Gravitational Instantons,

Weyl Curvature, &

Conformally Kähler Geometry

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en géométrie et topologie
Université du Québec à Montréal
6 octobre, 2023

Joint work with

Joint work with

Olivier Biquard

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Olivier Biquard
Sorbonne Université

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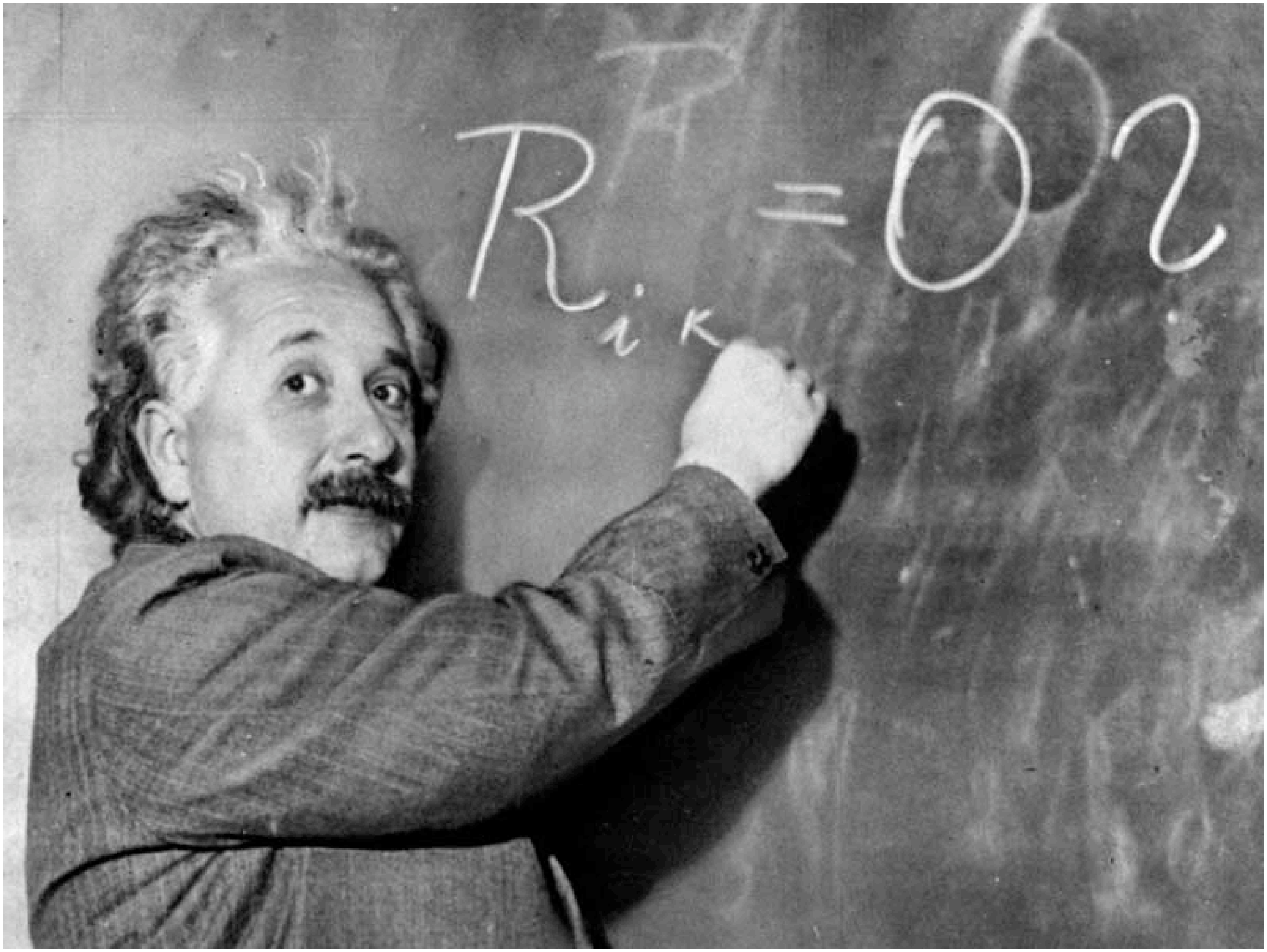

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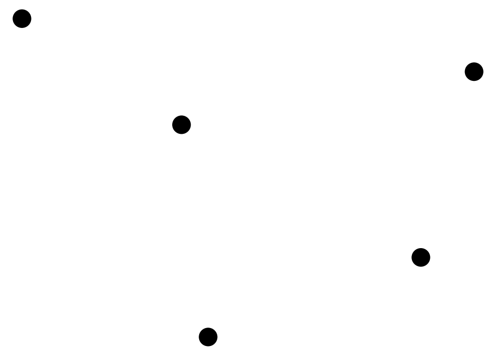

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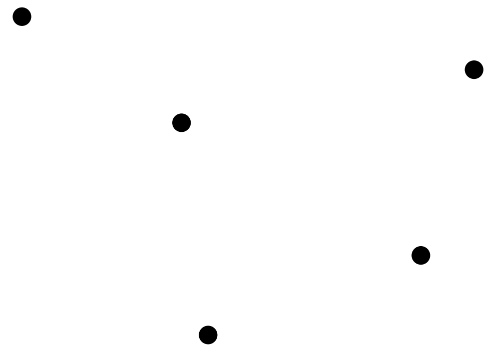

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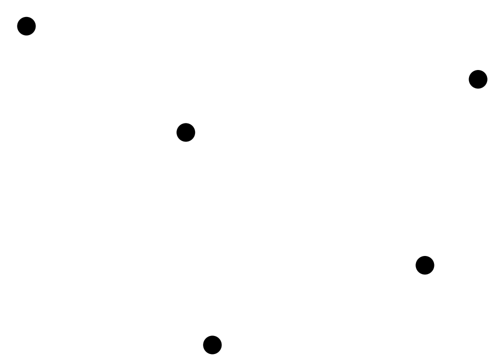

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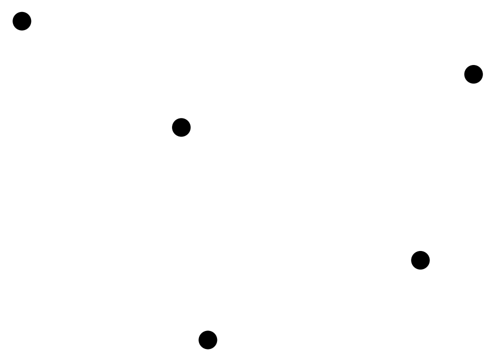

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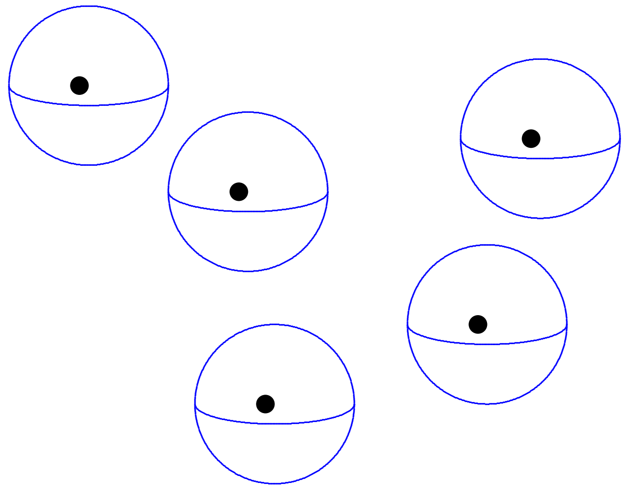

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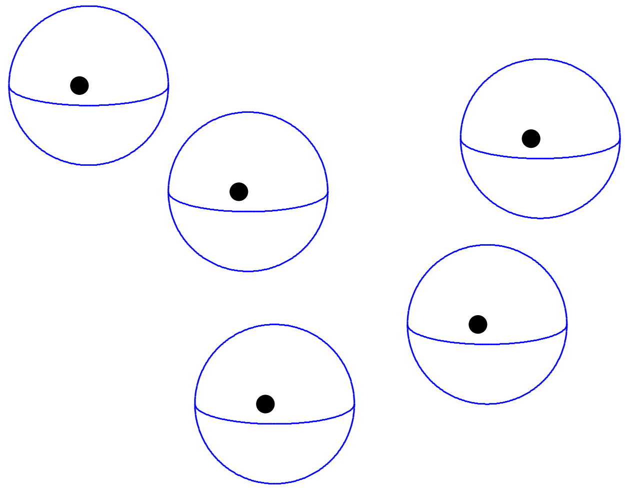

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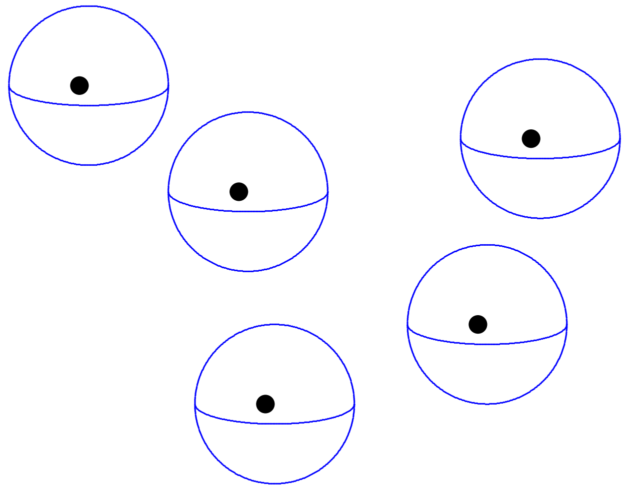

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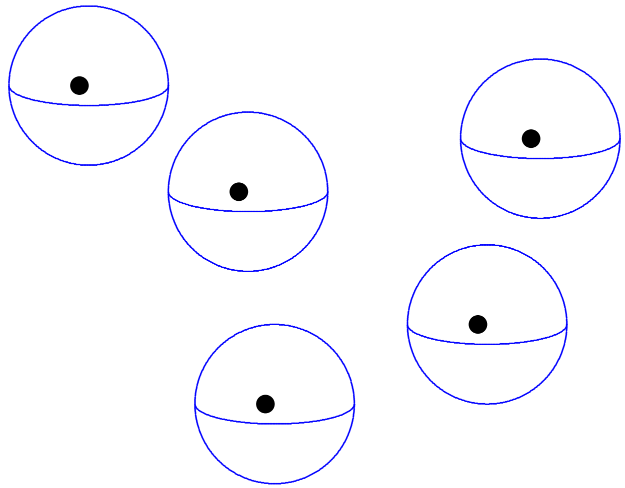

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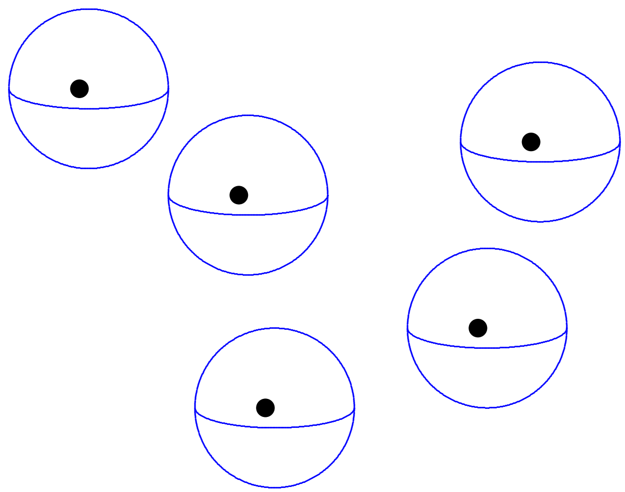

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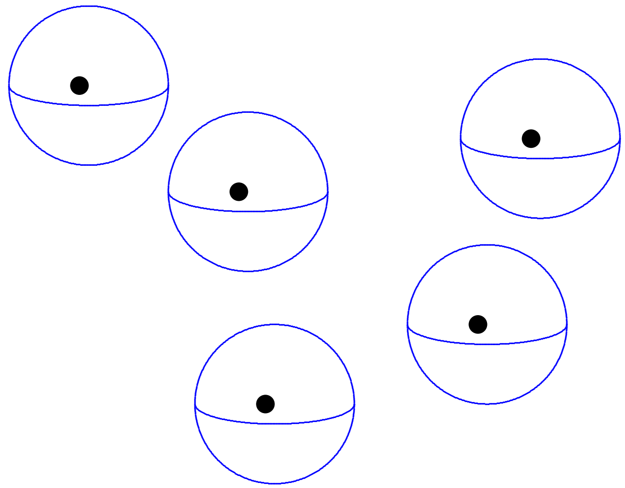

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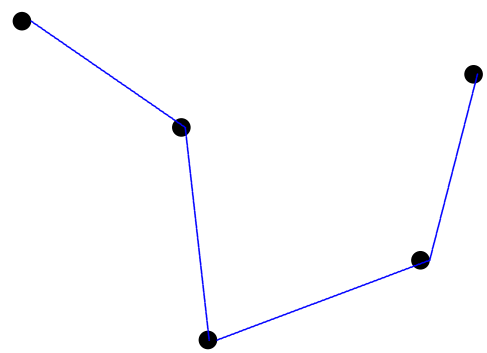

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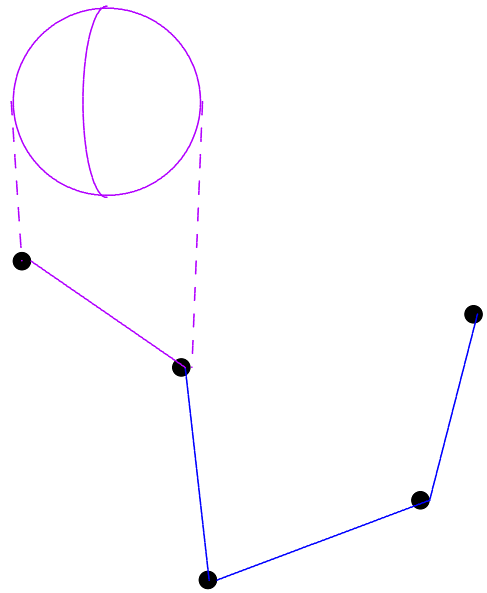

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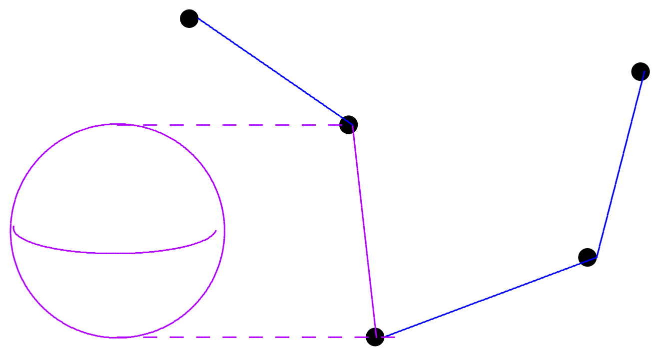

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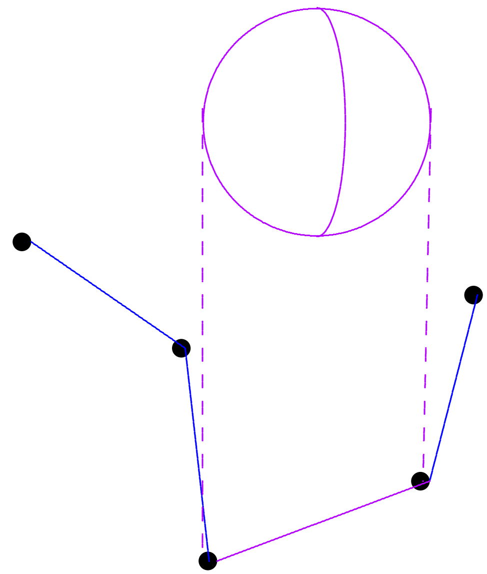

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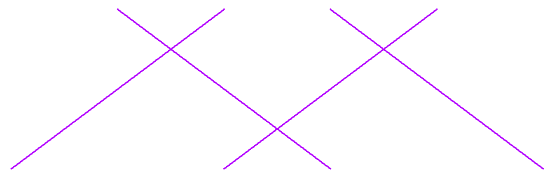

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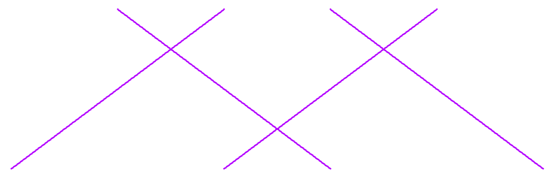

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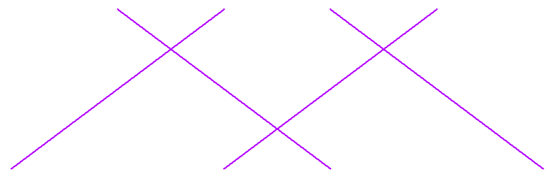

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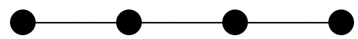

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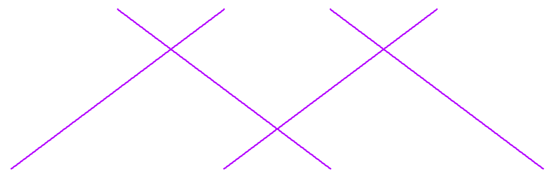

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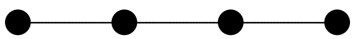

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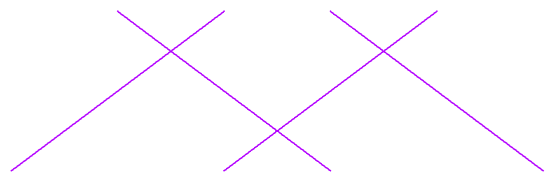

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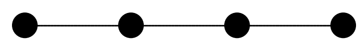

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Plumb together k copies of T^*S^2
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cf. Bishop-Gromov inequality!

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This last property distinguishes the ALF spaces from other classes of gravitational instantons:

ALG, ALH, ALG*, ALH*, . . .

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Taub-NUT becomes Hermitian metric on $\mathbb{C}^2$.

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for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Taub-NUT becomes Hermitian metric on $\mathbb{C}^2$.

This J determines opposite orientation from the hyper-Kähler complex structures.

Example. Taub-NUT metric:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

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Non-Kähler, but conformally Kähler!

Hawking also explored non-hyper-Kähler examples. . .

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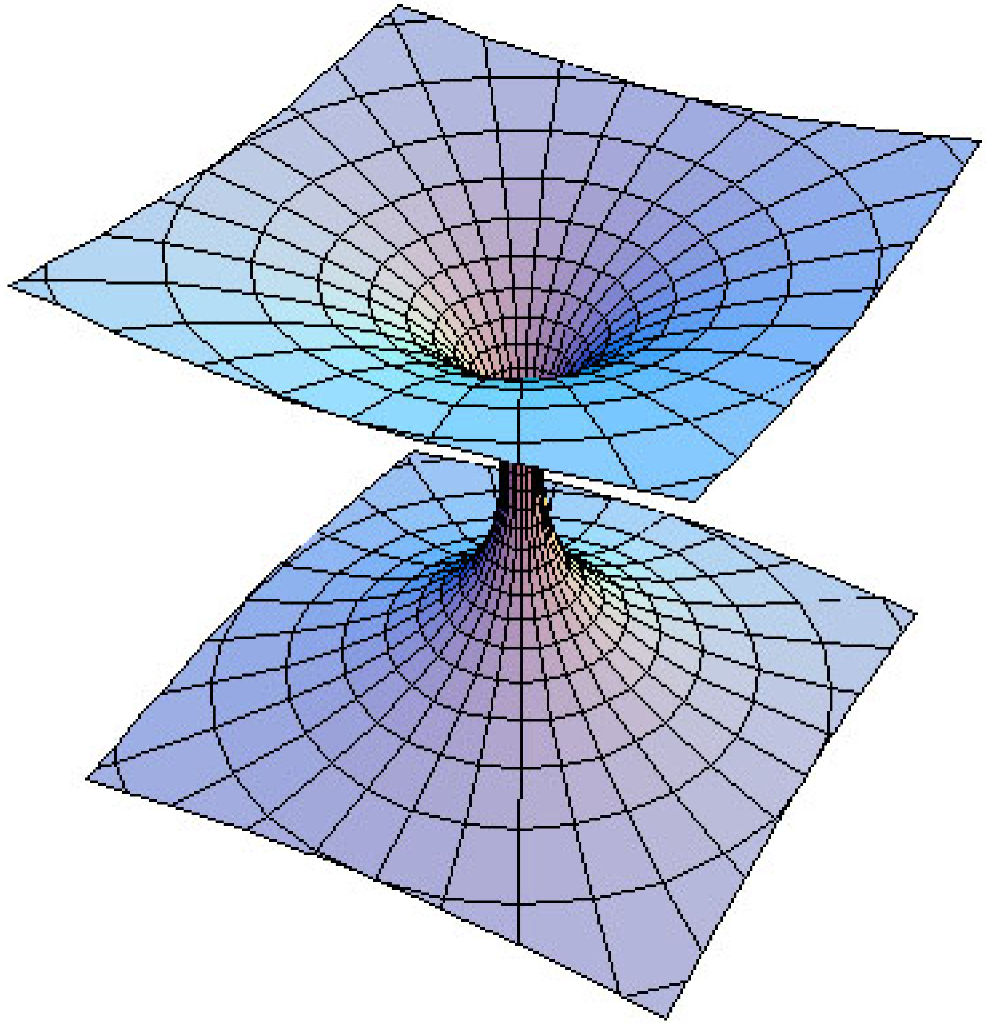
Makes h into extremal Kähler metric on $\mathbb{C} \times \mathbb{CP}_1$.



$$\mathbb{R} \times S^2 \subset \mathbb{R}^2 \times S^2$$

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This might lend some credence to the aphorism. . .

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$$\Longrightarrow \quad \mathrm{Vol}(B_\rho) \sim \mathrm{const} \cdot \rho^3$$

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Y. Chen & E. Teo, 2011

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It is false in all higher dimensions!

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Theorem B. *Let* (M, g_0) *be one of the ALF toric Hermitian gravitational instantons featured in Biquard-Gauduchon classification. Then any other Ricci-flat Riemannian metric* g *on* M *which is sufficiently* C^3_1*-close to* g

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However, if g_0 is Kerr or Taub-bolt, we can deduce definitive regidity result by combining our work with recent results of Aksteiner, Andersson, et al.

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We next hope to be able to show that the existence of a 2-torus $\mathbb{T}^2 \subset \mathrm{Iso}_0(M, g)$ is robust for complex-geometric reasons.

Merci de m'avoir invité!







C'est un grand plaisir d'être ici!