete Ricci-Flat Kähler Metrics on $\mathbf{C}^n$ Need Not Be Flat

CLAUDE LEBRUN

Dedicated to N. J. Hitchin, R. Penrose, and G. A. J. Sparling

Abstract. We observe that $\mathbf{C}^2$ admits a complete Ricci-flat Kähler metric which is not flat[1]. The resulting Riemannian manifold is isometric to the Euclidean Taub-NUT metric dicovered by Hawking [H]. Thus $\mathbf{C}^n$ admits nonflat complete Ricci-flat Kähler metrics for all $n \geq 2$. We also give other examples of complex manifolds (some Stein, some not) which carry families of distinct complete Ricci-flat metrics with identical volume form and Kähler class. This gives a negative answer to a question posed by Yau [Y].

The Euclidean Taub-NUT metric of Hawking [H] is a complete Ricci-flat Riemannian metric on $\mathbf{R}^4$ about which a certain amount of phony folk-lore persistently clings. On $S^3 \times \mathbf{R}^+$, the metric may be given explicitly in the form

$$g = \frac{\rho+1}{4\rho} d\rho^2 + \rho(1+\rho)\left[\sigma_1^2 + \sigma_2^2\right] + \frac{\rho}{\rho+1}\sigma_3^2, \tag{1}$$

where $\{\sigma_j : j = 1, 2, 3\}$ is a left invariant orthonormal coframe for S^3 and where $\rho \in \mathbf{R}^+$. A common erroneous assertion maintains that this metric is not Kähler, the reasoning being that, with respect to the most obvious integrable almost complex structure ($\sigma_1 \mapsto \sigma_2$, $\sigma_3 \mapsto -(1+\rho)d\rho/2\rho$), the metric is Hermitian but not Kähler. However, the metric has self-dual curvature tensor, and so has holonomy $\mathrm{SU}(2)$; thus, not only is there a parallel complex structure—there's actually an entire 2-sphere's worth of such creatures!

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[1]In fact, it has come to the attention of the author that Theorem 1 was previously known to Nigel Hitchin, who anonymously published a description of this example in [B], using a somewhat different approach. When queried about this, Hitchin in turn ascribed the key ideas to Roger Penrose and George Sparling! I therefore dedicate this article to Hitchin, Penrose, and Sparling, in the hope that this example will prove as stimulating for others as it has been for me.

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These complex structures, with respect to which g becomes Kähler, determine the orientation opposite to that of the "red herring" complex structure alluded to above. Notice that the volume of a large ball of geodesic radius R about the origin is asymptotically given by

$$\mathrm{Vol}(R) \sim \frac{8}{3}\pi^2 R^3, \tag{2}$$

so that this complete simply-connected 4-manifold is not asymptotically Euclidean, and thus, in particular, has nonzero curvature.

To obtain a better working picture of this manifold, we employ a different realization of the metric via the Gibbons-Hawking ansatz **[GH]**. To this end, let $V := 1 + 1/2r : \mathbf{R}^3 - \{0\} \to \mathbf{R}^+$, and let $\pi : M_0 \to \mathbf{R}^3 - \{0\}$ denote the S^1-bundle of Chern class -1, which we equip with the connection of curvature $\mathbf{F} := *dV$. Let ω denote the connection form of this connection, and let ds^2 denote the Euclidean metric on $\mathbf{R}^3$. Then

$$g = V\pi^* ds^2 + V^{-1}\omega^2, \tag{3}$$

gives a metric on M_0 isometric to that of Equation 1, as may be seen by setting $\rho = 2r$, and we may complete this Riemannian manifold by adding a single point, sent to 0 by π, so that $M := M_0 \cup \{pt\} \approx \mathbf{R}^4$ has an isometric action of S^1 with isolated fixed point for which the smooth map $\pi : M \longrightarrow \mathbf{R}^3$ becomes the projection to the space of orbits. The promised 2-sphere of complex structures now corresponds to the set of unit vectors in $\mathbf{R}^3$; for each such unit vector, thought of as a constant vector field on $\mathbf{R}^3$, there is an integrable almost complex structure tensor sending its horizontal lift to the vertical Killing field while preserving g. To be more explicit, let's take our unit vector to be $\partial/\partial x$, and take some local trivialization of $\pi : M_0 \to \mathbf{R}^3$ with vertical coordinate t, so that $\omega = dt + \theta$ for some 1-form θ on $\mathbf{R}^3$ satisfying $d\theta = *dV$ which represents the connection in this gauge. Then our complex structure J is given on 1-forms by

$$dx \mapsto V^{-1}(dt + \theta), \qquad dy \mapsto dz, \tag{4}$$

which is indeed integrable because

$$\begin{aligned} d[Vdx + i(dt + \theta)] &= dV \wedge dx + id\theta \\ &= (V_y dy + V_z dz) \wedge dx + i * dV \\ &= V_y dy \wedge dx + V_z dz \wedge dx \\ &\quad + i(V_x dy \wedge dz + V_y dz \wedge dx + V_z dx \wedge dy) \\ &= [(-V_y + iV_z)dx - iV_x dz] \wedge (dy + idz), \end{aligned}$$

showing that the differential ideal generated by $dy + idz$ and $Vdx + i(dt + i\theta)$ is closed. Moreover, the metric has Kähler form

$$\Omega = dx \wedge (dt + \theta) + Vdy \wedge dz, \tag{5}$$

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with respect to this complex structure; and since

$$\begin{aligned} d\Omega &= -dx \wedge d\theta + dV \wedge dy \wedge dz \\ &= -dx \wedge (V_x dy \wedge dz) + V_x dx \wedge dy \wedge dz \\ &= 0, \end{aligned}$$

it follows that, with respect to this structure, the metric is Kähler. Now since we obtain a parallel complex structure in this manner for each unit vector in $\mathbf{R}^3$, it follows that the holonomy is contained in $\mathrm{Sp}(1) = \mathrm{SU}(2)$, verifying that our metric is Ricci-flat.

(Note that, in these last calculations, the only property of V that we have used is the fact that $\triangle V = *d*dV = 0$; this, and this alone, makes the ansatz tick. The intrinsic meaning of the projection π is enhanced by the fact that it is the *hyper-Kähler moment map* [**HKLR**] of the S^1-action, and using this fact one may easily show that a hyper-Kähler 4-manifold with a Killing field preserving each of its 2-sphere's worth of of parallel complex structures is locally isometric to one given by (3) for a suitable positive harmonic function V. For another recent application of the ansatz, see [**AKL**].)

Consider the vector field

$$\begin{aligned} \xi &= \frac{-i}{2}\left(\frac{\partial}{\partial t} - iJ\frac{\partial}{\partial t}\right) \\ &= \frac{1}{2}\left(V^{-1}\frac{\hat{\partial}}{\partial x} - i\frac{\partial}{\partial t}\right), \end{aligned} \tag{6}$$

where

$$\frac{\hat{\partial}}{\partial x} = \frac{\partial}{\partial x} - \theta\left(\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t},$$

denotes the horizontal lift of $\partial/\partial x$. Since $\partial/\partial t$ preserves both the metric and the complex structure, it follows that ξ is a holomorphic vector field. Moreover, the flow of ξ is complete because

$$\int_{-\infty}^{0} V(x, y, z)dx = \int_{0}^{\infty} V(x, y, z)dx \equiv \infty, \tag{7}$$

so that ξ generates a holomorpic action of $\mathbf{C} - \{0\}$ on $\mathbf{M}$. The orbit structure of this action is easily read off from our picture. Namely, for each line

$$L_u = \{(x, y, z) : y + iz = u\},$$

parallel to the x-axis, the holomorphic curve $\mathbf{C}_u := \pi^{-1}(L_u)$ is a union of orbits. Indeed, for $u \neq 0$, $\mathbf{C}_u$ is a single orbit. By contrast, $\mathbf{C}_0$ consists of 3 different orbits, coresponding to $x > 0$, $x < 0$, and $x = 0$.

Since ξ also preserves the parallel (hence holomorphic) (2,0)-form

$$\Psi := -i(dy \wedge \omega + V dz \wedge dx) + (dz \wedge \omega + V dx \wedge dy), \tag{8}$$

we may, in the spirit of symplectic geometry, ask for a "complex Hamiltonian" $H : M \to \mathbf{C}$ for ξ, whereby we mean a holomorphic function such

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that

$$dH = \xi \rfloor \Psi. \tag{9}$$

Such a function is given explicitly by

$$H = y + iz,$$

and, of course, the simplicity of this formula is directly related to the fact that π is the hyper-Kähler moment map of $\partial/\partial t$.

Let L_+ denote $\{(x, 0, 0) : x \geq 0\}$, and let L_- denote $\{(x, 0, 0) : x \leq 0\}$. Cover $M_0 = M - \{\pi^{-1}(0)\}$ by two open sets defined by

$$\mathscr{U}_\pm := \pi^{-1}(\mathbf{R}^3 - L_\mp).$$

Then

$$H : \mathscr{U}_\pm \longrightarrow \mathbf{C},$$

are holomorphic principal $\mathbf{C}_*$-bundles over $\mathbf{C}$, where $\mathbf{C}_* := \mathbf{C} - \{0\}$. Since $\mathbf{C}$ is Stein and 2-connected, it follows that $\mathscr{U}_\pm$ are both biholomorphic to $\mathbf{C} \times \mathbf{C}_*$. Moreover, Equation 9 tells us that, under this biholomorphism,

$$\Psi \longleftrightarrow \frac{dv \wedge du}{v},$$

where u is the standard coordinate on $\mathbf{C}$ and v is the standard coordinate on $\mathbf{C}_*$. (Keep in mind that $\xi \leftrightarrow v\partial/\partial v$.)

Now notice that the induced bundle structures agree on

$$H : \mathscr{U}_+ \cap \mathscr{U}_- \to \mathbf{C}_*,$$

so that M_0 is biholomorphic to

$$[(\mathbf{C} \times \mathbf{C}_*) \coprod (\mathbf{C} \times \mathbf{C}_*)]/\sim,$$

where the equivalence relation $\sim$ is of the form

$$(u, v)_+ \sim (u, f(u)v)_-, \tag{10}$$

for some holomorphic function $f : \mathbf{C}_* \to \mathbf{C}_*$. (The subscripts $\pm$ will be used to distinguish between two coordinate systems.) On the other hand, $f(u)$ gives rise to a quotient which is biholomorphically equivalent to that induced by $f(u)g(u)$ for $g : \mathbf{C} \to \mathbf{C}_*$ any holomorphic function, since g corresponds exactly to the possibility of making holomorphic changes in our two local trivializations of H.

I now claim that M_0 is biholomorphic to $\mathbf{C}^2 - \{0\}$ in a manner converting $H : M_0 \to \mathbf{C}$ into

$$\begin{aligned} \mathbf{C}^2 - \{0\} &\longrightarrow \mathbf{C} \\ (\zeta_1, \zeta_2) &\longmapsto \zeta_1\zeta_2, \end{aligned}$$

and simultaneously converting Ψ into $d\zeta_1 \wedge d\zeta_2$. Since the latter is constructed from the transition function $\hat{f}(u) = 1/u$ (as may be seen by taking

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our basic trivializing sections to be $u \mapsto (1, u)$ and $u \mapsto (u, 1)$), it suffices to check that M_0 can be constructed by using a meromorphic function $f(u)$ with precisely a simple pole at $u = 0$, since any such function can be uniquely written as $f(u) = g(u)/u$ for $g(u)$ an entire nonzero function. But the nature of the singularity of $f(u)$ at $u = 0$ is independent of the fact that our trivializations were defined on all of $\mathbf{C}$; we would obtain the same answer using trivializations over any neighborhood $\mathscr{V}$ of 0, since the answer is independent of trivialization and our original trivializations can be restricted to $\mathscr{V}$.

Let us begin, then, by observing that $H : M \to \mathbf{C}$ has a nondegenerate critical point at $p := \pi^{-1}(0)$. (Indeed, M is given its smooth structure in the vicinity of p by introducing a new radial polar coordinate $\hat{\rho} := \sqrt{\rho} = \sqrt{2r}$.) Thus p has a neighborhood on which we may find complex coordinates (w_1, w_2) such that $H = w_1 w_2$. In terms of such coordinates, we have $\Psi = \psi\, dw_1 \wedge dw_2$ for some nonvanishing holomorphic function $\psi(w_1, w_2)$. Thus

$$\xi = \psi^{-1}\left(w_1 \frac{\partial}{\partial w_1} - w_2 \frac{\partial}{\partial w_2}\right),$$

in these coordinates. Remember, however, that ξ generates a $\mathbf{C}_*$-action; this periodicity requirement gives us the condition

$$\int_0^{2\pi} \psi\left(e^{i\theta} w_1, e^{-i\theta} w_2\right) d\theta \equiv 2\pi.$$

Thus, if

$$\psi = \sum_{j,k=0}^{\infty} a_{jk} w_1^j w_2^k,$$

we have $a_{00} = 1$ and $a_{jj} = 0$ for $j > 0$. Hence we may solve the singular ordinary differential equation

$$\left(w_1 \frac{\partial}{\partial w_1} - w_2 \frac{\partial}{\partial w_2}\right) \alpha = \psi - 1,$$

by setting

$$\alpha := \sum_{j \neq k} \frac{a_{jk}}{j - k}\, w_1^j w_2^k,$$

which will converge absolutely on some polydisk. Now set

$$s_1 := e^{\alpha} w_1, \qquad s_2 := e^{-\alpha} w_2.$$

Then (s_1, s_2) gives us a set of coordinates around p for which

$$H = s_1 s_2,$$

and

$$\Psi = ds_1 \wedge ds_2.$$

In these coordinates,

$$\xi = s_1 \frac{\partial}{\partial s_1} - s_2 \frac{\partial}{\partial s_2}.$$

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Thus, near such a critical point, the action is necessarily standard, and the "standard" sections $(s_1(u), s_2(u)) = (\epsilon, u/\epsilon)$ and $(s_1(u), s_2(u)) = (u/\epsilon, \epsilon)$ are related by the transition function $f(u) = \epsilon^2/u$.

It follows that the transition functions for $M_0 \to \mathbf{C}$ and $\mathbf{C}^2 \to \mathbf{C}$ are equivalent. Hence there is a biholomorphism

$$\Phi : M_0 \longrightarrow (\mathbf{C}^2 - \{0\}),$$

with

$$\Phi^* d\zeta_1 \wedge d\zeta_2 = \Psi.$$

By Hartog's theorem, this map extends holomorphically across the isolated point p, giving us a a volume-preserving biholomorphism between M and $\mathbf{C}^2$. This proves our main result:

THEOREM 1. *There is a complete Ricci-flat Kähler metric on $\mathbf{C}^2$ which is not flat, and yet which, none the less, has the the same volume form as the standard metric.*

As an immediate consequence, we have

COROLLARY 1. *For all $n \geq 2$, $\mathbf{C}^n$ admits complete Ricci-flat Kähler metrics which are not flat.*

PROOF. Take the Cartesian product with $\mathbf{C}^{n-2}$ equipped with its usual metric. □

EXERCISE. The Taub-NUT metric can be described in terms of a Kähler potential on $\mathbf{C}^2$ as follows: let $m \geq 0$ be a nonnegative constant, and let the functions u and v of $|z_1|$ and $|z_2|$ be defined implicitly by

$$|z_1| = e^{m(u^2 - v^2)} u,$$
$$|z_2| = e^{m(v^2 - u^2)} v.$$

Then

$$\varphi := u^2 + v^2 + m\left(u^4 + v^4\right),$$

is a potential for a complete Ricci-flat metric. When $m = 0$, we obtain the usual flat metric, whereas $m > 0$ gives a metric agreeing isometric to the above Taub-NUT metric up to dilation and rescaling.

Because our method depended upon so little of the fine detail of the formula for the metric, it can also be be applied to analyze more general metrics arising from (3). For instance, let a finite collection

$$\mathscr{S} := \{X_1 = (x_1, y_1, z_1), \ldots, X_N = (x_N, y_N, z_N)\},$$

of points in $\mathbf{R}^3$ be given, and let

$$V := \kappa^2 + \frac{1}{2}\sum_{j=1}^{N} \frac{1}{r_j}, \tag{11}$$

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where $r_j = \|X - X_j\|$ denotes the Euclidean distance from the j^{th} point. Choose the direction of the x-axis to be generic in the sense that $u = y + iz$ is one-to-one on $\mathscr{S}$. Then, letting

$$\hat{\mathscr{S}} = u[\mathscr{S}] = \{a_j = y_j + iz_j : j = 1, \dots, N\},$$

the same argument shows that the complex manifold built from our data (with the N points $\pi^{-1}(X_j)$ deleted) is biholomorphically equivalent to the one constructed in the form

$$[(\mathbf{C} - \hat{\mathscr{S}}) \times \mathbf{C}_* \coprod (\mathbf{C} - \hat{\mathscr{S}}) \times \mathbf{C}_*]/\sim, \tag{12}$$

where

$$(u, v)_+ \sim \left(u, \frac{v}{(u-a_1)\cdots(u-a_N)}\right)_- ,$$

but this then means that our complete Ricci-flat Kähler manifold is biholomorphic to the Stein manifold $\Sigma \subset \mathbf{C}^3$ defined by

$$\zeta_1 \zeta_2 = (\zeta_3 - a_1)\cdots(\zeta_3 - a_N), \tag{13}$$

as may be seen by setting $\zeta_1 = v_-$, $\zeta_2 = 1/v_+$ and $\zeta_3 = u$. Moreover, the volume form depends on only the set $\hat{\mathscr{S}}$. Now, since the second homology of such a manifold is generated by the 2-spheres $S_j = \pi^{-1}\left[\overline{X_j X_{j+1}}\right]$, where $\overline{X_j X_{j+1}}$ denotes the line segment from the j^{th} point to the $(j+1)^{st}$ point, our formula (5) for the Kähler form tells us that the Kähler class is completely characterized by the numbers

$$\int_{S_j} \Omega = 2\pi(x_{j+1} - x_j). \tag{14}$$

On the other hand, our construction tells us that the (2,0)-form Ψ is independent of the parameters x_j and κ; it is always given by

$$\Psi = \begin{cases} d\zeta_1 \wedge d\zeta_2 / P'(\zeta_3) & \text{if } P'(\zeta_3) \neq 0, \\ -\zeta_1 d\zeta_2 \wedge d\zeta_3 / P(\zeta_3) & \text{if } P(\zeta_3) \neq 0, \end{cases}$$

where $P(u) = (u - a_1)\cdots(u - a_N)$, so that $\gcd(P(u), P'(u)) = 1$. Since the metric specified by (3) and (11) is asymptotically Euclidean for $\kappa = 0$, but has only cubic asymptotic volume growth for $\kappa \neq 0$, we have the following result:

THEOREM 2. *The Stein manifolds defined by equation* (13) *carry a one-parameter family of distinct complete Ricci-flat Kähler metrics with identical volume forms in each Kähler class.*

When the projection u is not generic, we also observe the same phenomenon. For example, the cotangent bundle of the Riemann sphere $\mathbf{C}P_1$ carries a complete Ricci-flat Kähler metric distinct from the Eguchi-Hansen metric, but having the same volume form and Kähler class. Namely, we

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may take $\mathscr{S}$ to consist of two points on the x-axis, and consider the two metrics that result from taking $\kappa = 0$ and $\kappa = 1$. The former is the Eguchi-Hansen metric, but the latter metric with nonzero "NUTtiness" κ^2 is not asymptotically locally Euclidean.

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