Einstein Manifolds

and

Extremal Kähler Metrics

Claude LeBrun
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— A. Einstein, to G. Gamow

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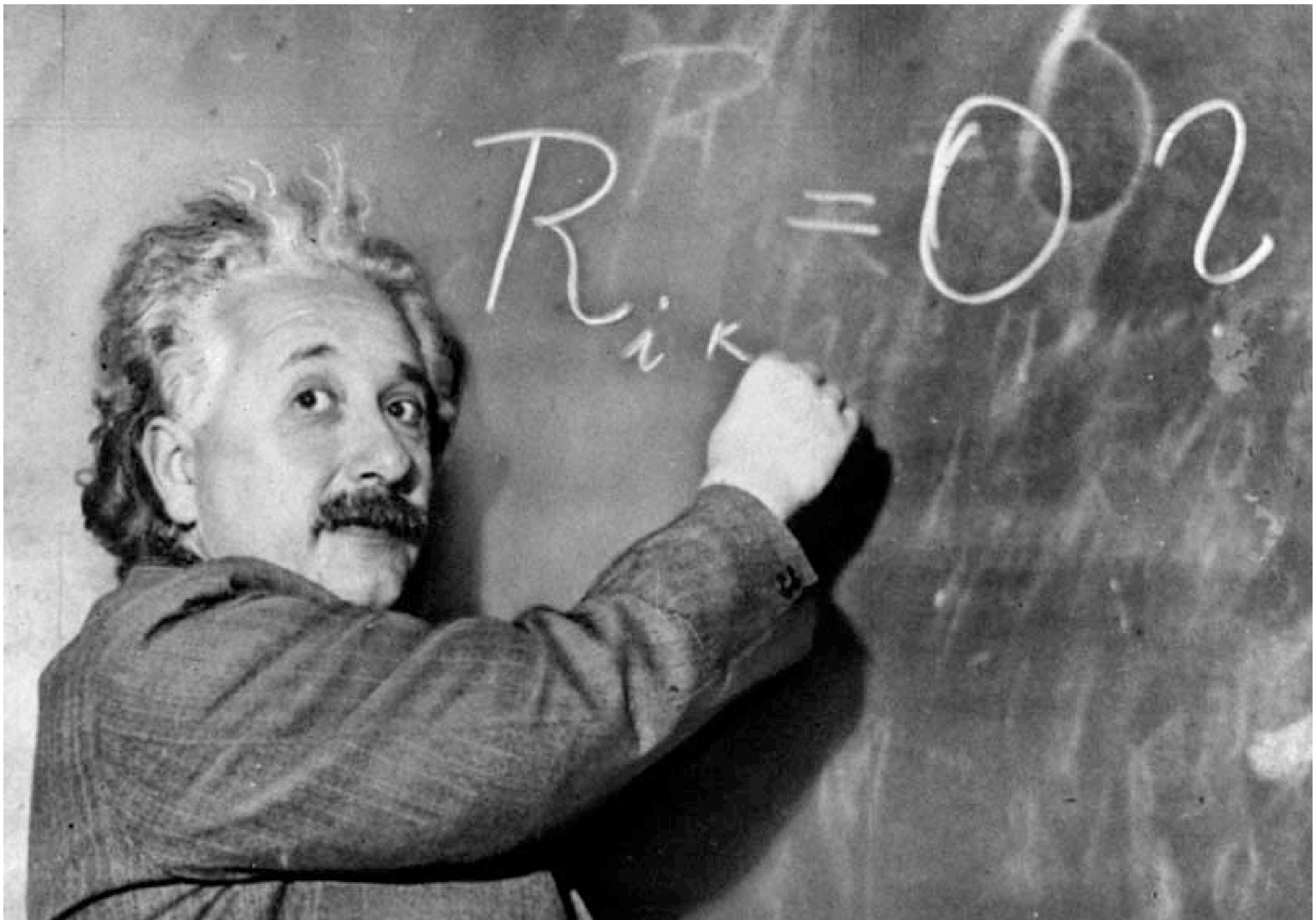

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Has same sign as the *scalar curvature*

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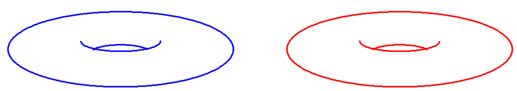

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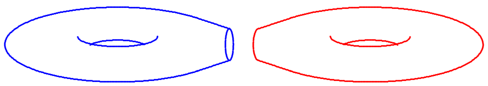

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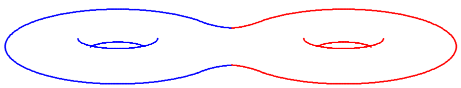

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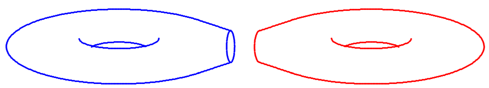

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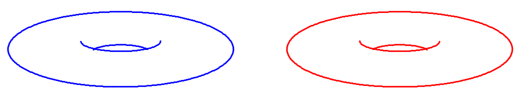

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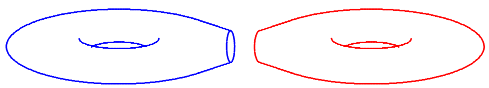

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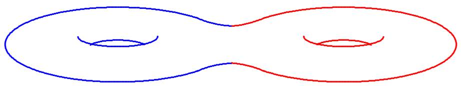

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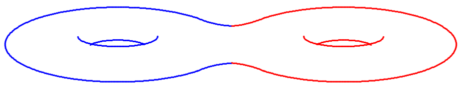

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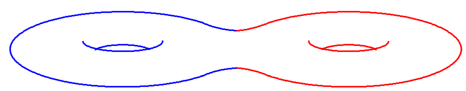

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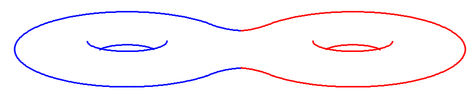

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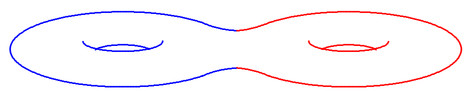

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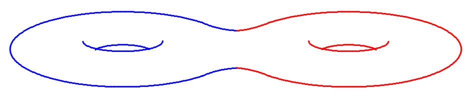

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in which new $\mathbb{CP}_1$ has self-intersection -1.

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Of course, $\mathbb{CP}_2$ and $S^2 \times S^2$ also admit K-E metrics with $\lambda > 0$ — namely, obvious homogeneous ones!

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Since $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ and $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$ have non-reductive automorphism groups, no K-E metrics.

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Note both of above Einstein metrics are Hermitian.

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- Automorphism group non-trivial, non-semi-simple.
  - g is extremal, s non-constant.

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X.X. Chen: always minimizers.

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Donaldson/Mabuchi/Chen-Tian:
unique in Kähler class, modulo bihomorphisms.

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$$\mathcal{A}([\omega]) := \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2}\|\mathcal{F}_{[\omega]}\|^2$$

where $\mathcal{F}$ is Futaki invariant.

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and corresponds to harmonic primitive $(1,1)$-form

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So the critical points of restriction of $\mathcal{W}$ to $\{$Kähler metrics$\}$ also have $B = 0$!

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Del Pezzo case: $s \neq 0$ everywhere!

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Necessary calculations also led to new existence proof. . .

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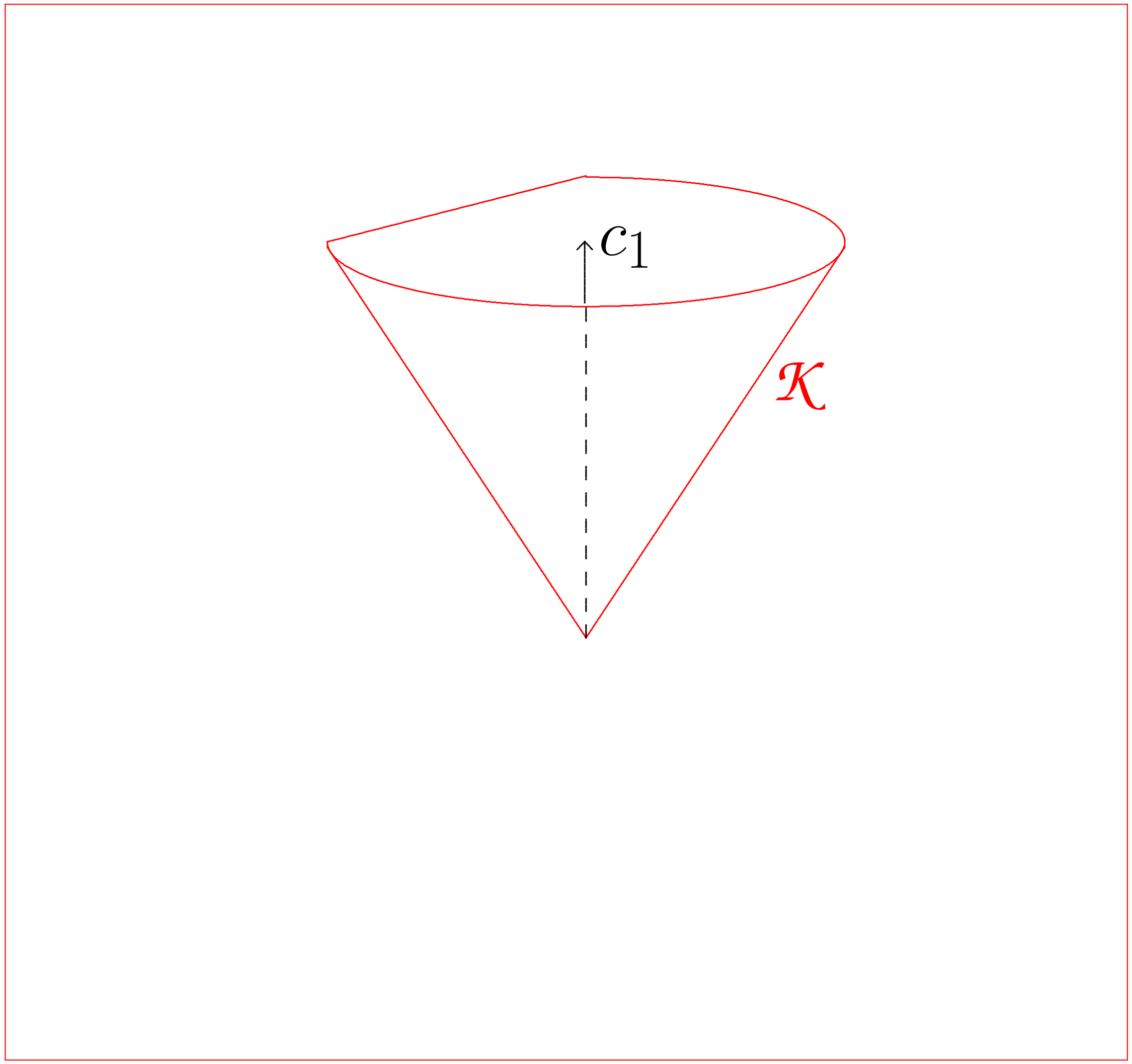
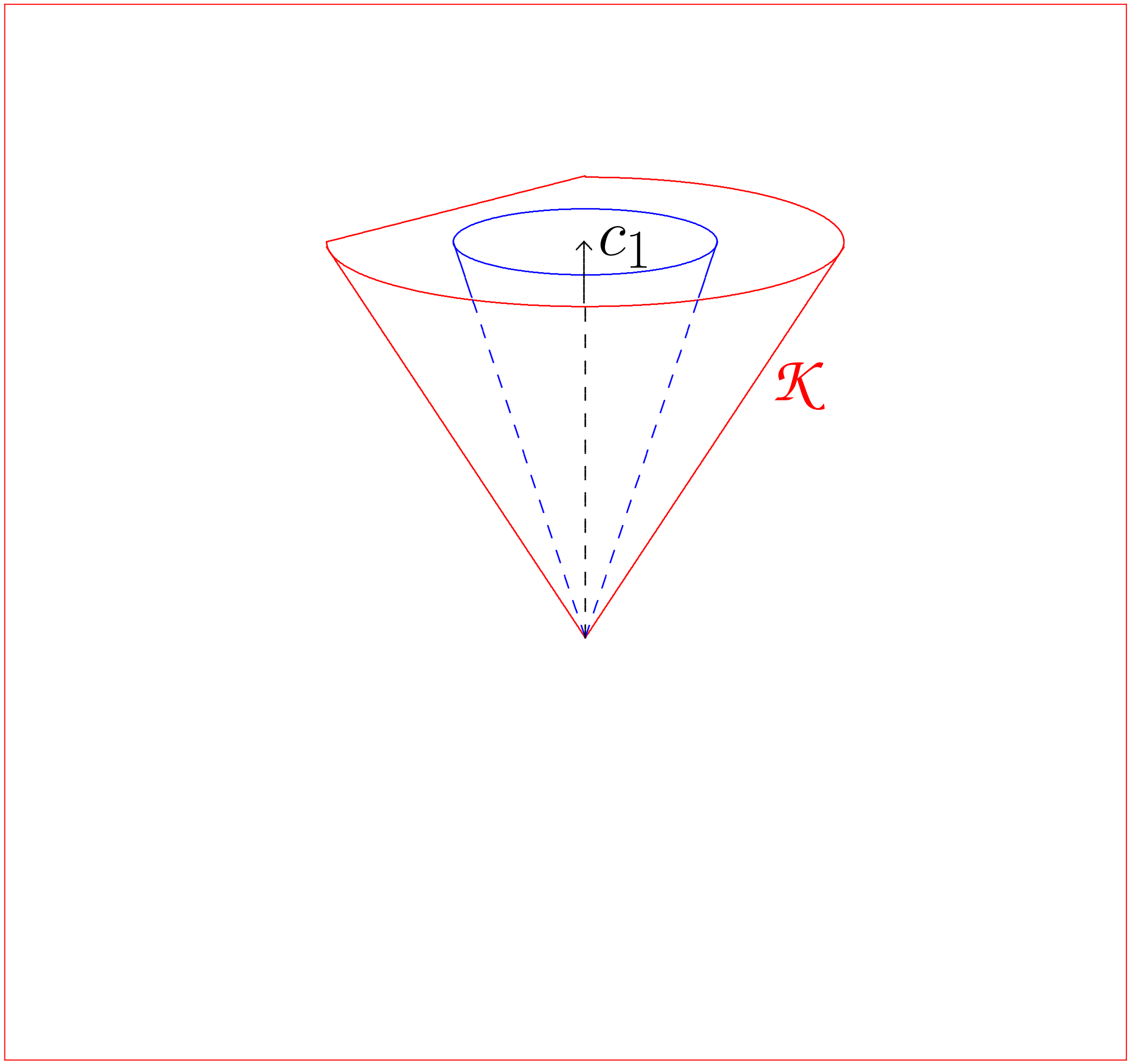
Then there is an extremal Kähler metric g *on* M *with Kähler form* $\omega \in [\omega]$.



$$\mathcal{K} \subset H^{1,1}(M, \mathbb{R}) = H^2(M, \mathbb{R})$$



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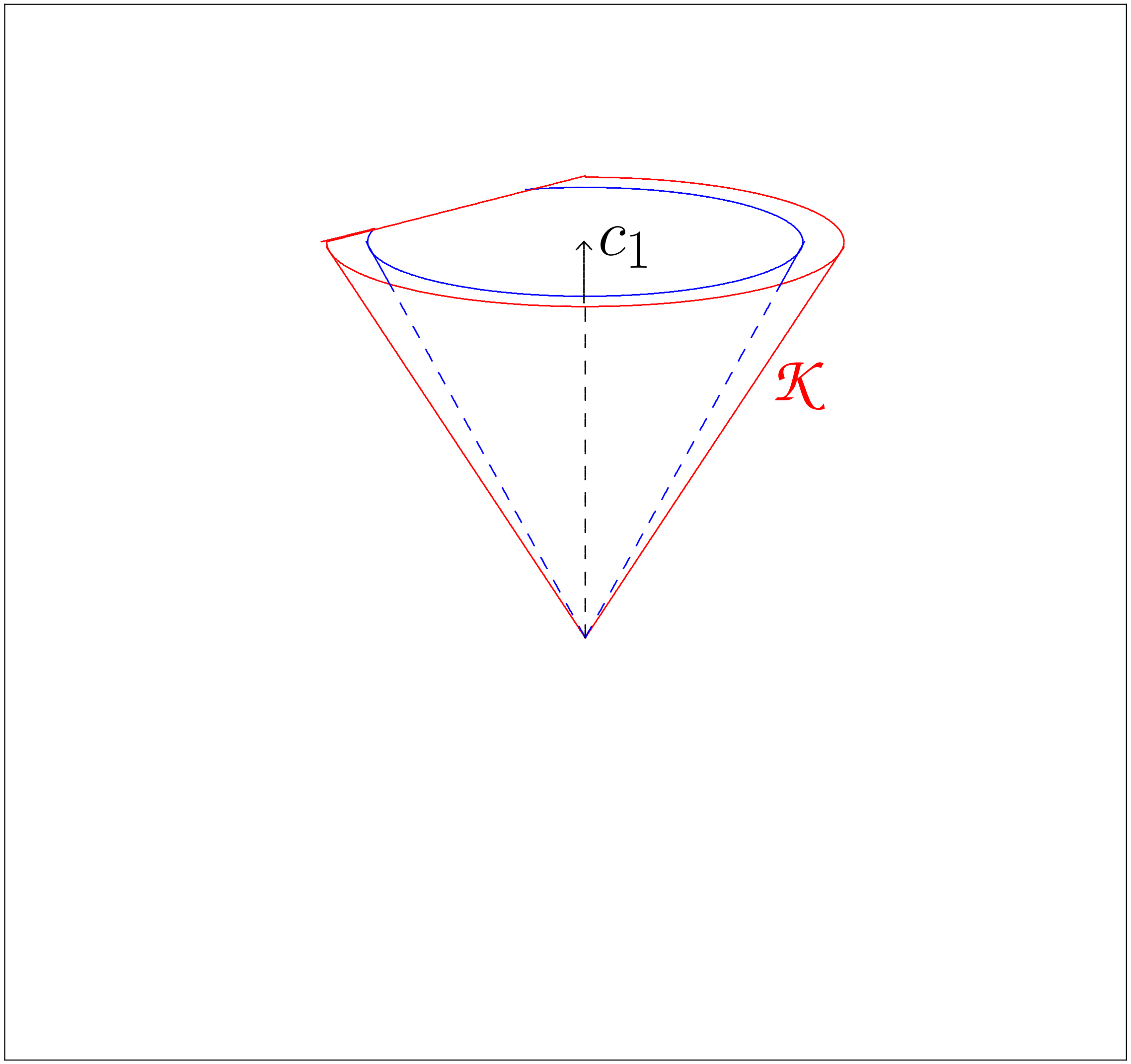

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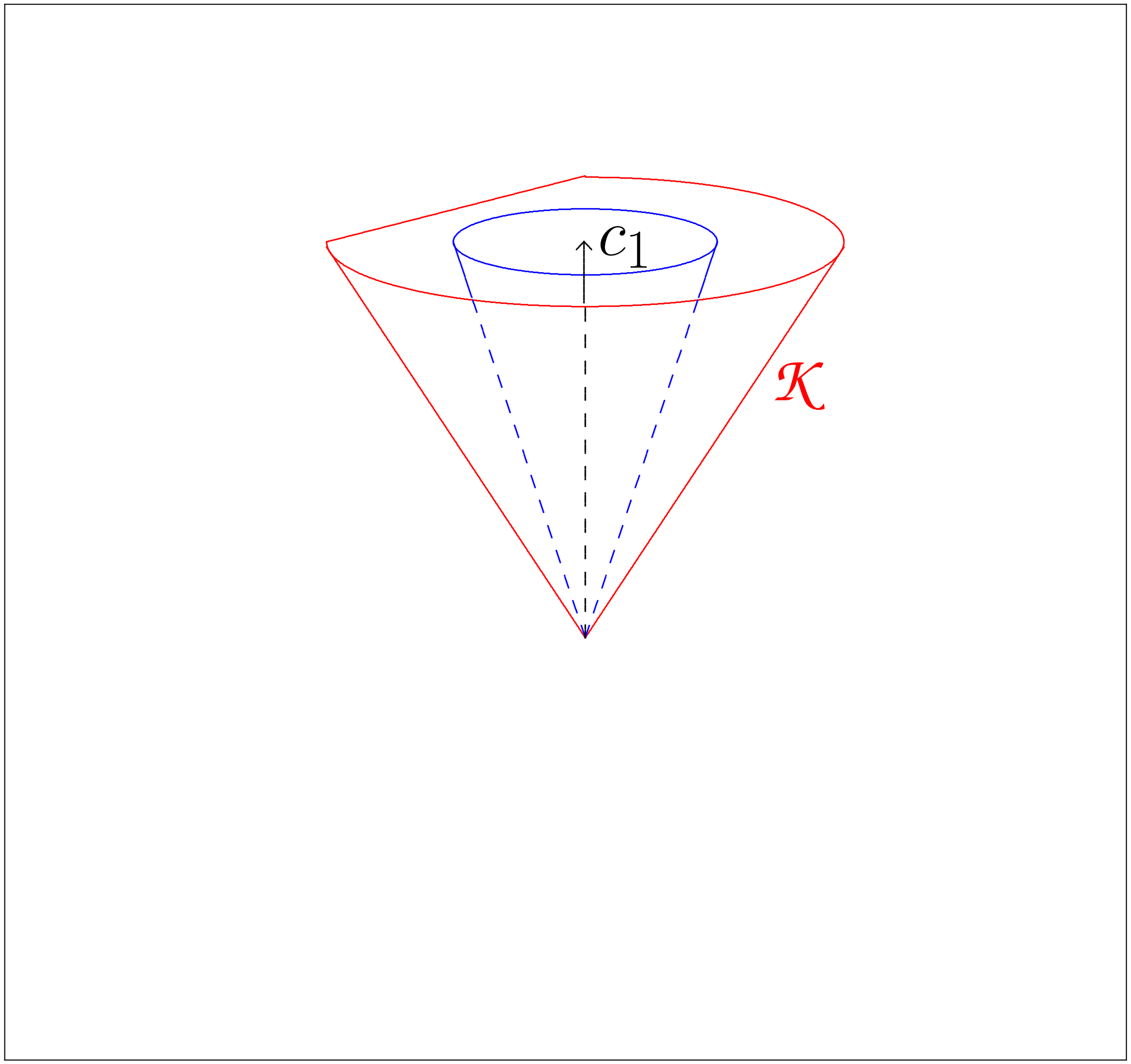

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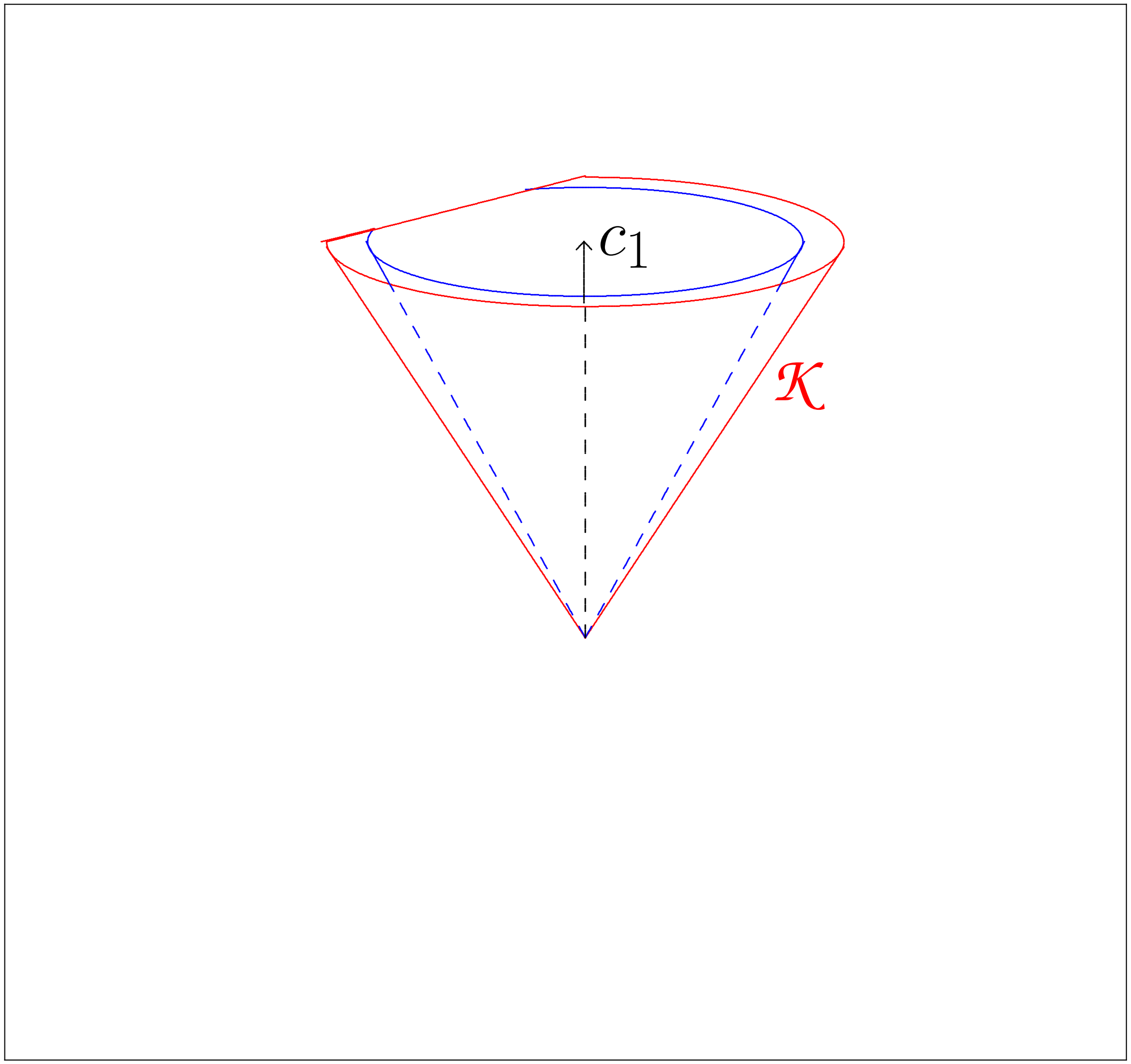

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