Einstein Metrics, Minimizing Sequences,

and the

Differential Topology of Four-Manifolds

Claude LeBrun
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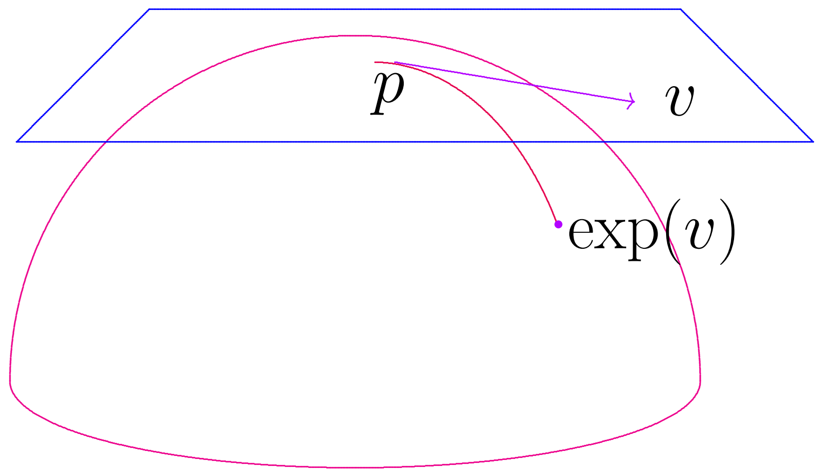
Ricci curvature measures
volume distortion by exponential map:



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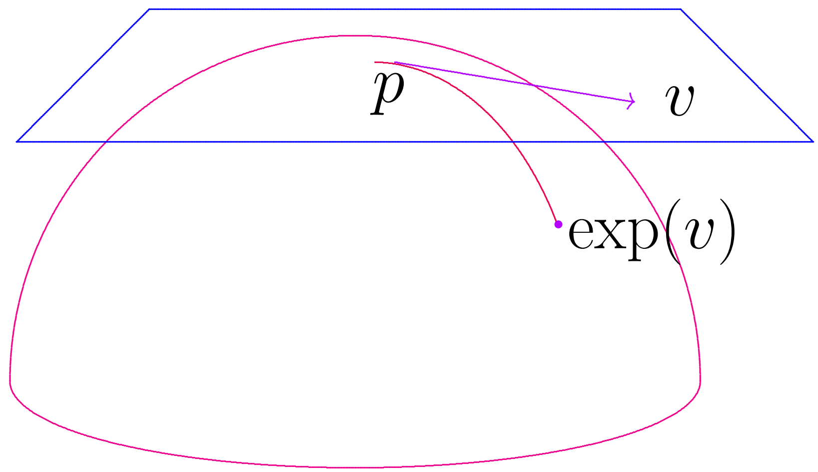

In "geodesic normal coordinates"
metric volume measure is

$$d\mu_g = \left[1 - \frac{1}{6}\, r_{jk}\, x^j x^k + O(|x|^3)\right] d\mu_{\text{Euclidean}},$$

where r is the *Ricci tensor* $r_{jk} = \mathcal{R}^i{}_{jik}$.

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- When $n = 5$: Yes?? (Boyer-Galicki-Kollár)

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- When $n \geq 6$, wide open. Maybe???

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But seems related to geometrizations of 4-manifolds by decomposition into Einstein and collapsed pieces.

By contrast, high-dimensional Einstein metrics too common, so have little to do with geometrization.

Variational Problems

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$$\begin{aligned} \mathcal{G}_M &\longrightarrow \mathbb{R} \\ g &\longmapsto V^{(2-n)/n} \int_M s_g d\mu_g \end{aligned}$$

where $V = \text{Vol}(M, g)$ inserted to make scale-invariant.

If $\nexists g \in \mathcal{G}_M$ with $s > 0$,

$\Longrightarrow$ any metric minimizing

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since

$$|r|_g^2 = \frac{s_g^2}{n} + |\mathring{r}|_g^2 \geq \frac{s_g^2}{n}$$

with $\equiv \iff$ Einstein.

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- compute these invariants for many 4-manifolds;
- describe minimizing sequences for functionals;
- show that above inequality often strict;
- provide context for Anderson's talk.

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$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

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Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

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$$\begin{array}{c|c|c|} & \Lambda^{+*} & \Lambda^{-*} \\ \hline \Lambda^{+} & W_+ + \frac{s}{12} & \mathring{r} \\ \hline \Lambda^{-} & \mathring{r} & W_- + \frac{s}{12} \\ \hline \end{array}$$

where

$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W_+ &= \text{self-dual Weyl curvature} \\ W_- &= \text{anti-self-dual Weyl curvature} \end{aligned}$$

(M, g) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu$$

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for Euler-characteristic $\chi(M) = \sum_j (-1)^j b_j(M)$.

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Here $b_\pm(M) = \max \dim$ subspaces $\subset H^2(M, \mathbb{R})$ on which intersection pairing

$$\begin{array}{ccc} H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) & \longrightarrow & \mathbb{R} \\ ([\varphi] , [\psi]) & \mapsto & \int_M \varphi \wedge \psi \end{array}$$

is positive (resp. negative) definite.

Associated 'square-norm'

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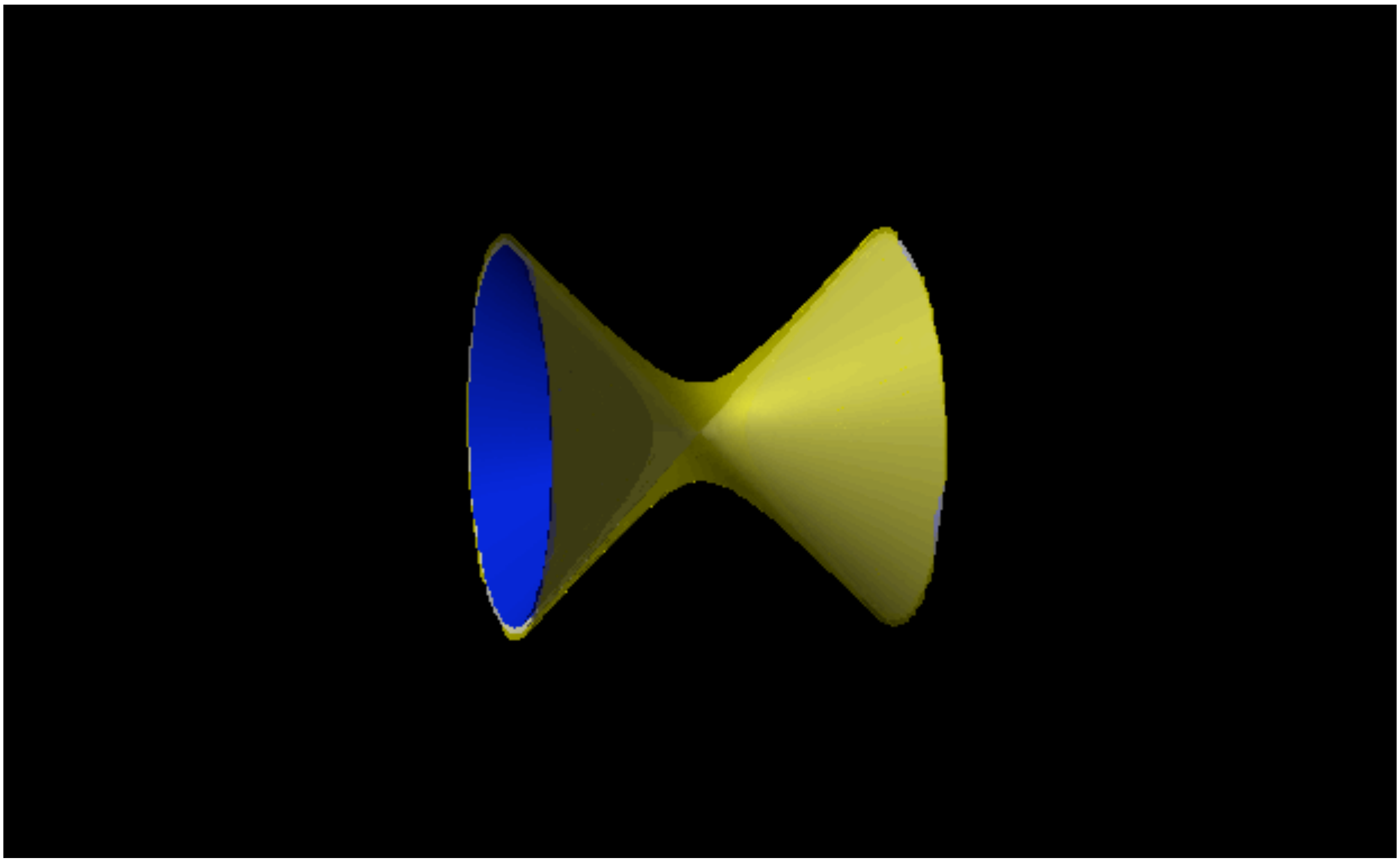

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- *both are spin, or both are non-spin.*

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Warning: "Exotic differentiable structures!"

No diffeomorphism classification currently known!

Typically, one homeotype $\longleftrightarrow$ ∞ many diffeotypes.

Hitchin-Thorpe Inequality:

$$(2\chi \pm 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_\pm|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

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Theorem (Hitchin-Thorpe Inequality). *If smooth compact oriented M^4 admits Einstein g, then*

$$(2\chi + 3\tau)(M) \geq 0$$

and

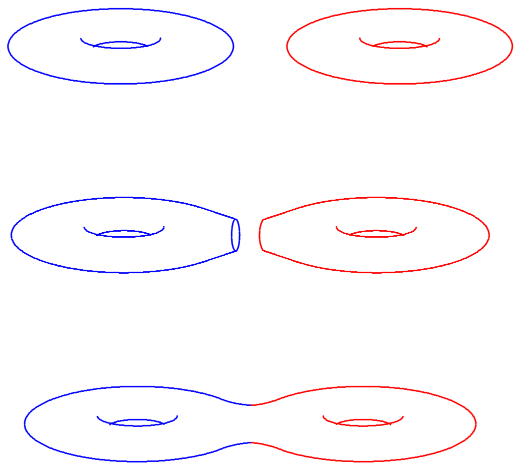
$$(2\chi - 3\tau)(M) \geq 0.$$

Example.

Let $\overline{\mathbb{CP}_2}$ = reverse-oriented $\mathbb{CP}_2$.

$$j\mathbb{CP}_2 \# k\overline{\mathbb{CP}_2} = \underbrace{\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2}_{j} \# \underbrace{\overline{\mathbb{CP}_2} \# \cdots \# \overline{\mathbb{CP}_2}}_{k},$$

Connected sum:



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has

$$2\chi + 3\tau = 4 + 5j - k$$

so $\nexists$ Einstein metric if $k \geq 4 + 5j$.

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Both inequalities strict unless finitely covered by flat T^4, Calabi-Yau $K3$, or Calabi-Yau $\overline{K3}$.

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Theorem (Yau)**.** $K3$ *admits Ricci-flat metrics.*

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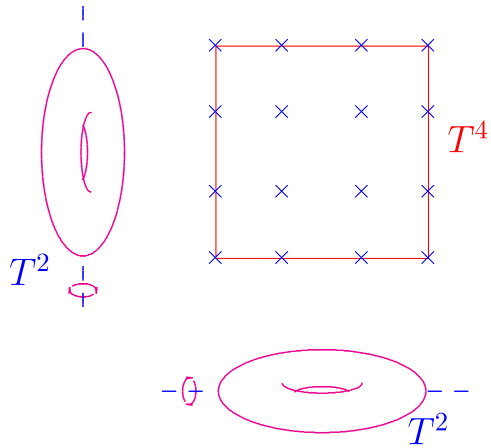
Begin with $T^4/\mathbb{Z}_2$:



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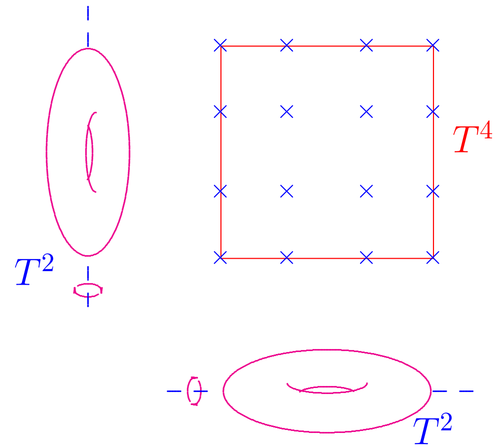
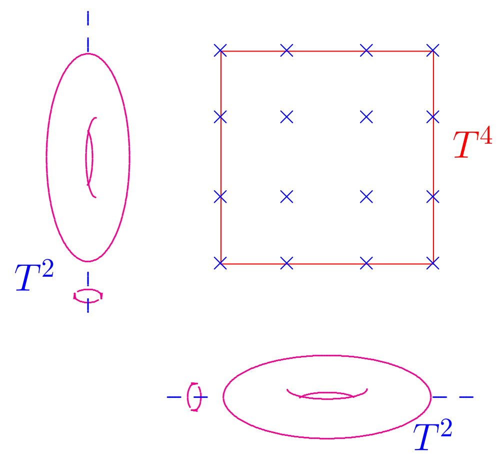


Replace $\mathbb{R}^4/\mathbb{Z}_2$ neighborhood of each singular point with copy of T^*S^2.

Approximate Calabi-Yau metric:

Replace flat metric on $\mathbb{R}^4/\mathbb{Z}_2$



with Eguchi-Hanson metric on T^*S^2:

$$g_{EH,\epsilon} = \frac{d\varrho^2}{1-\epsilon\varrho^{-4}} + \varrho^2\left(\theta_1^2 + \theta_2^2 + \left[1-\epsilon\varrho^{-4}\right]\theta_3^2\right)$$

(Page, Kobayashi-Todorov, LeBrun-Singer)

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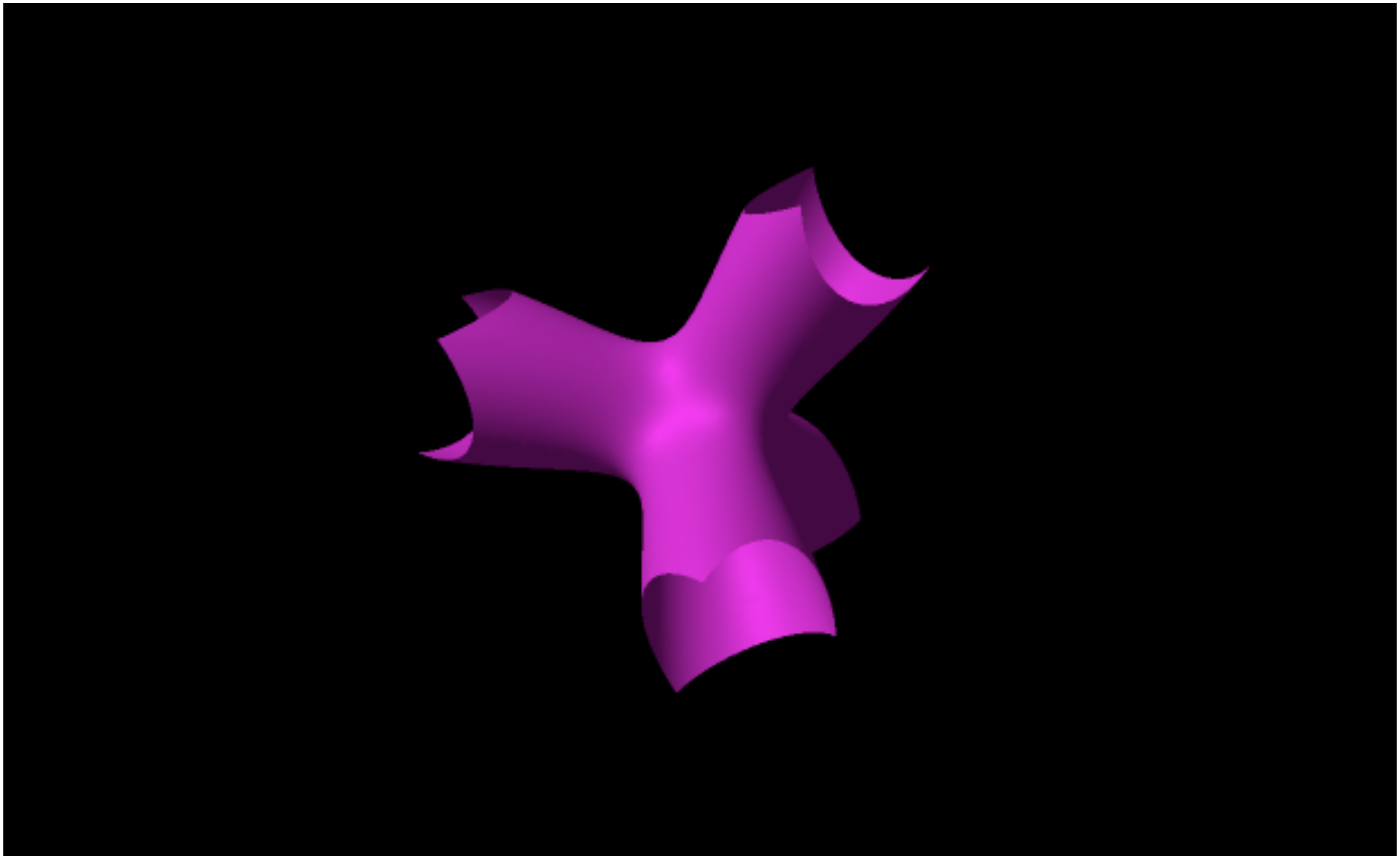
such that $c_1(M)$ *is negative multiple of* $j^* c_1(\mathbb{CP}_k)$*.*

Corollary. *For any $\ell \geq 5$, the degree ℓ surface*

$$t^\ell + u^\ell + v^\ell + w^\ell = 0$$

in $\mathbb{CP}_3$ admits $s < 0$ Kähler-Einstein metric.



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such that $c_1(M)$ *is negative multiple of* $j^*c_1(\mathbb{CP}_k)$.

Remark. This happens $\Leftrightarrow -c_1(M)$ is a Kähler class. Short-hand: $c_1(M) < 0$.

Examples of Einstein 4-Manifolds

Richest source: Kähler geometry.

Theorem (Aubin/Yau). *Compact complex manifold* (M^{2m}, J) *admits compatible Kähler-Einstein metric with* $s < 0 \Longleftrightarrow \exists$ *holomorphic embedding*

$$j : M \hookrightarrow \mathbb{CP}_k$$

such that $c_1(M)$ *is negative multiple of* $j^* c_1(\mathbb{CP}_k)$.

Remark. This happens $\Leftrightarrow -c_1(M)$ is a Kähler class. Short-hand: $c_1(M) < 0$.

Remark. When $m = 2$, such M are necessarily minimal complex surfaces of general type.

Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

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Any complex surface M can be obtained from a minimal surface X by blowing up a finite number of times:

$$M \approx X \# k \overline{\mathbb{CP}}_2$$

One says that X is minimal model of M.

Compact complex surface (M^4, J) general type if

$$\dim \Gamma(M, \mathcal{O}(K^{\otimes \ell})) \sim a\ell^2, \qquad \ell \gg 0,$$

where $K = \Lambda^{2,0}$ is canonical line bundle.

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If $\ell \geq 5$, then $\Gamma(M, \mathcal{O}(K^{\otimes \ell}))$ gives holomorphic map

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Pluricanonical model X is a complex orbifold with $c_1 < 0$ and singularities $\mathbb{C}^2/G$, $G \subset SU(2)$.

Aubin-Yau proof $\Longrightarrow$

Corollary (R. Kobayashi). *The pluricanonical model X of any compact complex surface M of general type admits and orbifold Kähler-Einstein metric with $s < 0$.*

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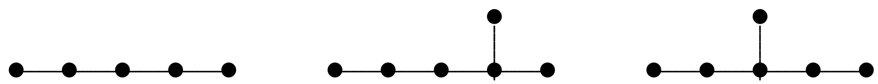

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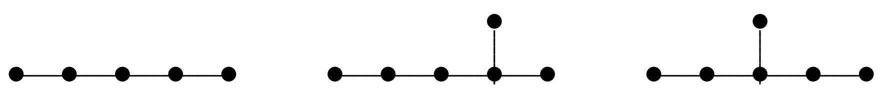

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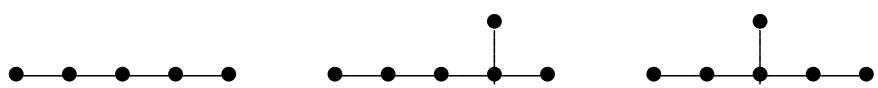

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   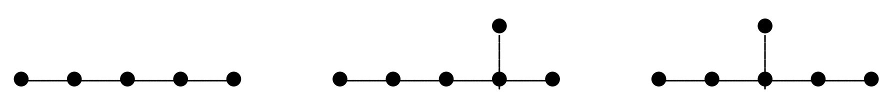

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Seiberg-Wittten theory:

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But $\bar{\partial} + \bar{\partial}^*$ does generalize:

spin^c Dirac operator, preferred connection on L.

Spin^c structures:

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where $\mathbb{S}_\pm$ are the (locally defined) left- and right-handed spinor bundles of (M, g).

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where $F_A{}^+$ = self-dual part curvature of A, and $\sigma : \mathbb{V}_+ \longrightarrow \Lambda^+$ is a natural real-quadratic map,

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Non-linear, but elliptic once 'gauge-fixing'

$$d^*(A - A_0) = 0$$

imposed to eliminate automorphisms of $L \to M$.

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When invariant is non-zero, solutions guaranteed.

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have a solution (Φ, A) for every metric g on M.

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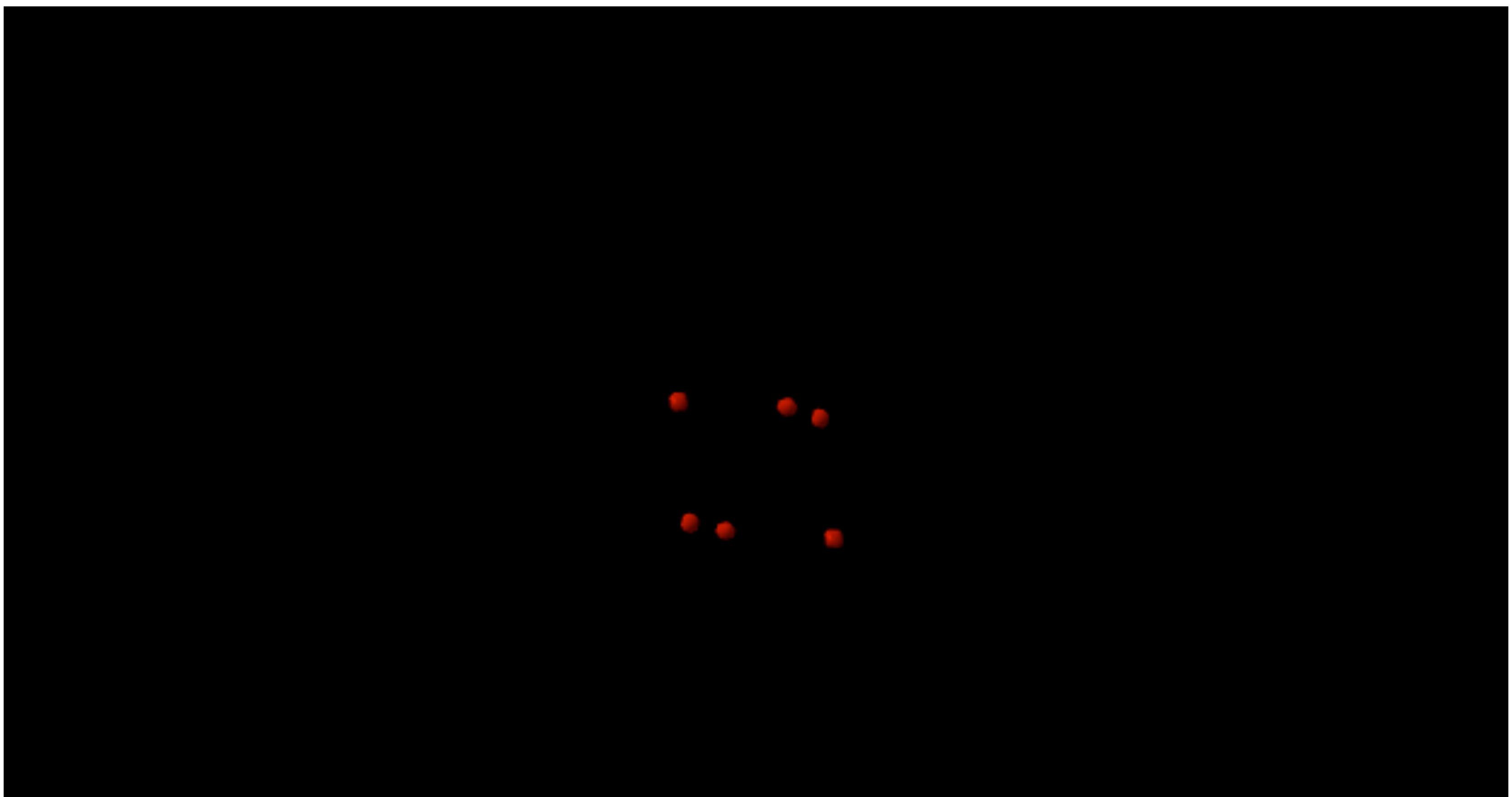

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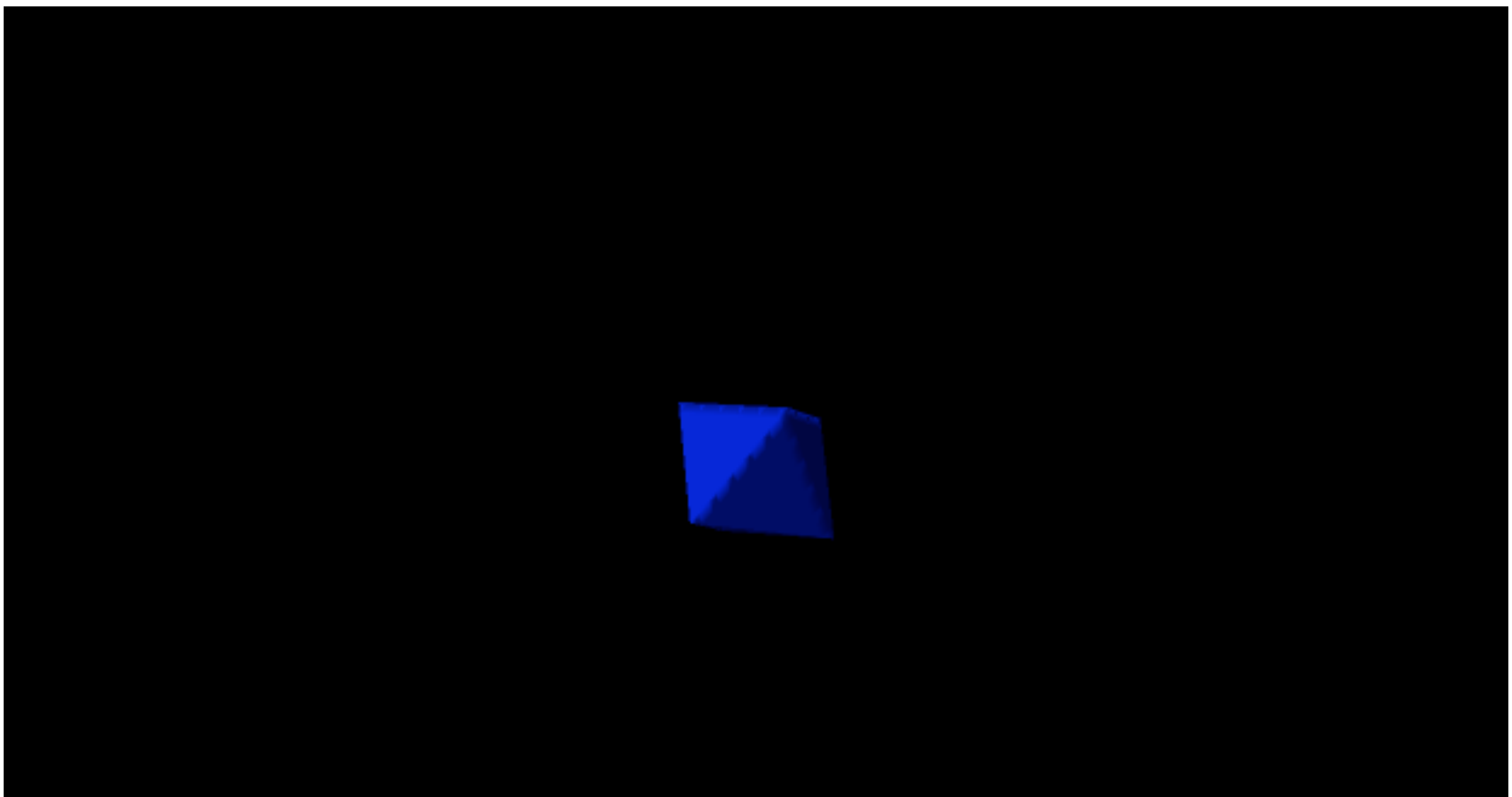

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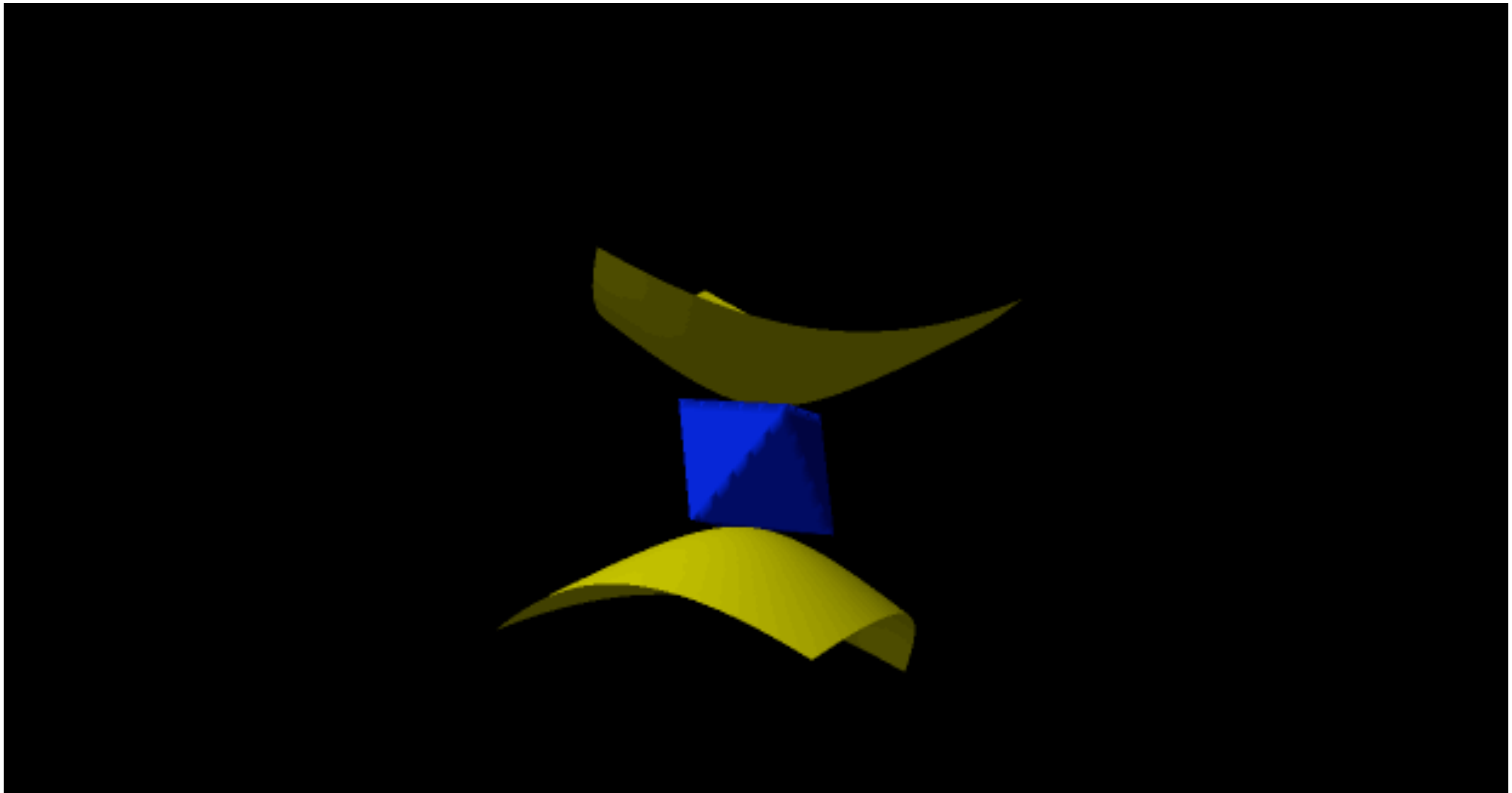

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Example If X is a minimal complex surface with $b_+ > 1$, and if

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then 'classical' Seiberg-Witten invariant allows one to show that

$$\beta^2(M) = {c_1}^2(X).$$

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Example If X, Y, Z are minimal complex surfaces with $b_1 = 0$ and $b_+ \equiv 3 \bmod 4$, and if

$$M = X\#Y\#Z\#\ell\overline{\mathbb{CP}}_2$$

Bauer-Furuta invariant allows one to show that

$$\beta^2(M) = {c_1}^2(X) + {c_1}^2(Y) + {c_1}^2(Z)$$

Similarly for 2 or 4... $\diamondsuit$

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Moreover, if $\beta^2(M) \neq 0$, equality holds in either case iff (M, g) is a Kähler-Einstein manifold with $s < 0$.

$$\frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_-|^2\right) d\mu_g \geq \frac{1}{4\pi^2}\int_M \frac{s^2}{24} d\mu_g$$

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$\Longrightarrow$ Einstein metric on $\mathbb{C}\mathcal{H}_2/\Gamma$ unique,. . .

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Theorem B. *Let M be a smooth compact oriented 4-manifold with $b_+(M) \geq 2$. If M admits an Einstein metric g, then*

$$\boxed{(2\chi+3\tau)(M) \geq \frac{2}{3}\beta^2(M)}$$

with equality only if both sides vanish,

Second curvature estimate implies

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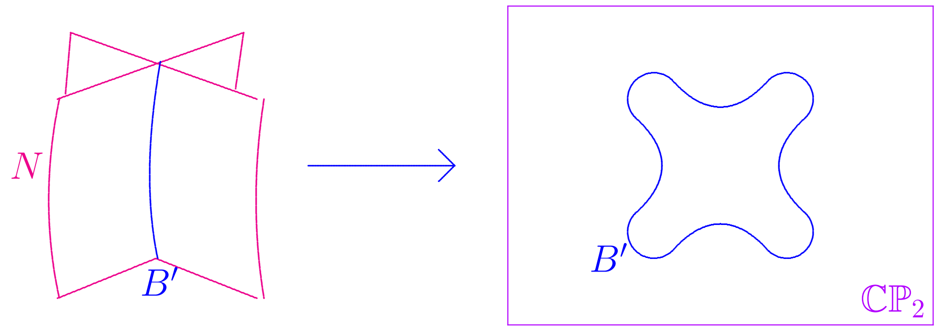
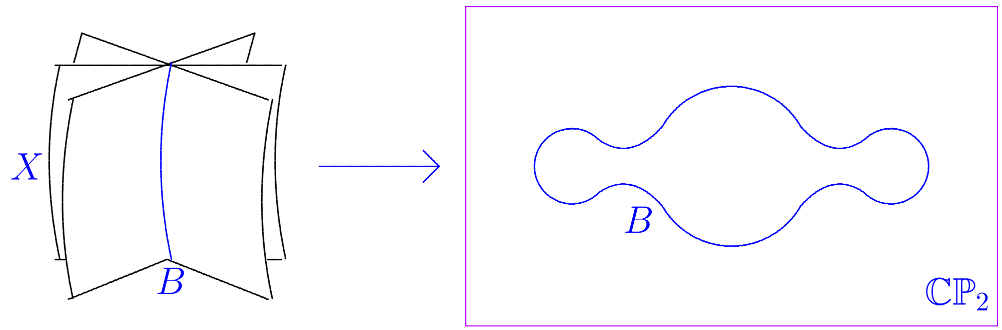
with equality only if both sides vanish, in which case g must be hyper-Kähler, and M must be diffeomorphic to either $K3$ or T^4.

Example Let N be double branched cover $\mathbb{CP}_2$, ramified at a smooth octic:



Aubin/Yau $\Longrightarrow$ N carries Einstein metric.

Now let X be a triple cyclic cover $\mathbb{CP}_2$, ramified at a smooth sextic



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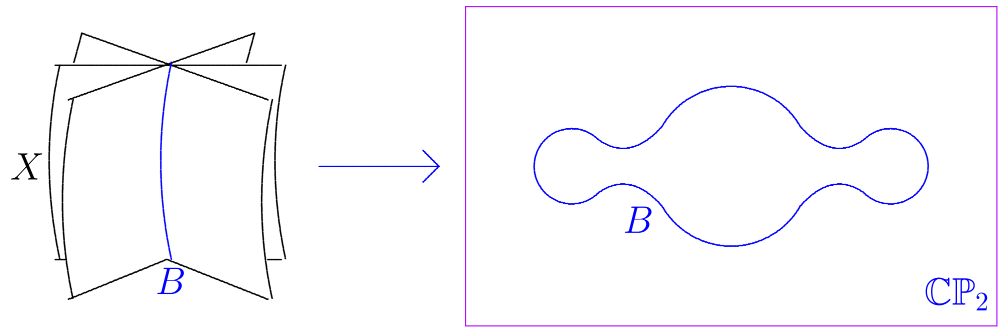

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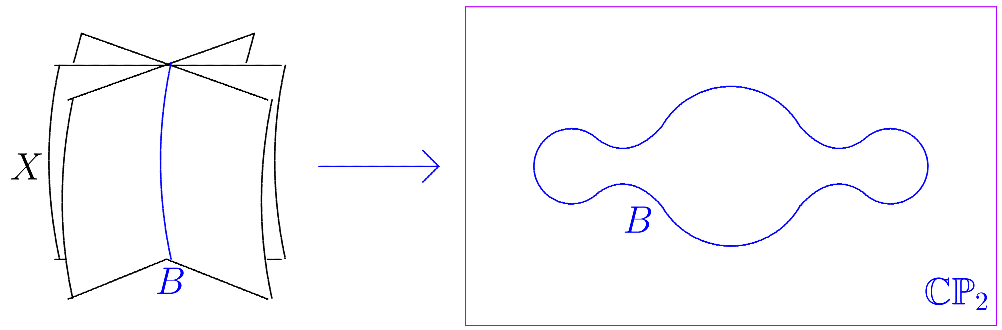

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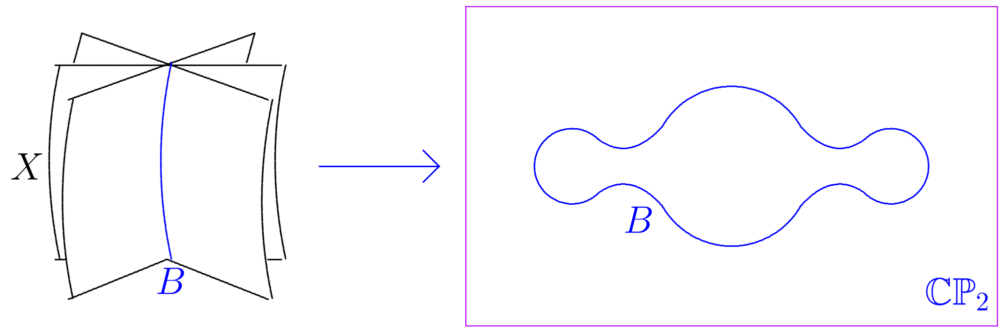

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Then

$$\begin{aligned} \beta^2(M) &= c_1^2(X) = 3 \\ (2\chi + 3\tau)(M) &= c_1^2(X) - 1 = 2 \end{aligned}$$

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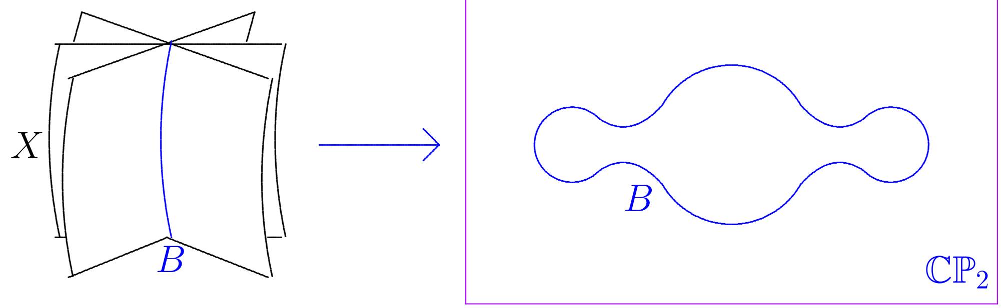
In example:

$$\beta^2(M) = 3$$
$$(2\chi + 3\tau)(M) = 2$$

X is triple cover $\mathbb{CP}_2$ ramified at sextic



$$M = X \# \overline{\mathbb{CP}}_2.$$

Theorem B $\Longrightarrow$ *no* Einstein metric on M.

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Moral: Existence depends on diffeotype!

Same ideas lead to infinitely many other examples.

Typically get non-existence for infinitely many smooth structures on fixed topological manifold.

Existence: look in Kähler-Einstein catalog.

Until now, discussed arbitrary Einstein metrics.

Instead, focus on Einstein metrics which minimize

$$g \longmapsto \int_M s_g^2 d\mu_g$$

Related to soft invariants

$$\mathcal{I}_s(M) = \inf_g \int_M s_g^2 d\mu_g$$

$$\mathcal{I}_r(M) = \inf_g \int_M |r|_g^2 d\mu_g$$

which satisfy

$$\boxed{\mathcal{I}_r(M) \geq \frac{1}{4}\mathcal{I}_s(M)}$$

with $= \Longleftarrow \exists$ Einstein minimizer.

Theorem (Curvature Estimates). *For any C^2 Riemannian metric g on any smooth compact oriented 4-manifold M with $b_+ \geq 2$, the following curvature bounds are satisfied:*

$$\int_M s^2 d\mu_g \geq 32\pi^2 \beta^2(M)$$

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Theorem. *Suppose M^4 diffeo to non-minimal compact complex surface with $b_+ > 1$. Then M does not admit a metric which minimizes either*

$$g \longmapsto \int_M s_g^2 d\mu_g \quad or \quad \int_M |r|_g^2 d\mu_g$$

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By hypothesis

$$M = X \# k\overline{\mathbb{CP}}_2$$

where X minimal and $k > 0$.

One shows

$$\begin{aligned} \mathcal{I}_s(M) &= 32\pi^2 {c_1}^2(X) \\ \mathcal{I}_r(M) &= 8\pi^2[{c_1}^2(X) + k] \end{aligned}$$

so that

$$\mathcal{I}_r(M) > \tfrac{1}{4}\mathcal{I}_s(M)$$

Theorem. *Let X, Y and Z be simply connected minimal complex surfaces with $b_+ \equiv 3 \bmod 4$. Then*

$$M = X\#Y\#Z\#k\overline{\mathbb{CP}}_2$$

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In fact,

$$\mathcal{I}_s(M) = 32\pi^2[{c_1}^2(X) + {c_1}^2(Y) + {c_1}^2(Z)]$$
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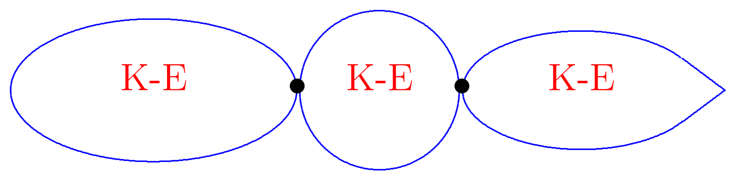
Similarly for # of 2 or 4 complex surfaces.

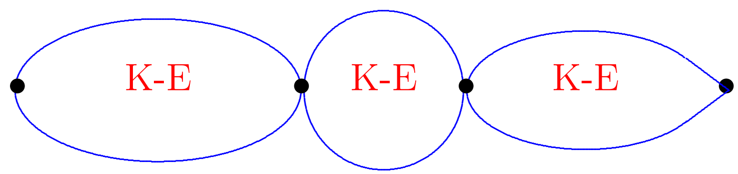
Mystery: More summands? $b_+ \equiv 1 \bmod 4$?



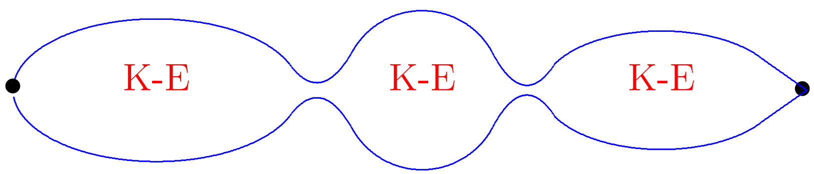
When X, Y and Z general type, however,

$\exists$ minimizing $\{g_j\}$ with Gromov-Hausdorff limit

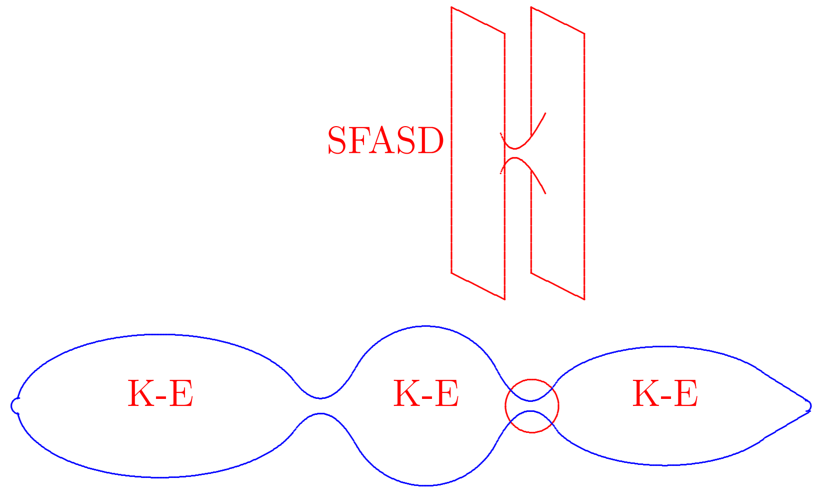
3 Kähler-Einstein orbifolds touching at points.



$\exists$ points where curvature has accumulated.



Predictable amount of $\mathring{r}$ accumulates on necks.

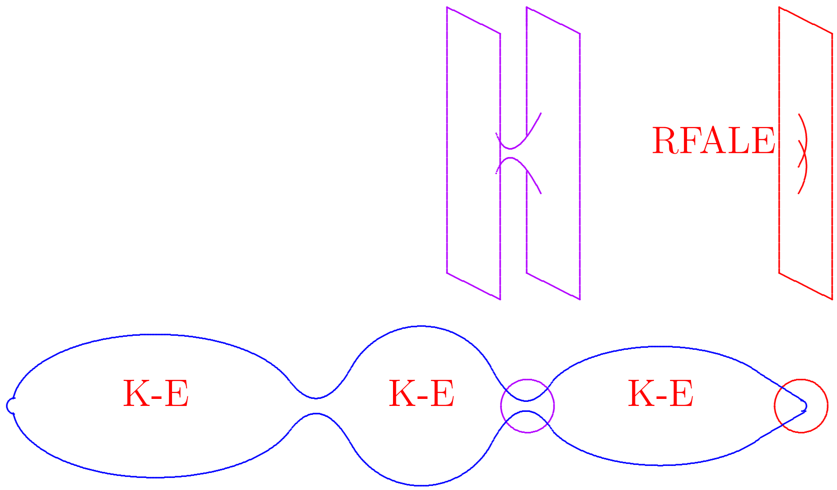


Rescaled limit of neck carries AE metric with

$$s = 0$$
$$W_+ = 0$$

Example:

$$g = (1 + \frac{1}{\varrho^2}) \quad g_{\text{Euclidean}}$$

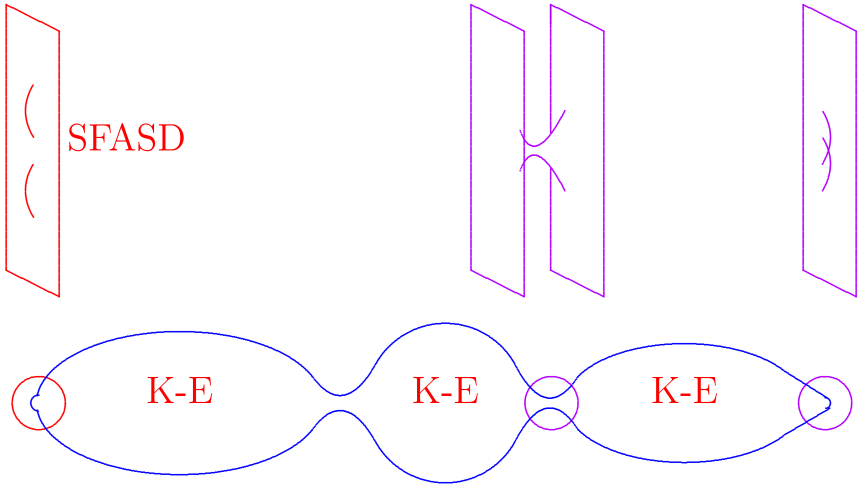


Orbifold singularities:

rescaled metric tends to gravitational instanton:

Asymptotically Locally Euclidean metric with

$$r = 0$$
$$W_+ = 0$$



Bubbling off $\overline{\mathbb{CP}}_2$'s:

Asymptotically Euclidean metric with

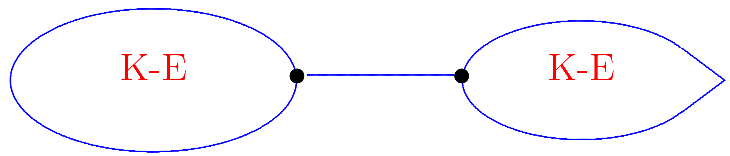
$$\begin{aligned} s &= 0 \\ W_+ &= 0 \end{aligned}$$

Basic example:

Burns metric on $\overline{\mathbb{CP}}_2 - \{\infty\}$:

$$g_{B,\epsilon} = \frac{d\varrho^2}{1 - \epsilon\varrho^{-2}} + \varrho^2 \left(\theta_1^2 + \theta_2^2 + \left[1 - \epsilon\varrho^{-2}\right] \theta_3^2\right)$$

Conformal Greens rescaling of Fubini-Study.



If one of X, Y and Z is elliptic,
collapses in limit to orbifold Riemann surface.

Typical example: