Kodaira Dimension

and the

Yamabe Problem,

Revisited

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Analysis, Geometry and Topology of Positive
Scalar Curvature Metrics.
Mathematisches Forschungsinstitut Oberwolfach,
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Joint work with

Joint work with

Michael Albanese

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Michael Albanese
Université du Québec à Montréal

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e-prints: arXiv:2106.14333 [math.DG]

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This talk focuses on the relationship between a complex-analytic invariant called the Kodaira dimension, and a diffeomorphism invariant called the Yamabe invariant (or sigma constant), which encodes information about the scalar curvature.

The new results concern complex surfaces which do not admit Kähler metrics, and thus are far-removed from the original context.

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$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

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where $V = \text{Vol}(M, g)$ inserted to make scale-invariant.

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Then restriction $\mathcal{S}|_\gamma$ is bounded below.

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Difficulty: $L^2_1 \hookrightarrow L^p$ bounded, but not compact.

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Has $s =$ constant.

Unique up to scale when $s \leq 0$.

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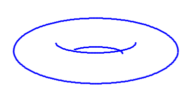
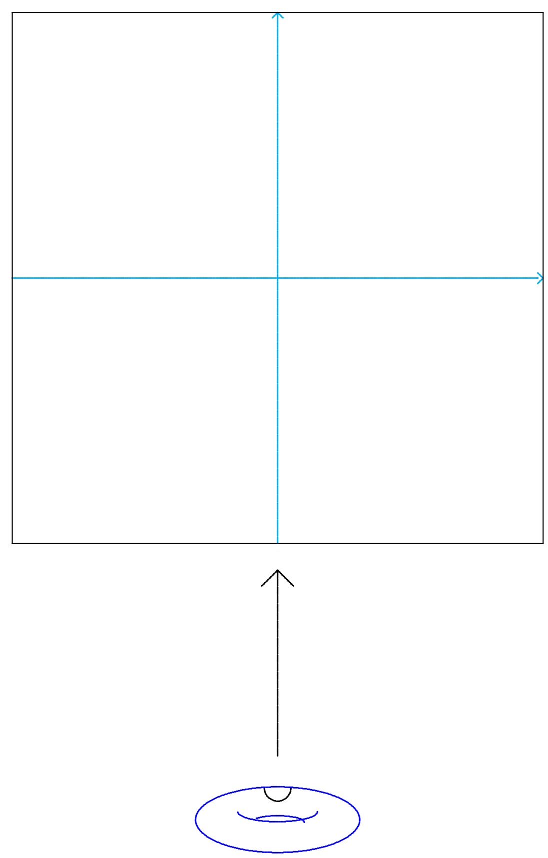
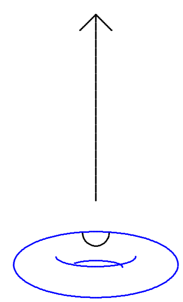
Aubin:

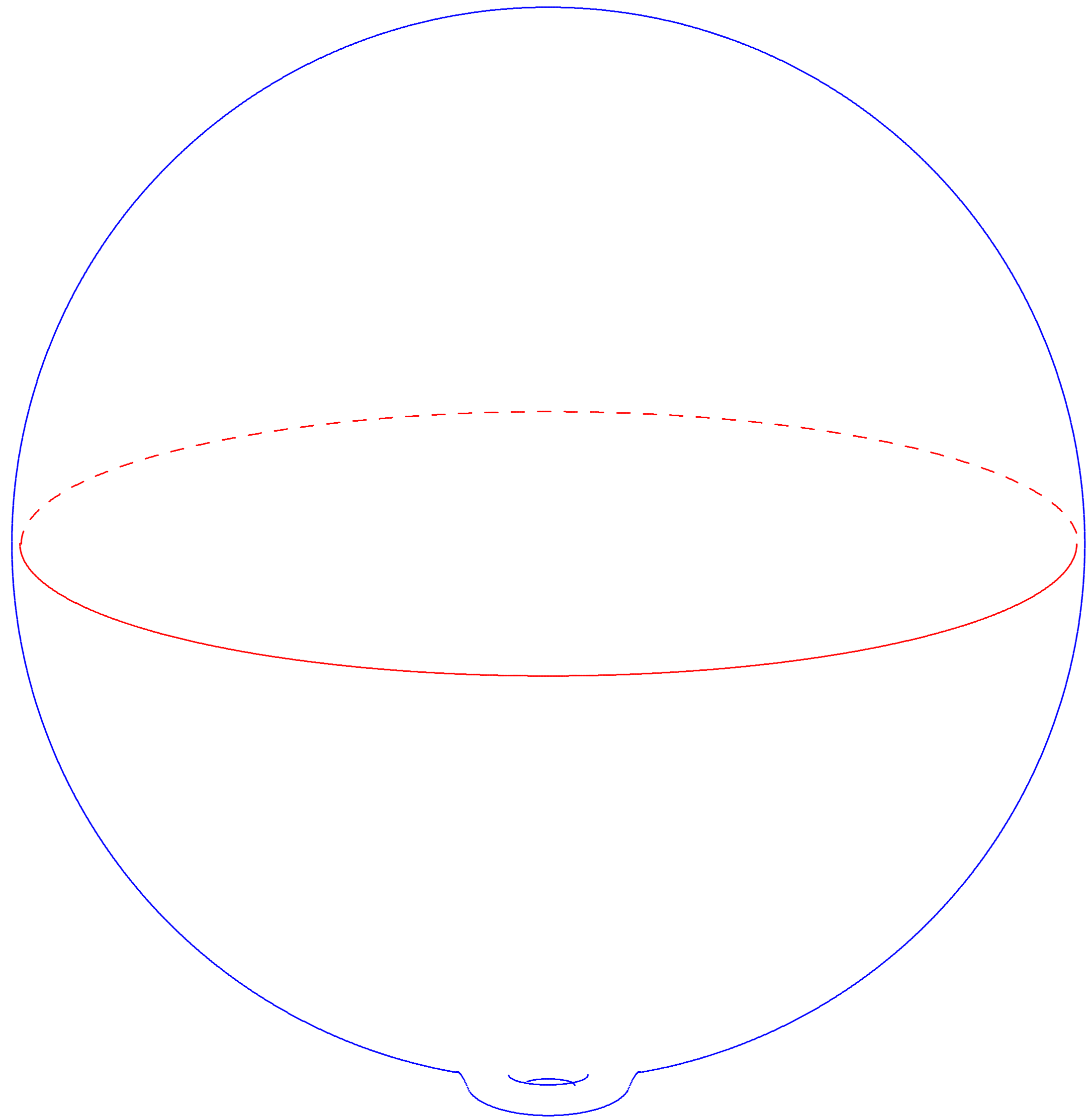
$$Y(M,\gamma) \leq \mathcal{S}(S^n, g_{\text{round}})$$





$$g_{jk} = \delta_{jk} + O(|x|^2)$$





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$$= \quad \text{only for round sphere.}$$

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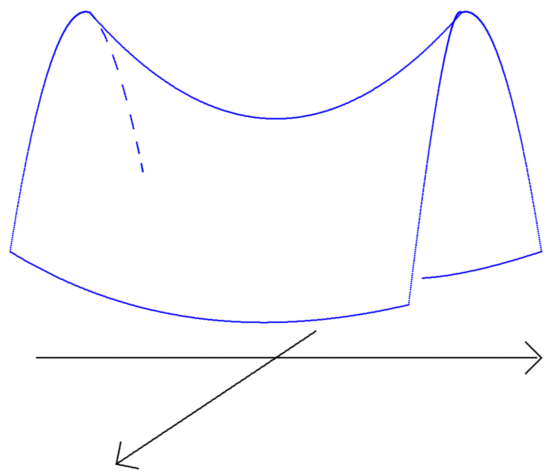

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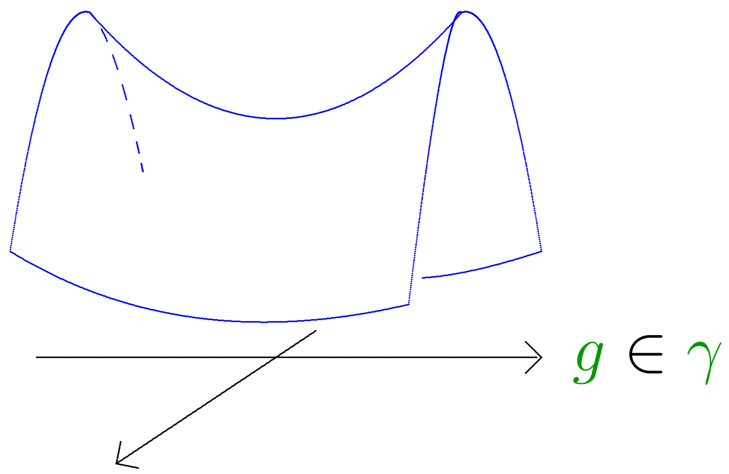

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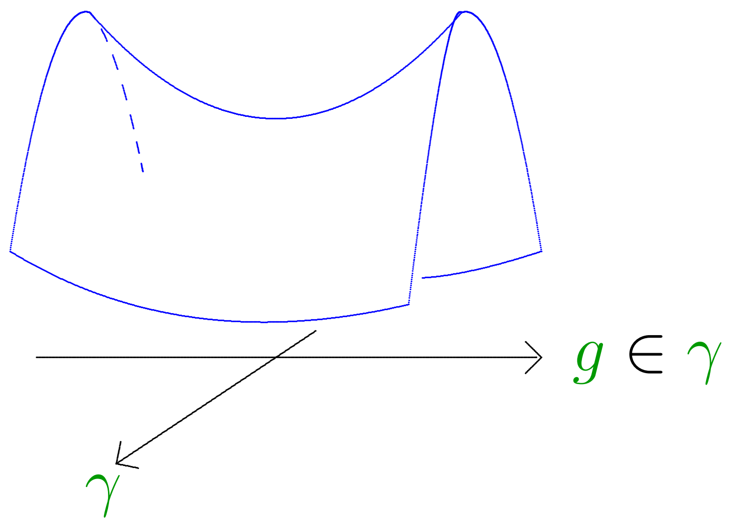

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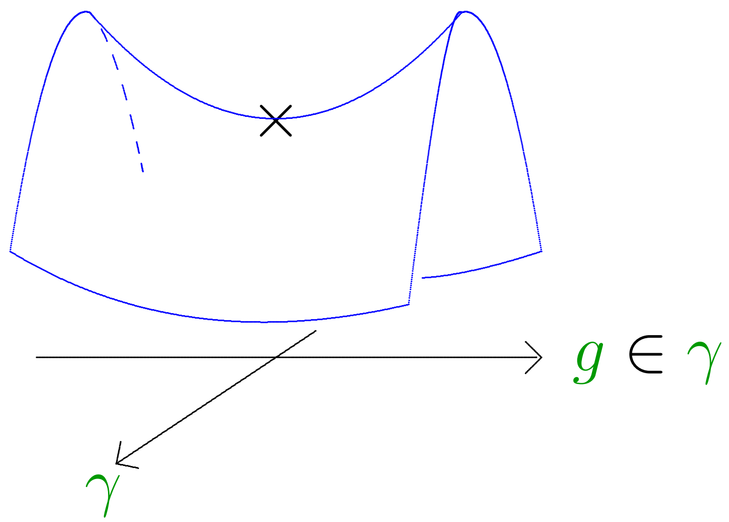

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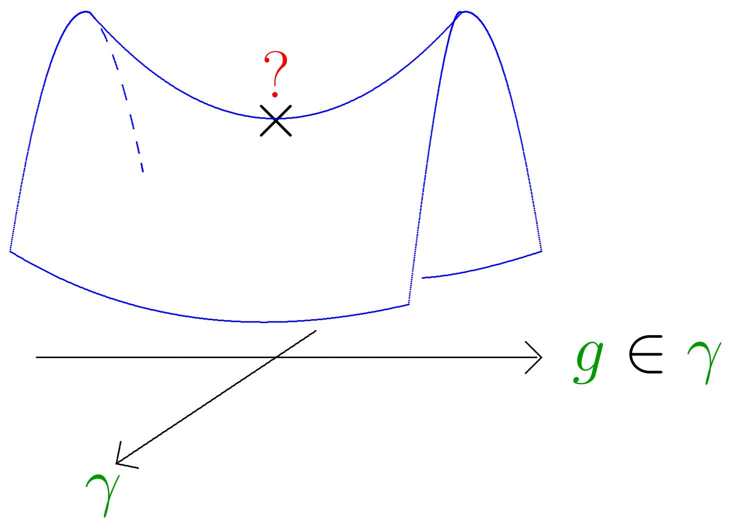

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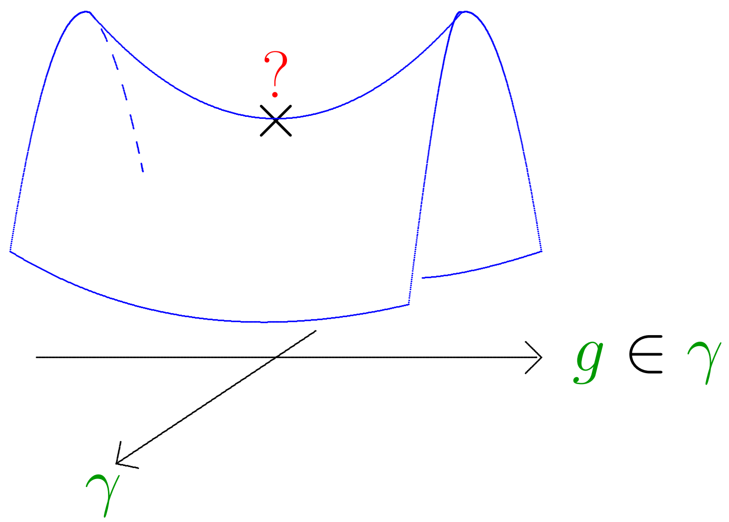

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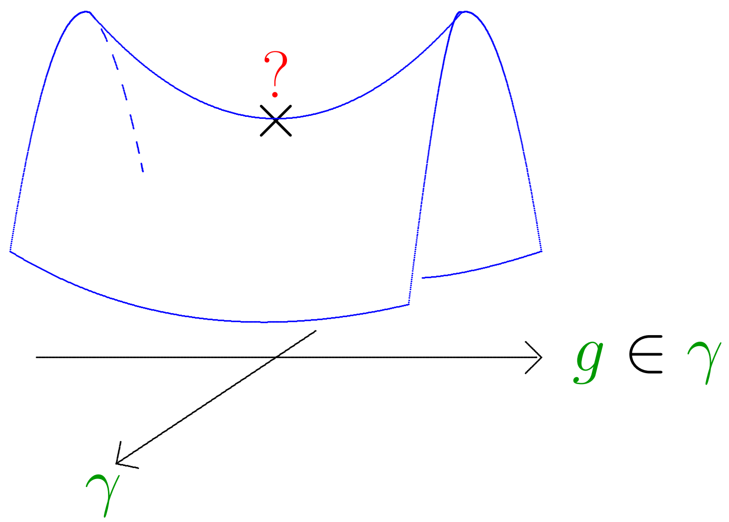

Too good to be true! But ...

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Problem. *Compute actual value of $\mathscr{Y}(M)$ for concrete, interesting manifolds.*

A Differential-Topological Invariant:

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Theorem (Petean et. al.). *Let M be a compact simply connected n-manifold, $n \neq 4$. Then*

$$\mathscr{Y}(M) \geq 0.$$

Theorem (L '96). *There exist compact simply connected 4-manifolds M_j with $\mathscr{Y}(M_j) \to -\infty$.*

Moreover, can choose M_j such that each $\mathscr{Y}(M_j)$ is realized by an Einstein metric g_j.

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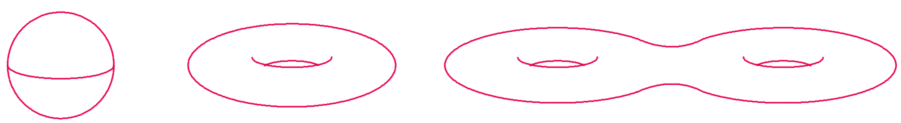

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By contrast, in complex dimension $m \geq 3$, Kod is not a diffeomorphism invariant, and has essentially nothing to do with $\mathscr{Y}(M)$.

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Today: what happens when $b_1(M)$ is odd?

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$$\max \ \dim_{\mathbb{C}} \ \mathrm{Image}(M \dashrightarrow \mathbb{CP}_N)$$

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where $K = \Lambda^{2,0}$ is canonical line bundle.

Then $\mathrm{Kod}(M, J) \in \{-\infty, 0, 1, 2\}$ is exactly

$$\max \ \dim_{\mathbb{C}} \mathrm{Image}(M \dashrightarrow \mathbb{CP}_N)$$

over maps defined by holomorphic sections of $K^{\otimes \ell}$.

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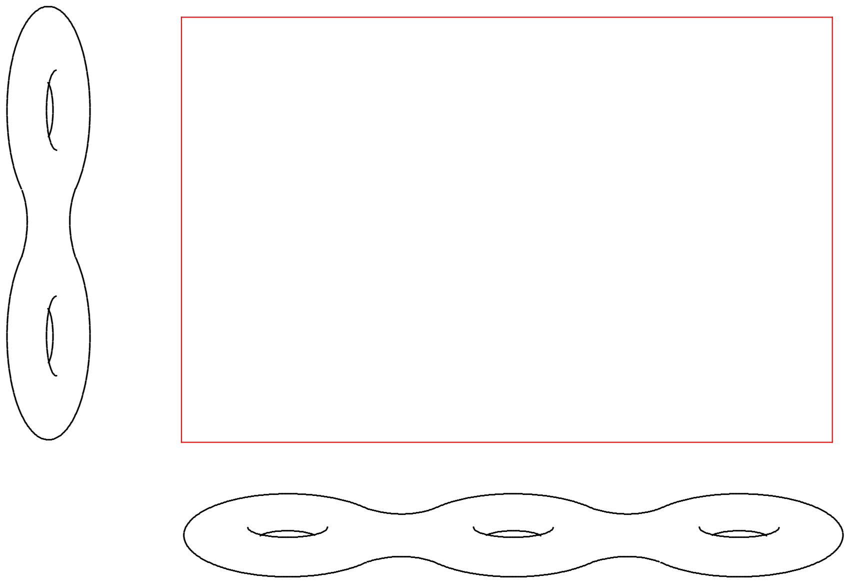

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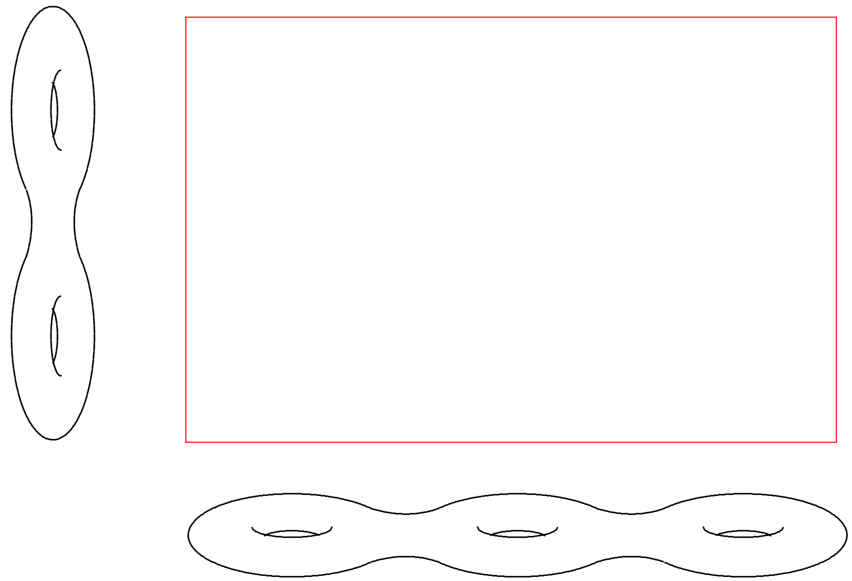

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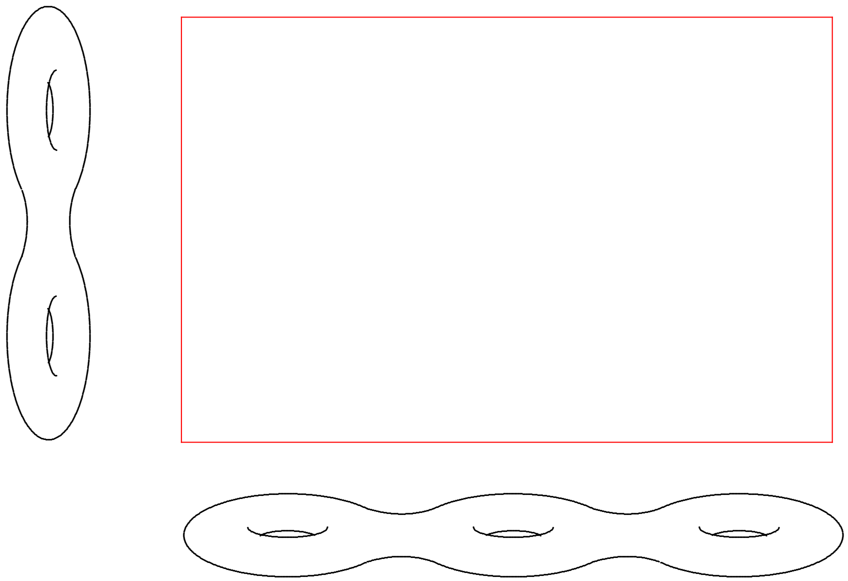

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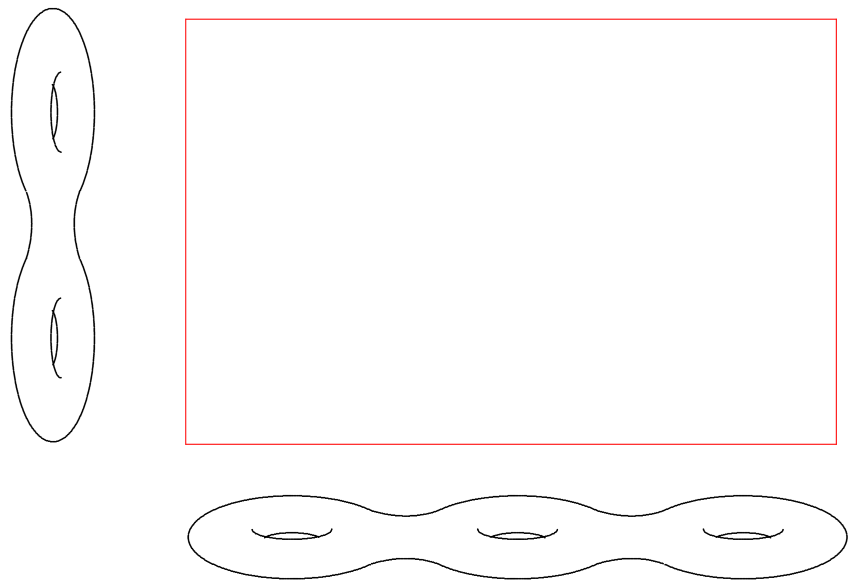

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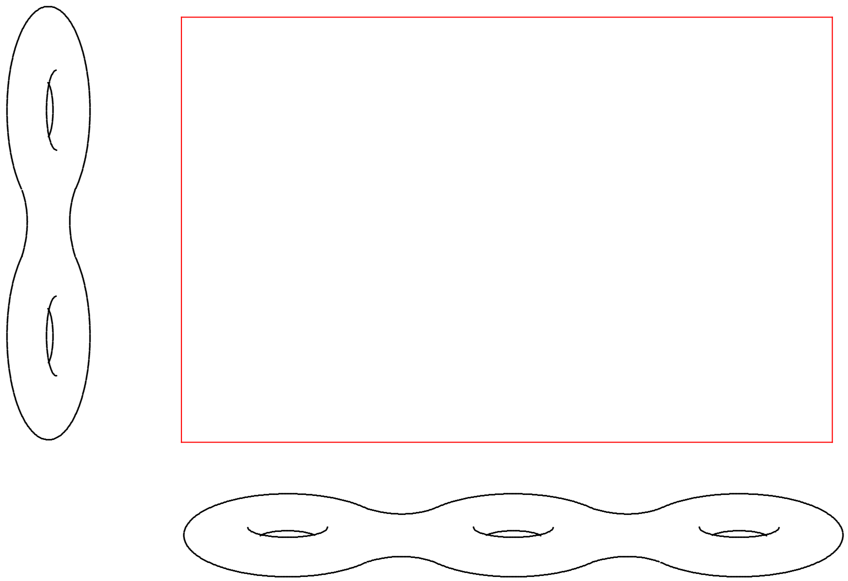

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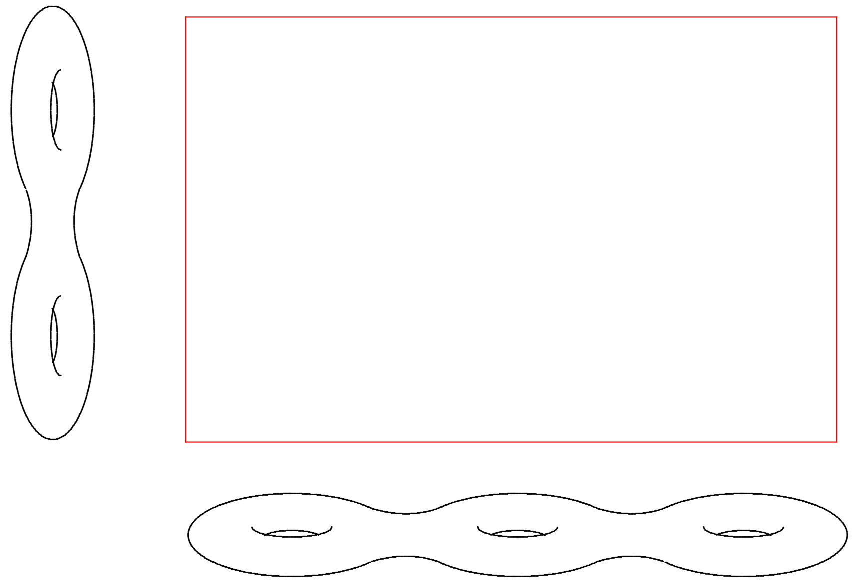

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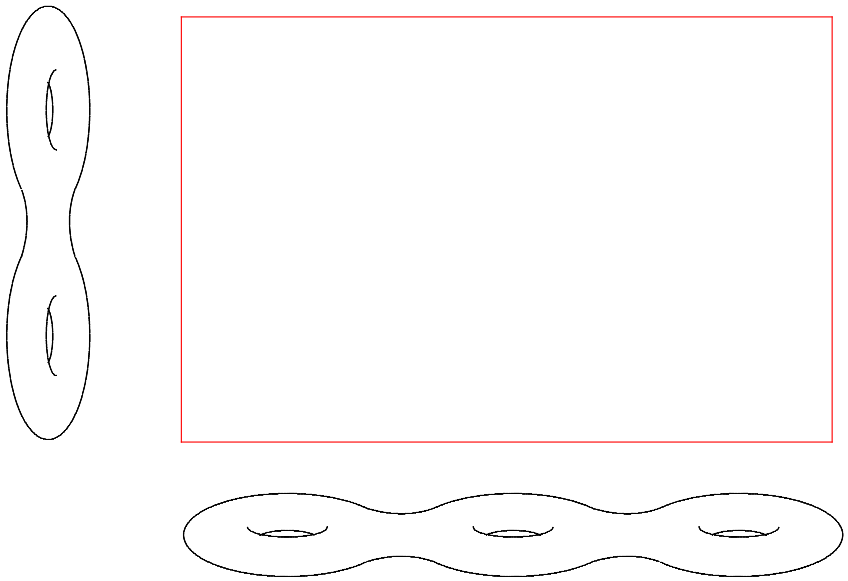
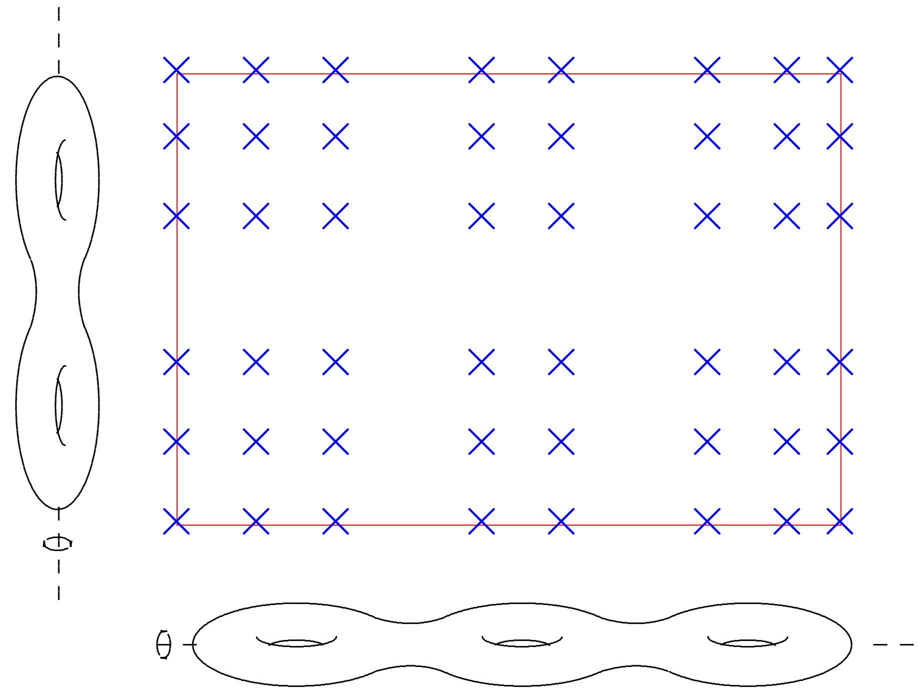


$$\mathrm{Kod}\,(\Sigma_1 \times \Sigma_2) = \mathrm{Kod}\,(\Sigma_1) + \mathrm{Kod}\,(\Sigma_2)$$

Examples. Simply connected examples:



$$M = (\widetilde{\Sigma_1 \times \Sigma_2})/\mathbb{Z}_2$$

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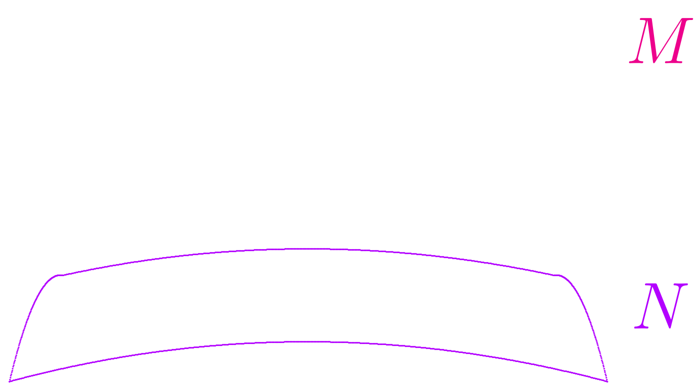

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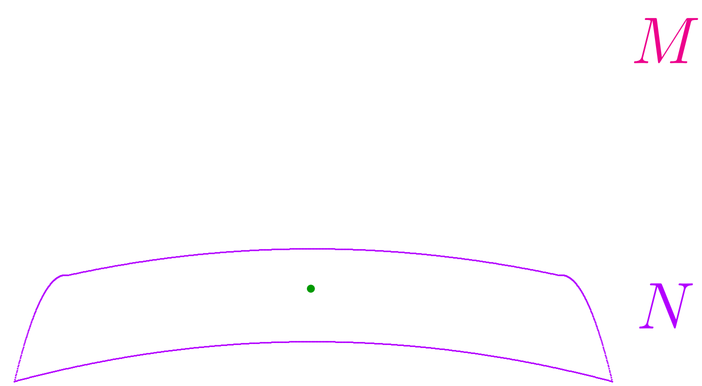

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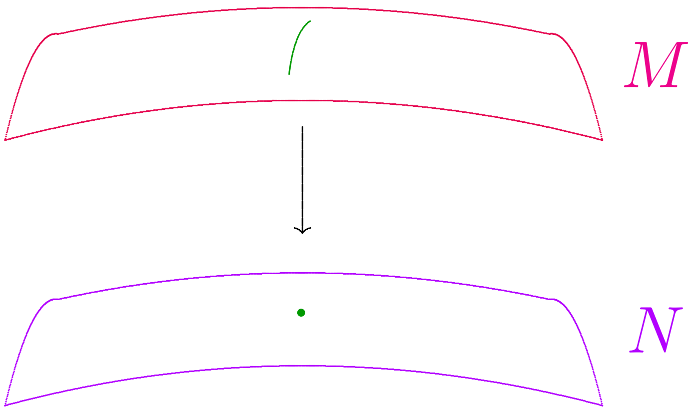

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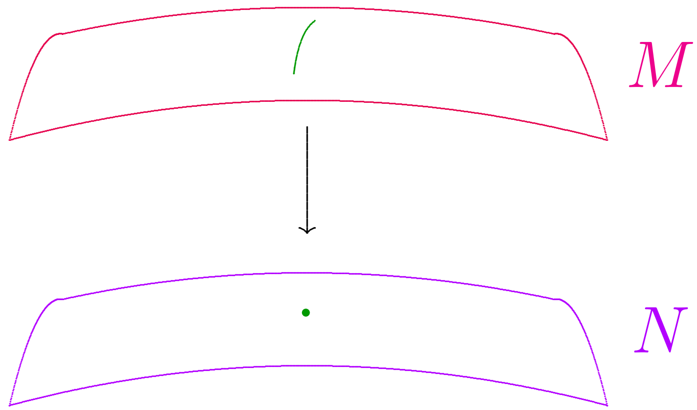

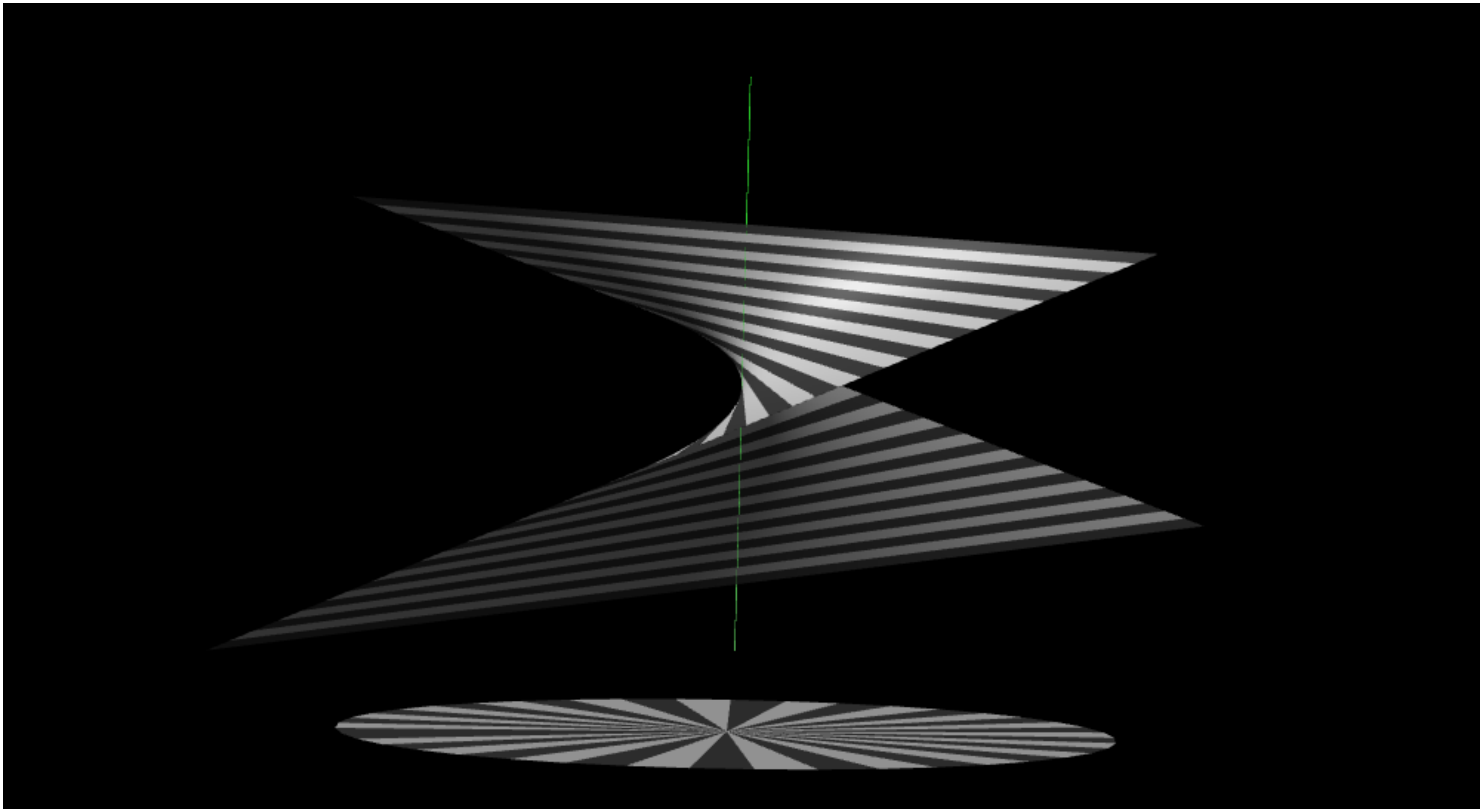

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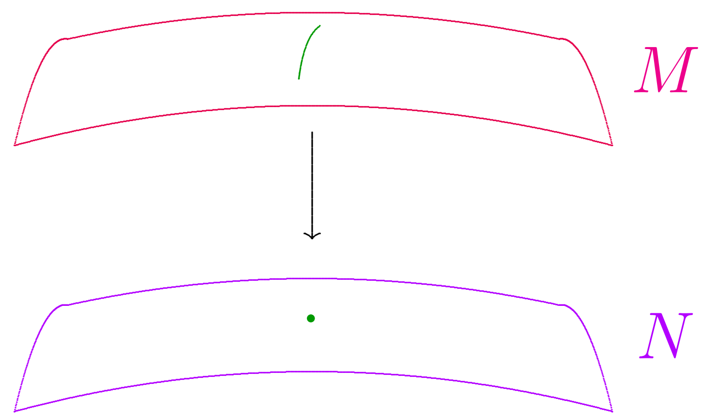

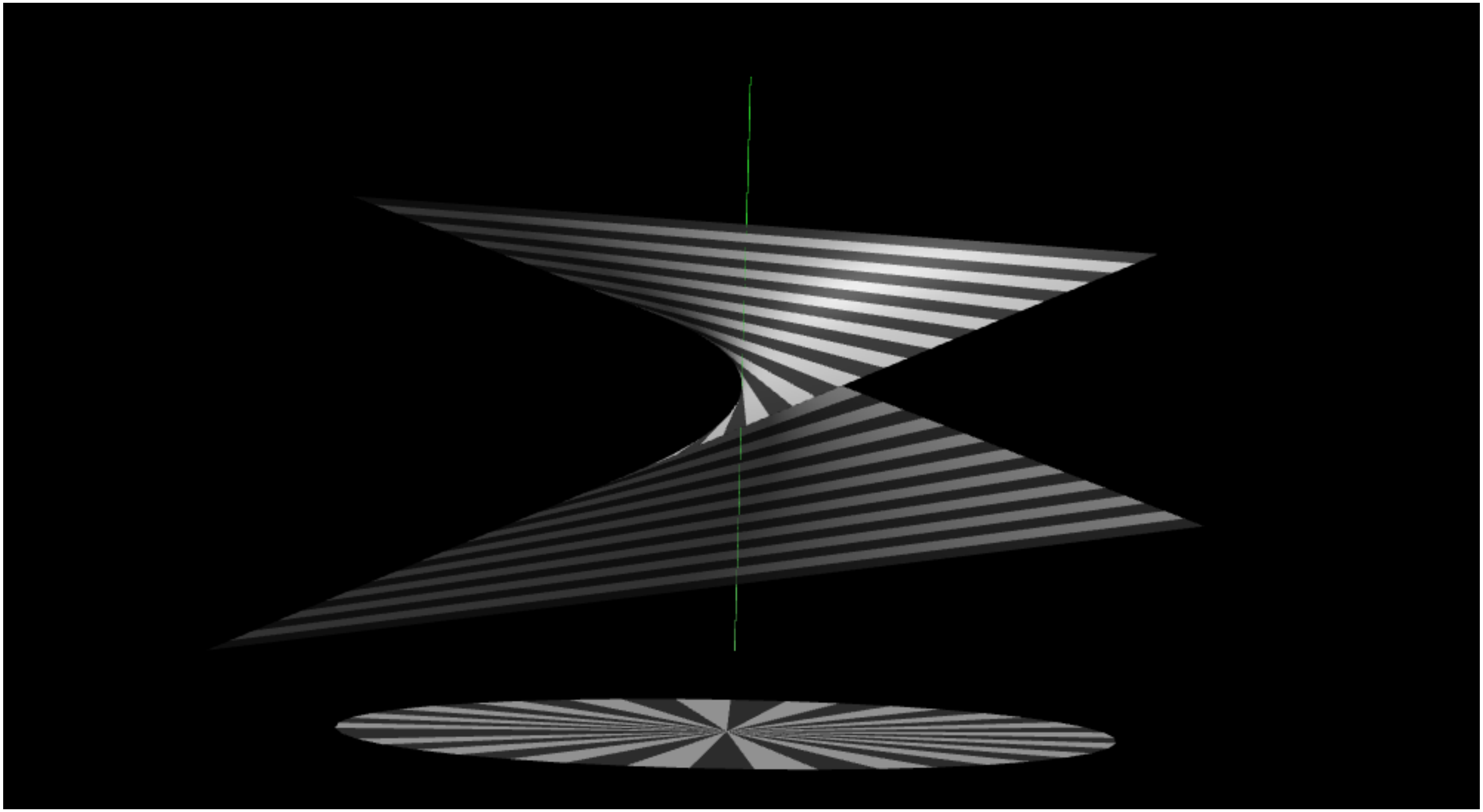

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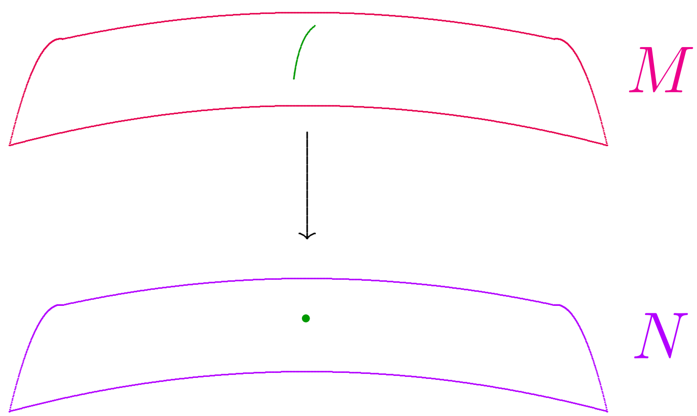

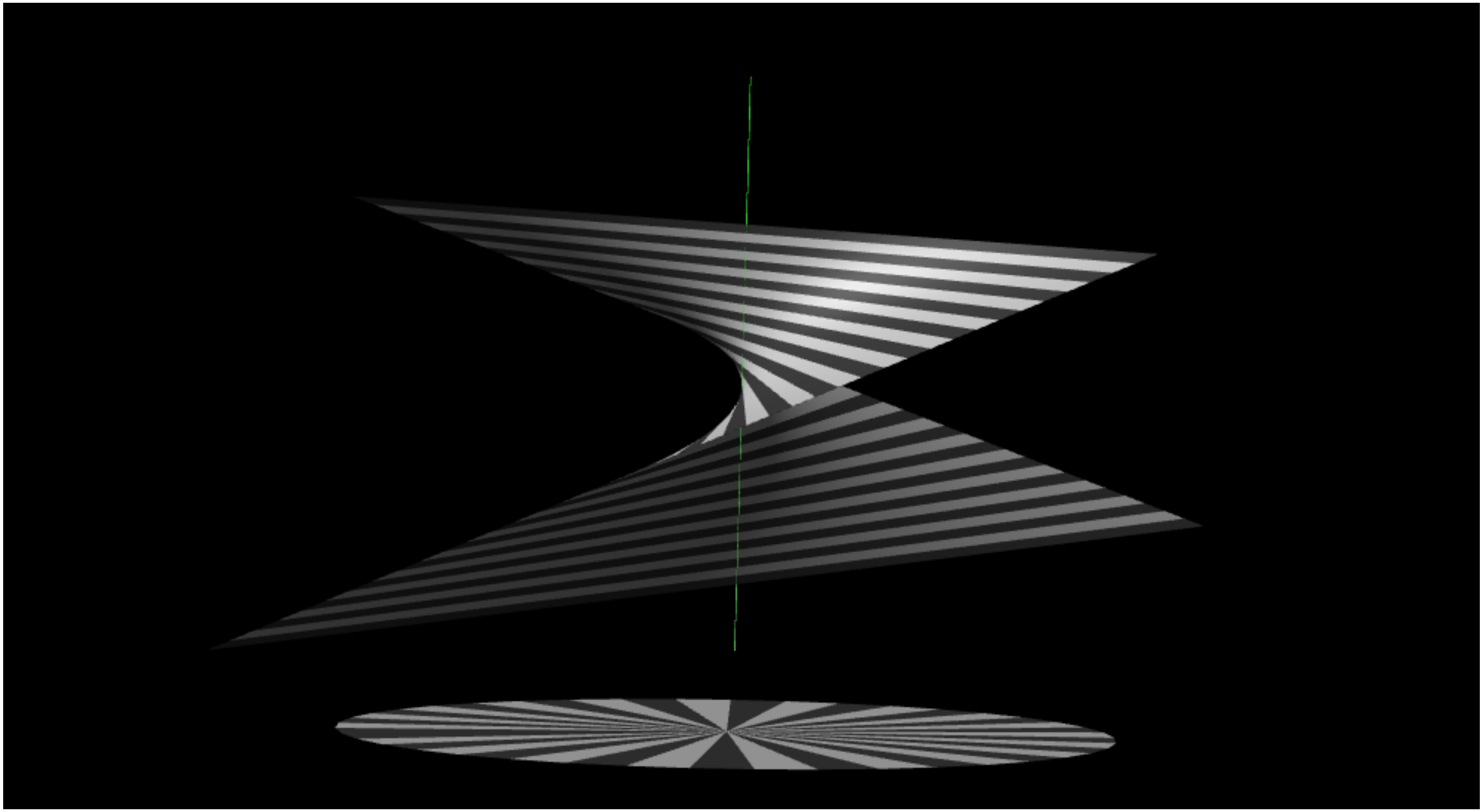

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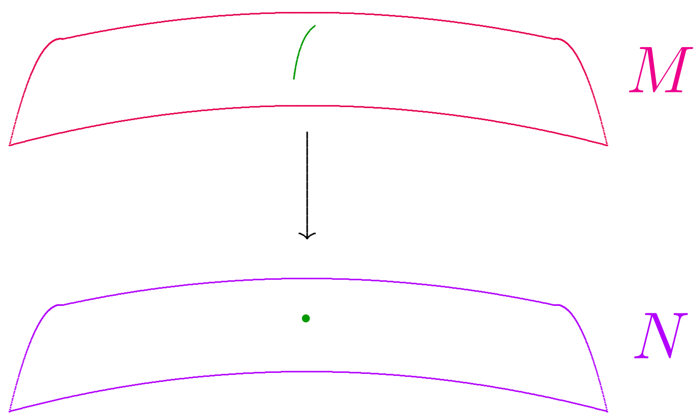

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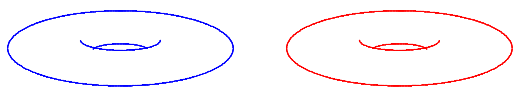

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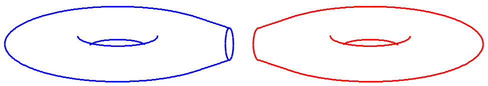

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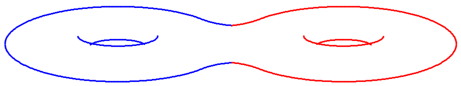

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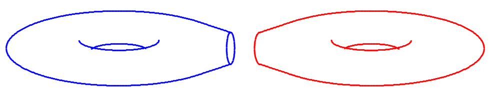

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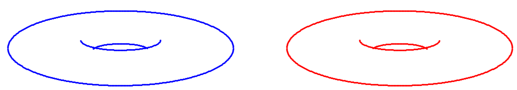

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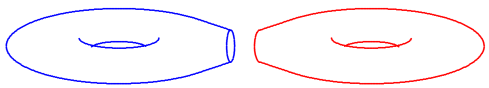

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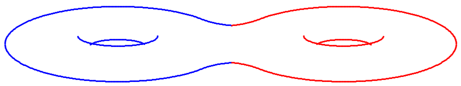

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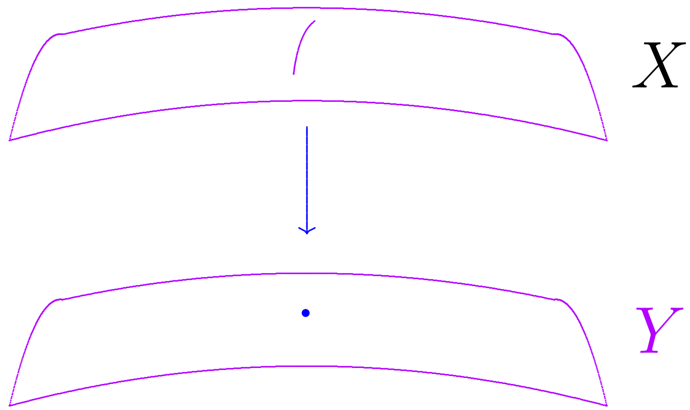

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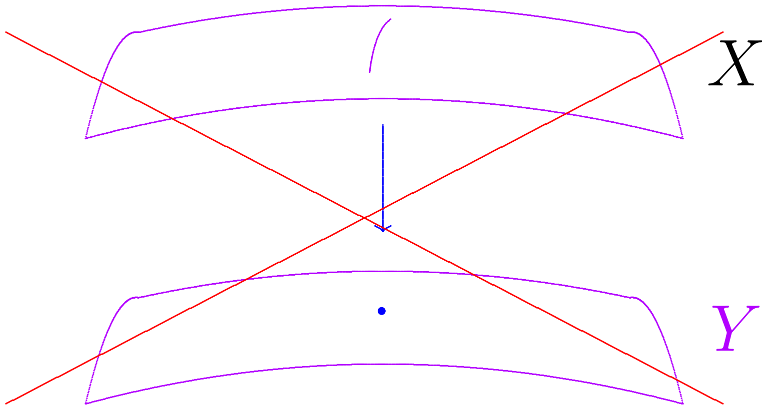

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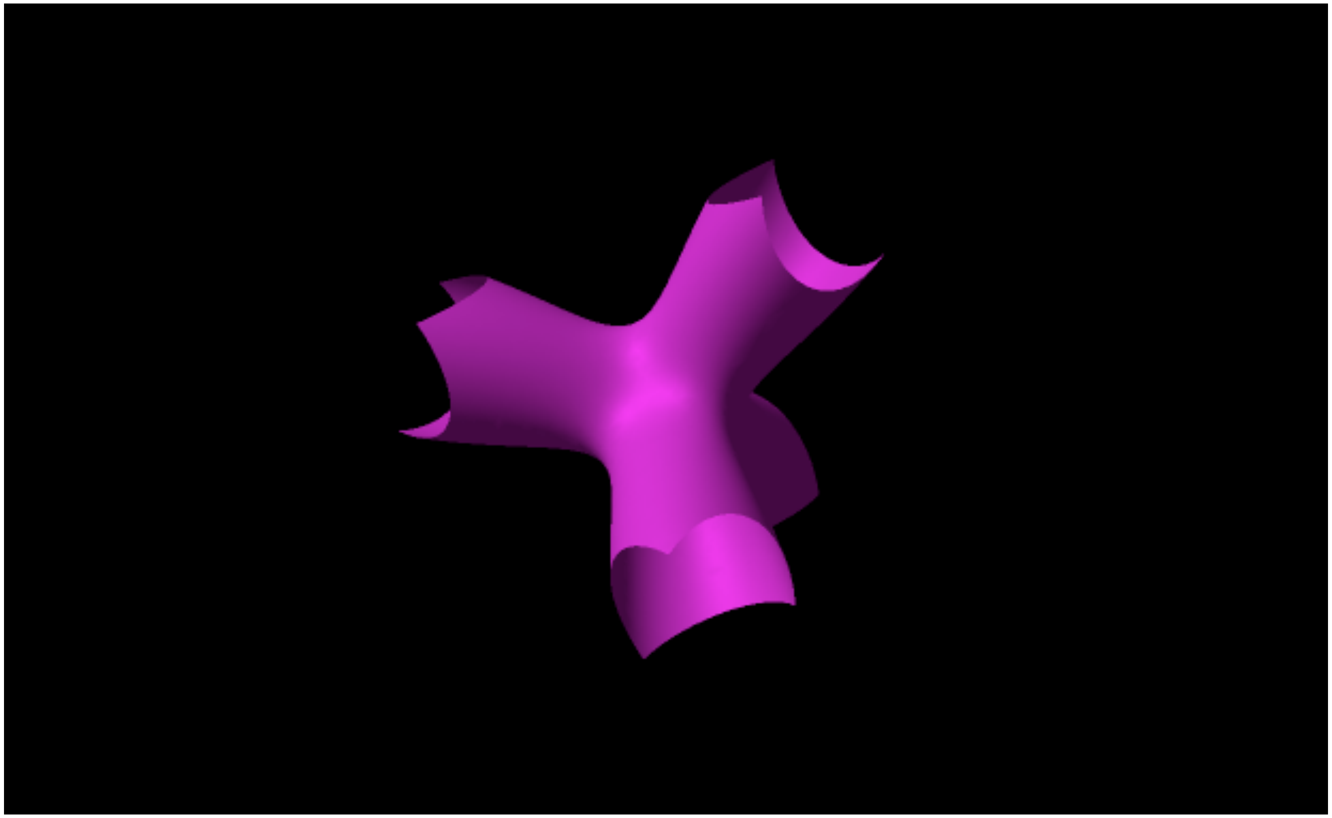

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"Fibration" allows singular fibers, so not fiber-bundle.

Theorem (L '98). *Let M be the smooth 4-manifold underlying any compact complex surface (M^4, J) of Kähler type. Then*

$$\mathscr{Y}(M) > 0 \iff Kod(M, J) = -\infty,$$

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In fact, if X admits K-E metric, achieves $\mathscr{Y}(X)$.

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We'll see that this isn't so when Kod $= -\infty$!

L '98 covers most pieces of Theorems **A** and **B**.

Covers the cases of Kod $= 0$ or 2.

Proves $\mathscr{Y}(M) \geq 0$ when Kod $= 1$.

Missing piece:
Prove $\mathscr{Y}(M) \leq 0$ when Kod $= 1$, b_1 odd.

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Hidden in plain sight: Every complex surface with Kod $= 1$ and b_1 odd has an (unbranched) covering of this form!

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Theorem.

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Proposition. *Let* N *be a compact oriented 3-manifold that admits a map* $\psi : N \to V$ *of non-zero degree to an aspherical manifold* V. *Then* $\mathscr{Y}(N) \leq 0$.

Crash course on Seiberg-Witten Theory. . .

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Any oriented M^4 admits spin^c structures $\mathfrak{c}$.

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$$w_2(TM^4) \in H^2(M, \mathbb{Z}_2)$$

in image of

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where $\mathbb{S}_\pm$ are the (locally defined) left- and right-handed spinor bundles of (M, g).

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where $F_\theta{}^+$ = self-dual part curvature of θ, and $\sigma : \mathbb{V}_+ \to \Lambda^+$ is a natural real-quadratic map,

$$|\sigma(\Phi)| = \frac{1}{2\sqrt{2}}|\Phi|^2.$$

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This leads to non-trivial scalar curvature estimates.

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$$s_- := \min(s, 0)$$

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where $c_1(L)_g^+ =$ self-dual part of harmonic rep.

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- Witten's SW invariant ("Basic classes")
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- Ozsváth-Szabo construction. . .

Definition. *Let M be a smooth compact oriented 4-manifold with $b_+ \geq 2$. A characteristic integral cohomology class $\mathbf{a} \in H^2(M, \mathbb{Z})/\mathrm{torsion}$ will be called a* mock-monopole class *of M if every Riemannian metric g on M satisfies the inequality*

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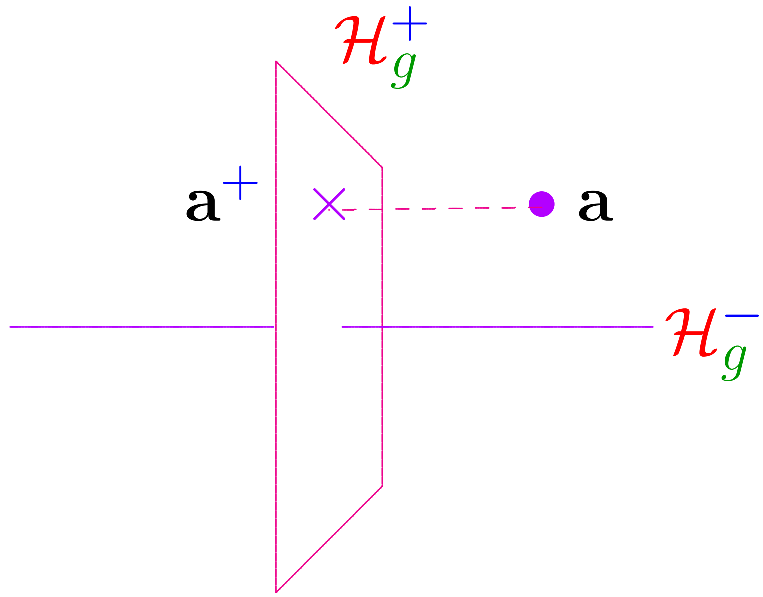
is the orthogonal projection of $\mathbf{a}$, *with respect to the intersection form* $\bullet$, *to the* $b_+(M)$*-dimensional subspace* $\mathcal{H}_g^+ \subset H^2(M, \mathbb{R})$ *represented by self-dual harmonic 2-forms with respect to g.*



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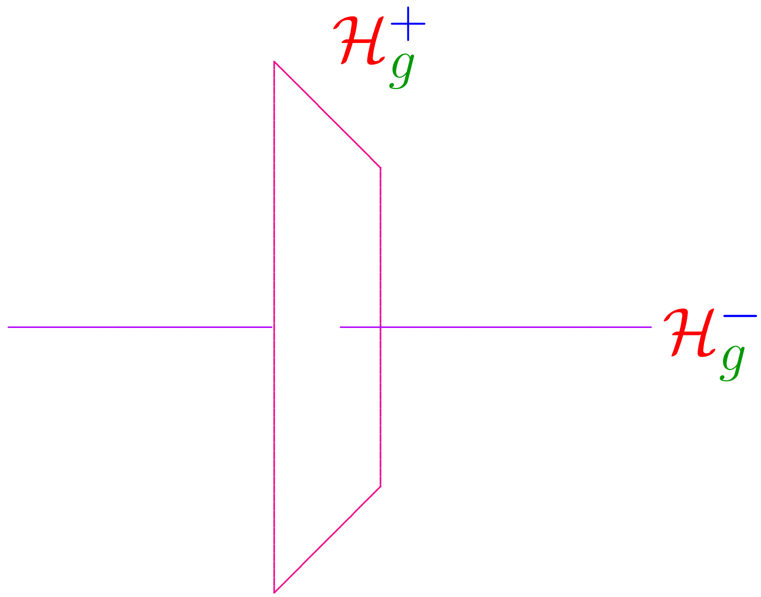

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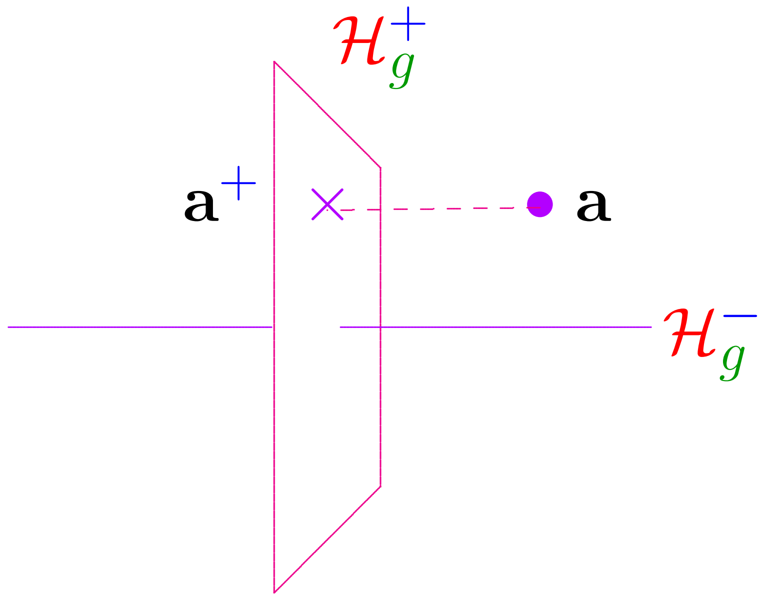

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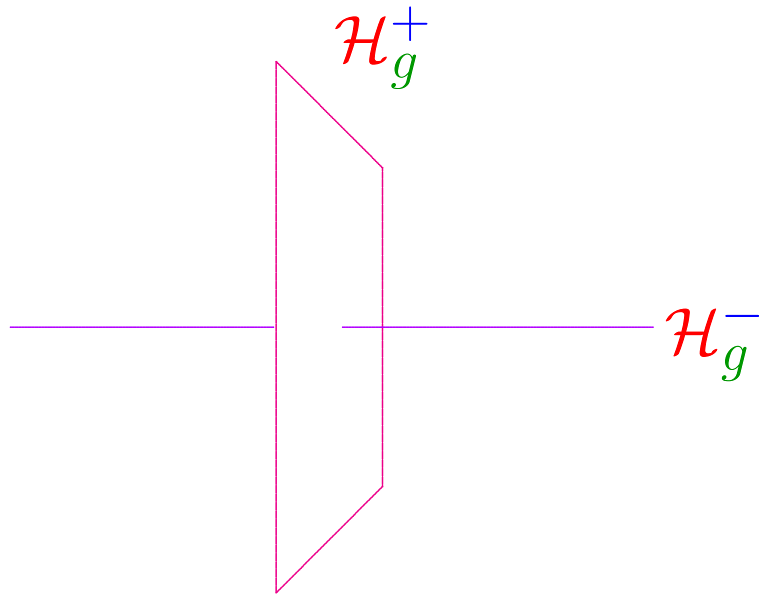

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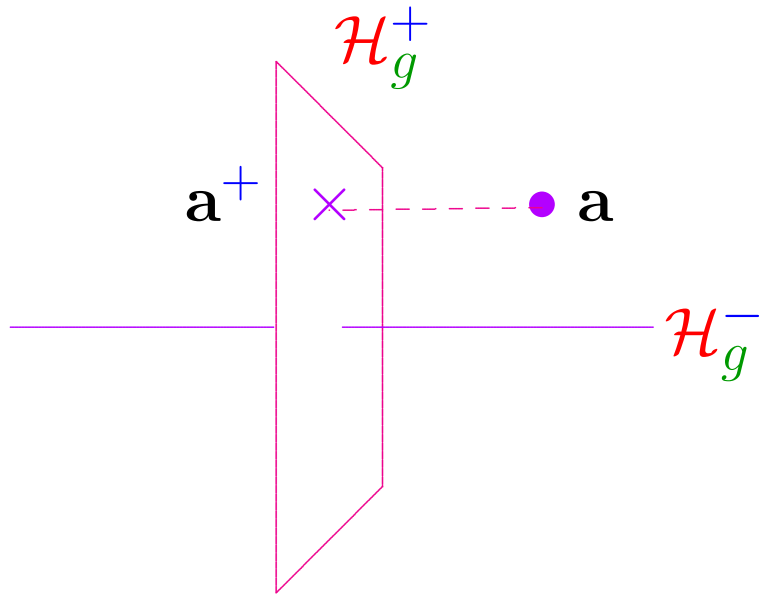

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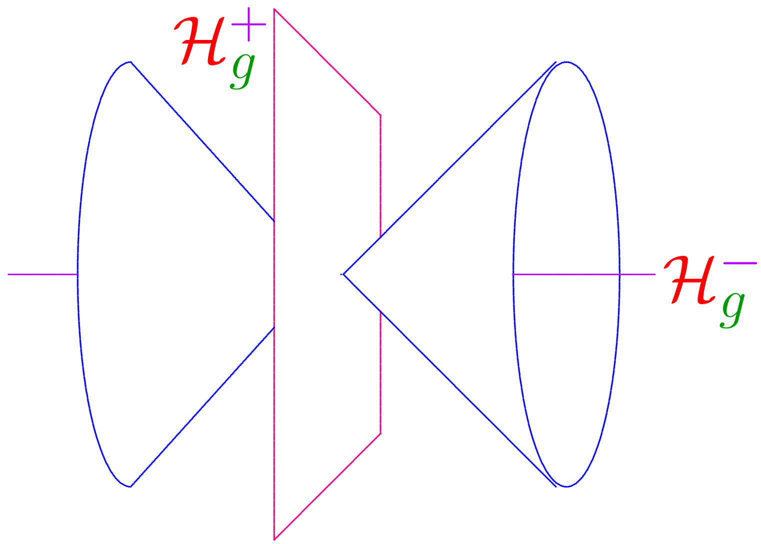

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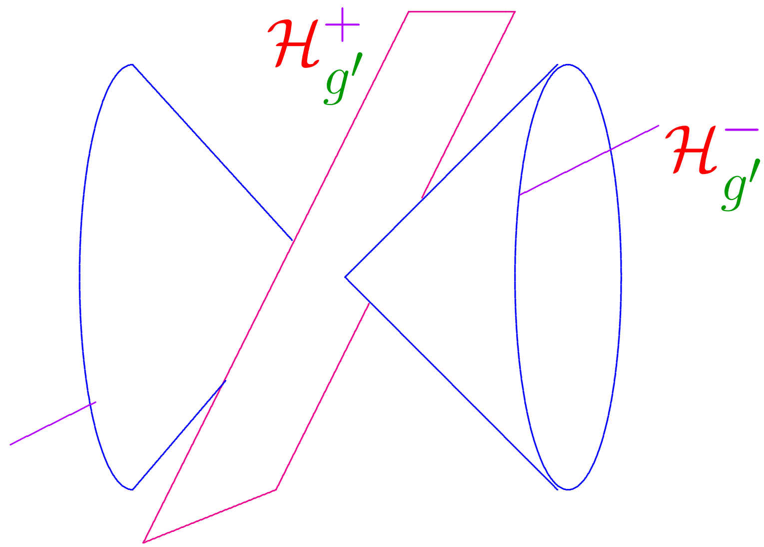

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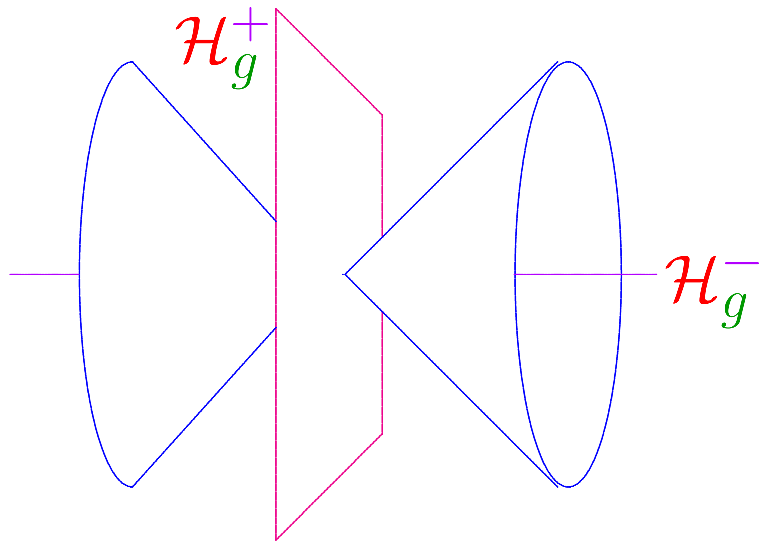

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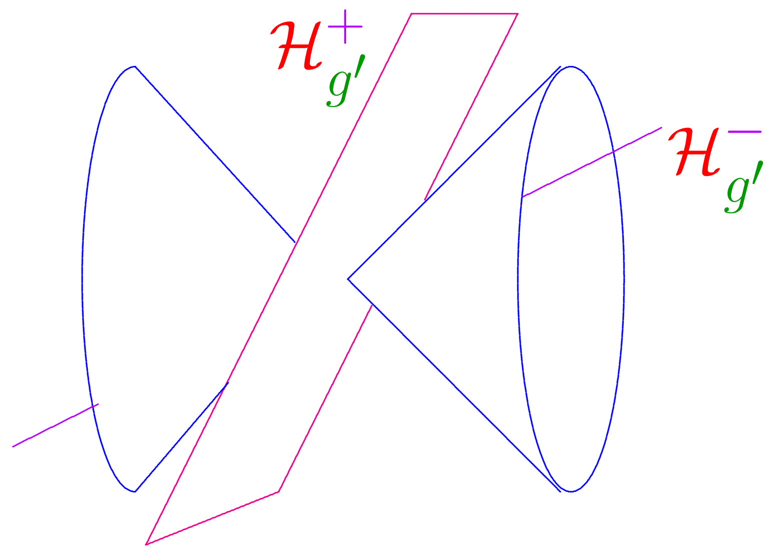

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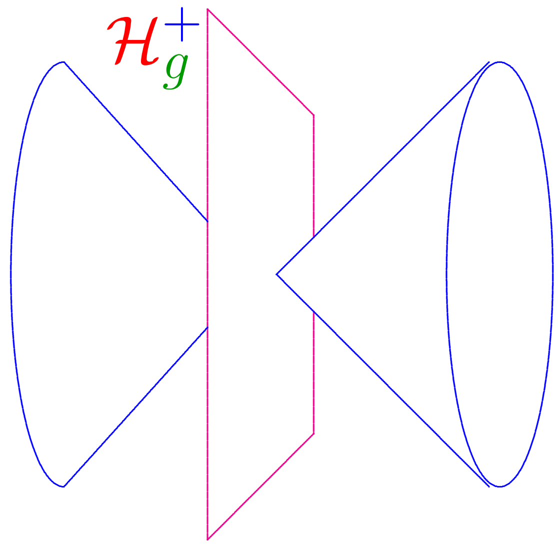

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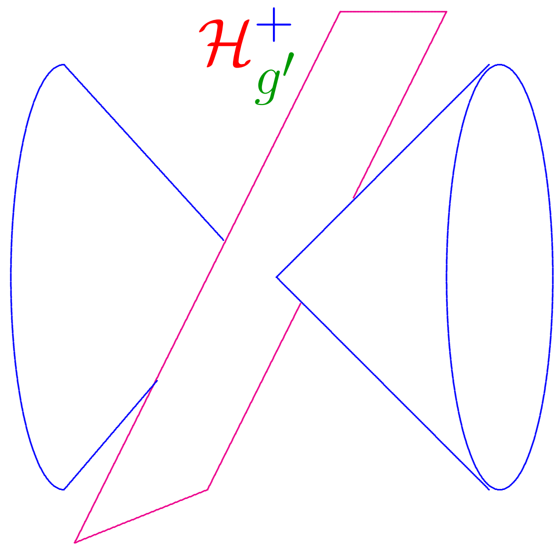

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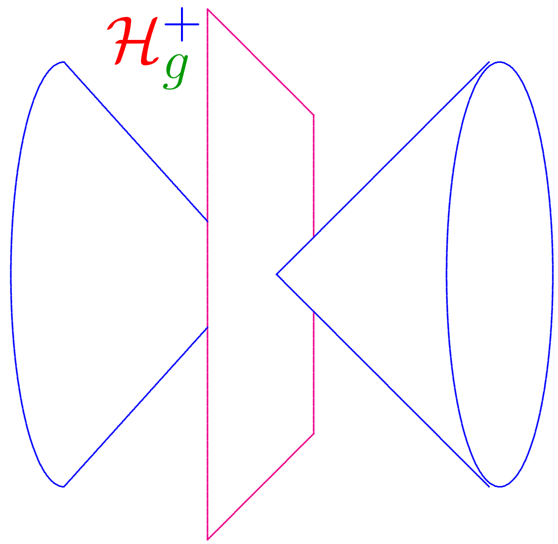

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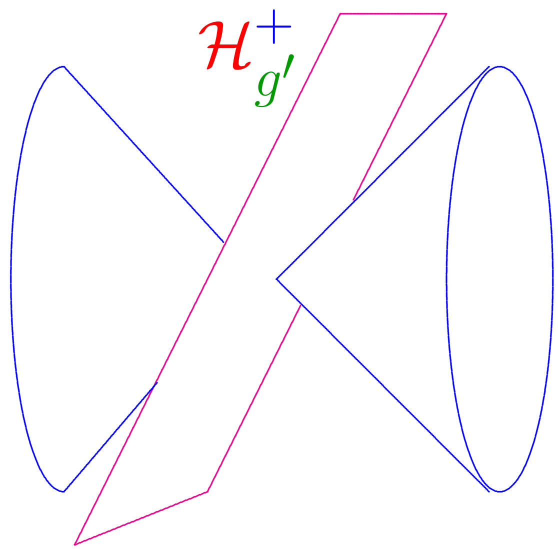

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Proposition.

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Key point: Metrics with $\mathbf{a}_g^+ \neq 0$ are dense.

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Corollary. *Let* X *be a smooth compact oriented 4-manifold with* $b_+ \geq 2$*, and let* $M = X \# k\overline{\mathbb{CP}}_2$ *for some* $k \geq 1$*. If* M *admits a mock-monopole class, then neither* M *nor* X *can admit metrics of positive scalar curvature.*

Proposition. *Let N be a compact oriented connected prime 3-manifold with $b_1(N) \geq 2$ that carries a taut foliation. Set $X = N \times S^1$, and equip $M = X \# k\overline{\mathbb{CP}}_2$. Then M carries a mock-monopole class.*

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Idea of the proof is hidden **Kronheimer '99**, without defining the concept or quite proving the estimate we need. His objective is instead to estimate

$$\int_M s^2 d\mu_g \geq \int_M (s_-)^2 d\mu_g.$$

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For known classes of examples, sign of $\mathscr{Y}(M)$ is left unchanged by blowing up.

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However, this **Conjecture** is very difficult, and has only been proved with $b_2(M) \leq 3$.

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So results in this talk prove...

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Vielen Dank an die Organisatoren und an das MFO für diese Einladung zur Teilnahme!