Gravitational Instantons,

Weyl Curvature, &

Conformally Kähler Geometry

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Analytic Methods in Complex Geometry
Westfälische Wilhelms-Universität Münster
7. August 2023

Joint work with

Joint work with

Olivier Biquard

Joint work with

Olivier Biquard
Sorbonne Université

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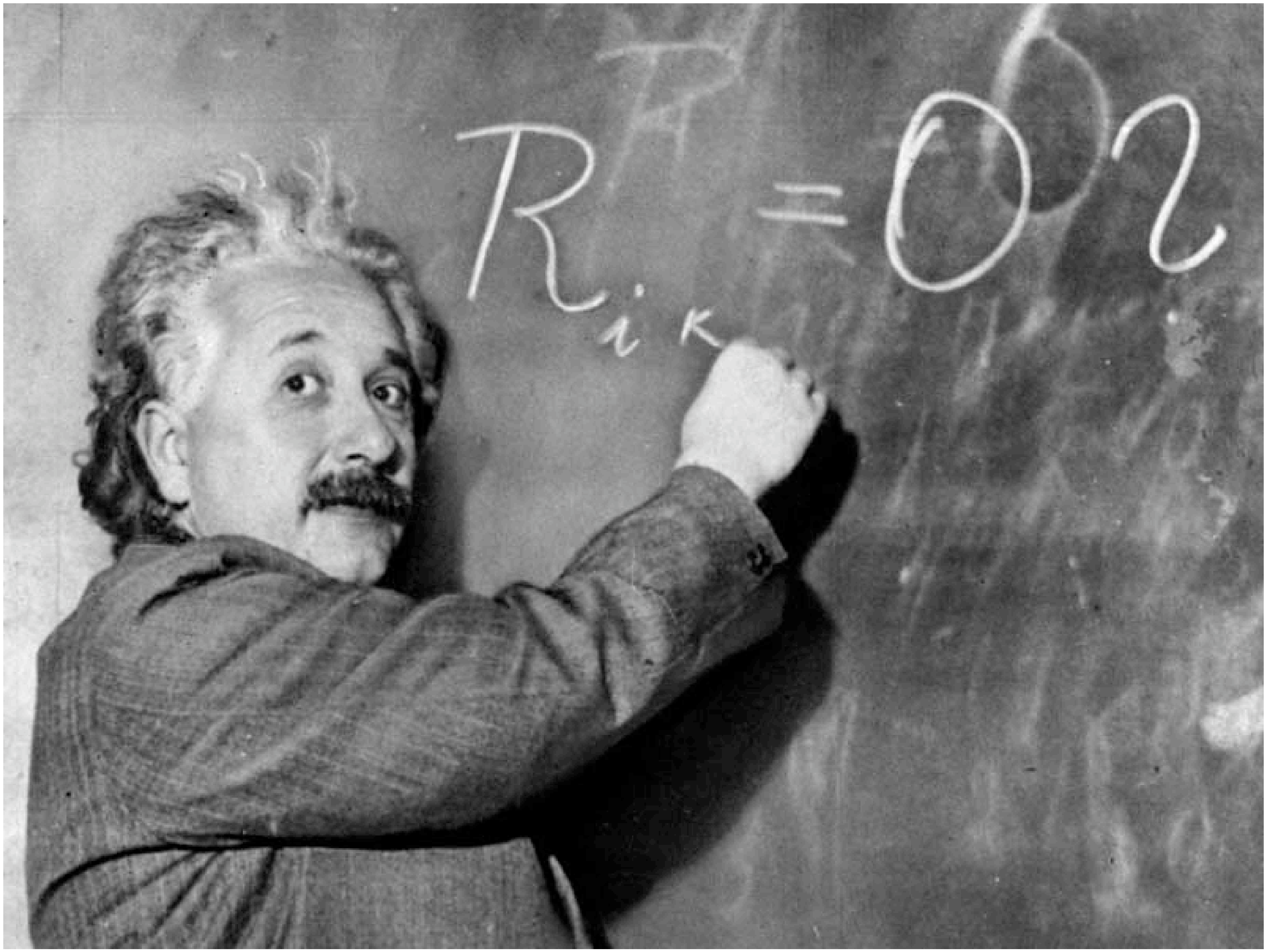

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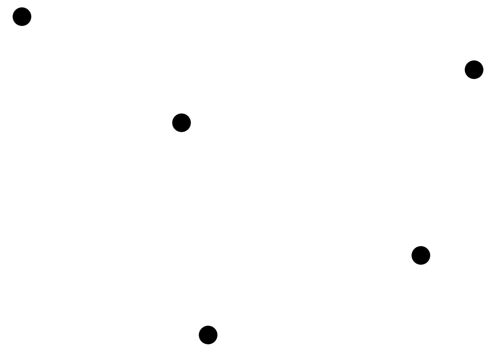

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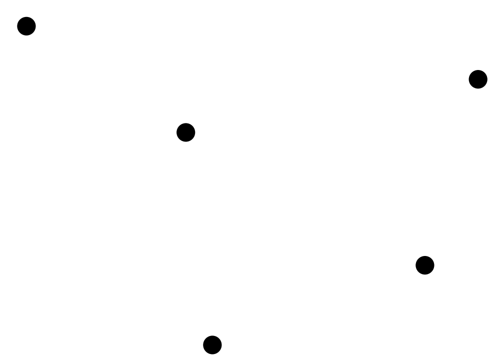

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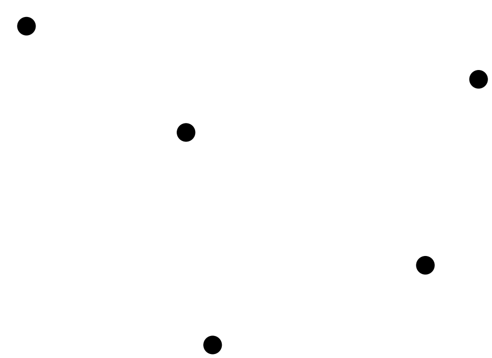

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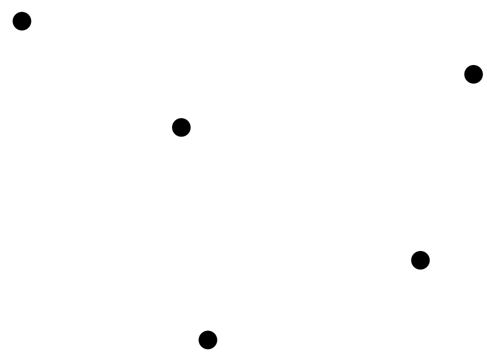

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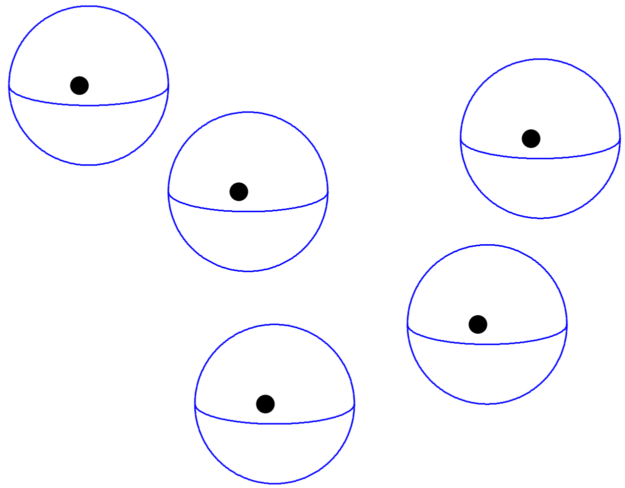

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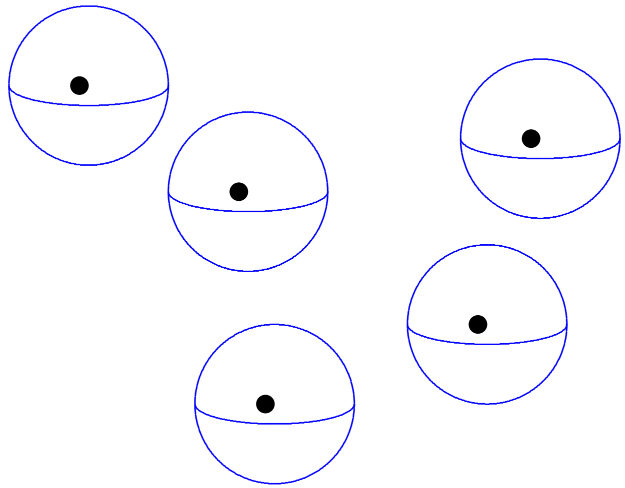

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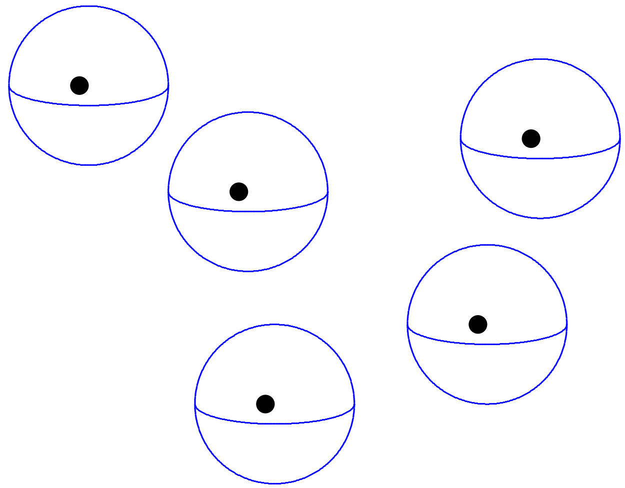

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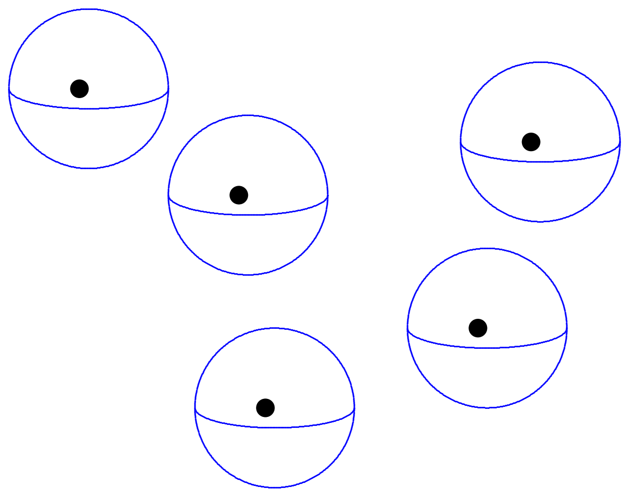

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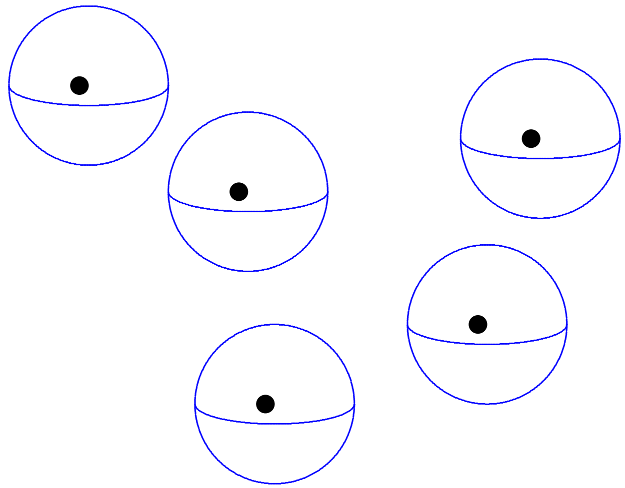

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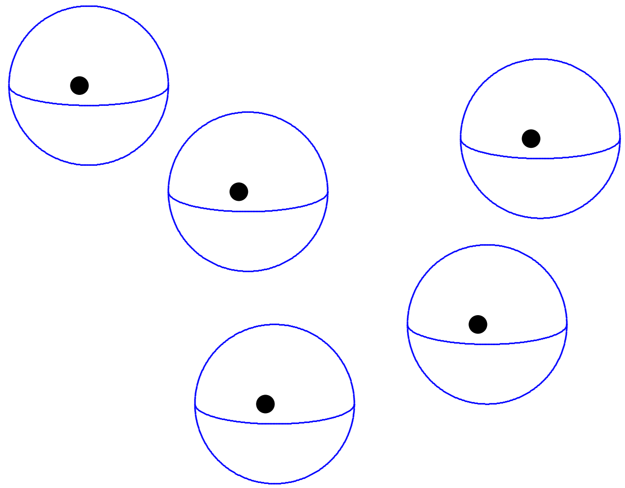

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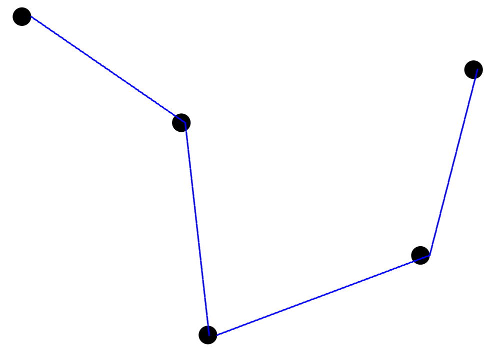

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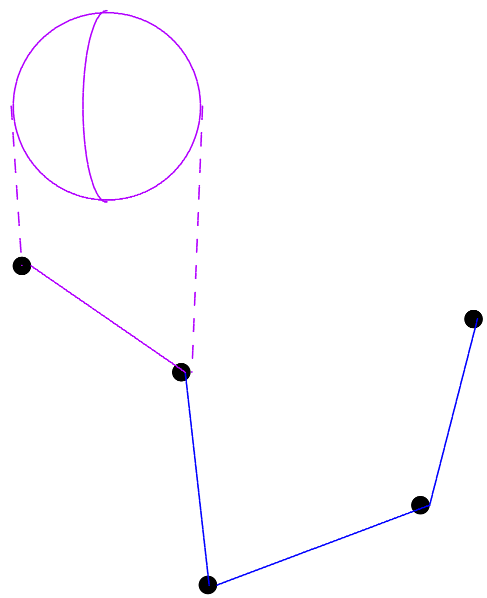

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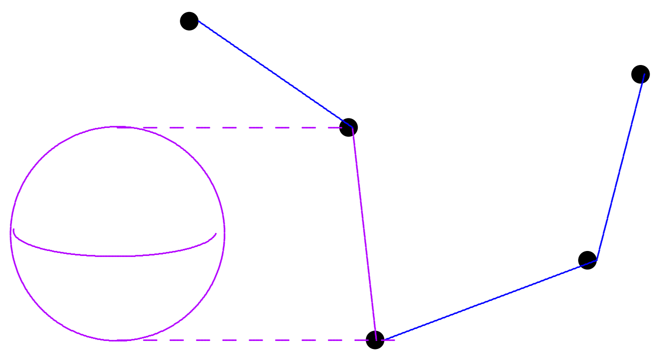

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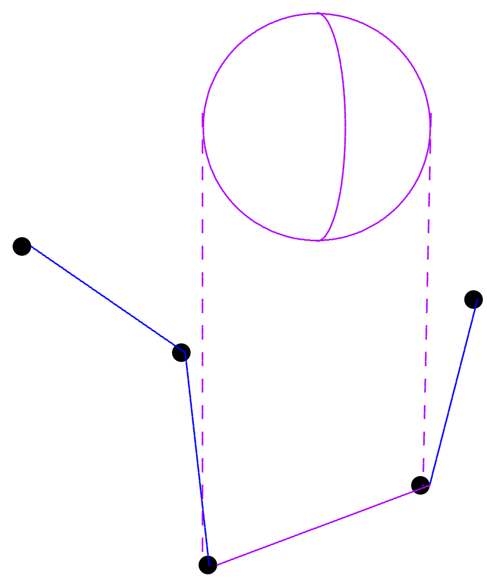

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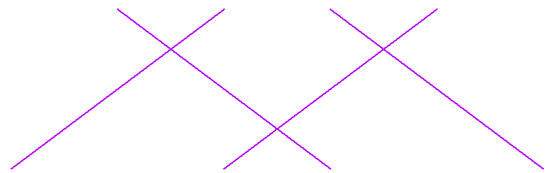

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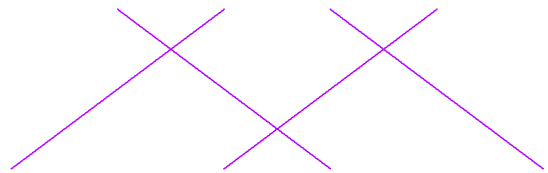

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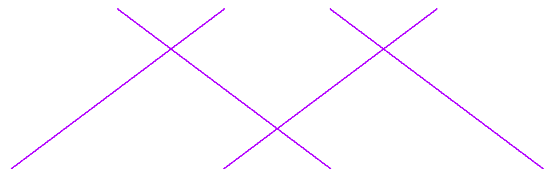

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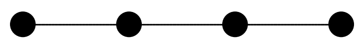

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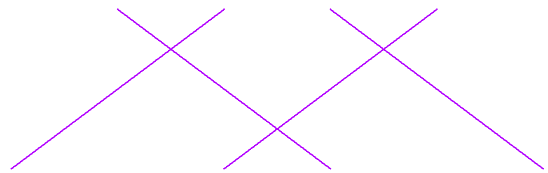

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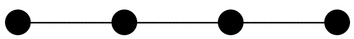

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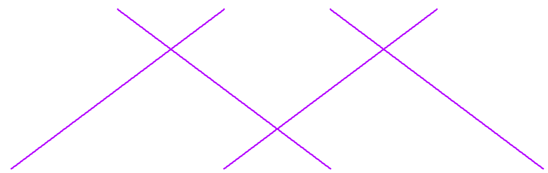

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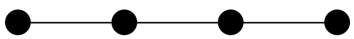

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Plumb together k copies of T^*S^2
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Gibbons and Hawking were unaware of all this!

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cf. Bishop-Gromov inequality!

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This last property distinguishes the ALF spaces from other classes of gravitational instantons:

ALG, ALH, ALG*, ALH*, ...

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Taub-NUT becomes Hermitian metric on $\mathbb{C}^2$.

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for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Taub-NUT becomes Hermitian metric on $\mathbb{C}^2$.

This J determines opposite orientation from the hyper-Kähler complex structures.

Example. Taub-NUT metric:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

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Non-Kähler, but conformally Kähler!

Hawking also discussed non-hyper-Kähler examples. . .

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Bach-flat Kähler metrics are extremal!

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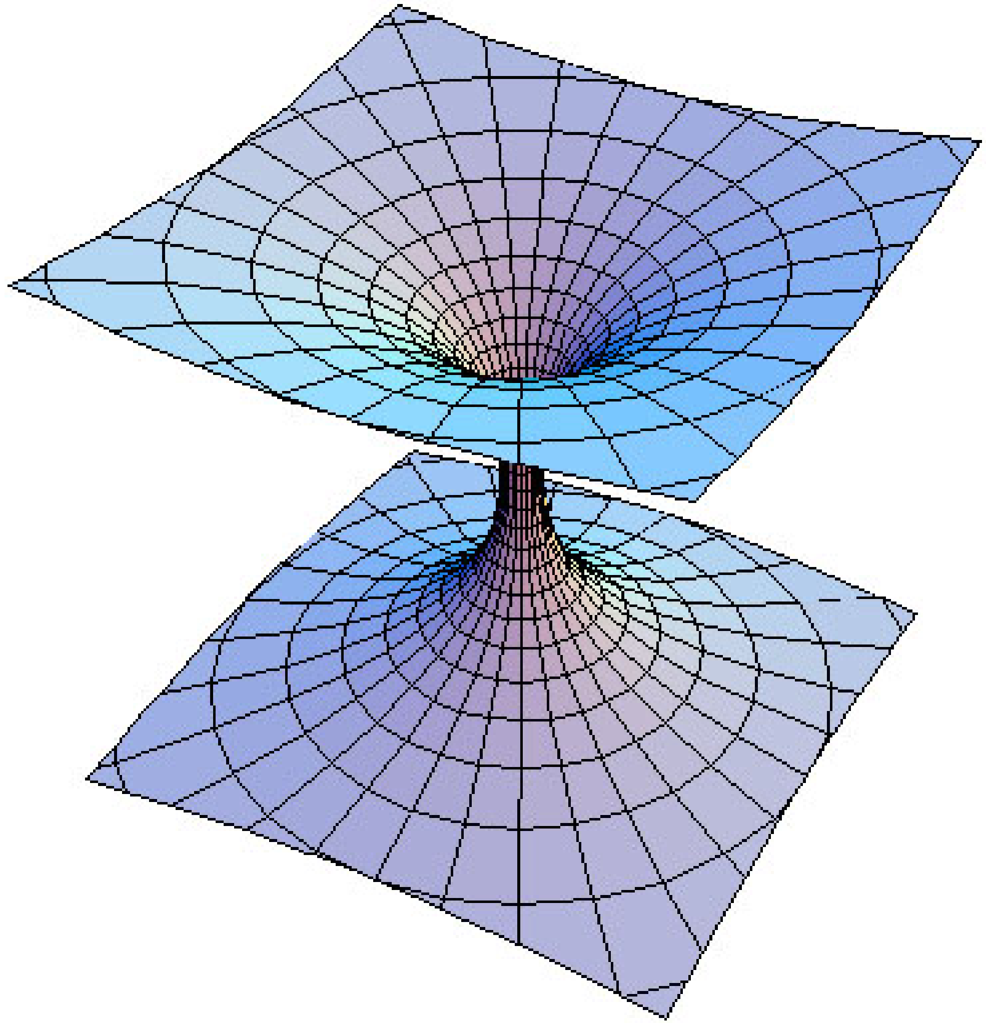

$$\mathbb{R} \times S^2 \subset \mathbb{R}^2 \times S^2$$

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This might lend some credence to the aphorism…

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By the Riemannian Goldberg-Sachs Theorem, the Hermitian assumption is equivalent to assuming that the Ricci-flat g is conformally Kähler.

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It is false in all higher dimensions!

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C^2_1 topology is induced by the natural weighted norm on symmetric 2-tensors σ that satisfy

$$\begin{aligned} |\sigma| &= O(\rho^{-1}) \\ |\nabla\sigma| &= O(\rho^{-2}) \\ |\nabla\nabla\sigma| &= O(\rho^{-3}) \end{aligned}$$

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For metrics of the type under discussion, we hope to show that the existence of a 2-torus $\mathbb{T}^2 \subset \mathrm{Iso}_0(M,g)$ is robust for quite different reasons. We hope to be able to announce a definitive result sometime soon.

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Thanks for inviting me!

Obrigado por me convidar!