Mass, Scalar Curvature,

Kähler Geometry, and All That

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Joint work with

Joint work with

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University of Maryland

Joint work with

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Fordham University

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Mass in Kähler Geometry

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Comm. Math. Phys. 347 (2016) 621–653.

Simple, natural problem:

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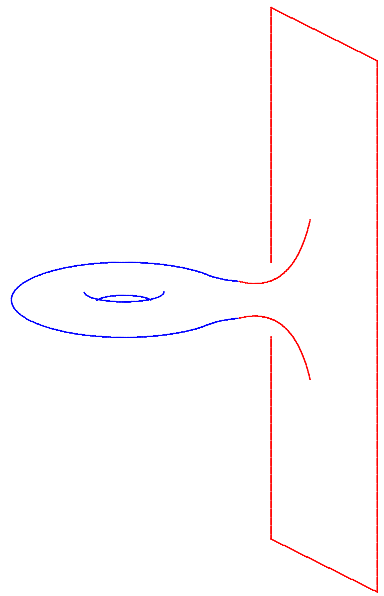
(M^n, g) complete non-compact Riemannian n-manifold.



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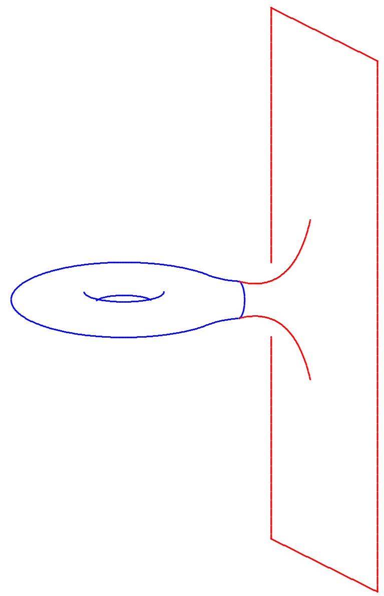
Suppose $\exists K \subset M$ compact



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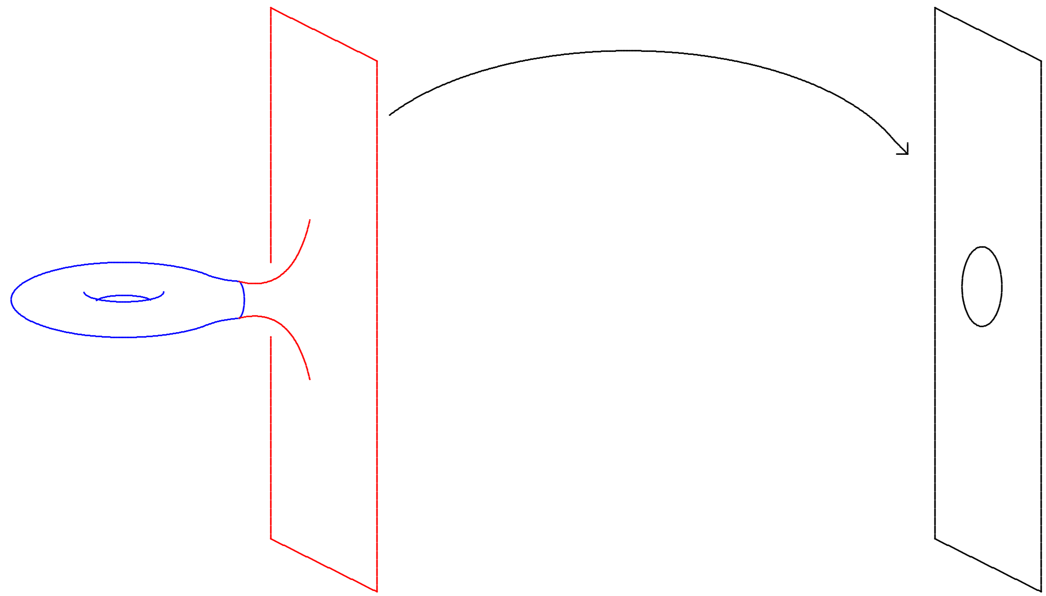

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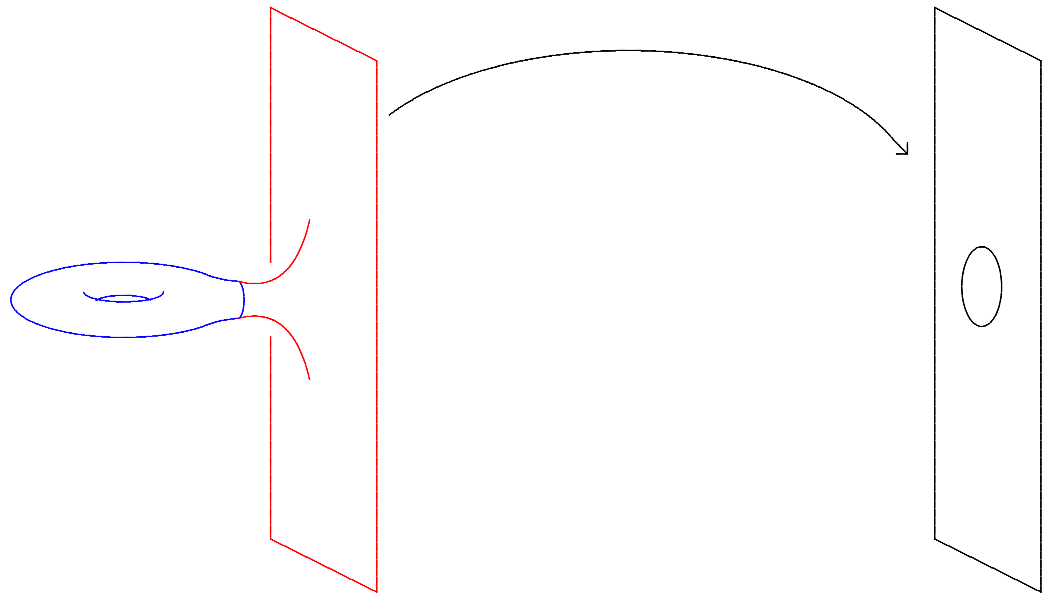

and an isometry $M - K \to \mathbb{R}^n - D^n$. (Euclidean)

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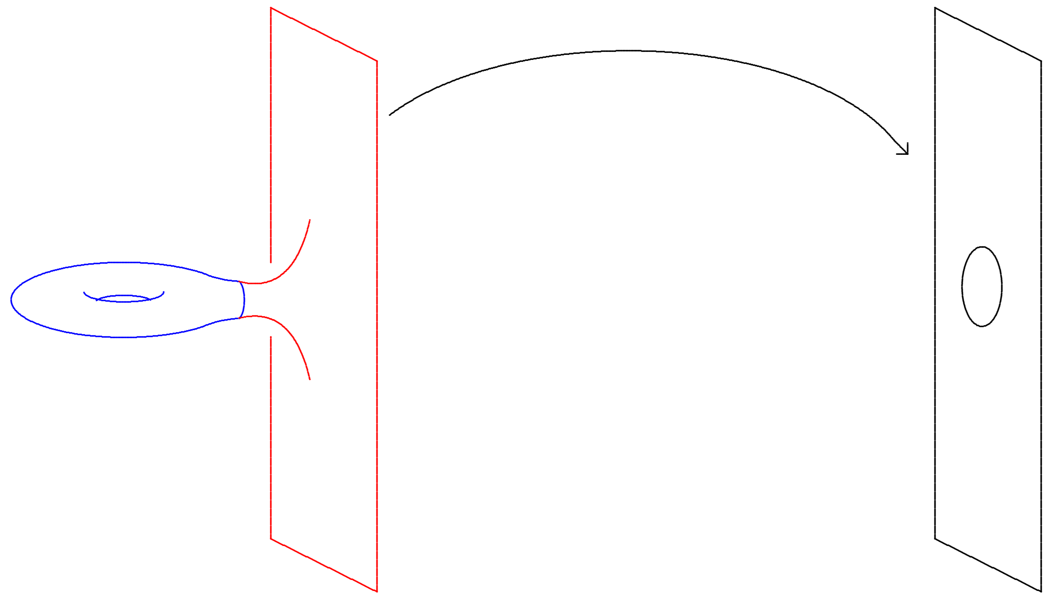

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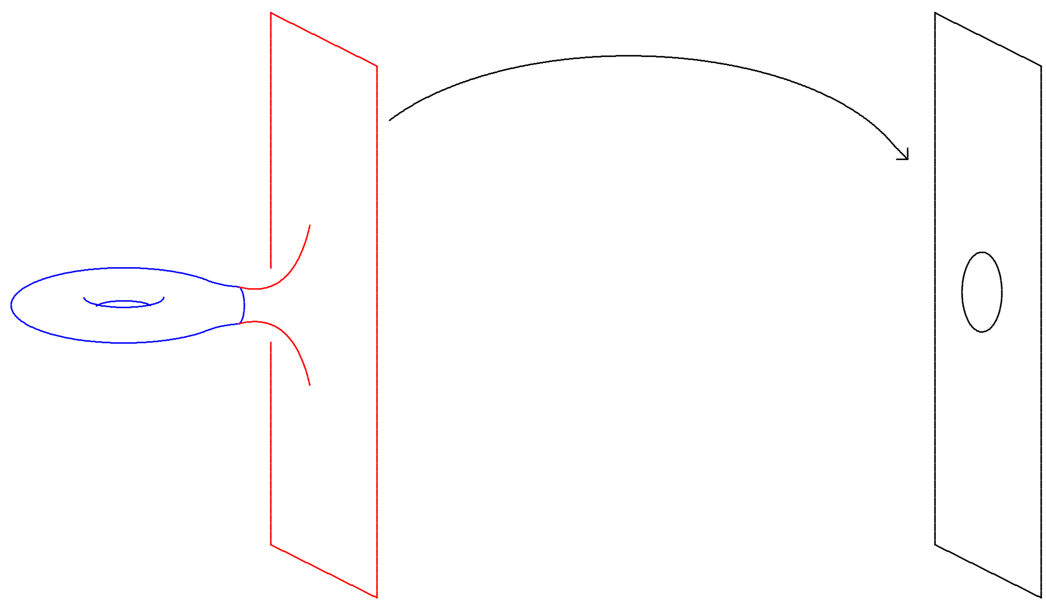

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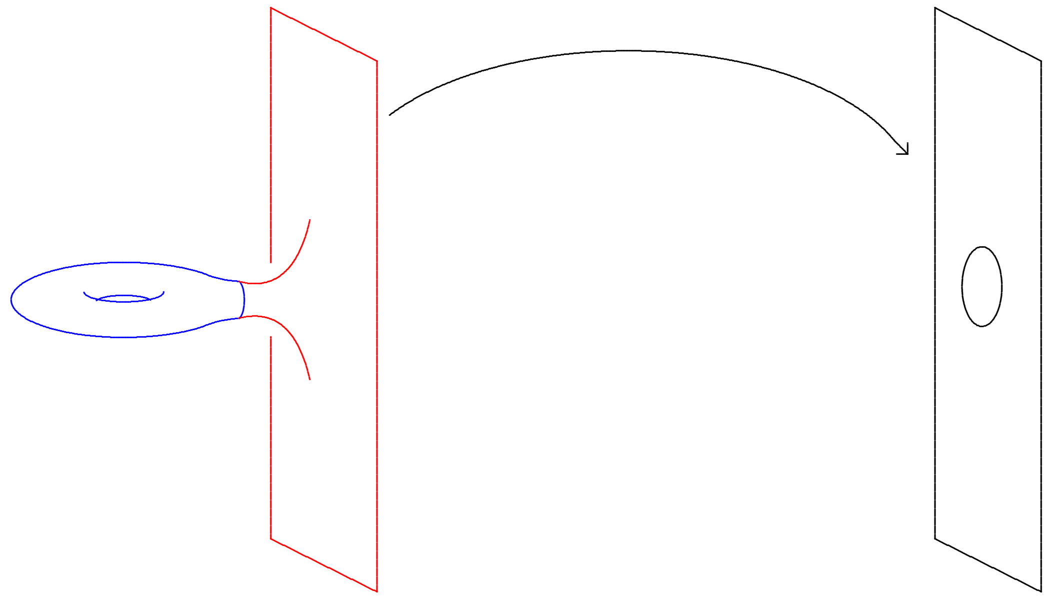

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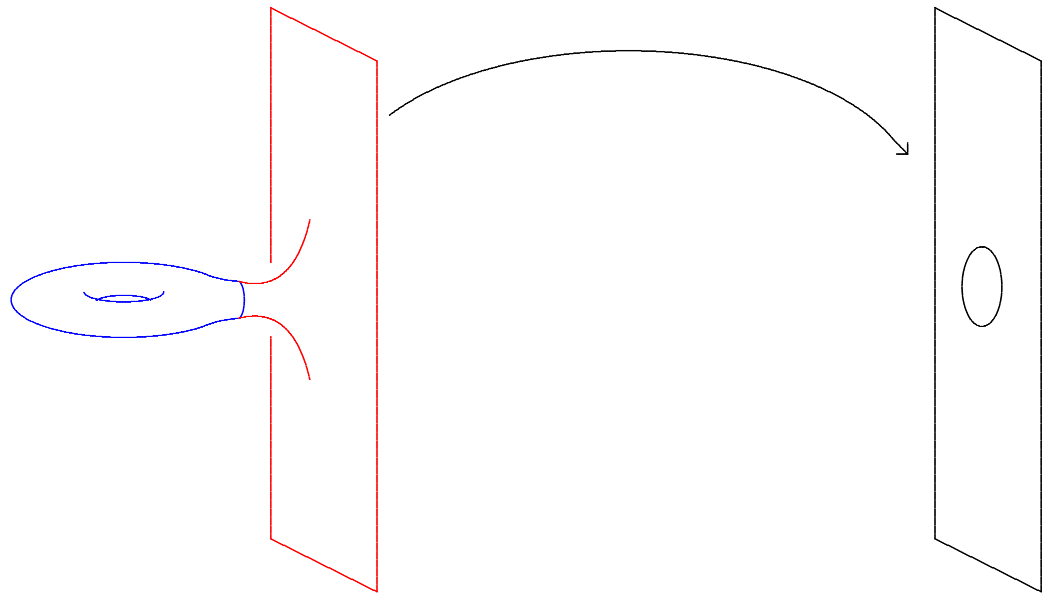

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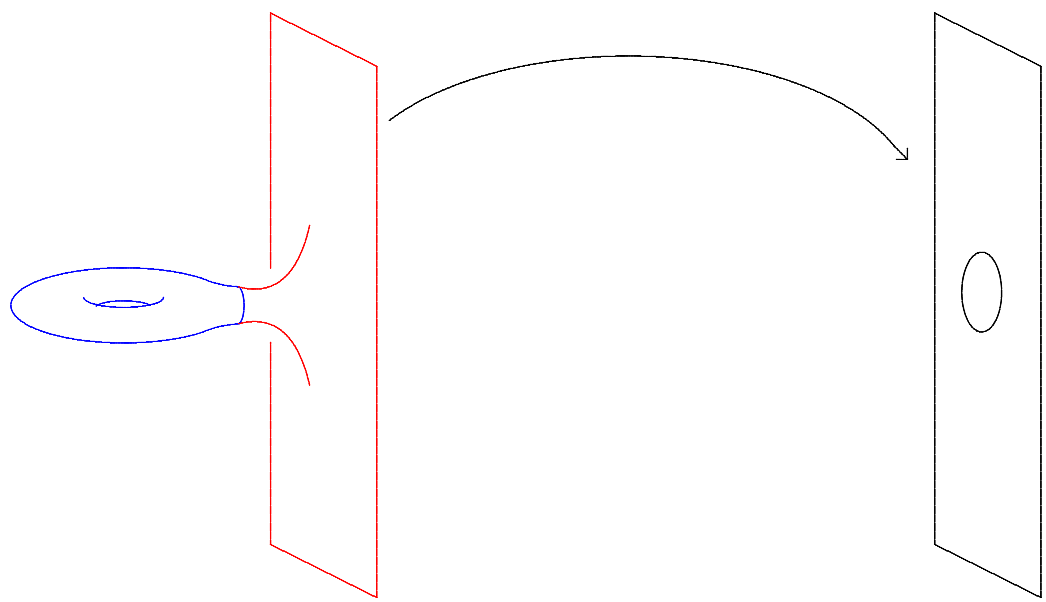

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Distance comparison...

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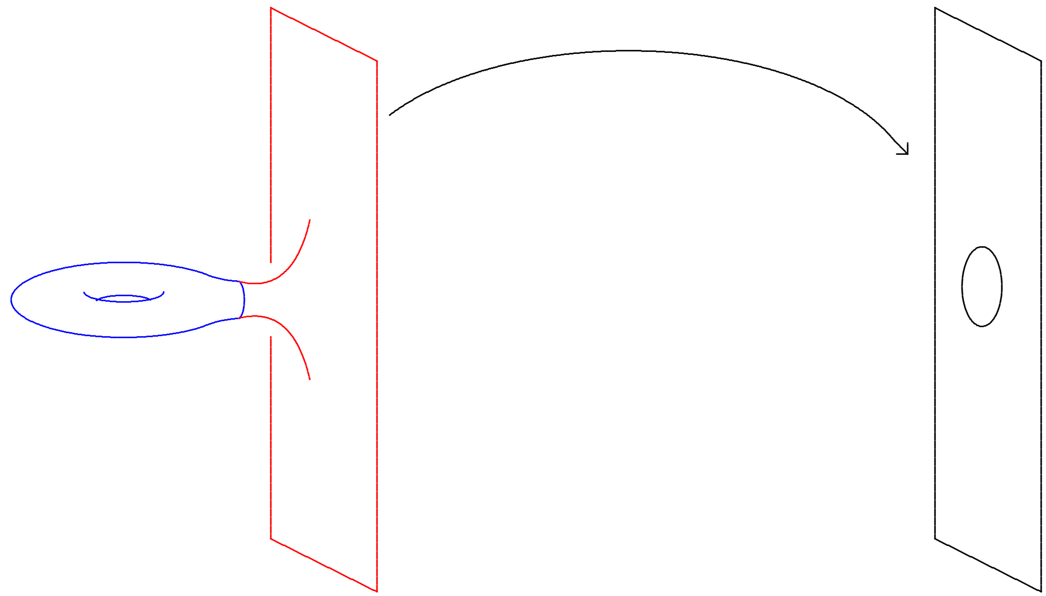

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If M has Ricci curvature ≥ 0, is it flat?

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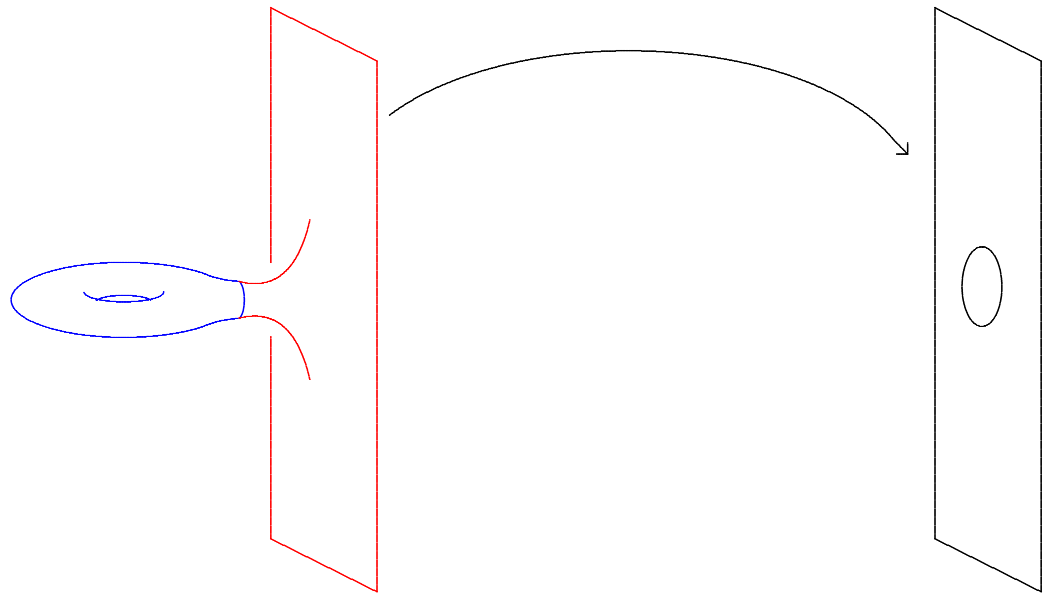

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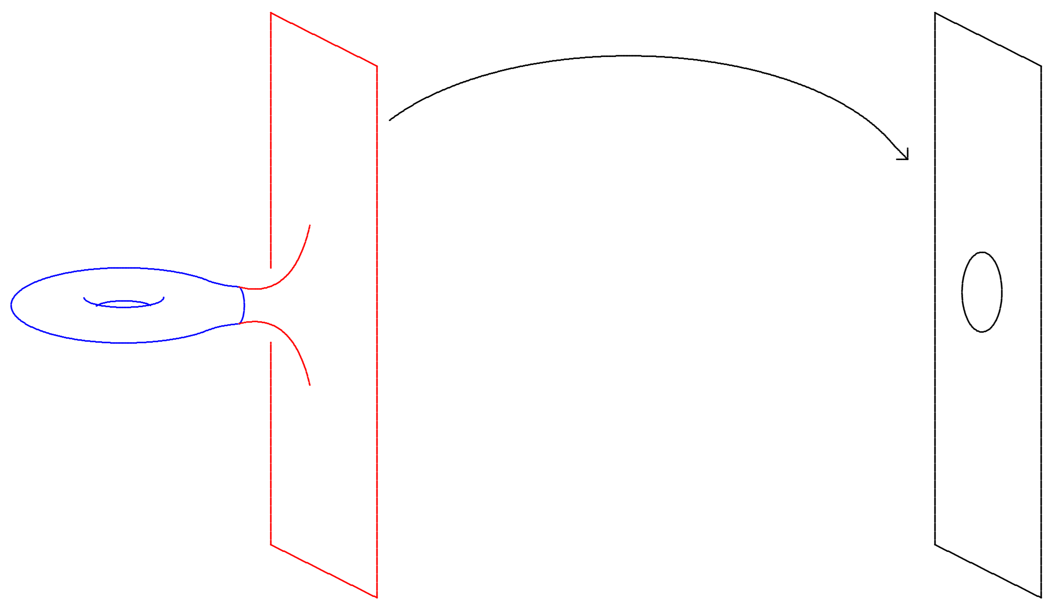

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Volume comparison...

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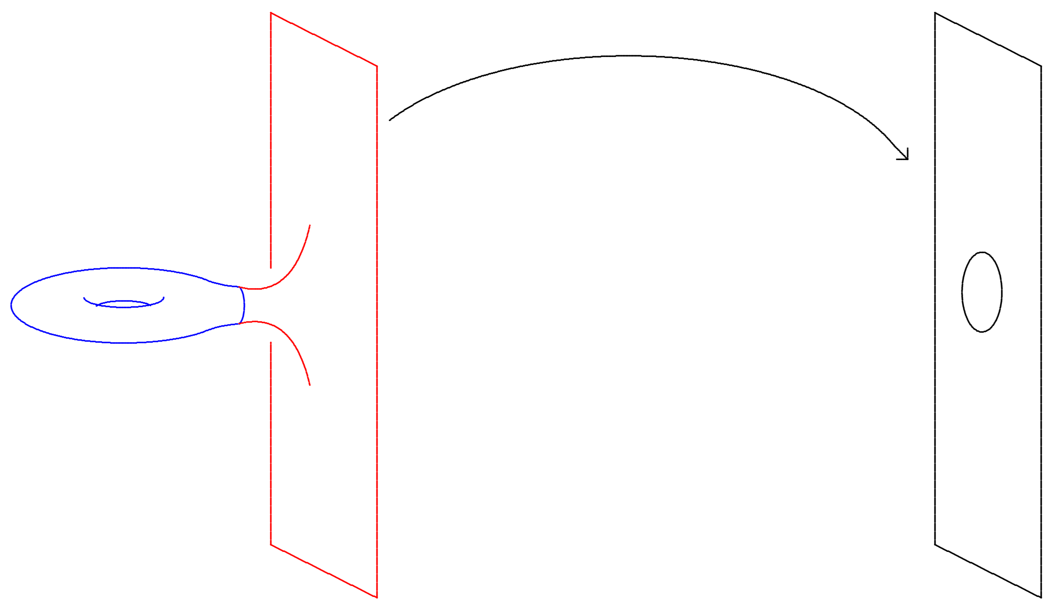

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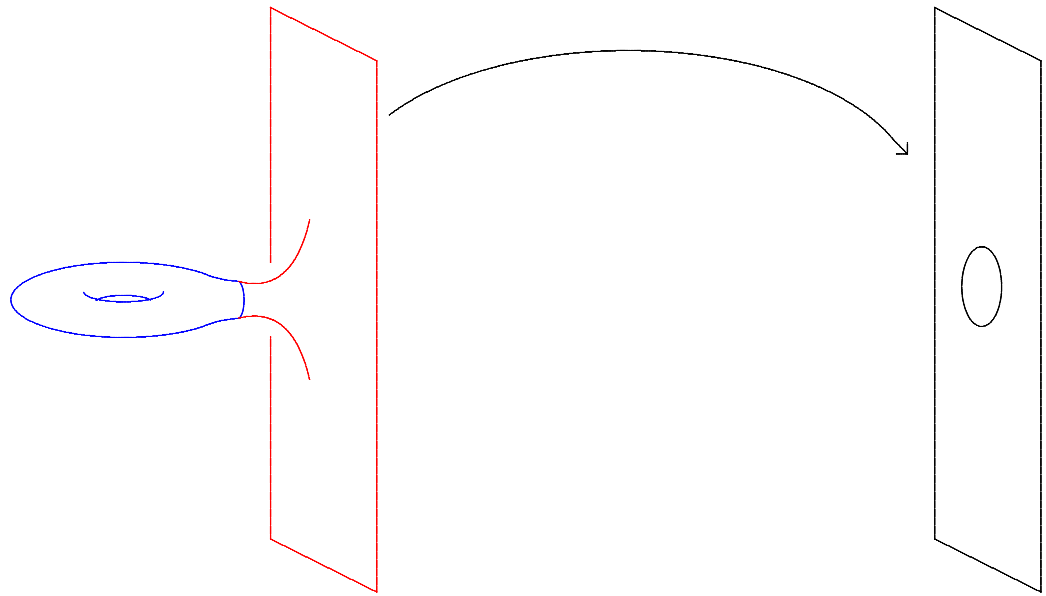

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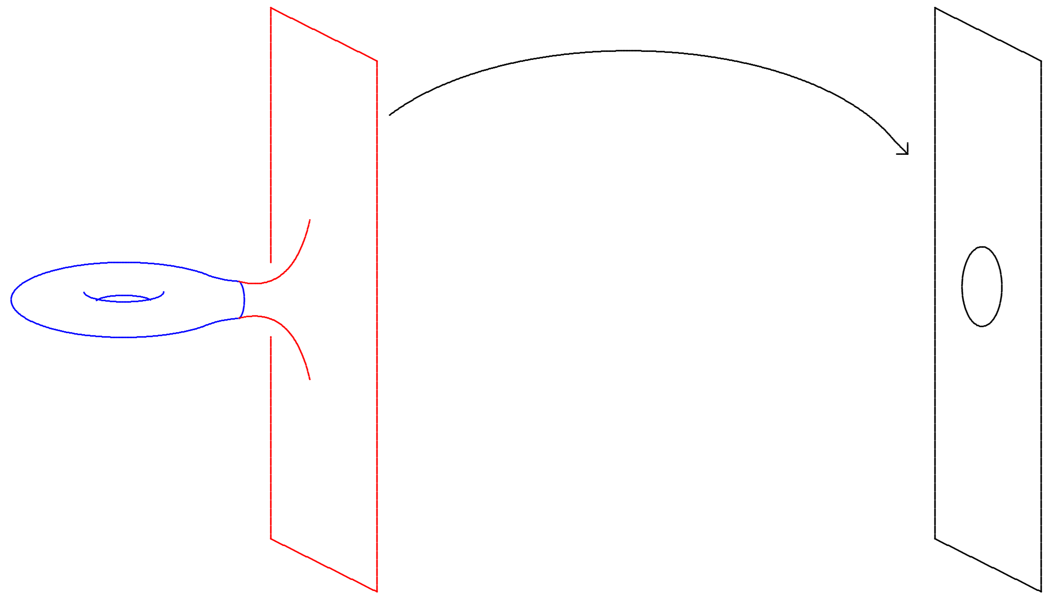

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If M has scalar curvature ≥ 0, is it flat? Yes...
at least if M spin, or if $n \leq 7$...

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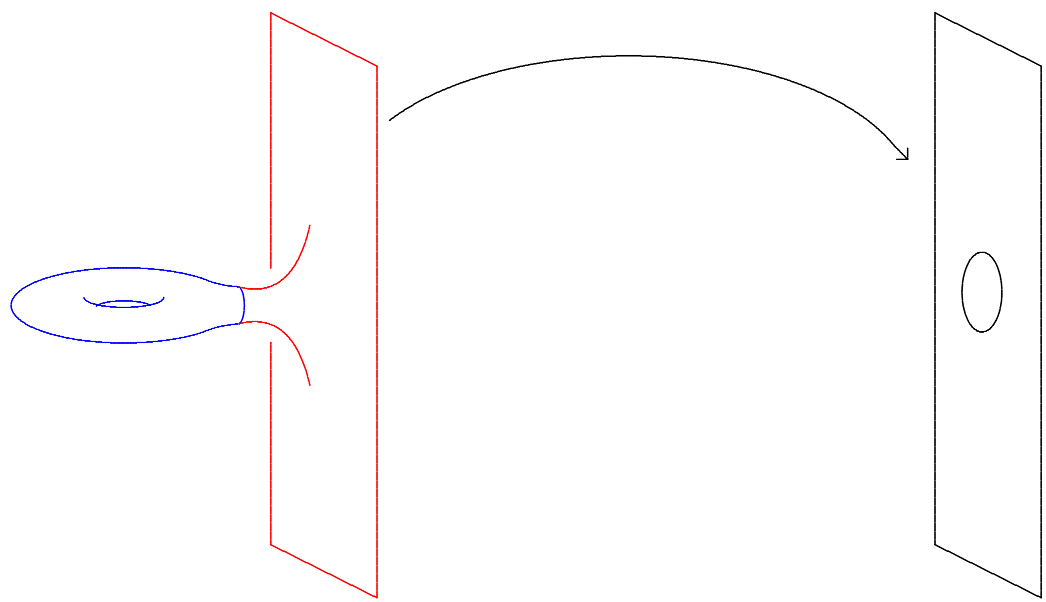

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Involves new idea from physics!

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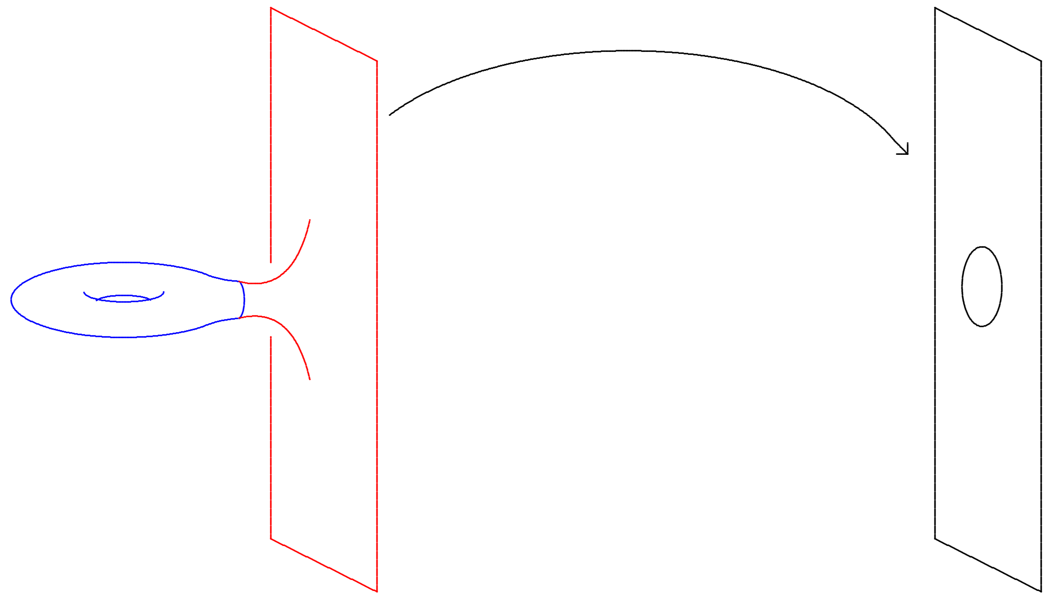

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If M has scalar curvature ≥ 0, is it flat?

Get result even with appropriate fall-off to Euclidean. . .

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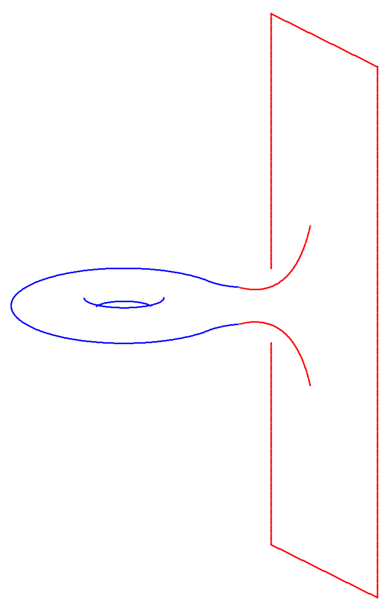

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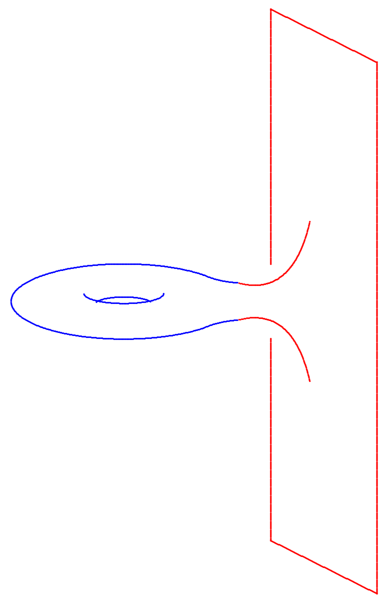

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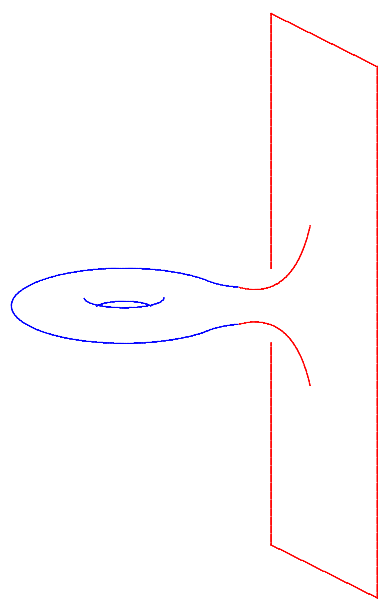

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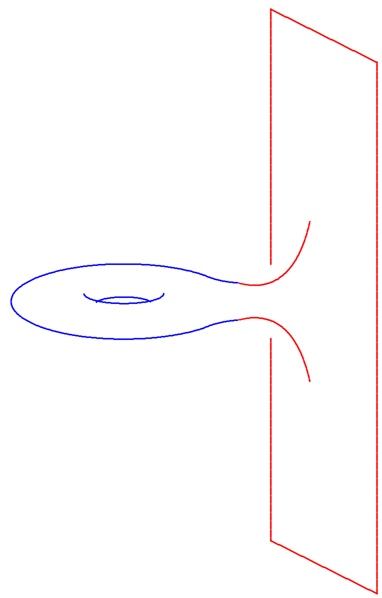

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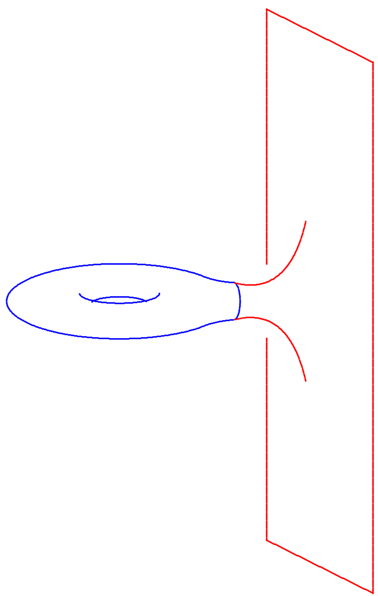

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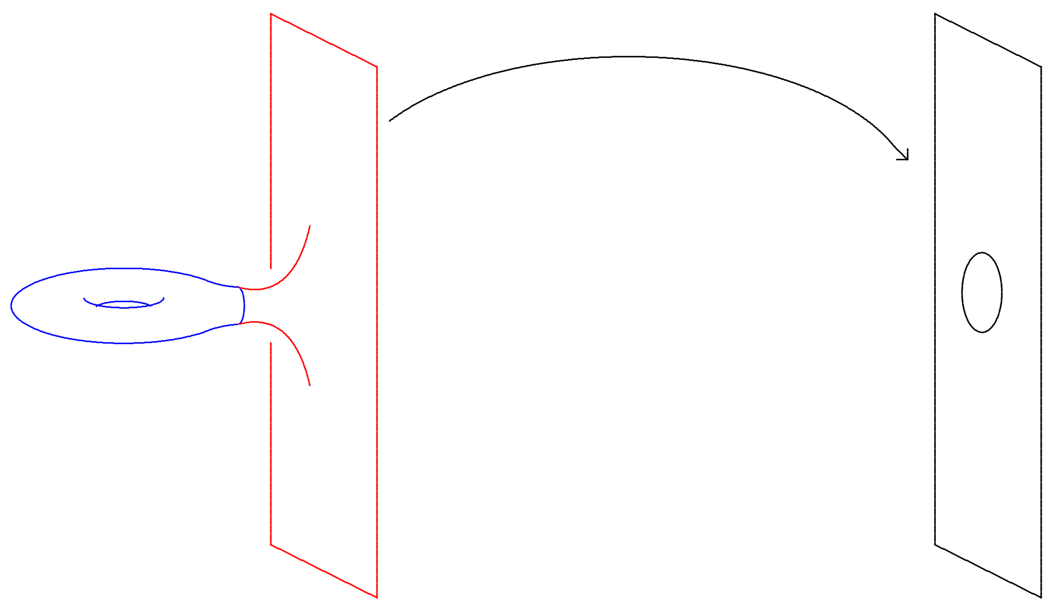

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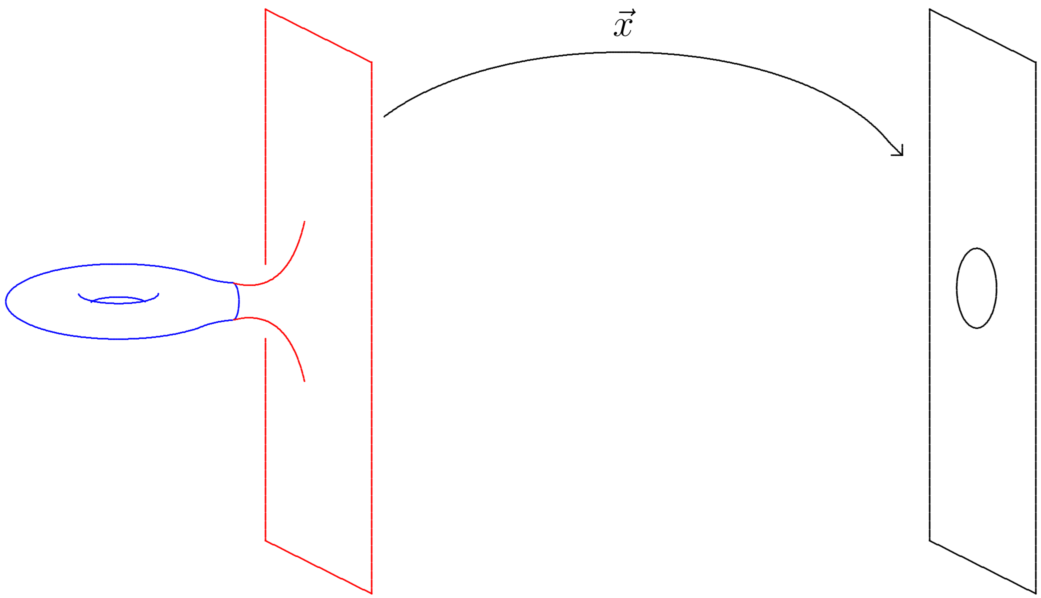

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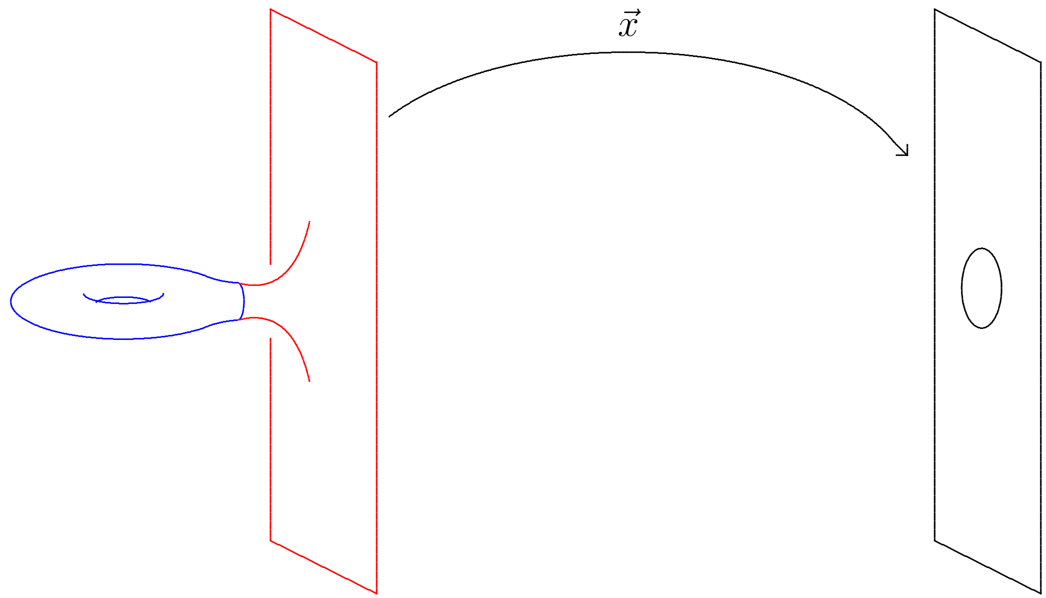

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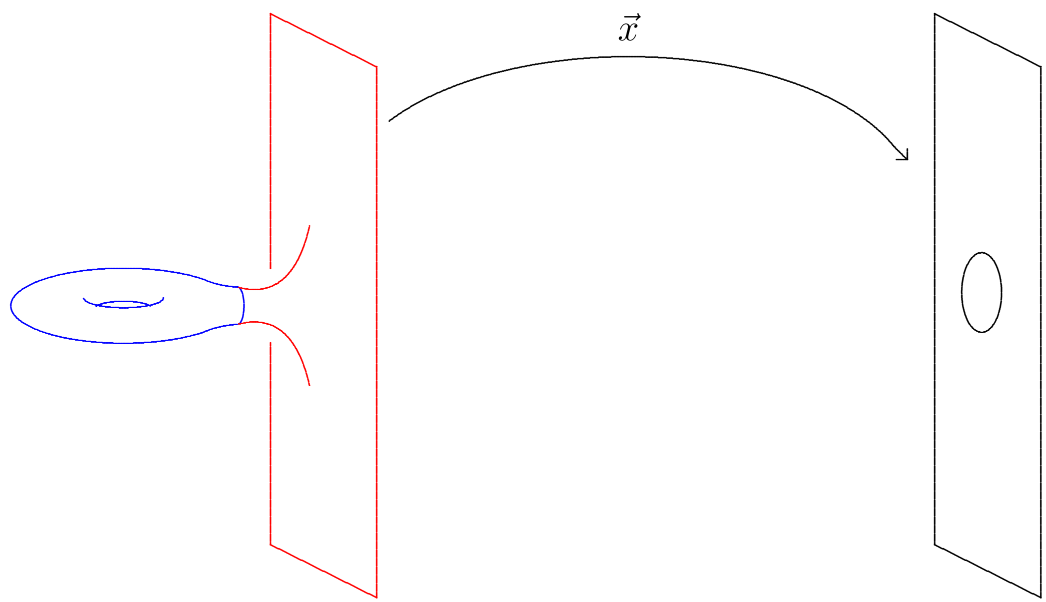

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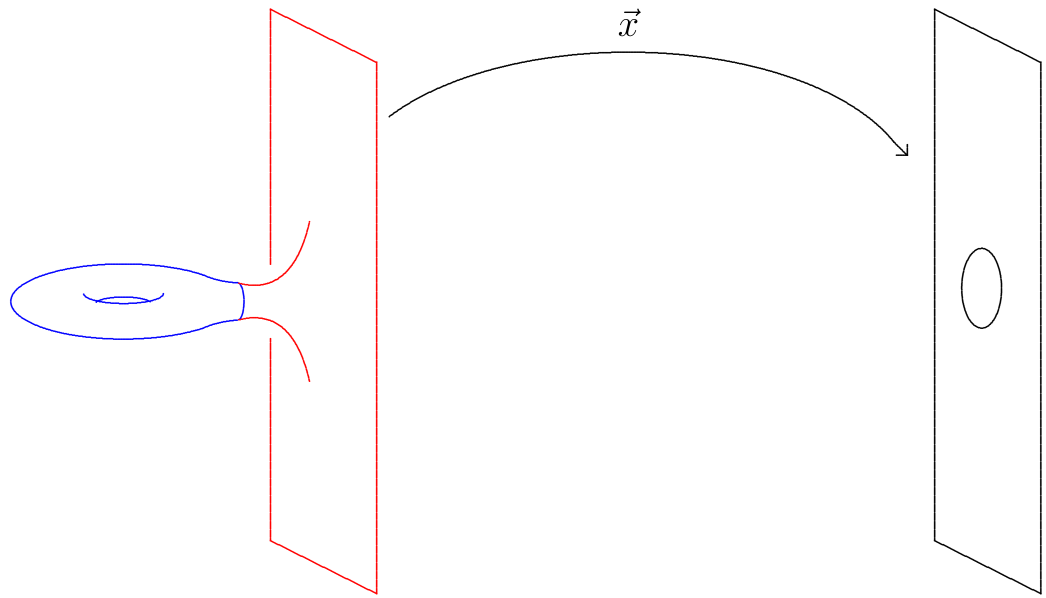

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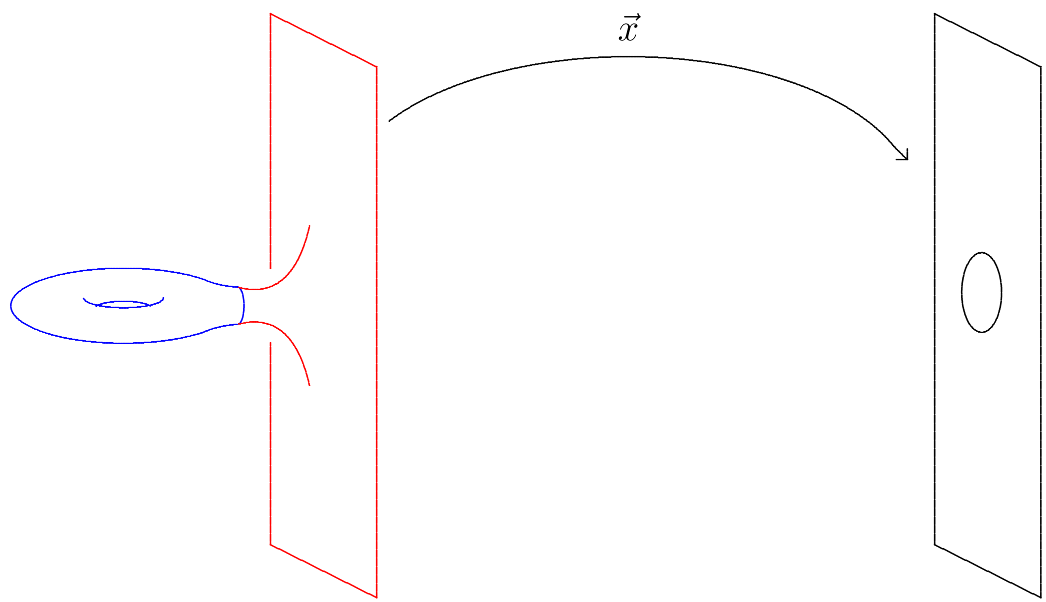

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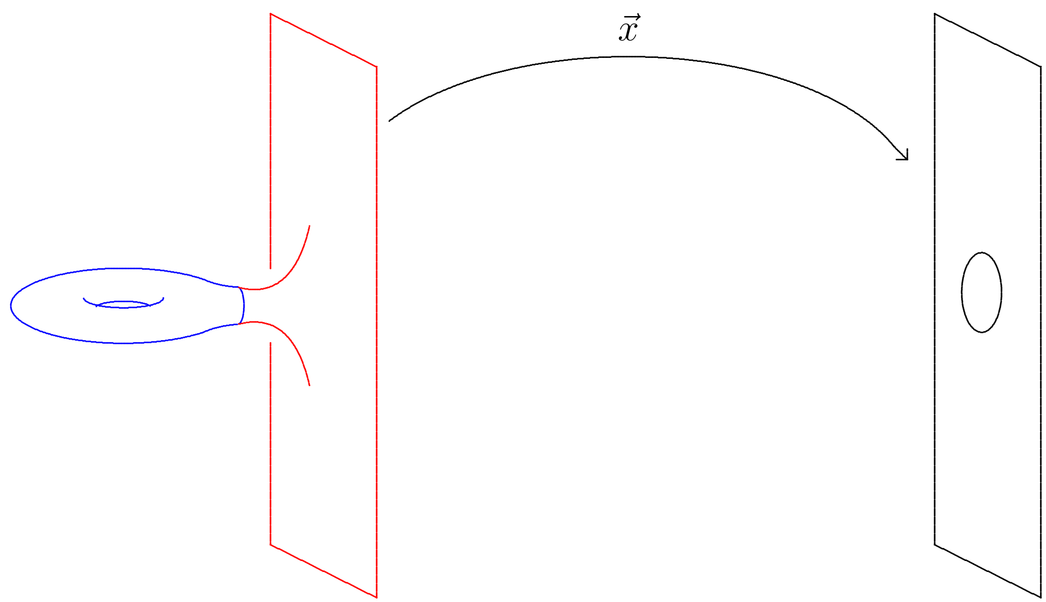

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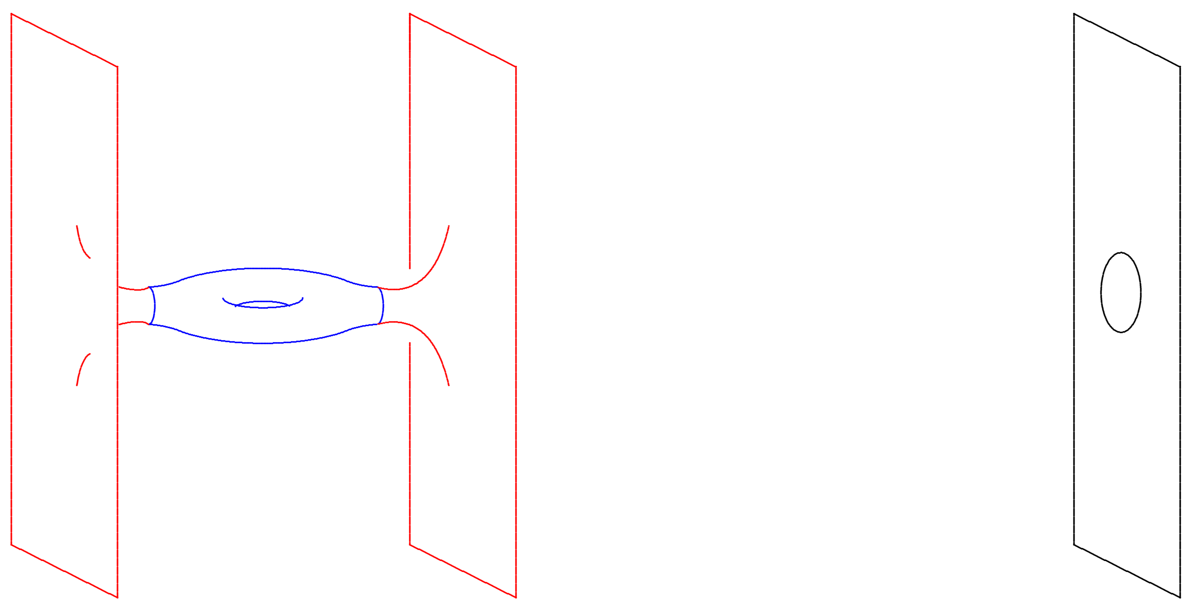

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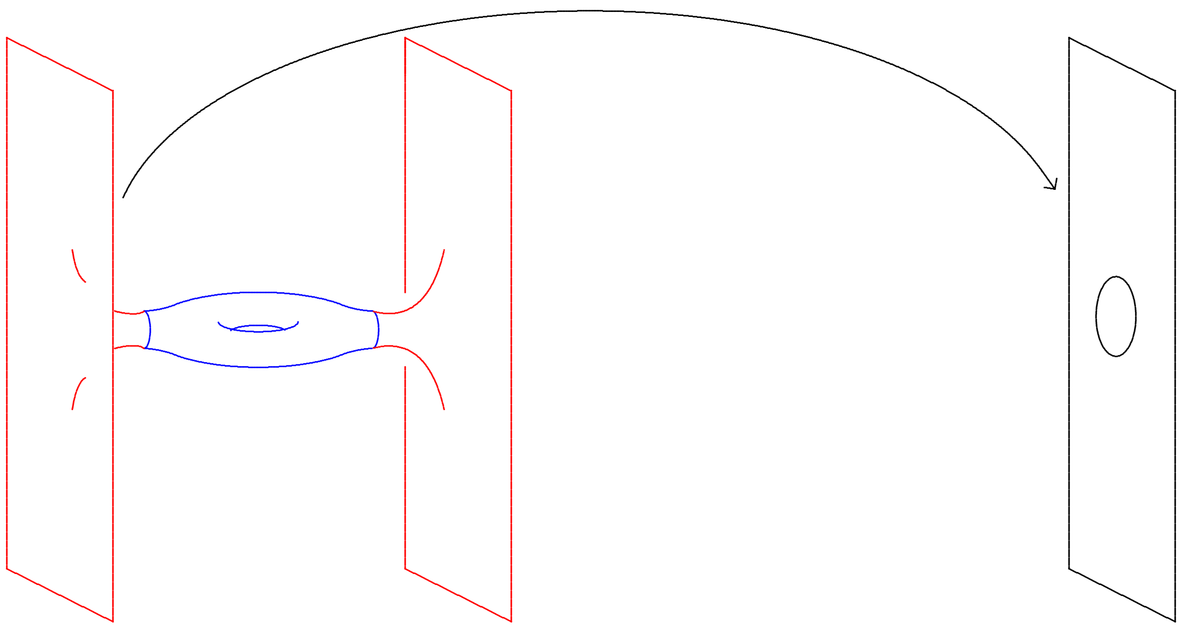

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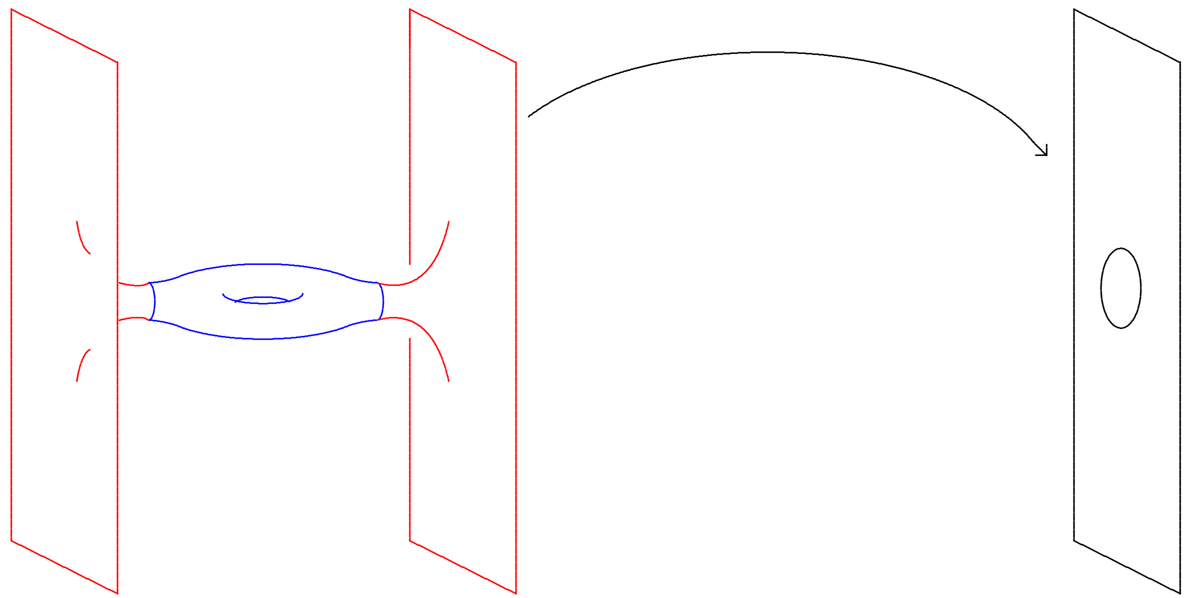

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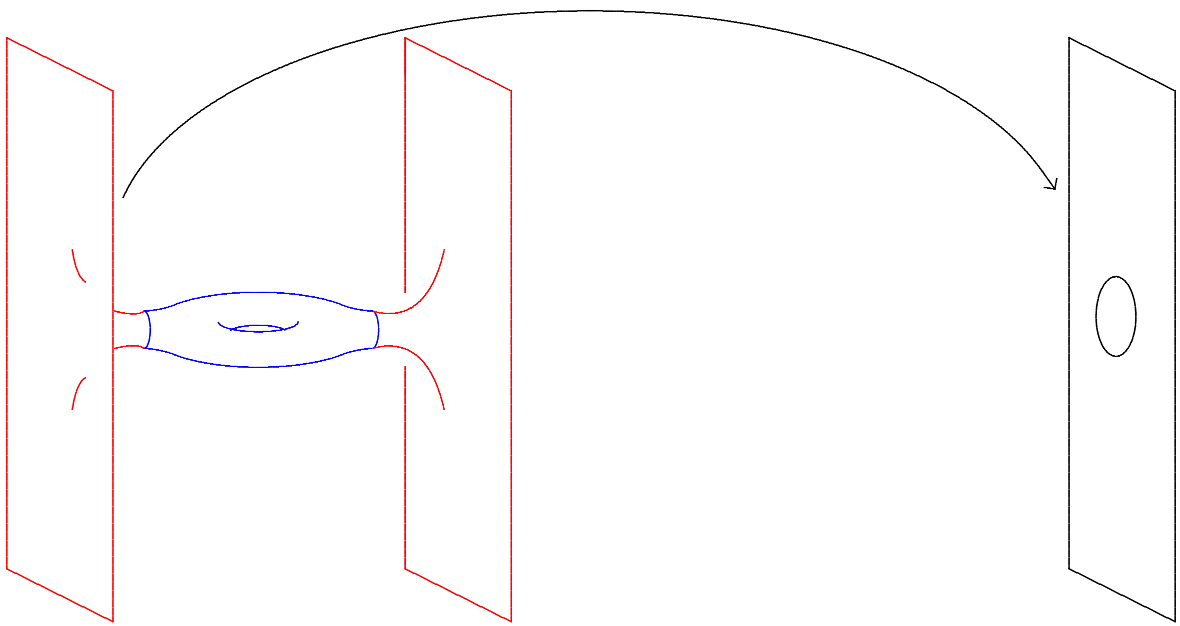

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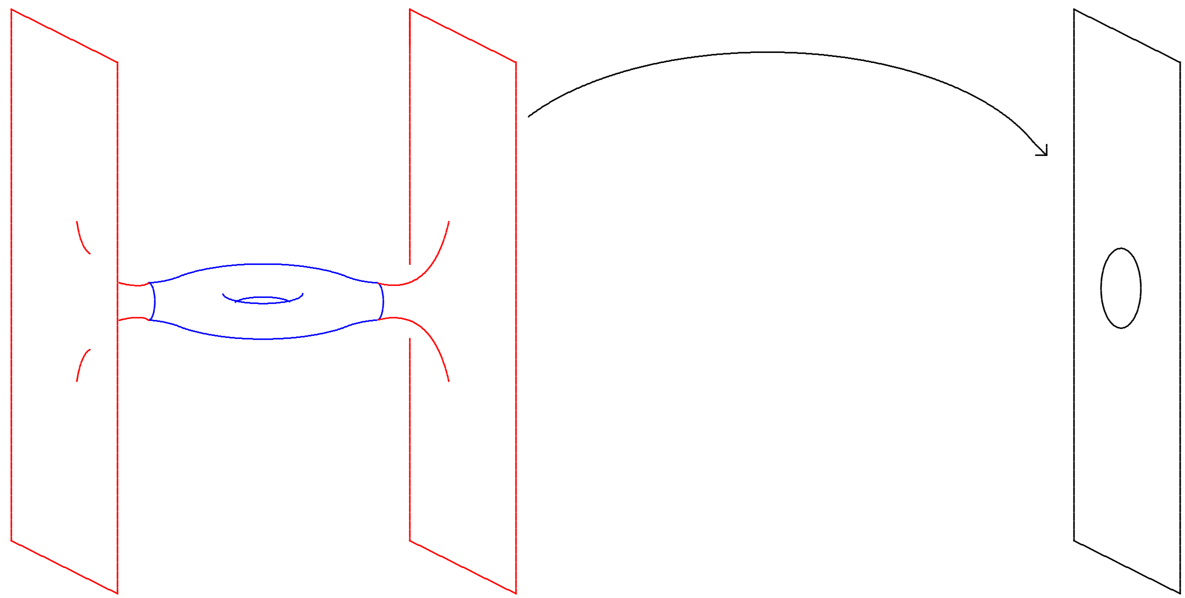

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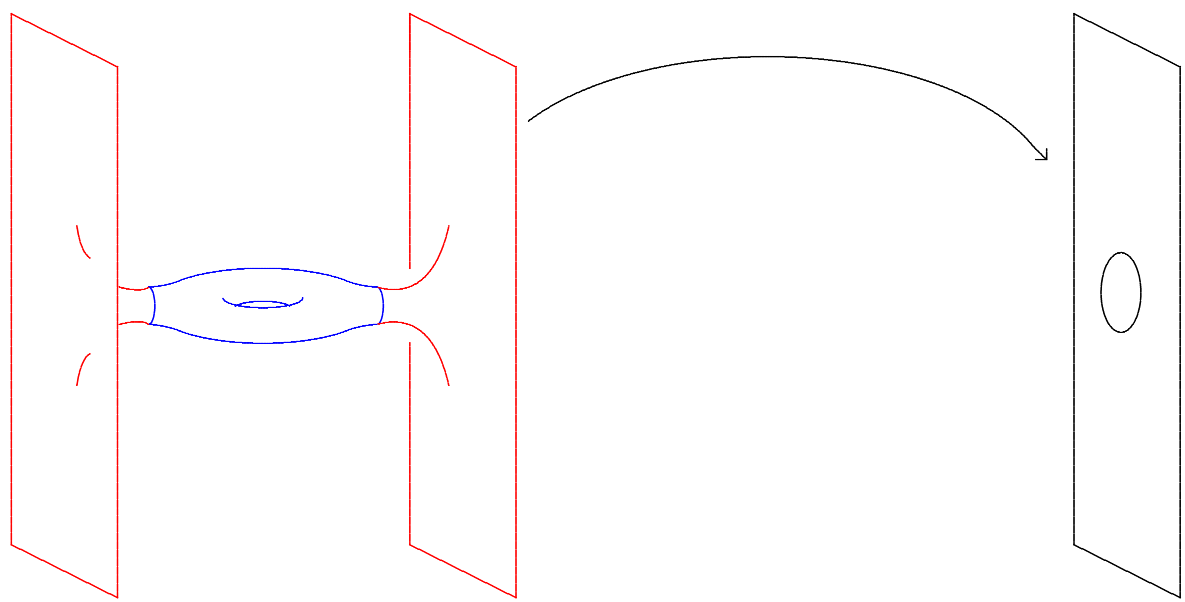

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Seems to depend on choice of coordinates!

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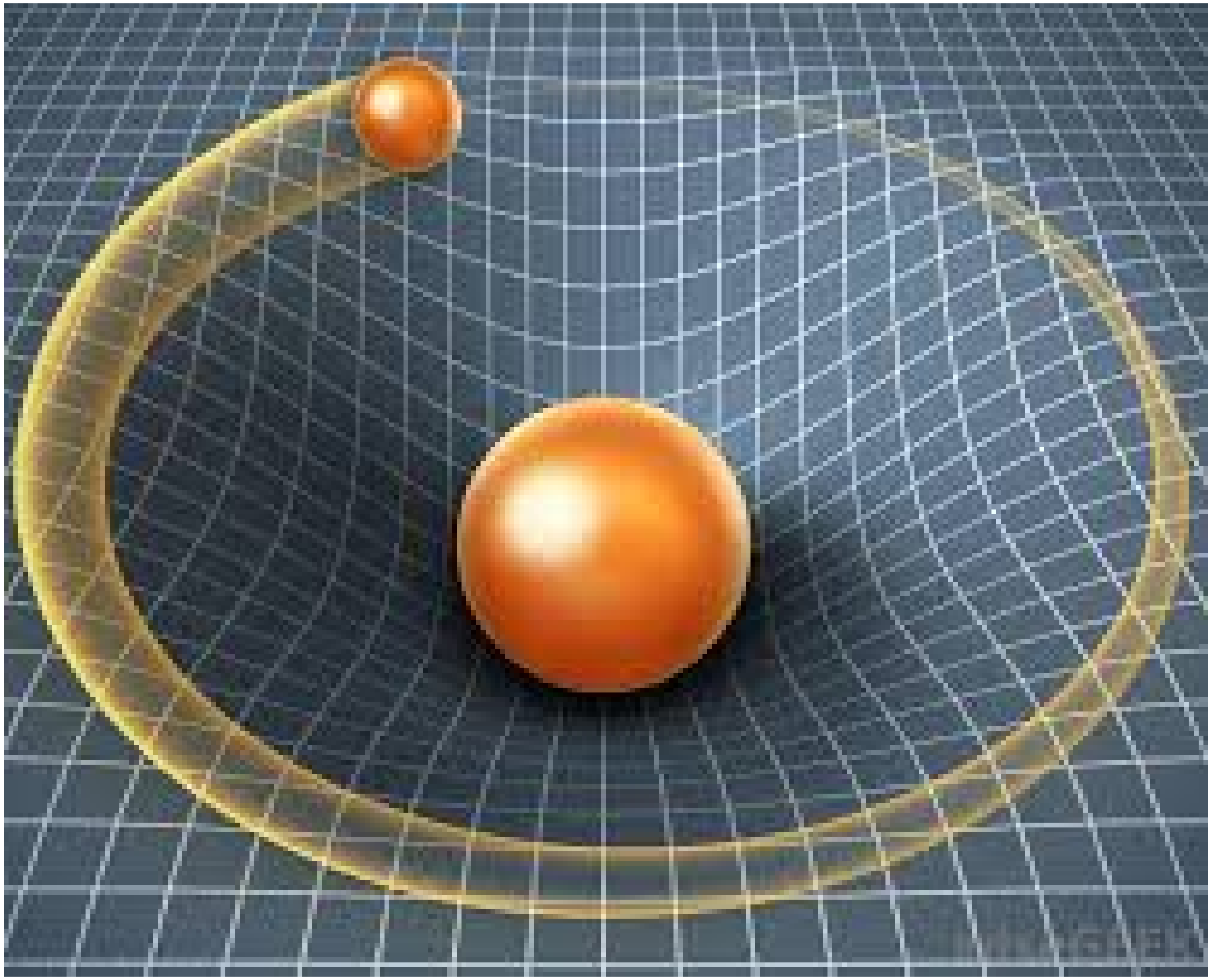

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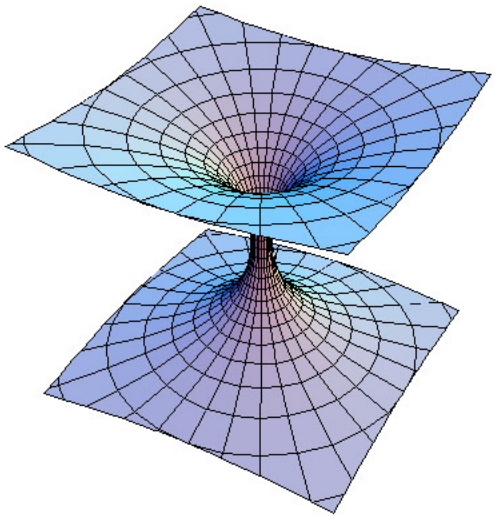

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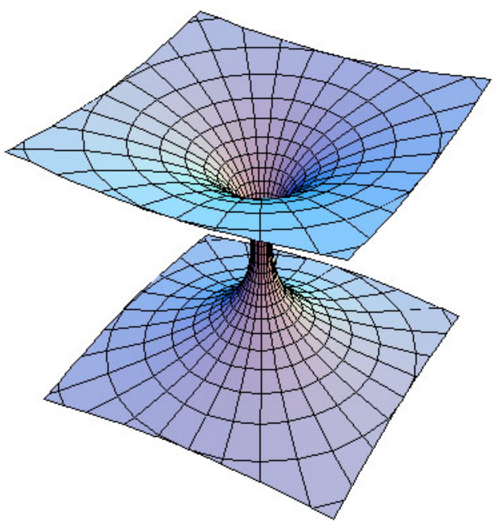

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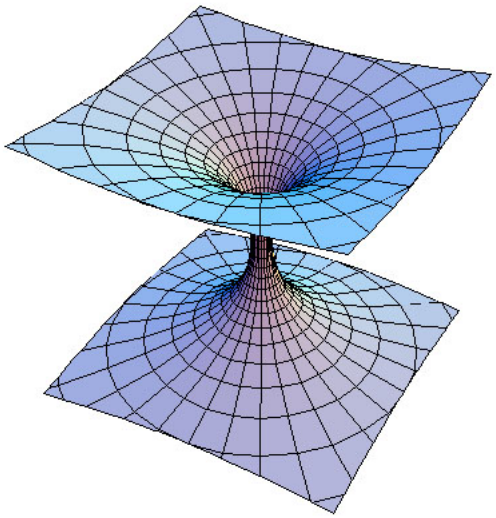

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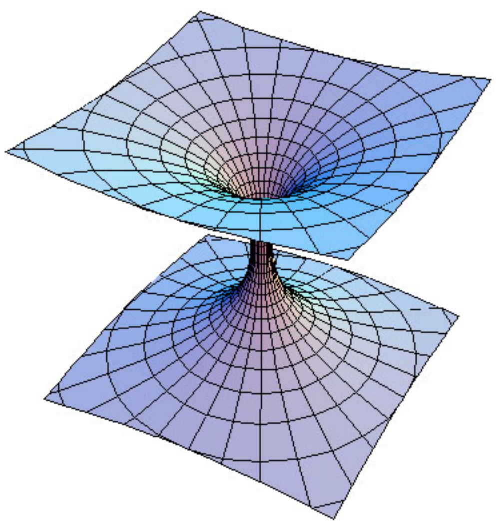

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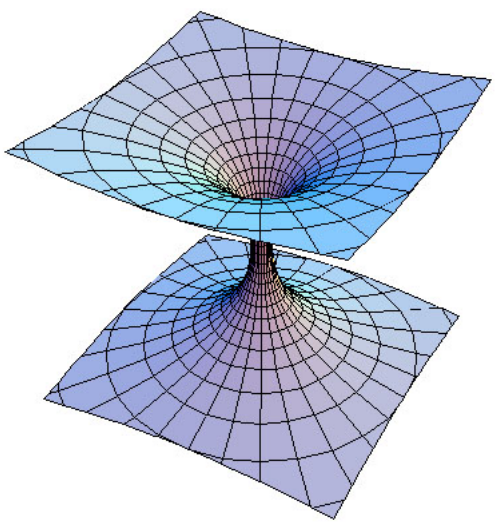

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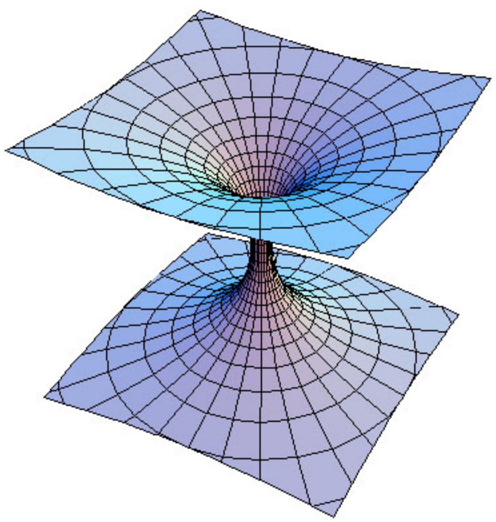

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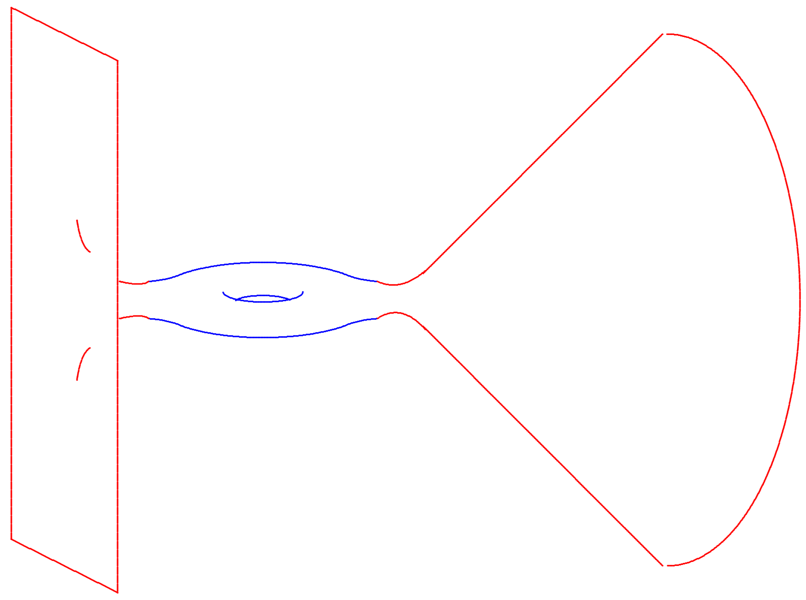
A Generalization. . .

Definition. *Complete, non-compact* n*-manifold* (M^n, g) *is* asymptotically locally Euclidean



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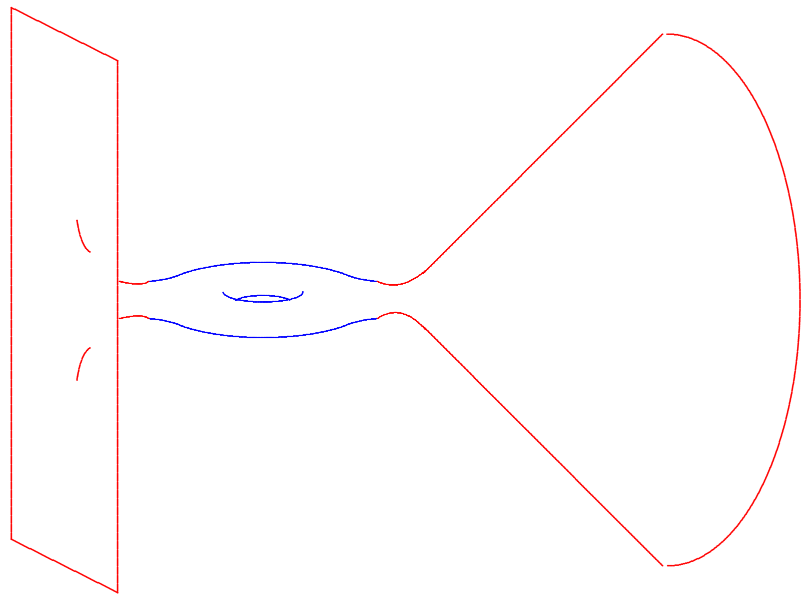

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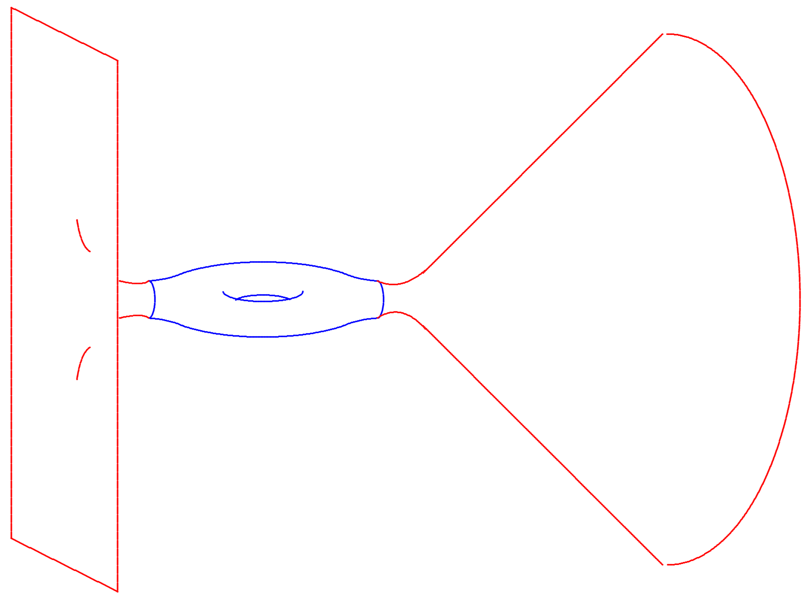

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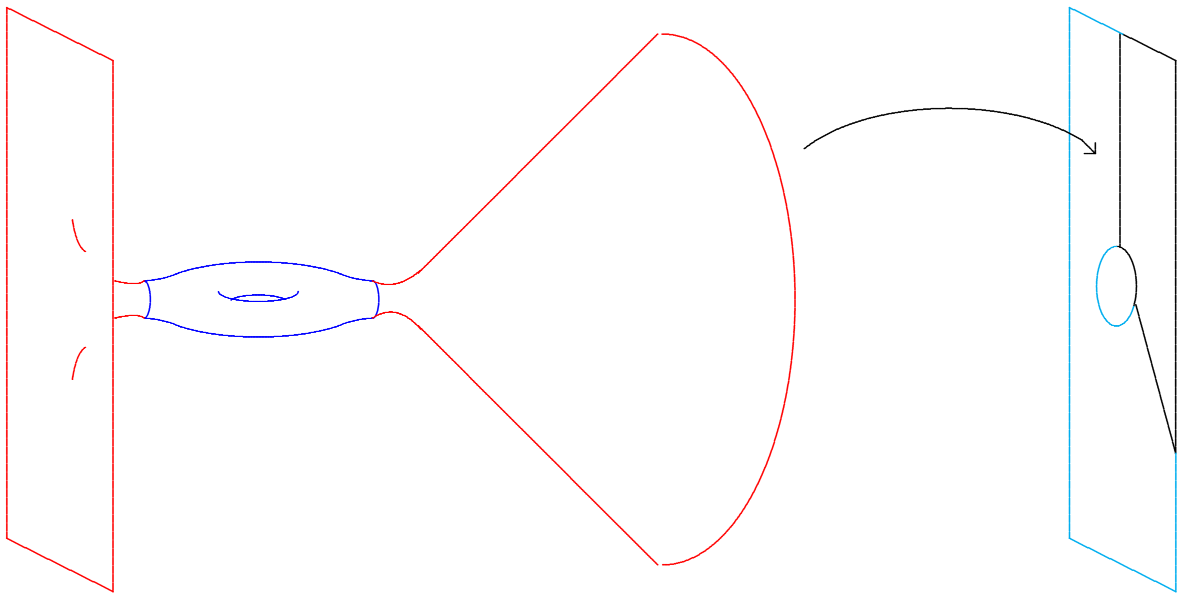

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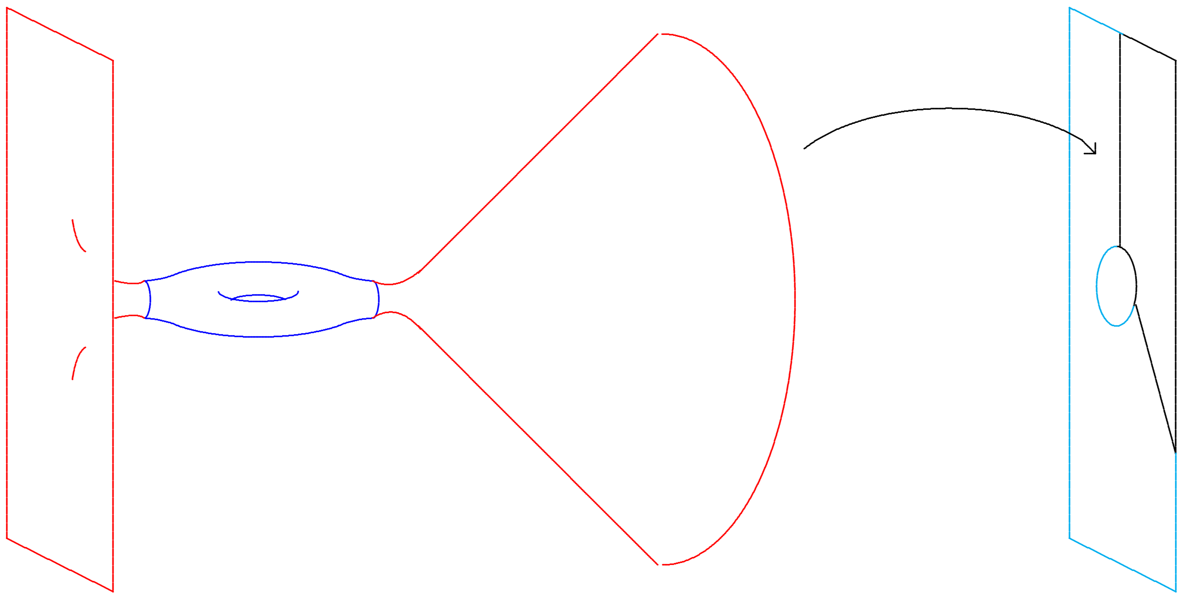

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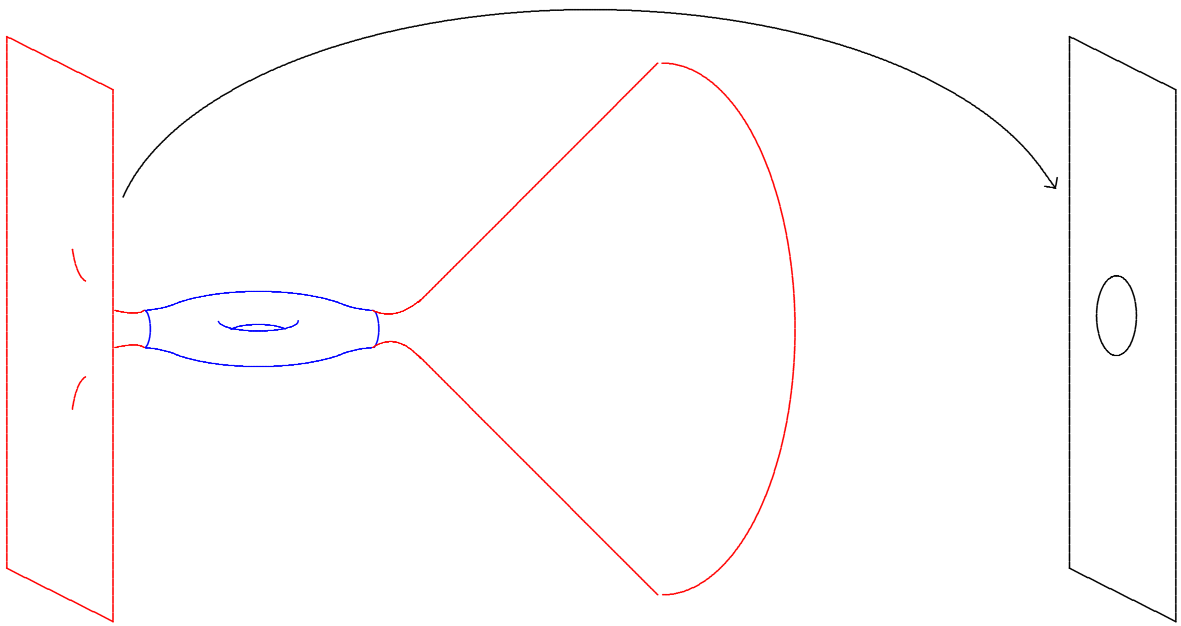

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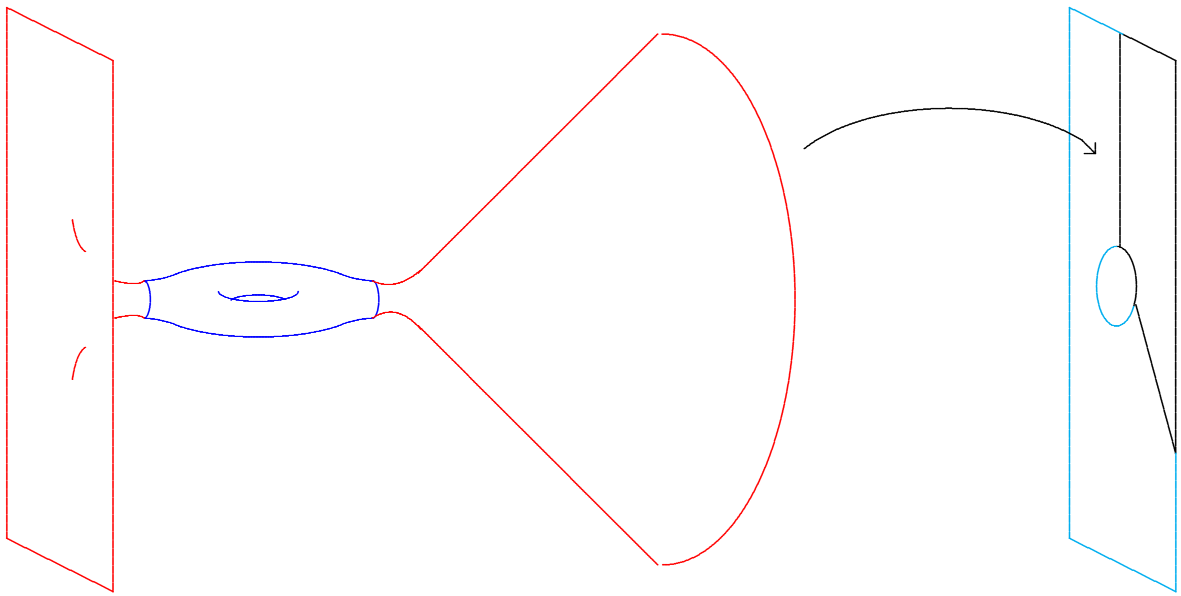

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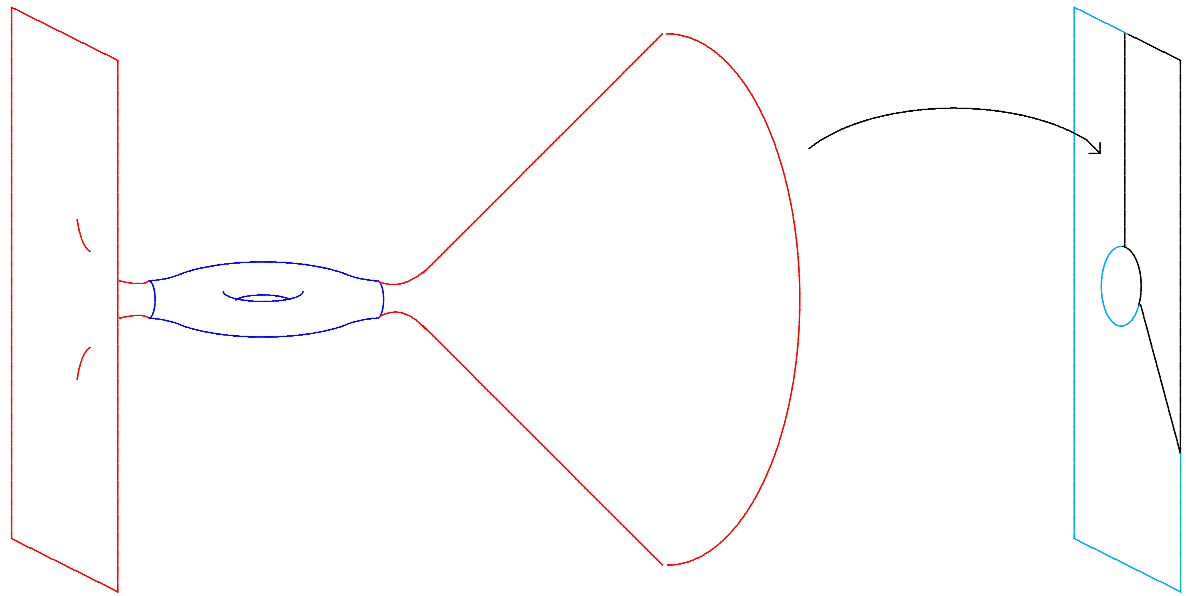

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Why consider ALE spaces?

Key examples:

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Gravitational Instantons

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Bubbling modes for sequences of Einstein metrics

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By contrast, any Ricci-flat AE manifold must be flat, by the Bishop-Gromov inequality…

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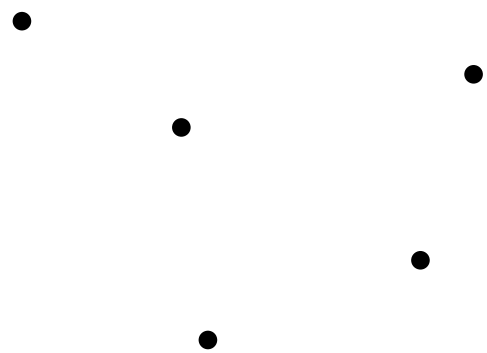

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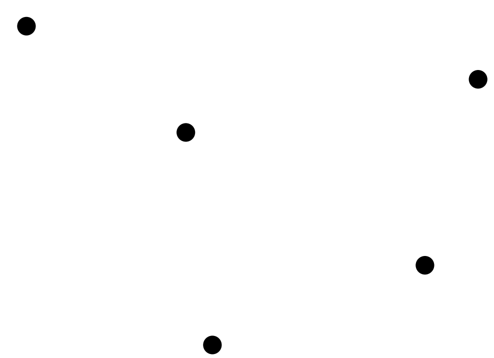

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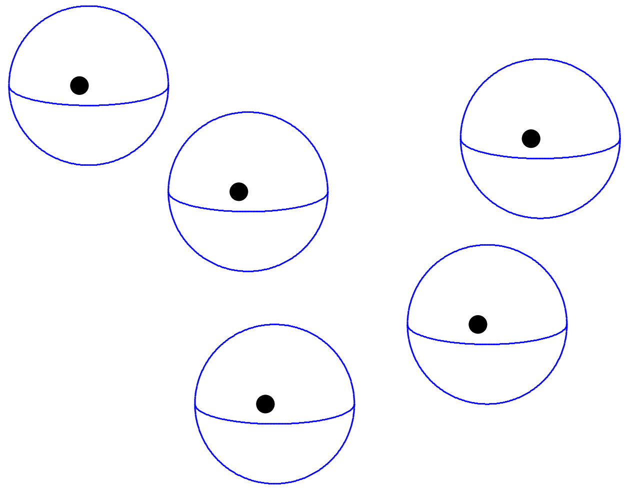

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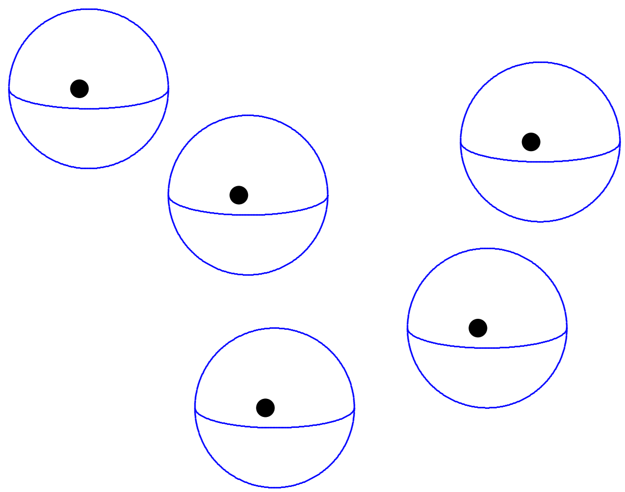

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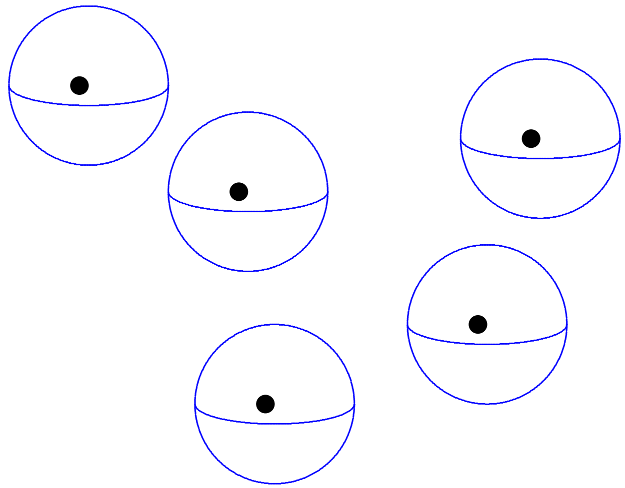

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The G-H metrics are hyper-Kähler, and were soon rediscovered independently by Hitchin.

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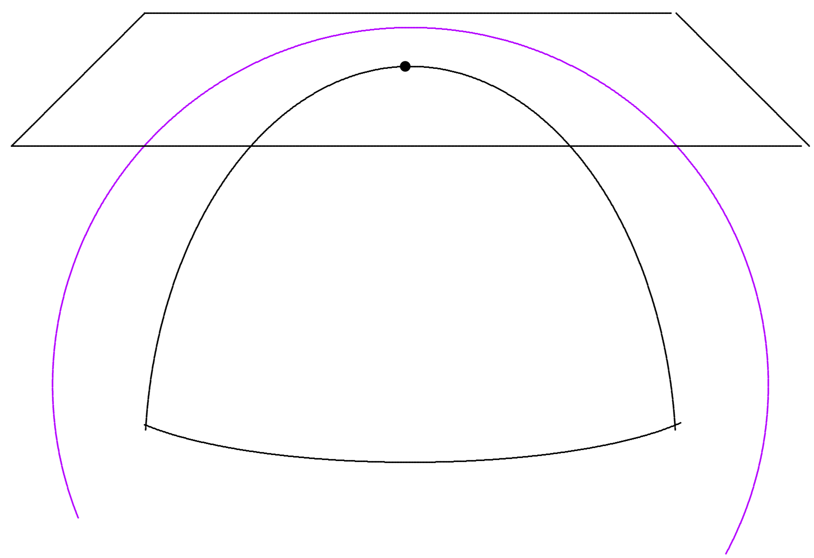

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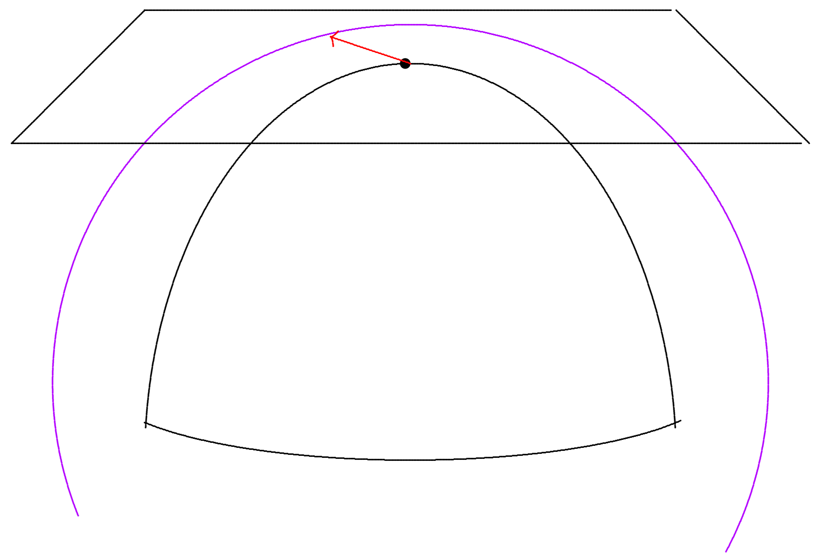

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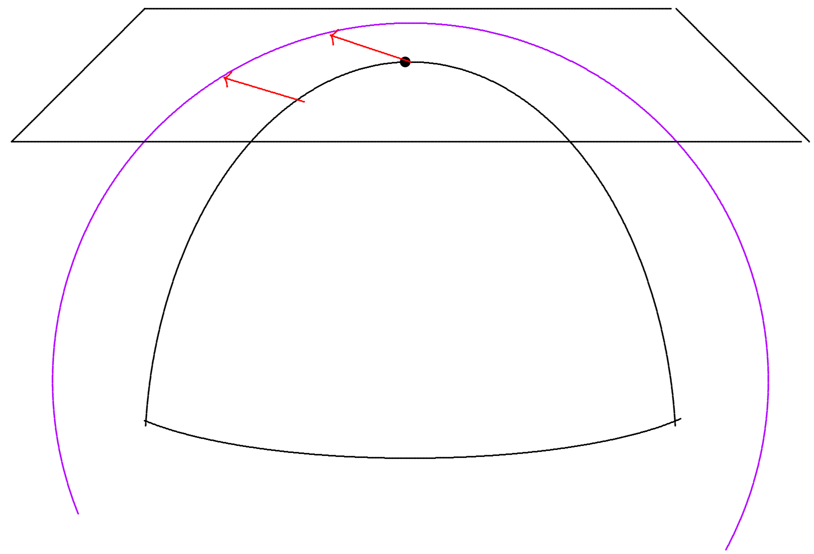

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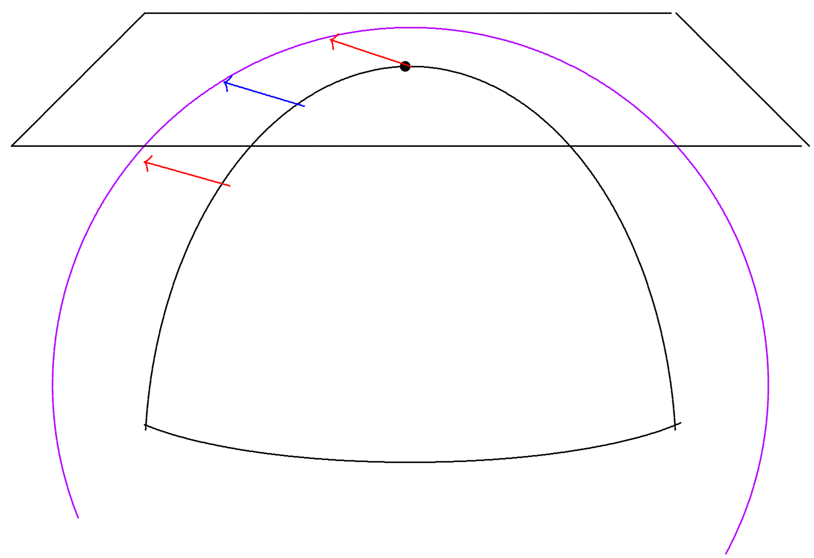

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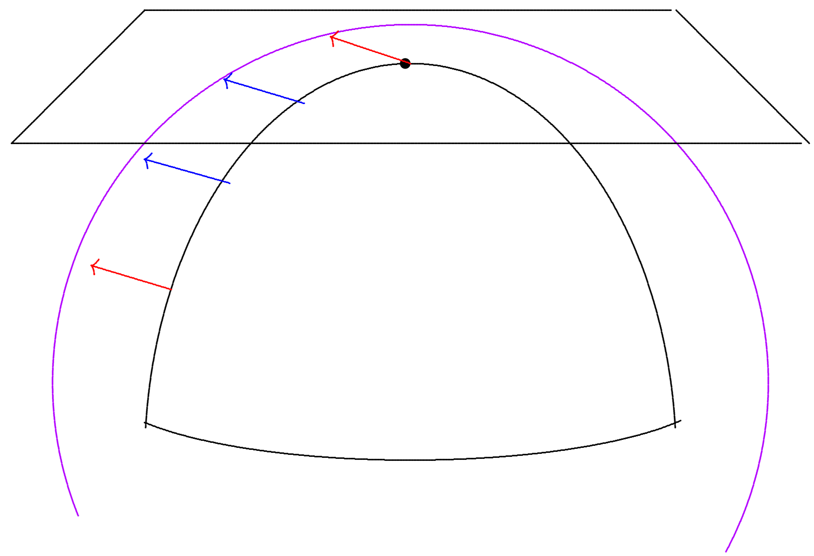

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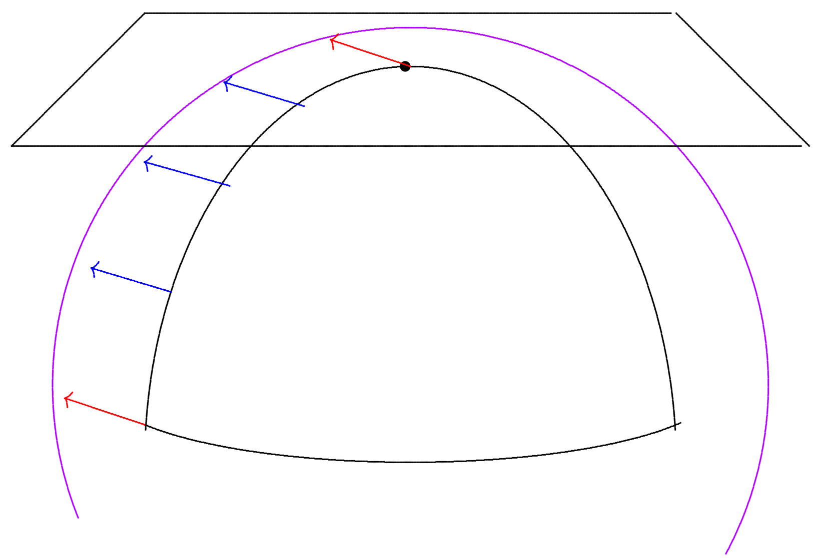

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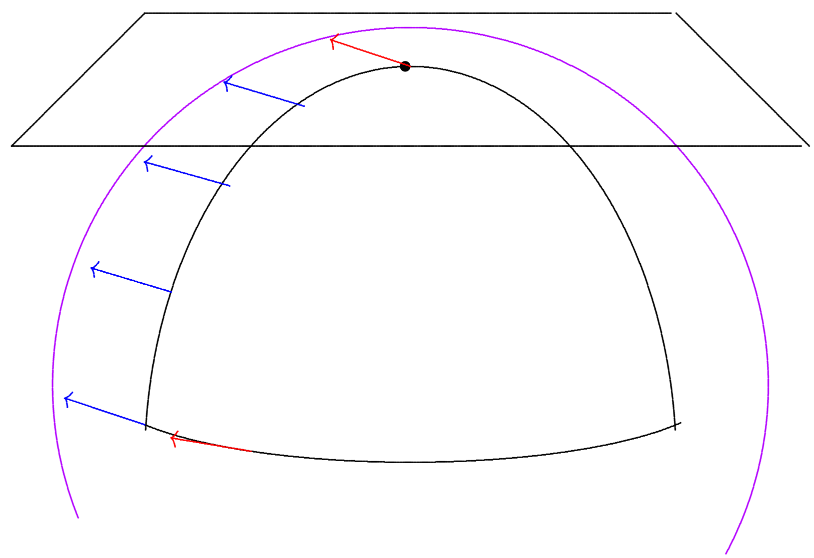

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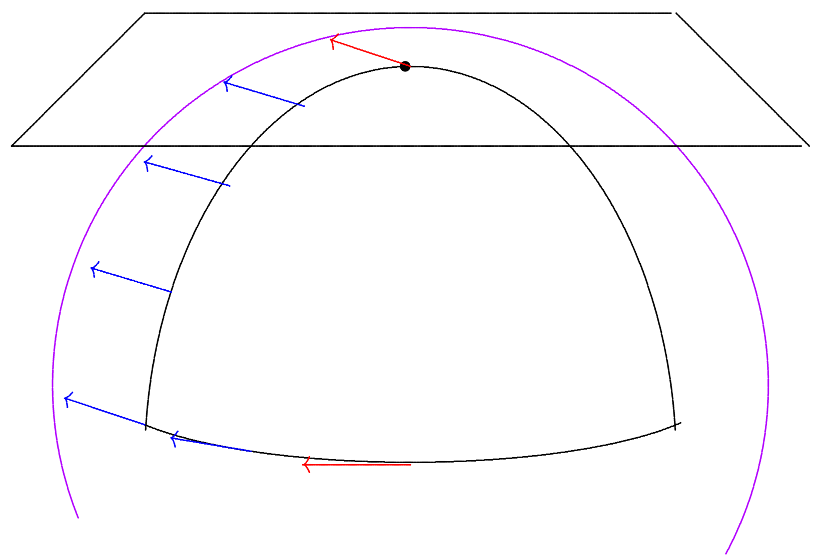

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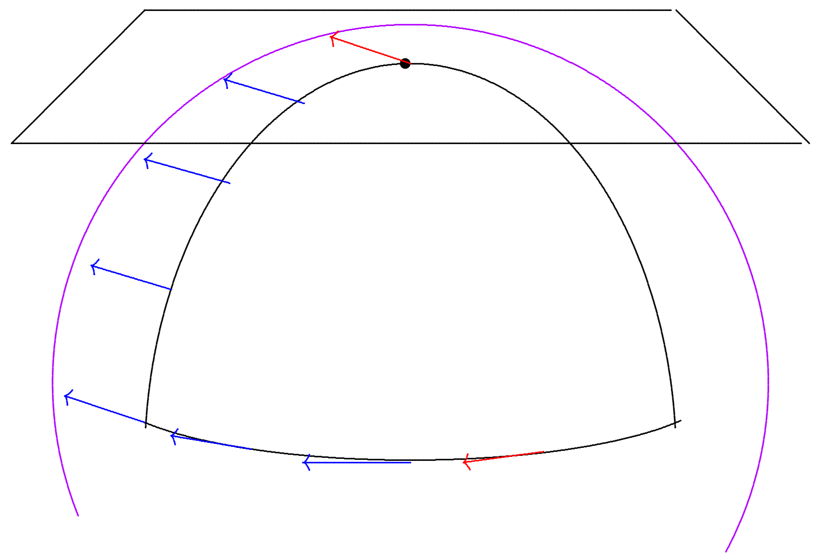

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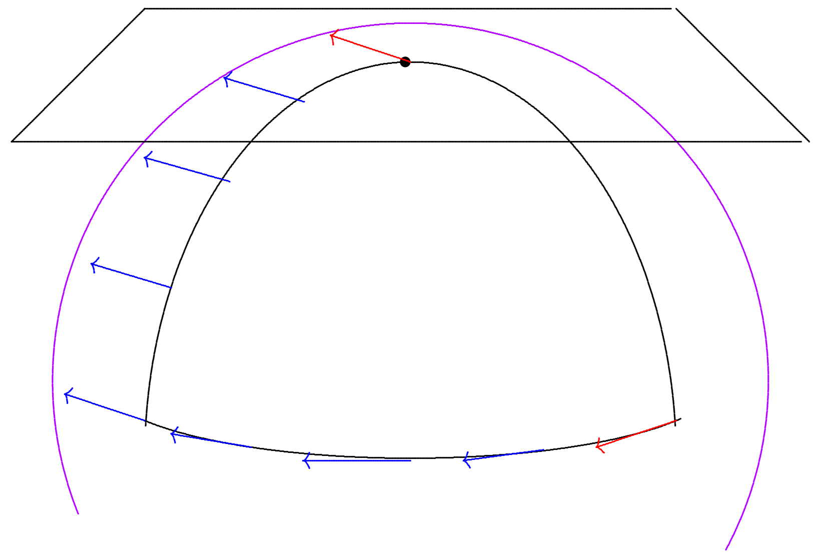

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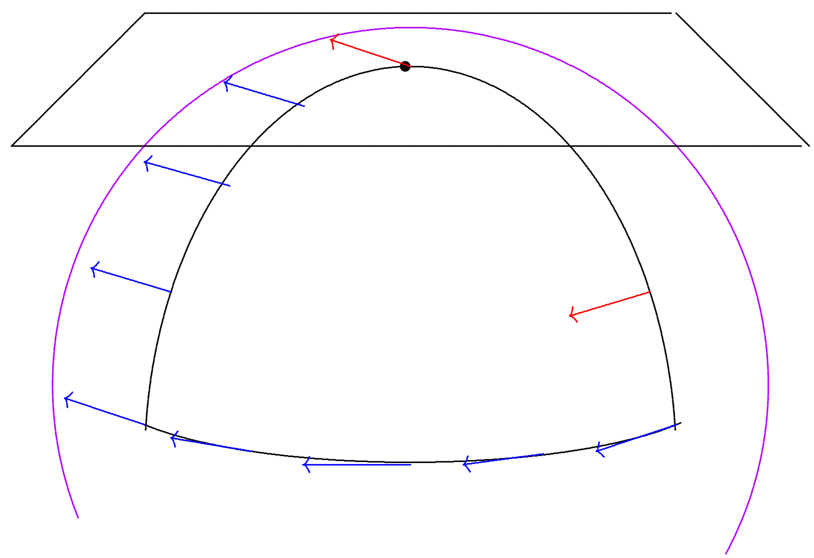

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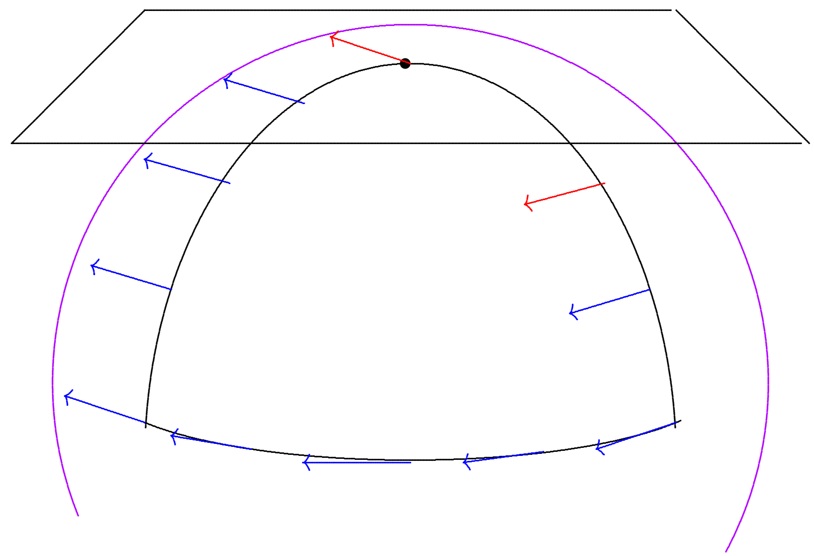

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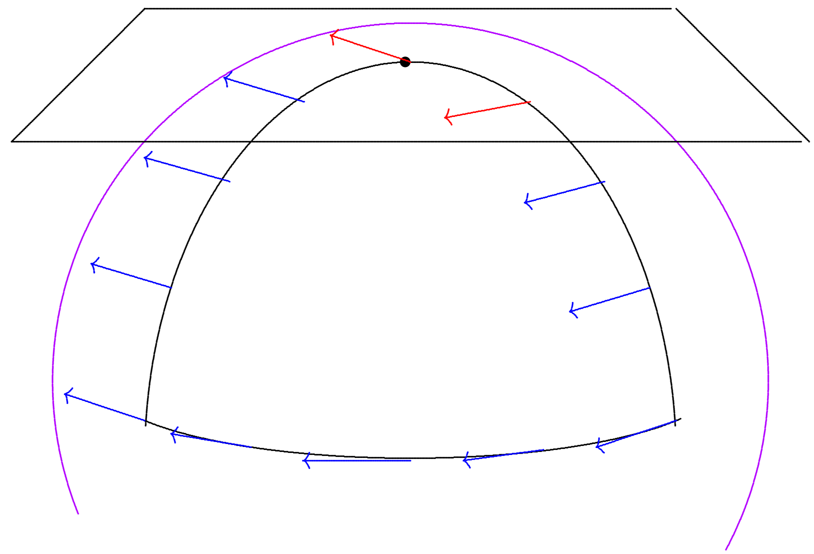

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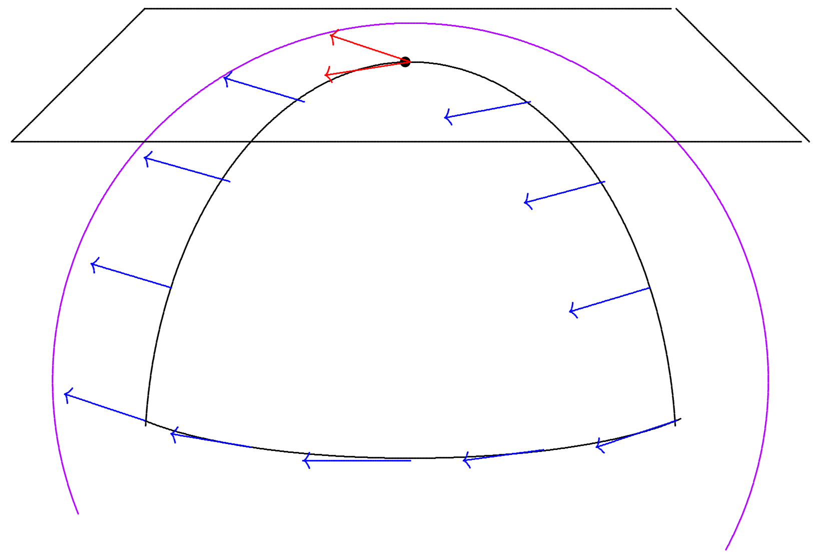

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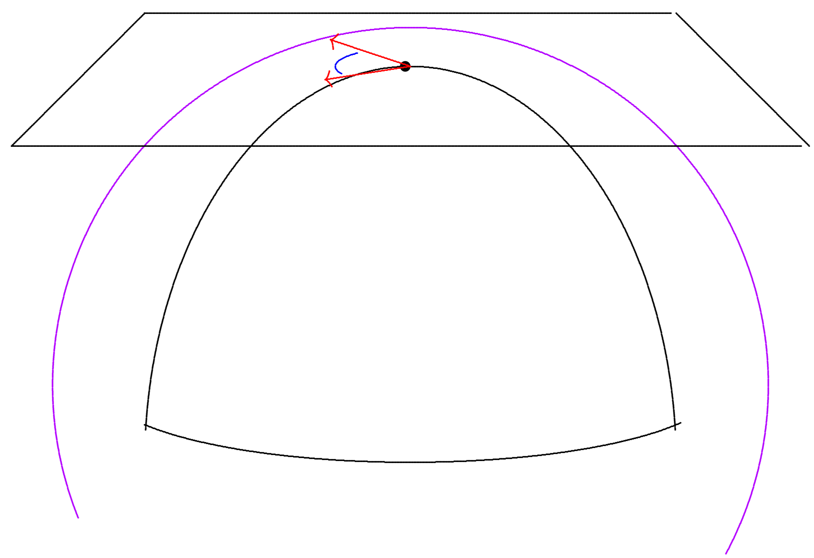

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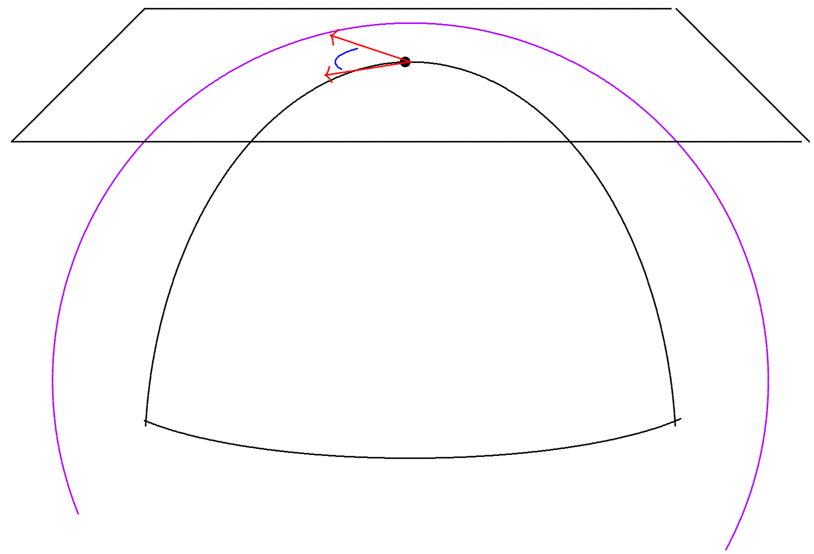

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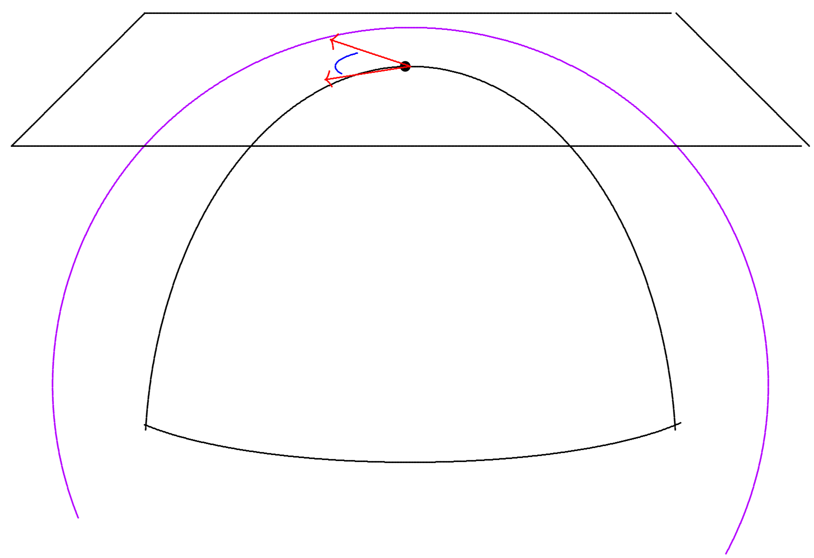

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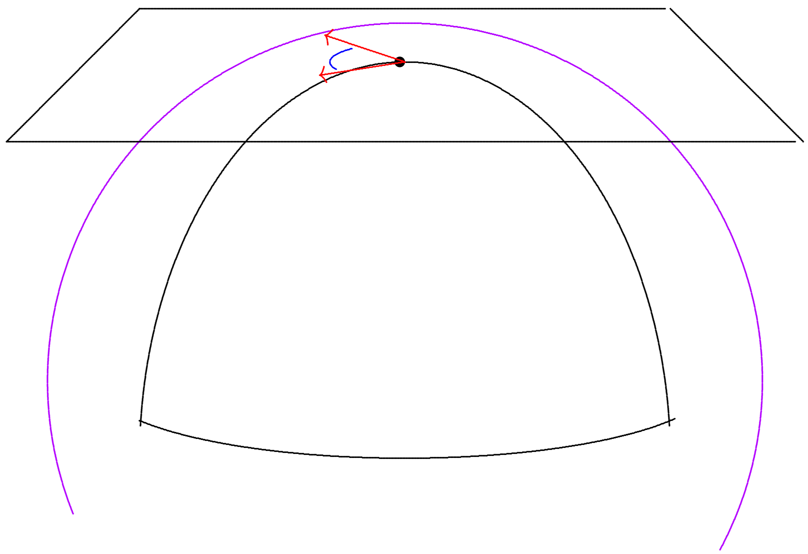

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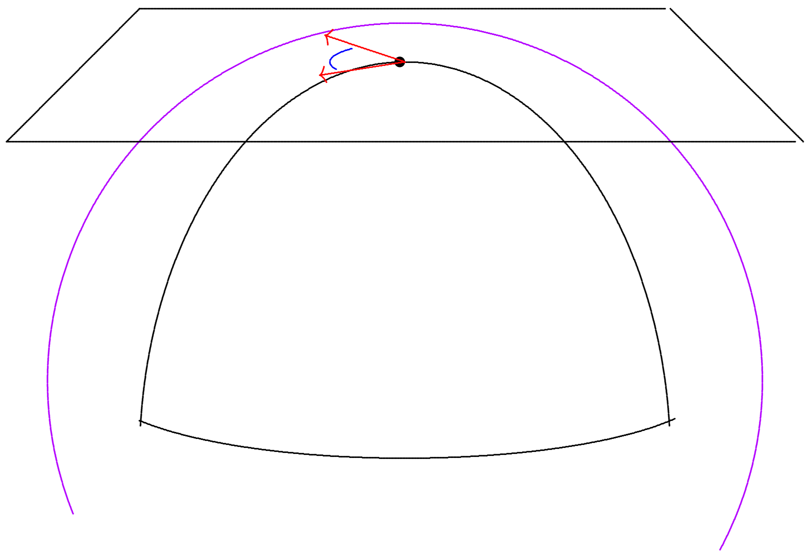

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Locally, $\Longleftrightarrow$ $s = 0, \quad \mathring{r} = 0, \quad W_+ = 0.$

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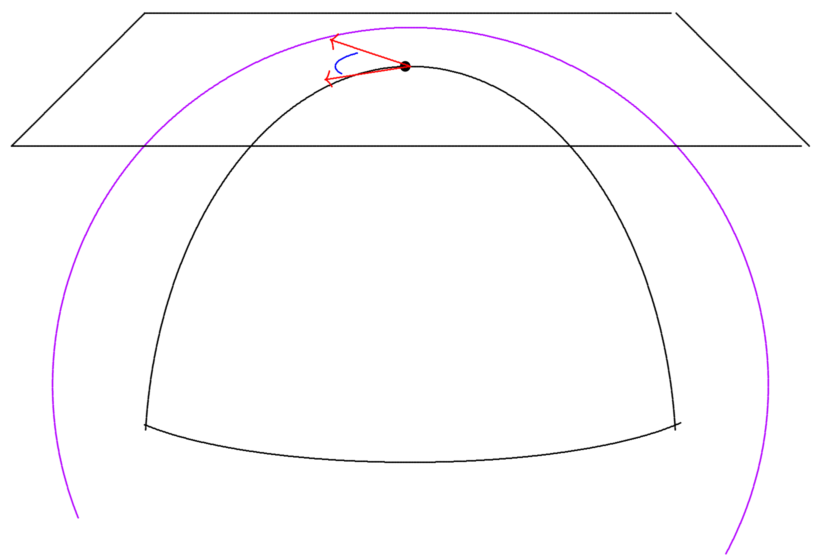

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Ricci-flat and Kähler,

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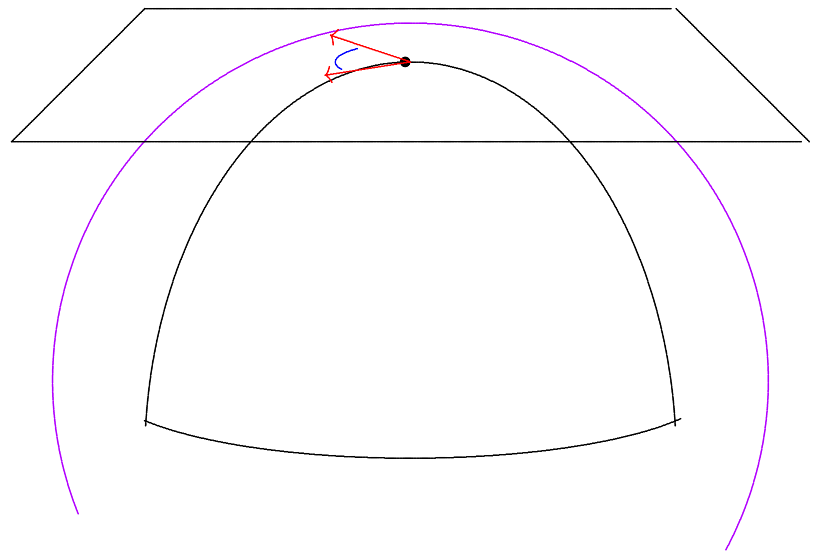

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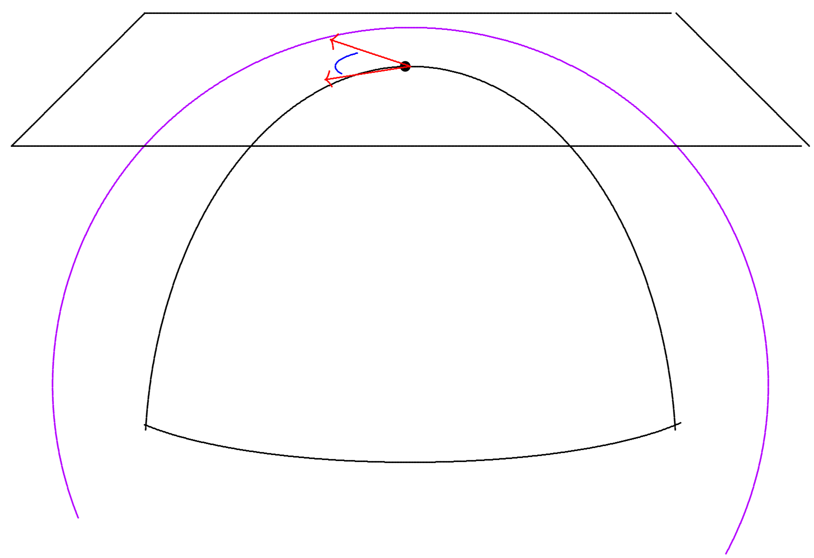

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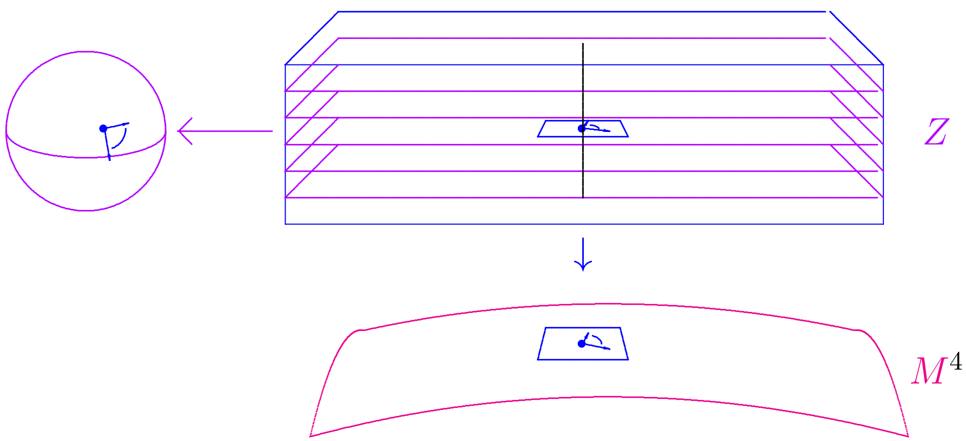

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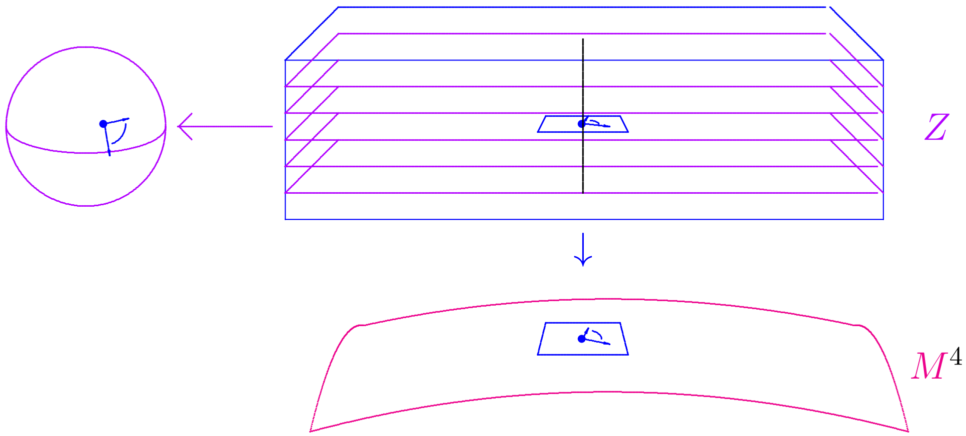

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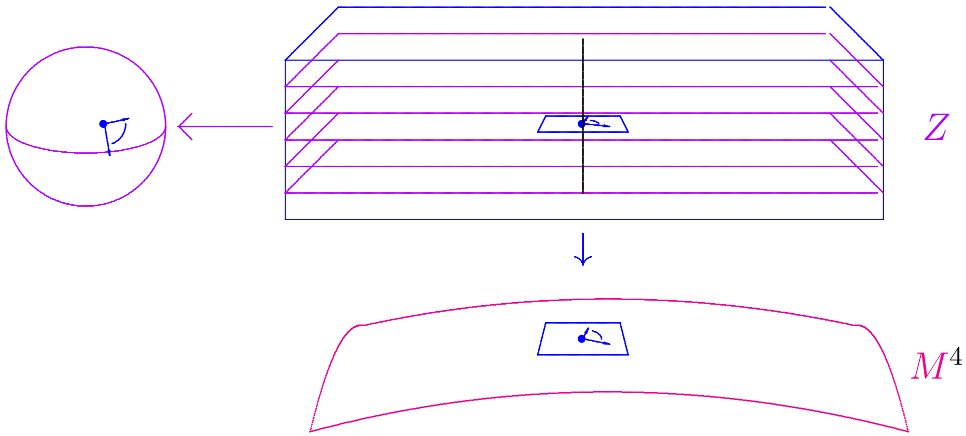

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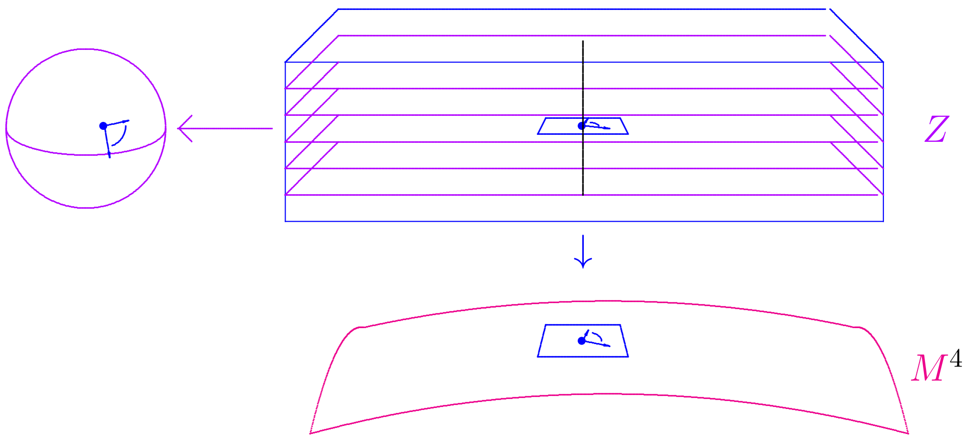

Riemannian non-linear graviton construction.

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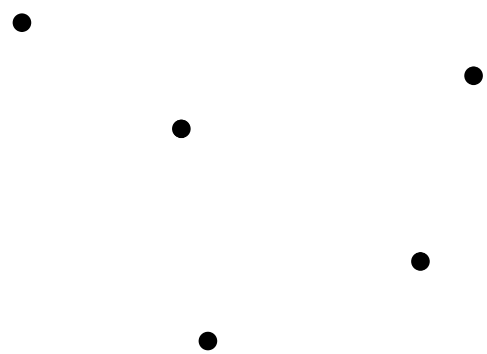

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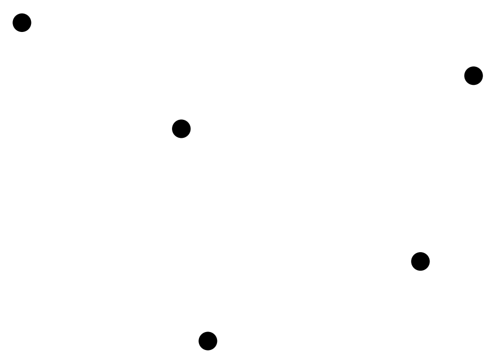

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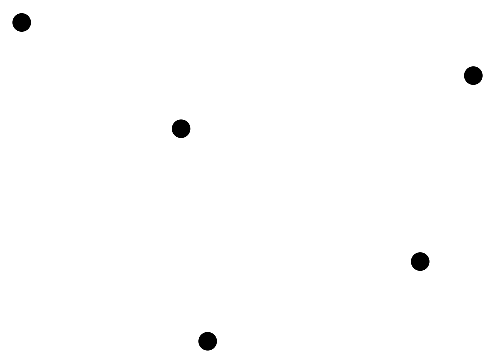

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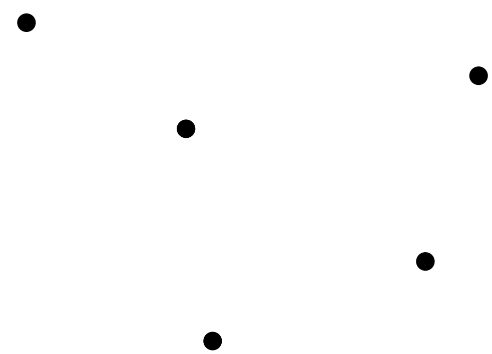

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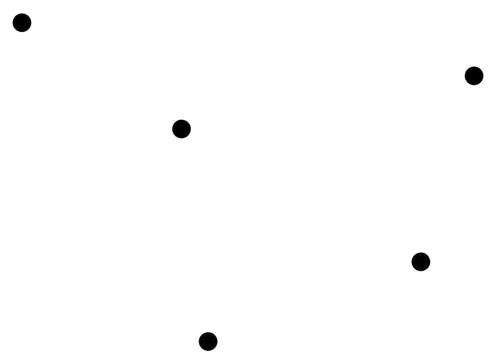

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is the twistor space of a Gibbons-Hawking metric.

Key examples:

Term ALE coined by Gibbons & Hawking, 1979.

They wrote down various explicit Ricci-flat ALE 4-manifolds they called gravitational instantons.

Their examples have just one end, with

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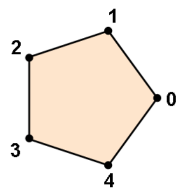

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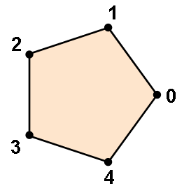

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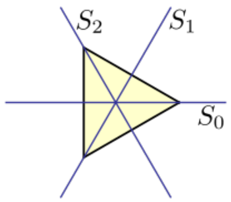

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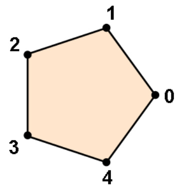

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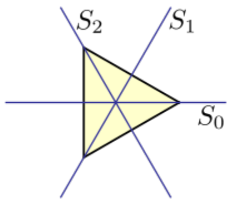

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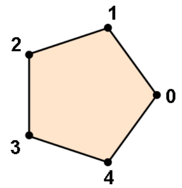

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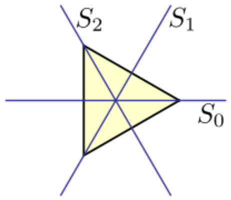

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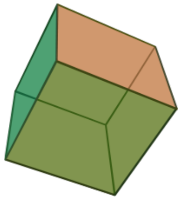

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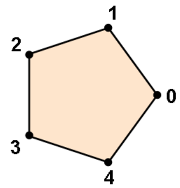

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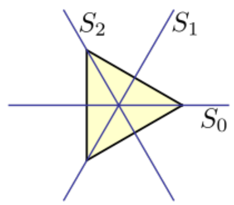

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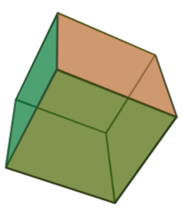

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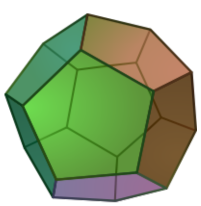
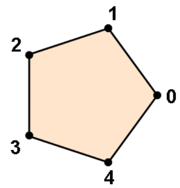


$I^* \longleftrightarrow x^2 + y^3 + z^5 = 0$



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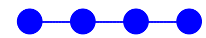

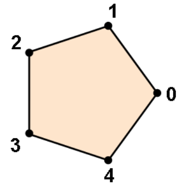

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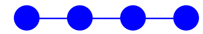

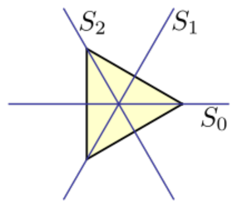

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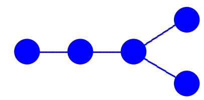

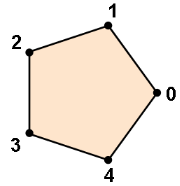

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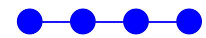

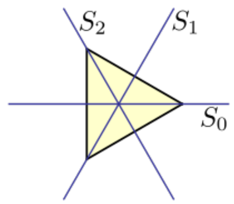

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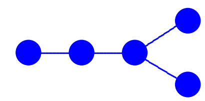

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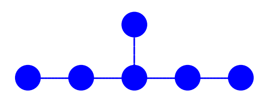

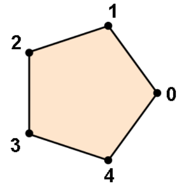

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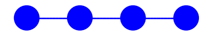

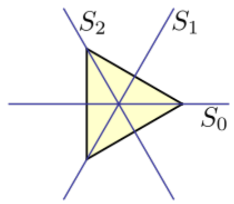

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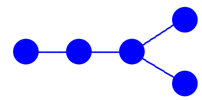

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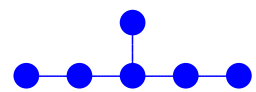

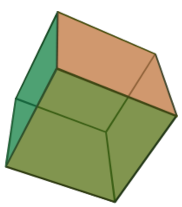

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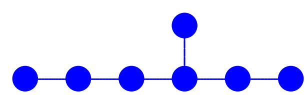

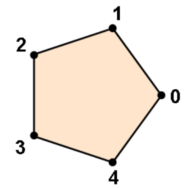

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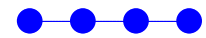

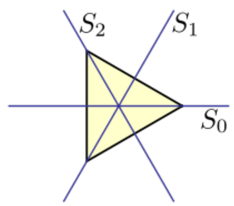

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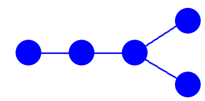

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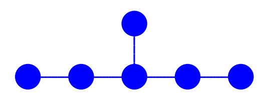

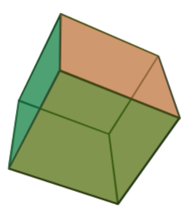

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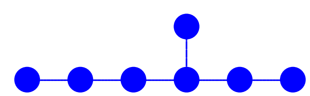

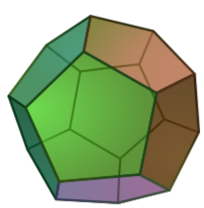

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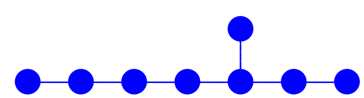

Mass still meaningful in this context. . .

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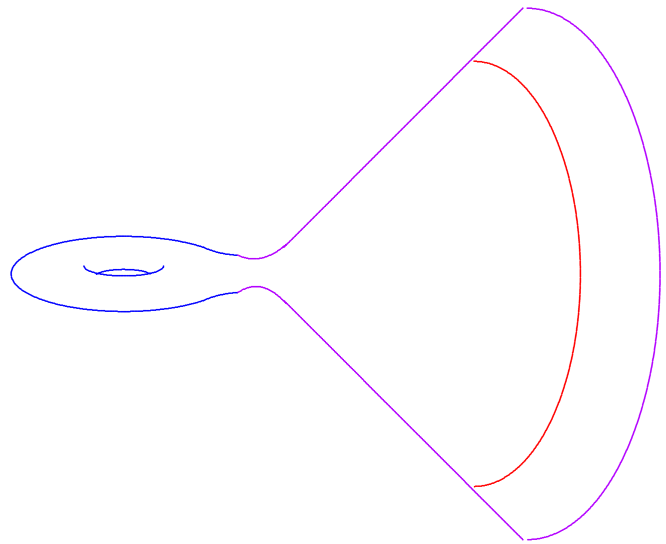

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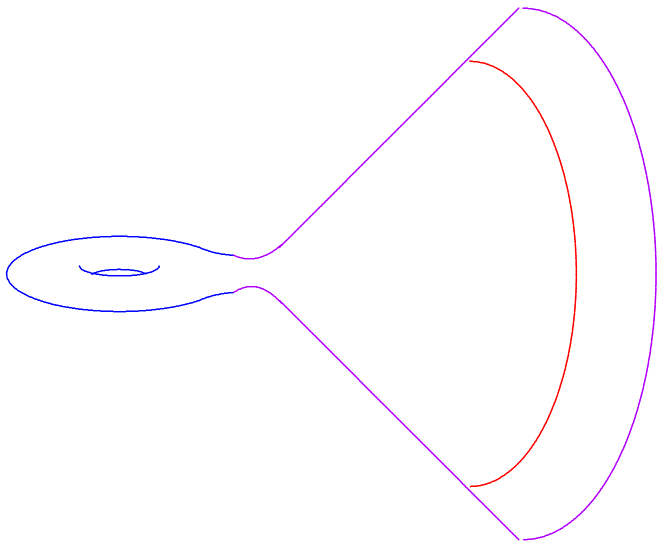

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$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

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Motivation:

When $n = 3$, ADM mass in general relativity.

Reads off "apparent mass" from strength of the gravitational field far from an isolated source.

In any dimension, reproduces "mass" of $t = 0$ hypersurface in $(n+1)$-dimensional Schwarzschild

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But there are other Kähler examples with $m \neq 0$.

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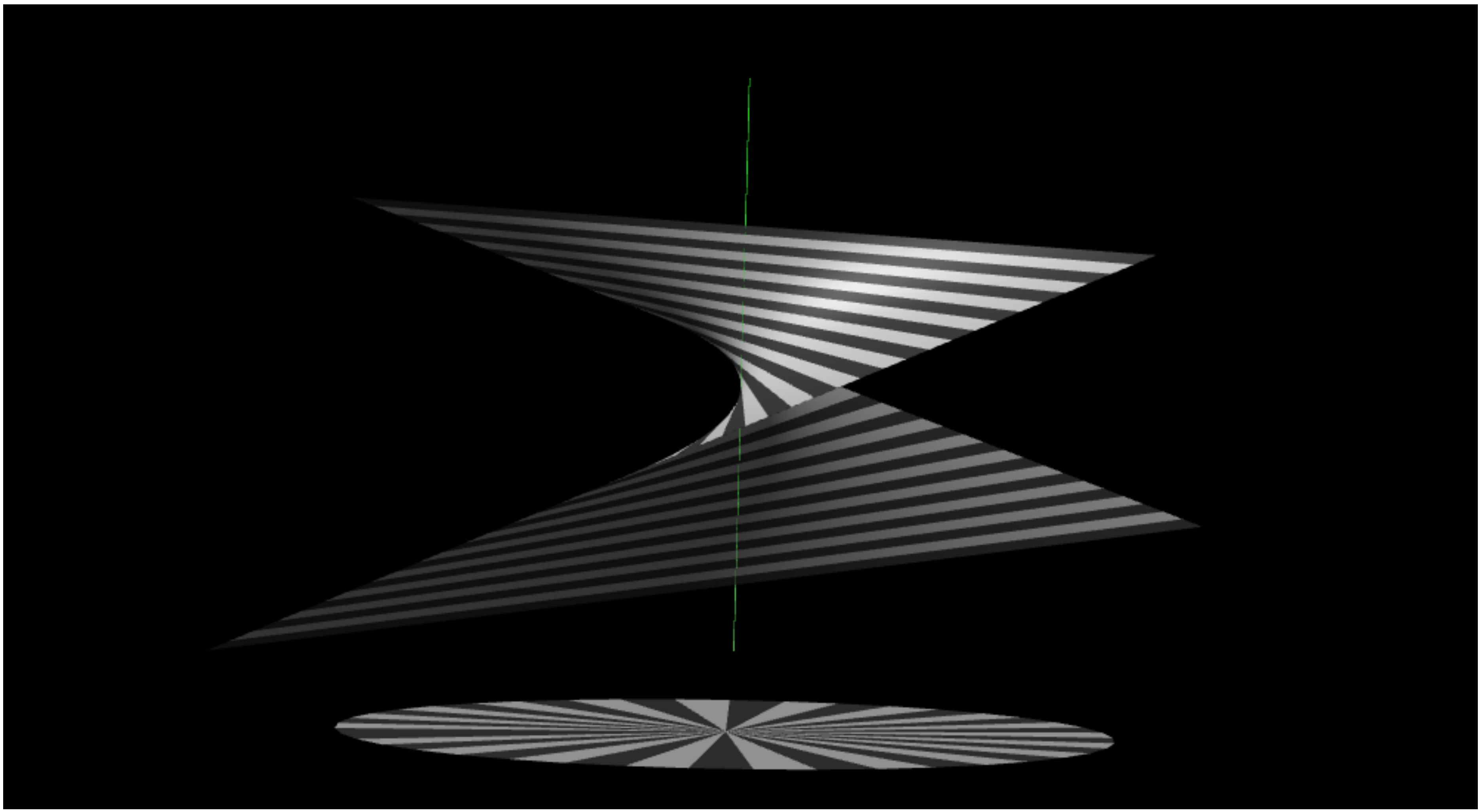

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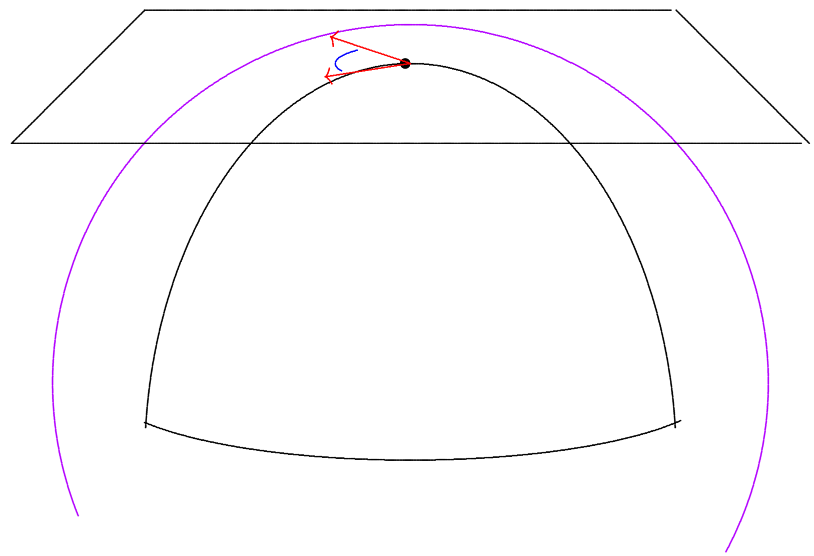
also has mass m.

Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset$ $\mathbf{U}(m)$



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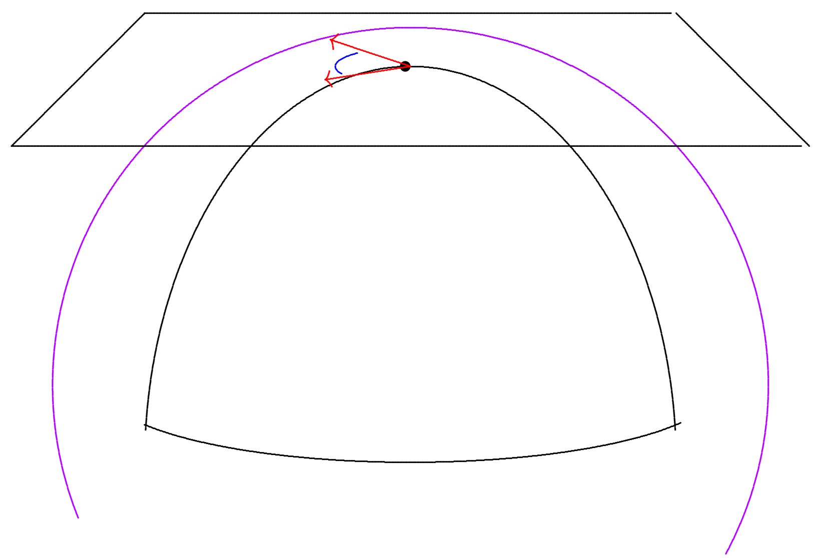

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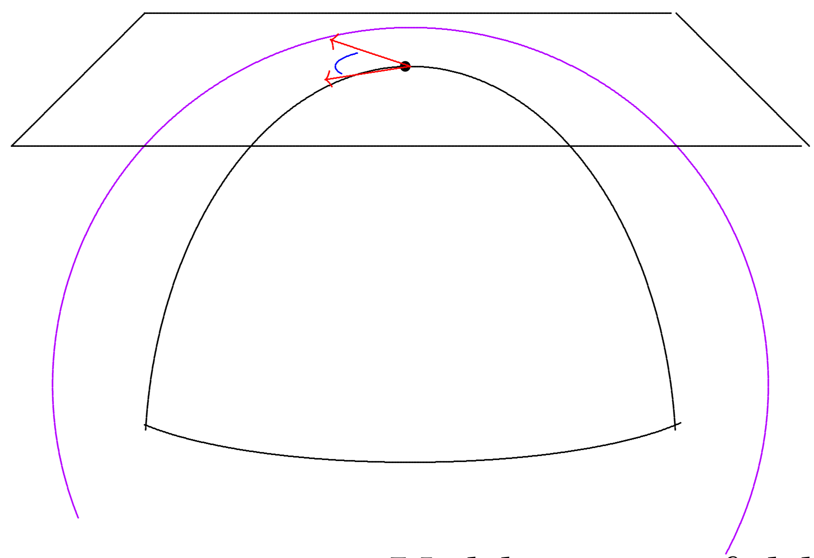

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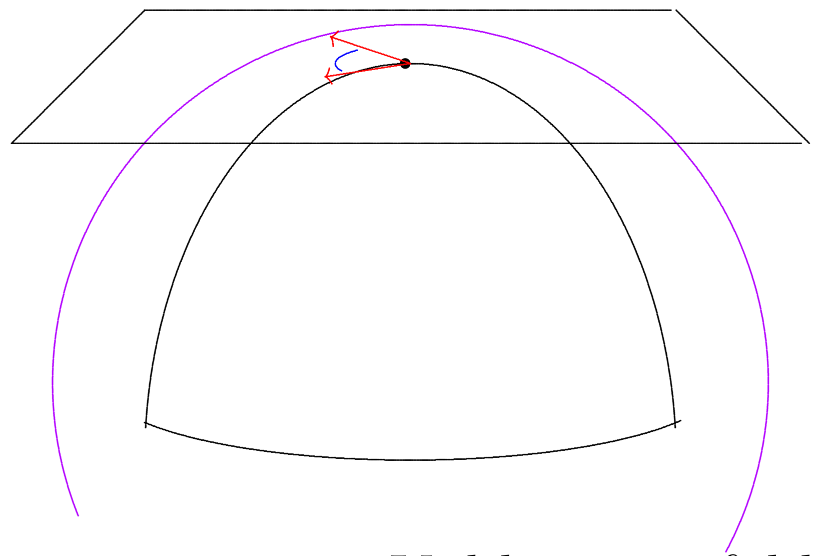

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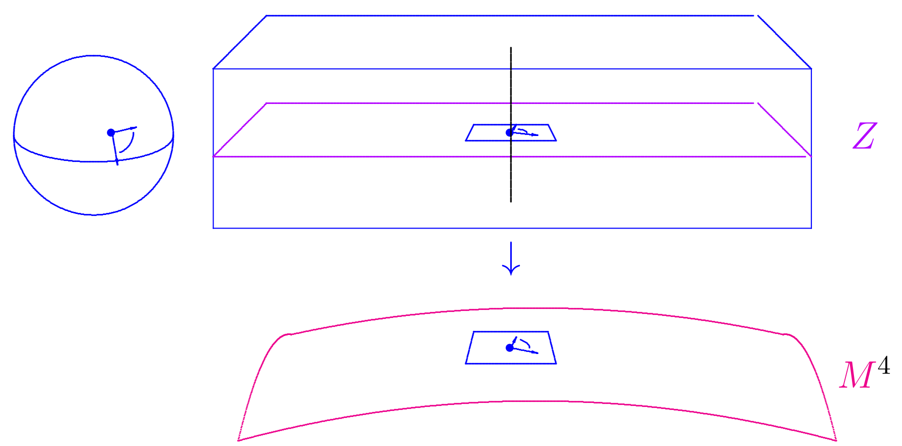
so that $W_+ = 0 \Longleftrightarrow s = 0$.

Any scalar-flat Kähler surface (M^4, g, J) has a

Penrose Twistor Space (Z, J),

which is once again a complex 3-manifold.



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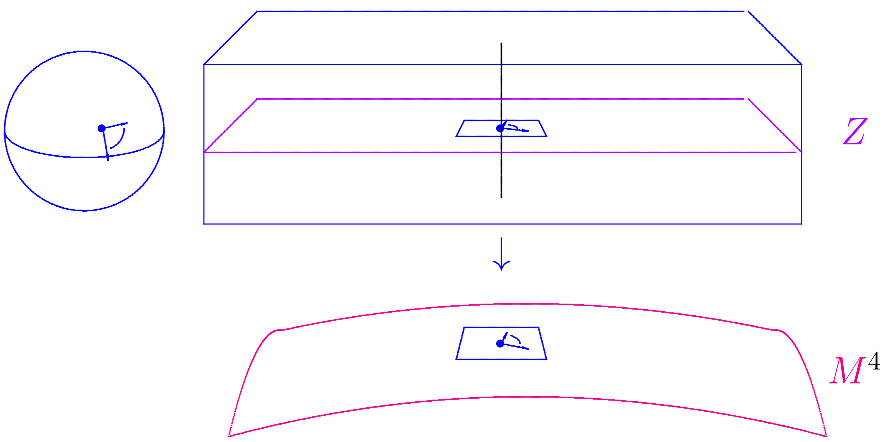

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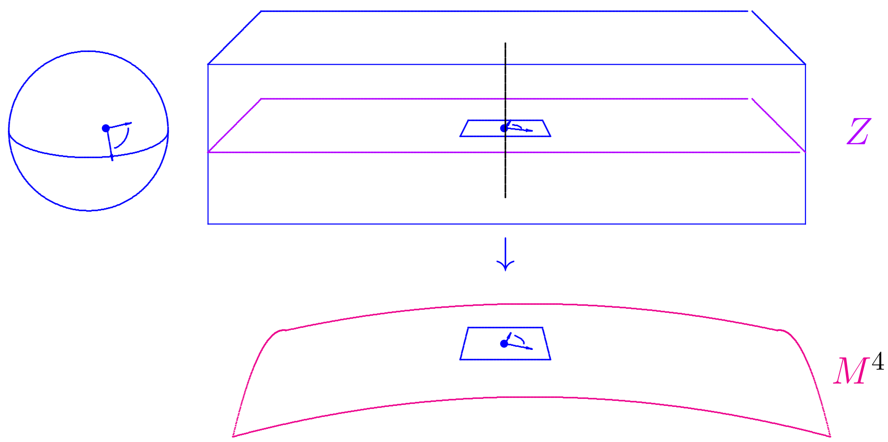

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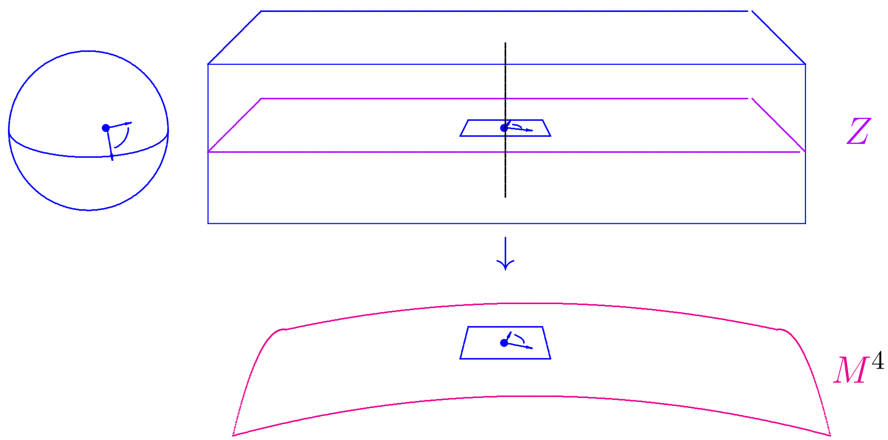

$$0 \to T^{1,0}M \to T^{1,0}Z|_M \to K_M{}^{-1} \to 0$$

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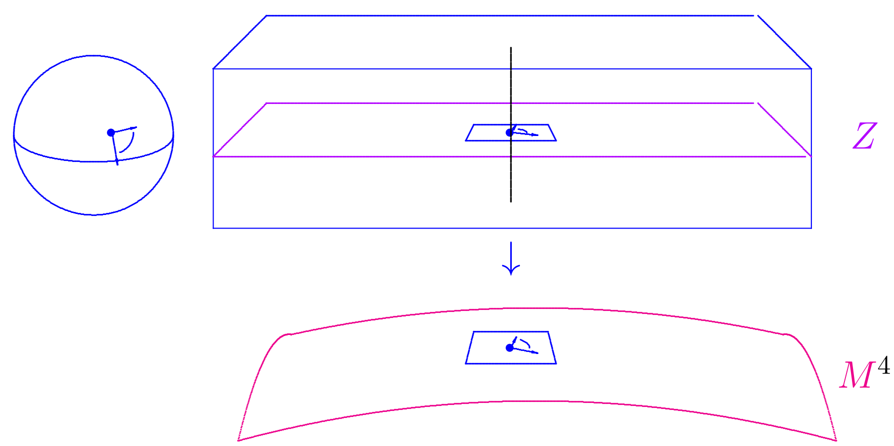

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Extension class: $\quad \in H^1(M, \mathcal{O}(K_M \otimes T^{1,0}M)).$

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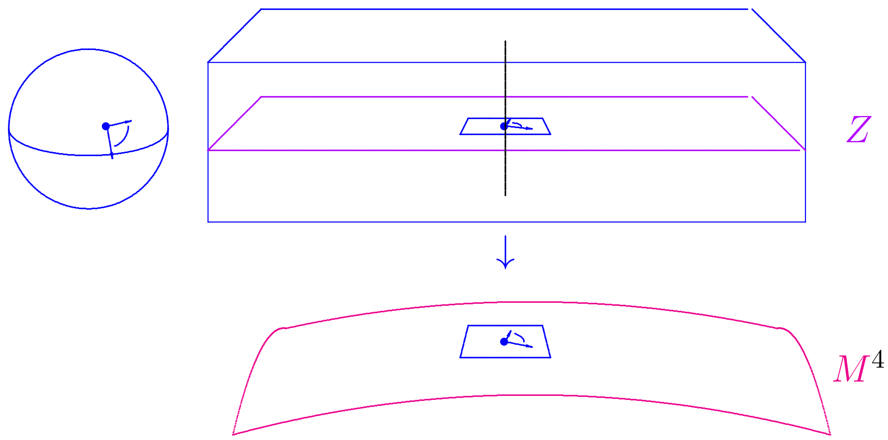

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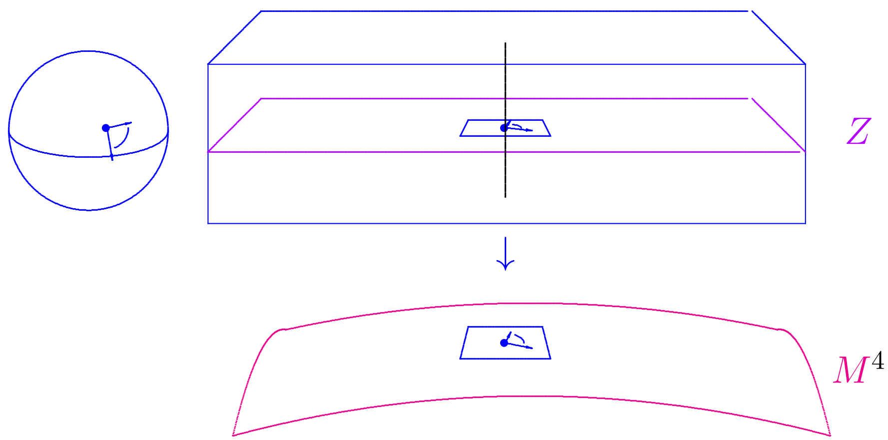

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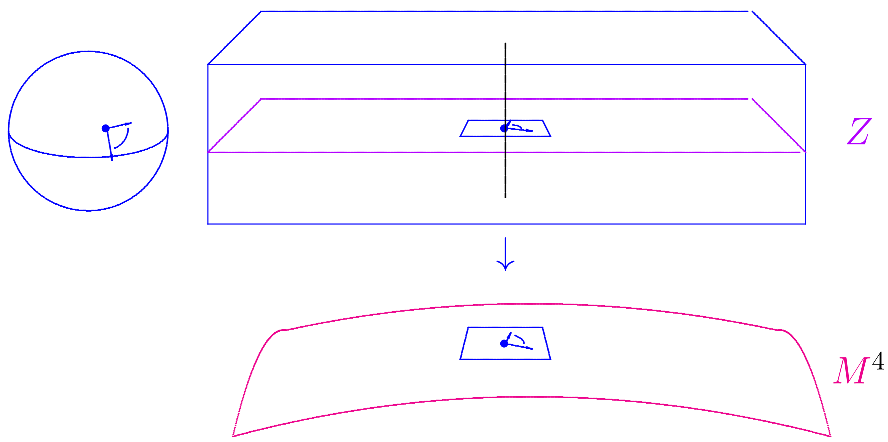

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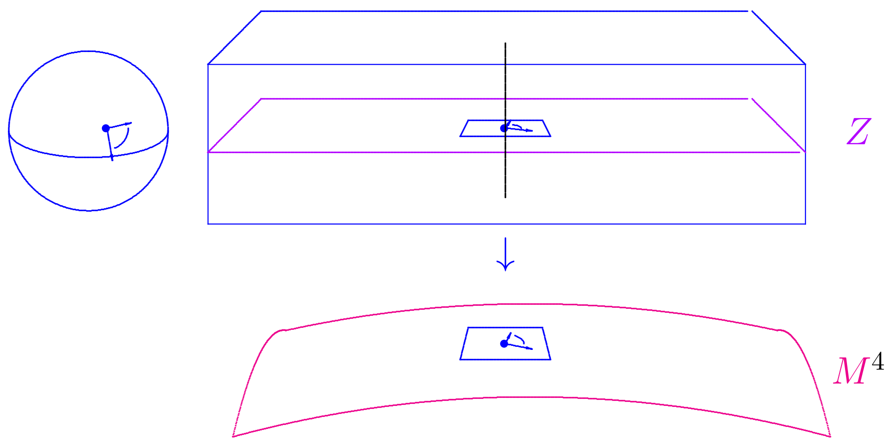

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Kähler form: $\omega = g(J\cdot, \cdot)$.

AE/ALE Scalar-Flat Kähler Surfaces:

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Key role in theory of Bach-flat 4-manifolds.

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(L '91)

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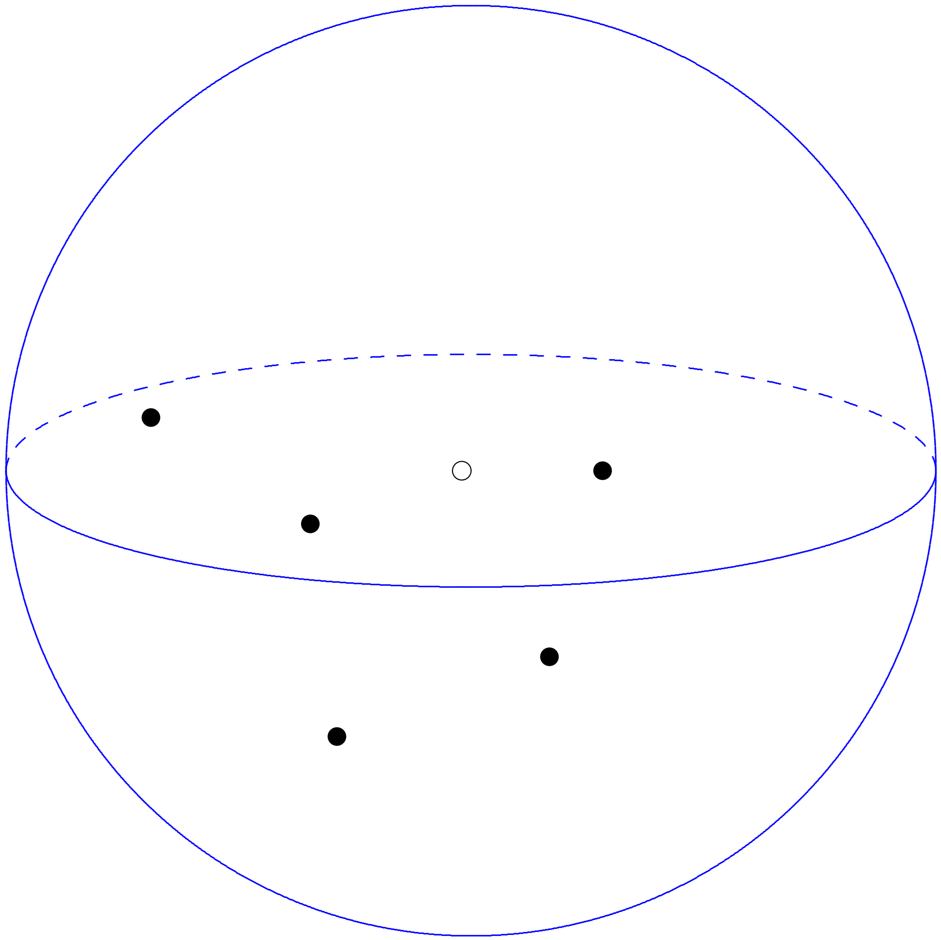

Data: $k+1$ points in $\mathcal{H}^3$.

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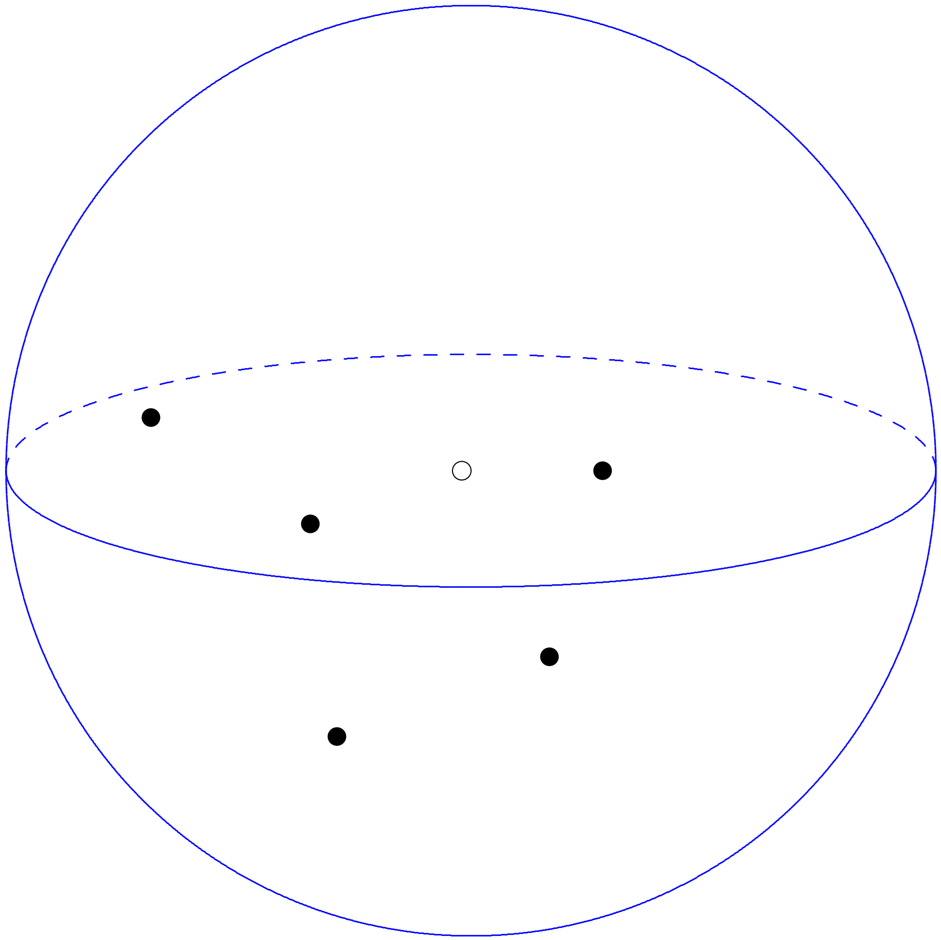

Data: $k+1$ points in $\mathcal{H}^3$. $\Longrightarrow V$ with $\Delta V = 0$

$$V = 1 + \frac{\ell}{e^{2\varrho_0} - 1} + \sum_{j=1}^{k} \frac{1}{e^{2\varrho_j} - 1}$$

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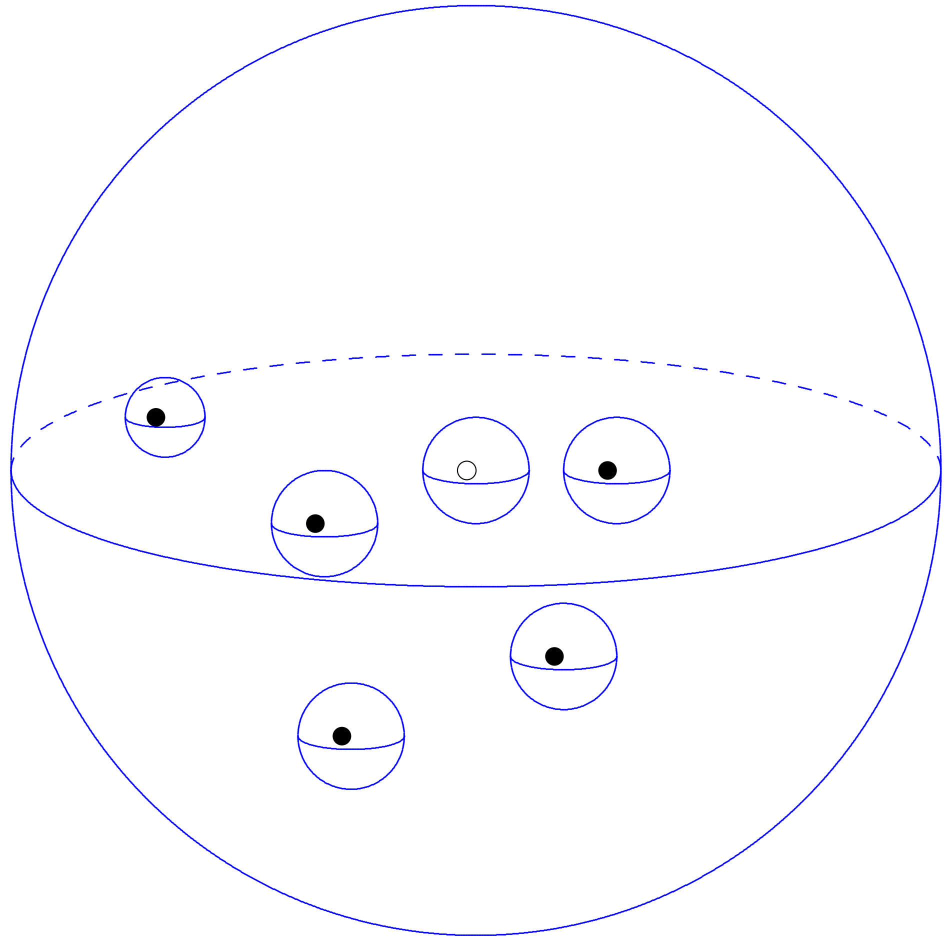

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$F = \star dV$ curvature θ on $P \to \mathcal{H}^3 - \{\text{pts}\}$.

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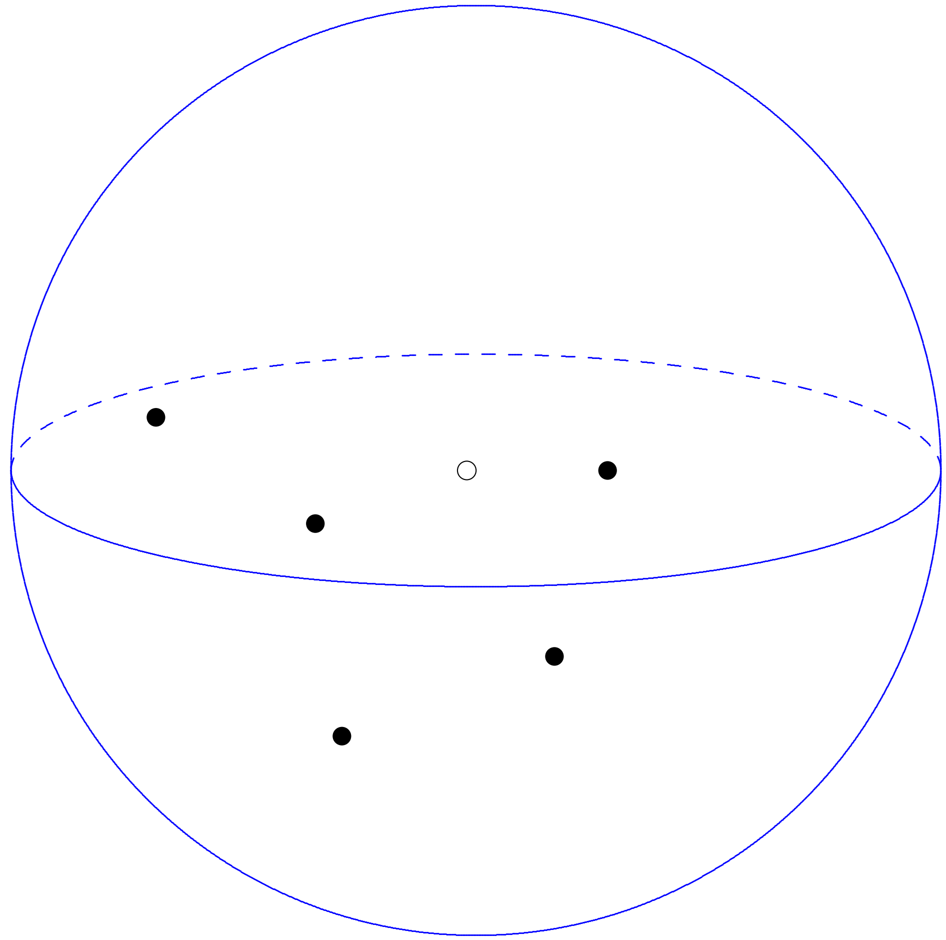

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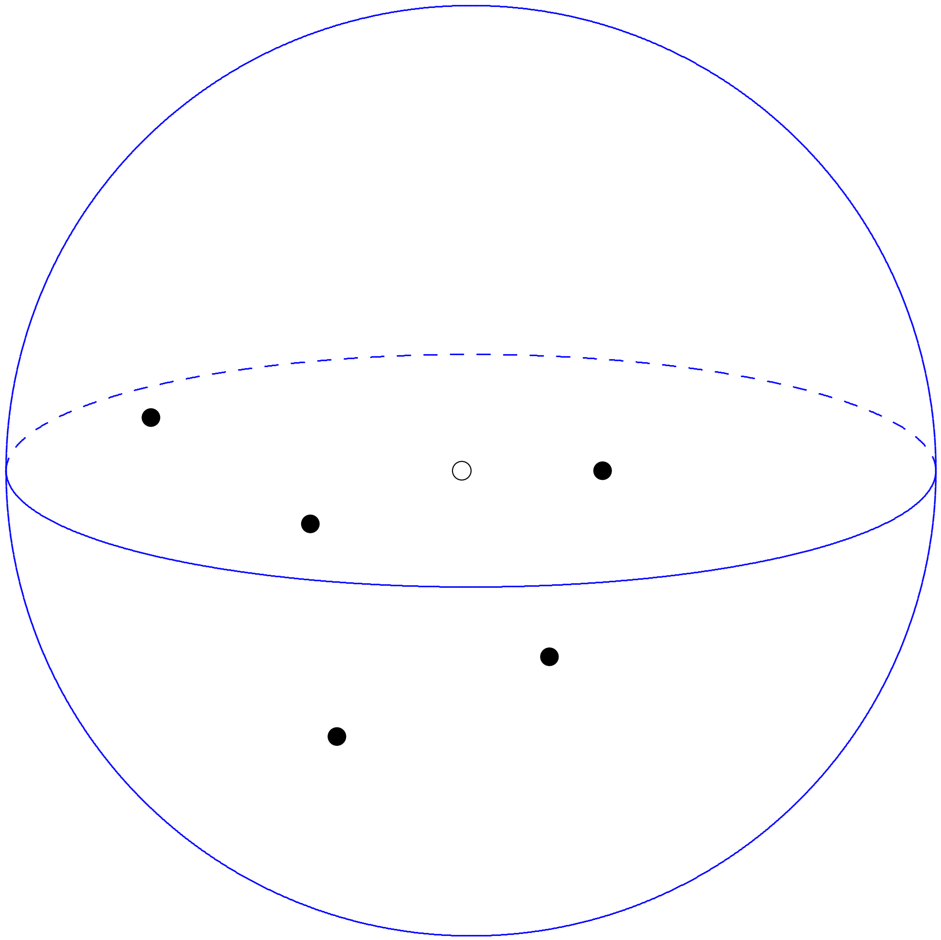

Riemannian completion is ALE scalar-flat Kähler.

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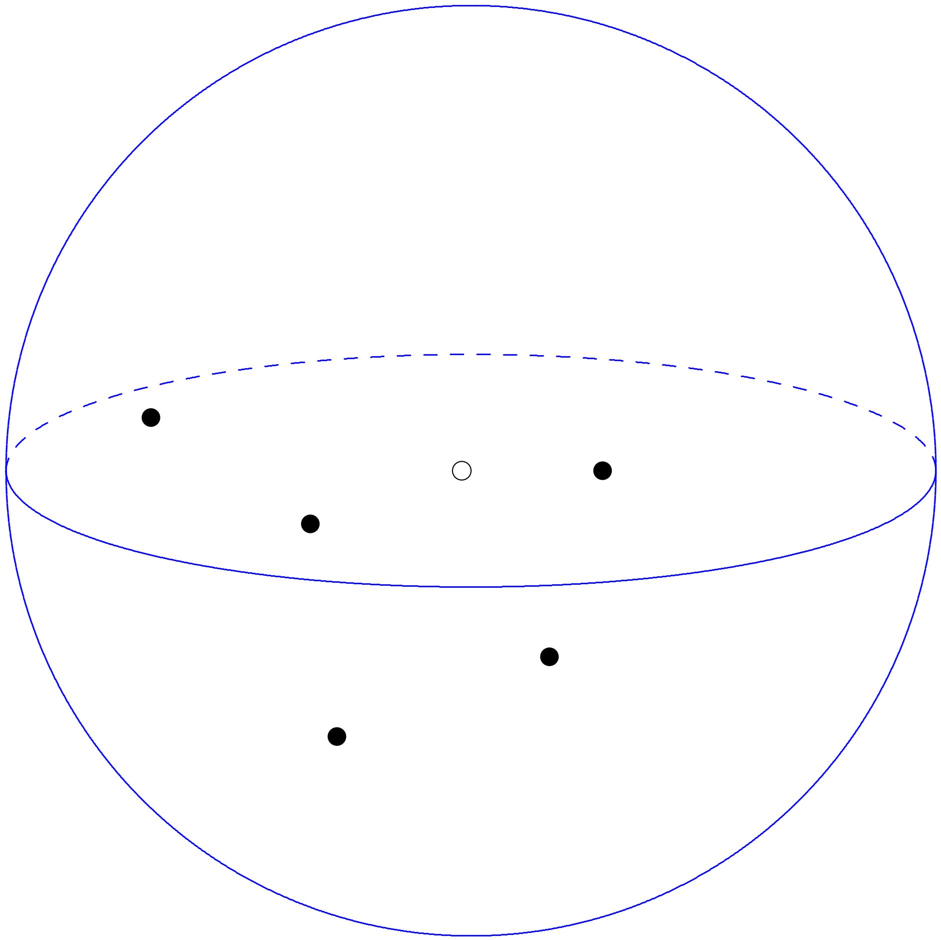

Riemannian completion is AE $\Longleftrightarrow$ $\ell = 1$:

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$$H^0(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(1,1)) = \mathbb{C}^4$$

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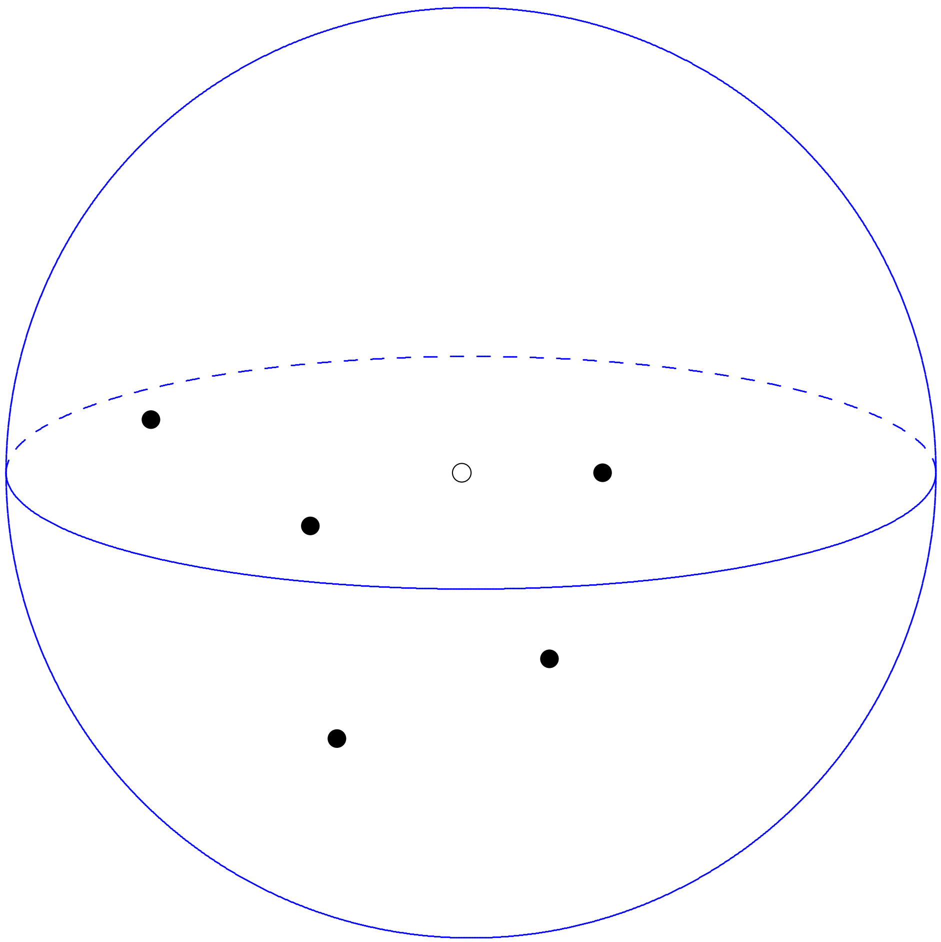

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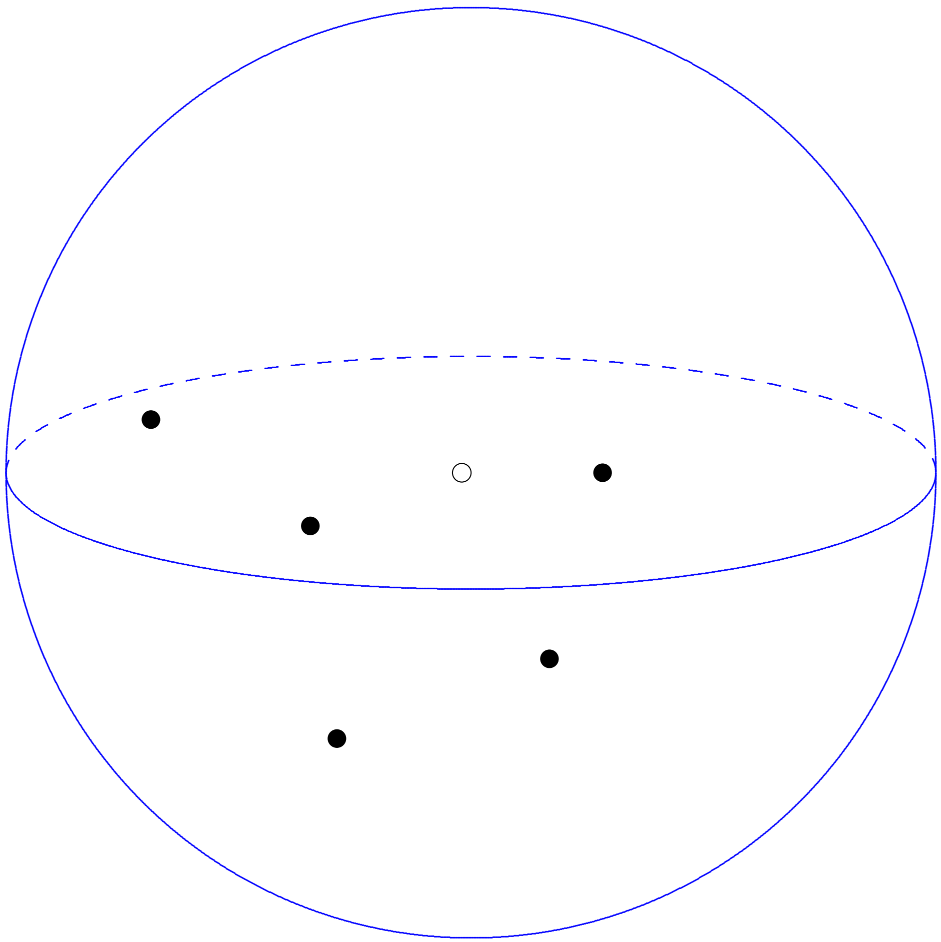

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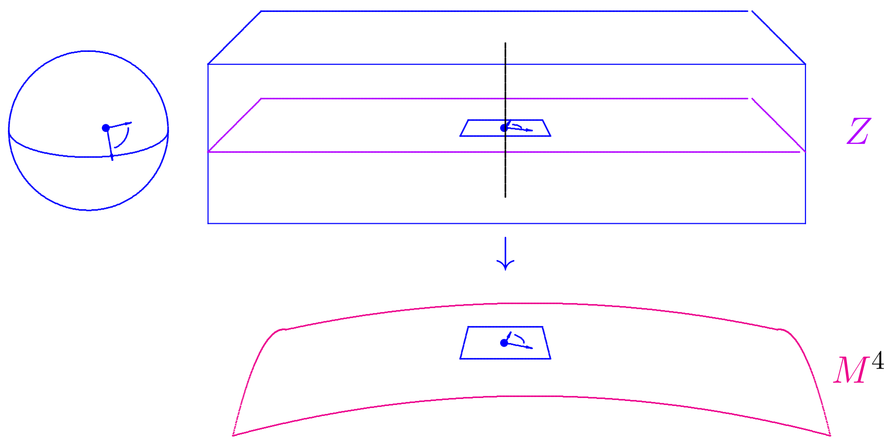
Penrose Twistor Space (Z, J),

which is once again a complex 3-manifold.



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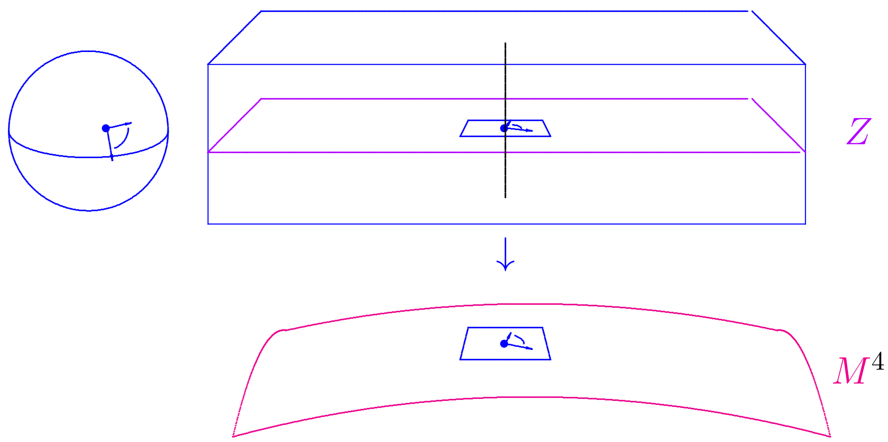

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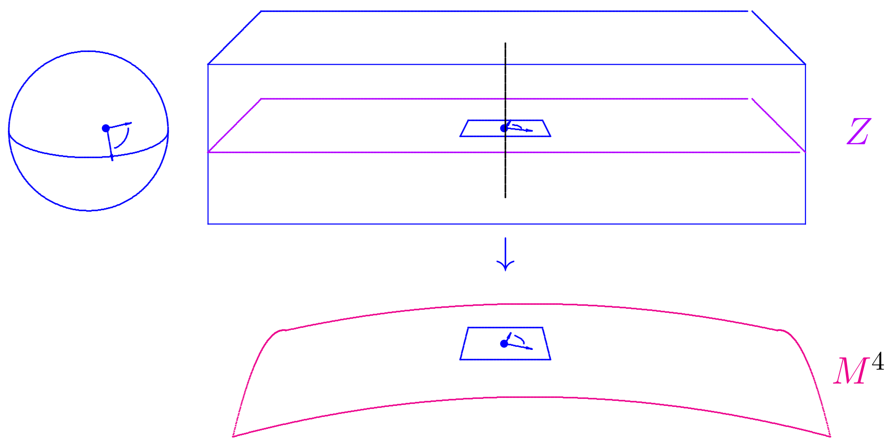

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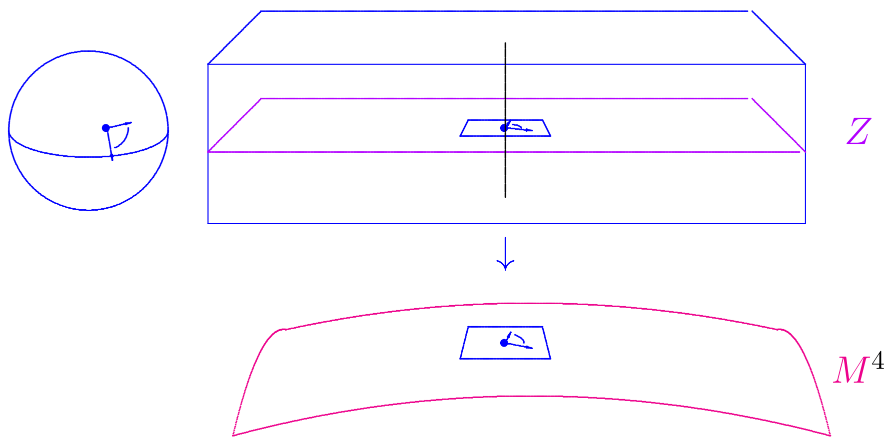

Lots more ALE scalar-flat Kähler surfaces now known:

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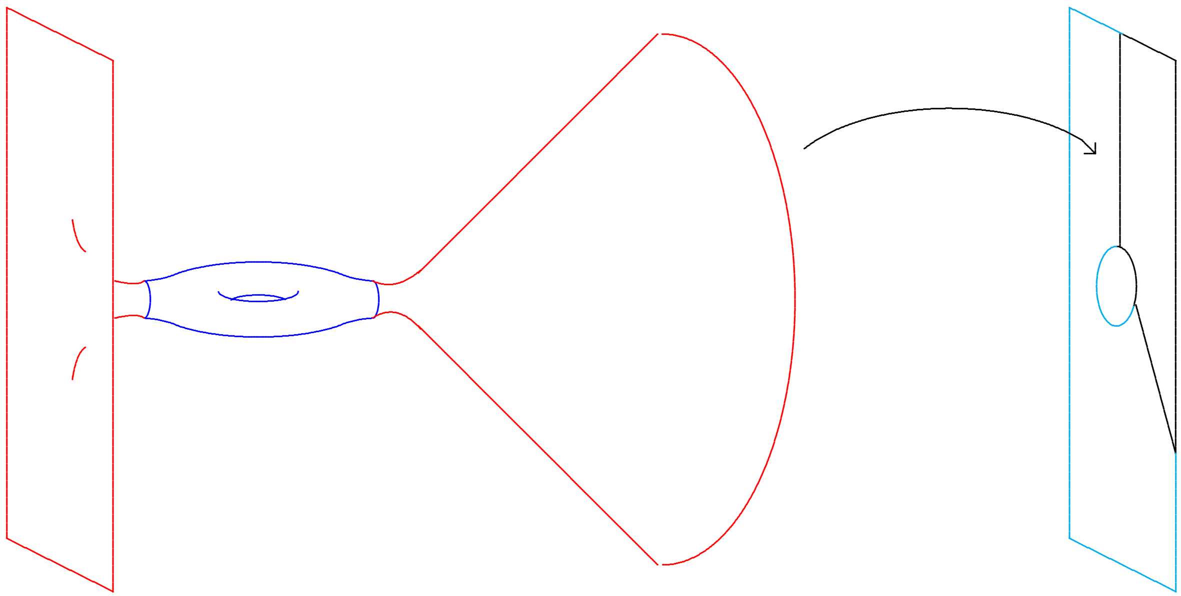
But full classification remains an open problem.

Definition. *Complete, non-compact* n*-manifold* (M^n, g) *is* asymptotically locally Euclidean *(ALE) if* $\exists$ *compact set* $K \subset M$ *such that* $M - K \approx \coprod_i (\mathbb{R}^n - D^n)/\Gamma_i$, *where* $\Gamma_i \subset \mathbf{O(n)}$, *such that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

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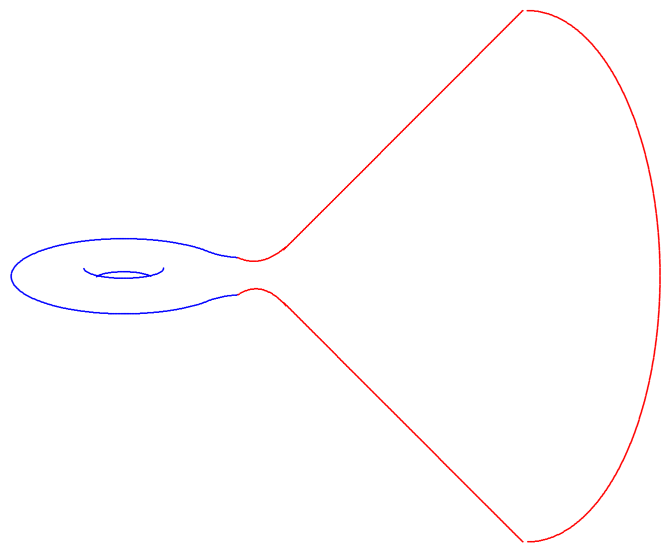

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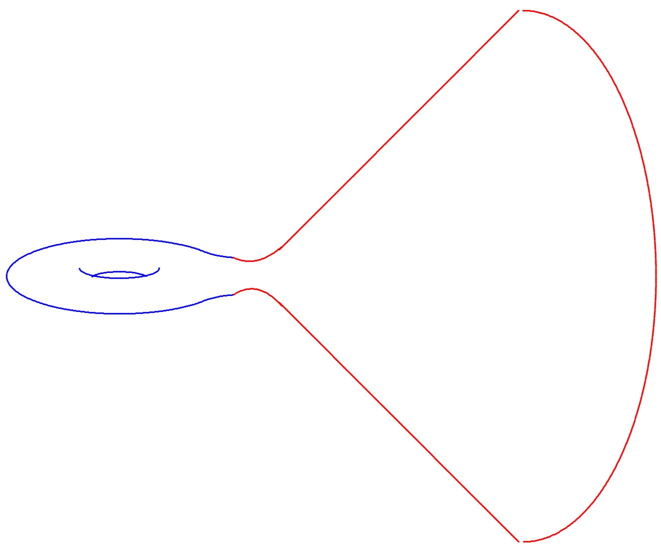

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

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Non-minimal resolutions typically admit families of such metrics for which the mass can be continuously deformed from negative to positive.

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So the mass is a "boundary correction" to the topological formula for the total scalar curvature.

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Corollary. *Any ALE scalar-flat Kähler manifold (M, g, J) of complex dimension m has mass given by*

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So **Theorem A** is an immediate consequence!

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$$d(\theta \wedge \omega) = \rho \wedge \omega = \frac{s}{4}\omega^2 = 0.$$

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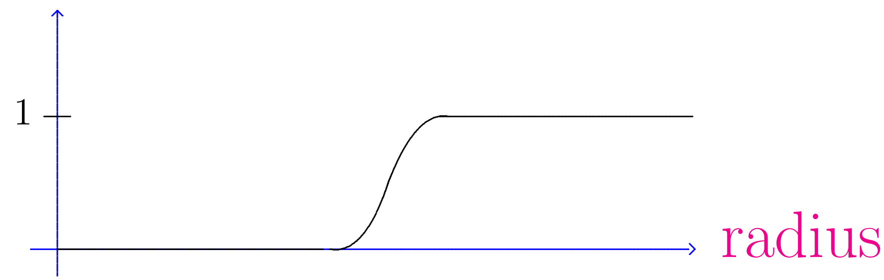

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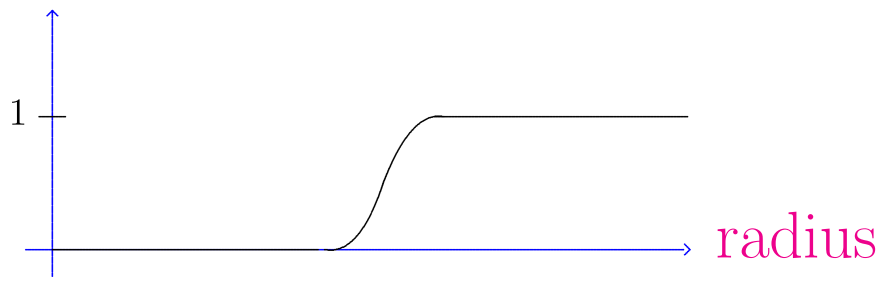

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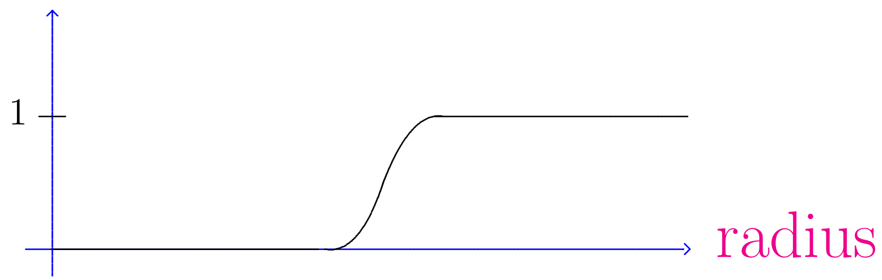

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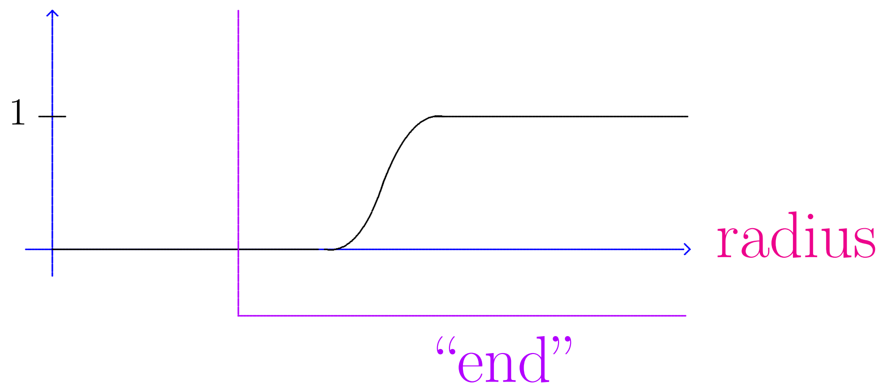

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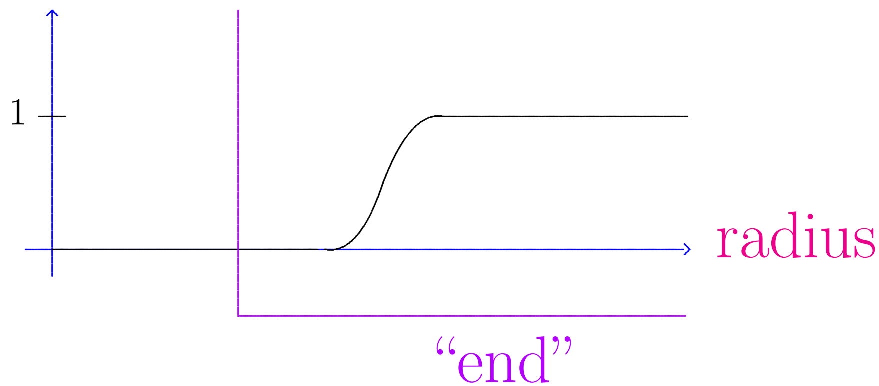

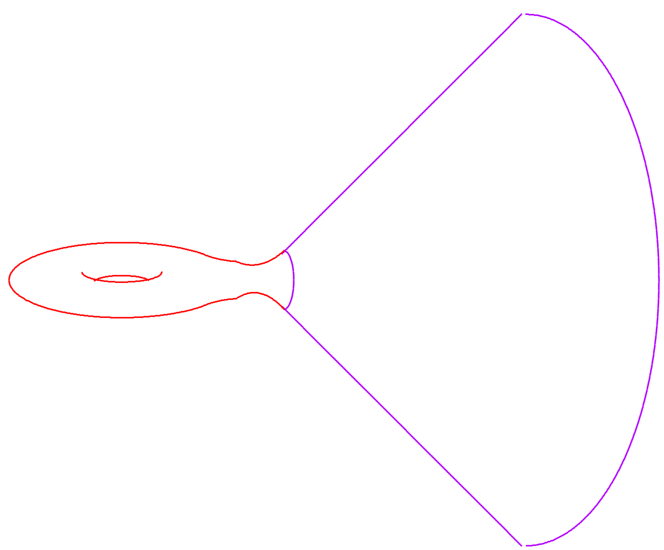

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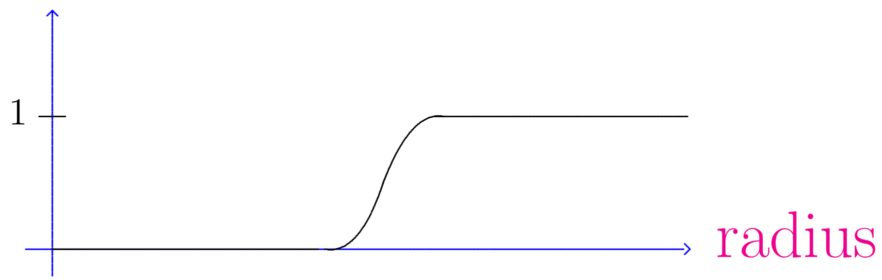

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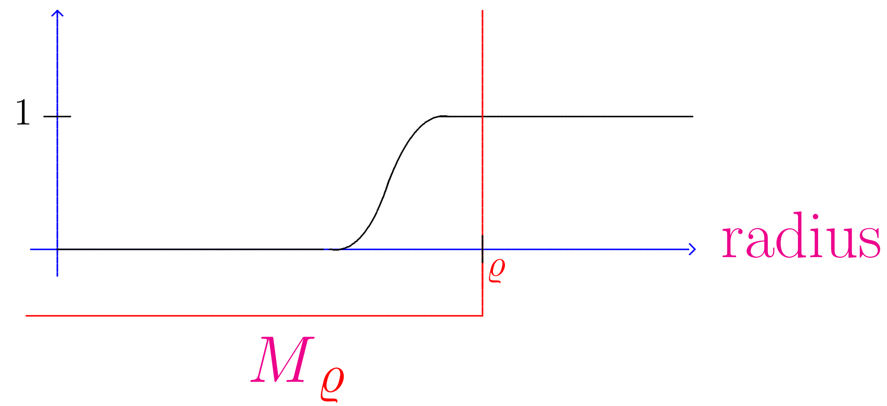

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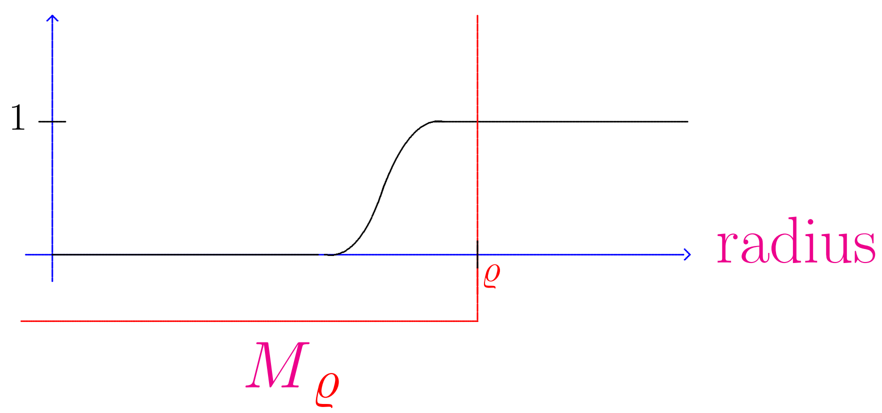

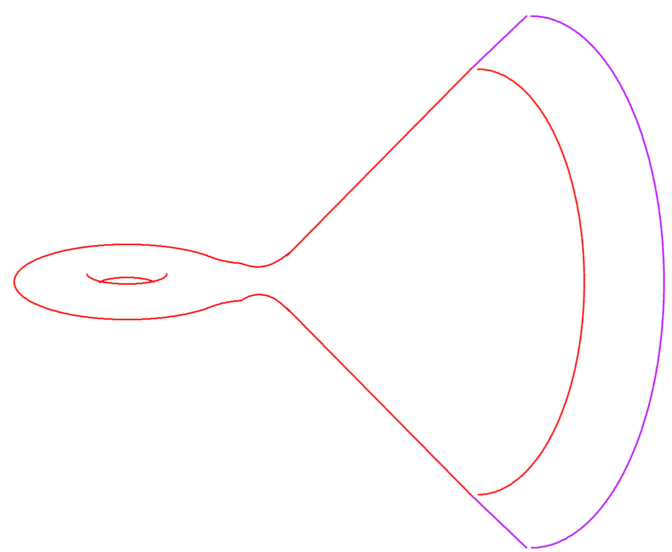

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Compactly supported, because $d\theta = \rho$ near infinity.

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where M_ϱ defined by radius $\leq \varrho$.

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Seen in "gravitational instantons"

and other explicit examples.

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$$J = J_0 + O(\varrho^{-3}), \qquad \nabla J = O(\varrho^{-4})$$

in suitable asymptotic coordinates adapted to g.

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Complete analytic family encodes info about J.

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Linear system of $\mathbb{CP}_{m-1}$ gives holomorphic map

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which is biholomorphism near $\mathbb{CP}_{m-1}$.

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This has some interesting consequences. . .

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Proof actually shows something stronger!

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so the mass formula implies the claim.

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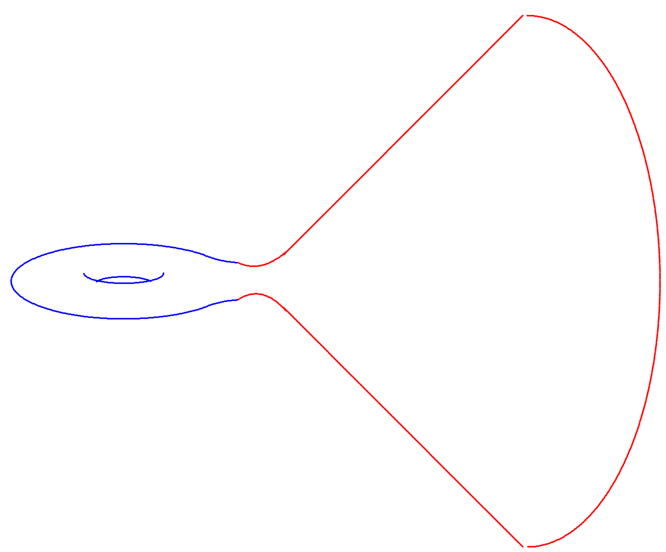

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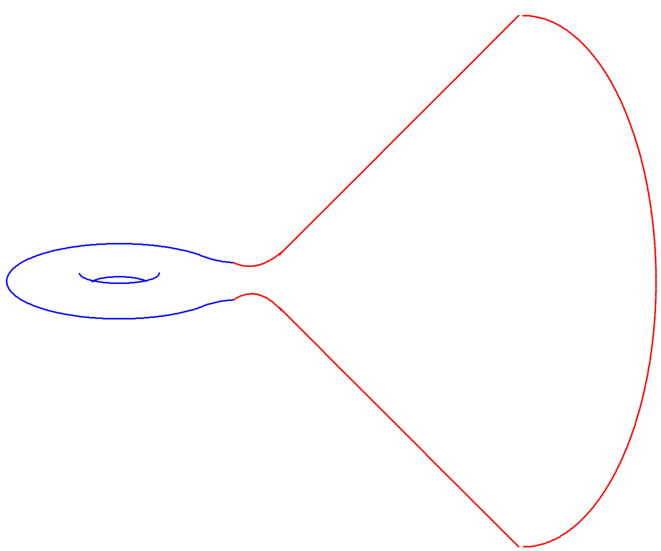

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