Einstein Manifolds,

Self-Dual Weyl Curvature, &

Conformally Kähler Geometry

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Definition. *A Riemannian metric* h

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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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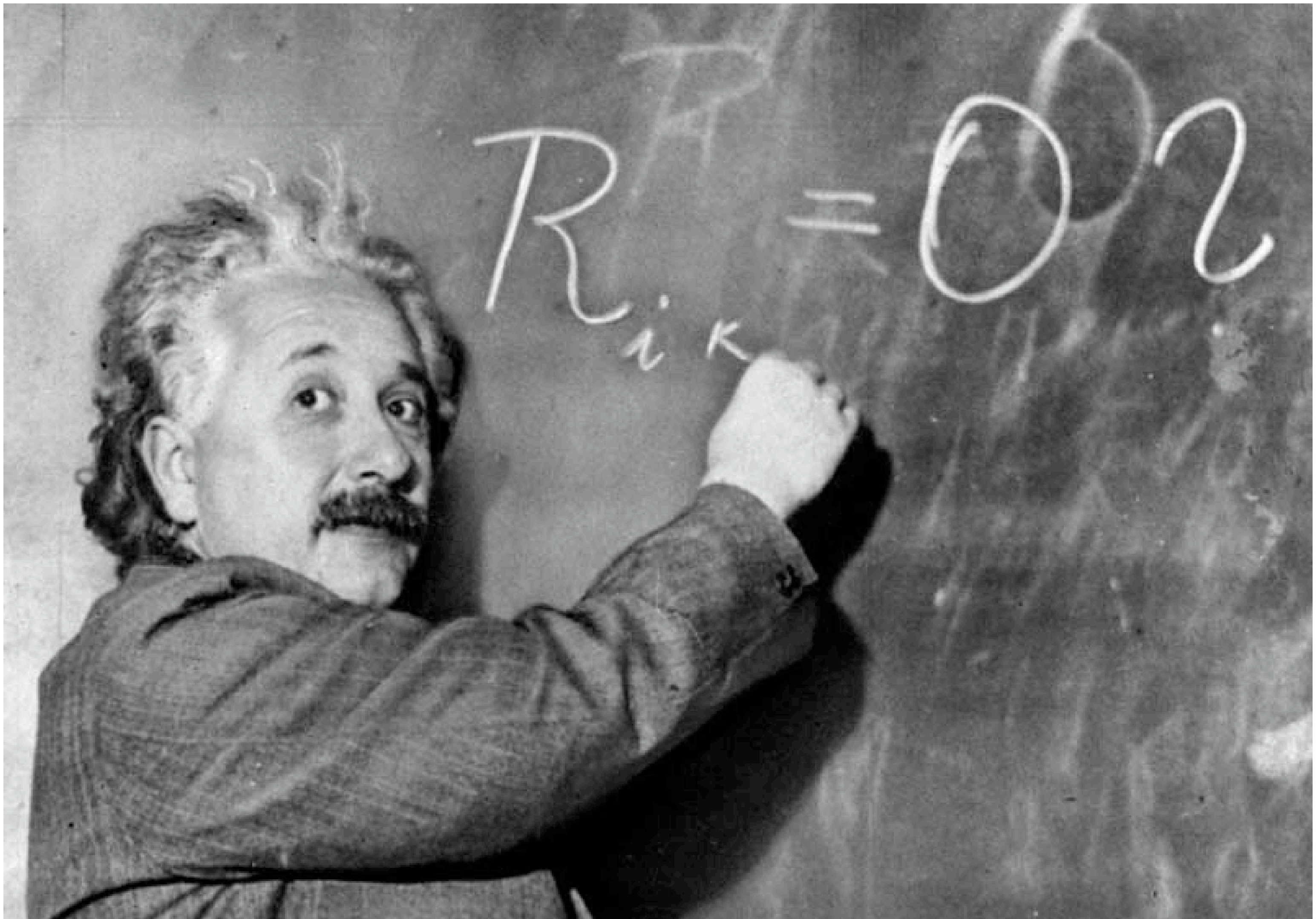

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Has same sign as the *scalar curvature*

$$s = r_j{}^j = \mathcal{R}^{ij}{}_{ij}.$$

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When $n = 4$, Einstein metrics satisfy a remarkable conformally-invariant condition.

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$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}{}_{[c}\delta^{b]}{}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c}\delta^{b}_{d]}$$

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$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

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Proposition. *Assume* $n \geq 4$. *Then* (M^n, g) *locally conformally flat* $\Longleftrightarrow W \equiv 0$.

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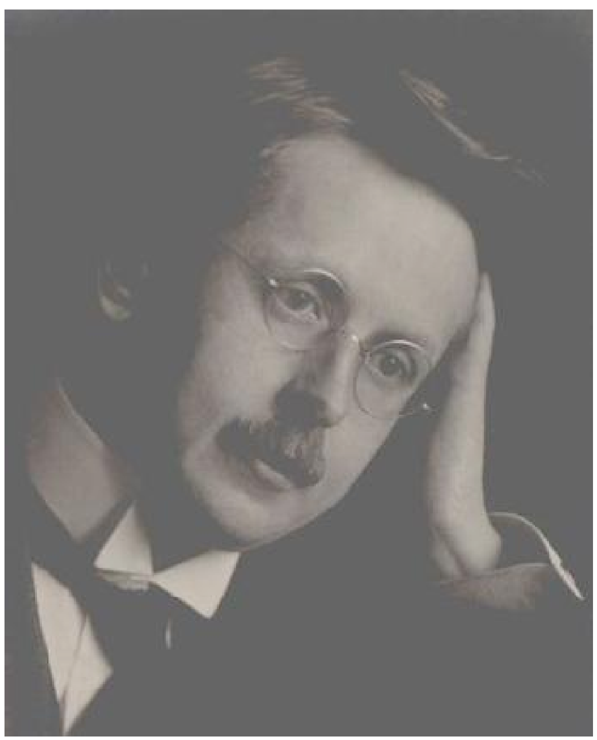

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Measures deviation $[g]$ from conformal flatness.

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Of course, conformally Einstein good enough!

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When $n = 4$, conf. Einstein $\Rightarrow$ critical for $\mathscr{W}$.

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$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array}\right)$$

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	Λ^{+*}	Λ^{-*}
Λ^{+}	$W_+ + \frac{s}{12}$	$\mathring{r}$
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$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

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Hence

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A case of special interest:

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(M^4, g, J) Kähler.

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$$|W_{+}|^2 = \frac{s^2}{24}$$

Restriction of $\mathscr{W}_+$ to Kähler metrics?

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Andrzej Derdziński : For Kähler metrics g,

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- $g_t = g + tB$ *is Kähler metric for small* t*.*

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where $\mathcal{F}$ is Futaki invariant.

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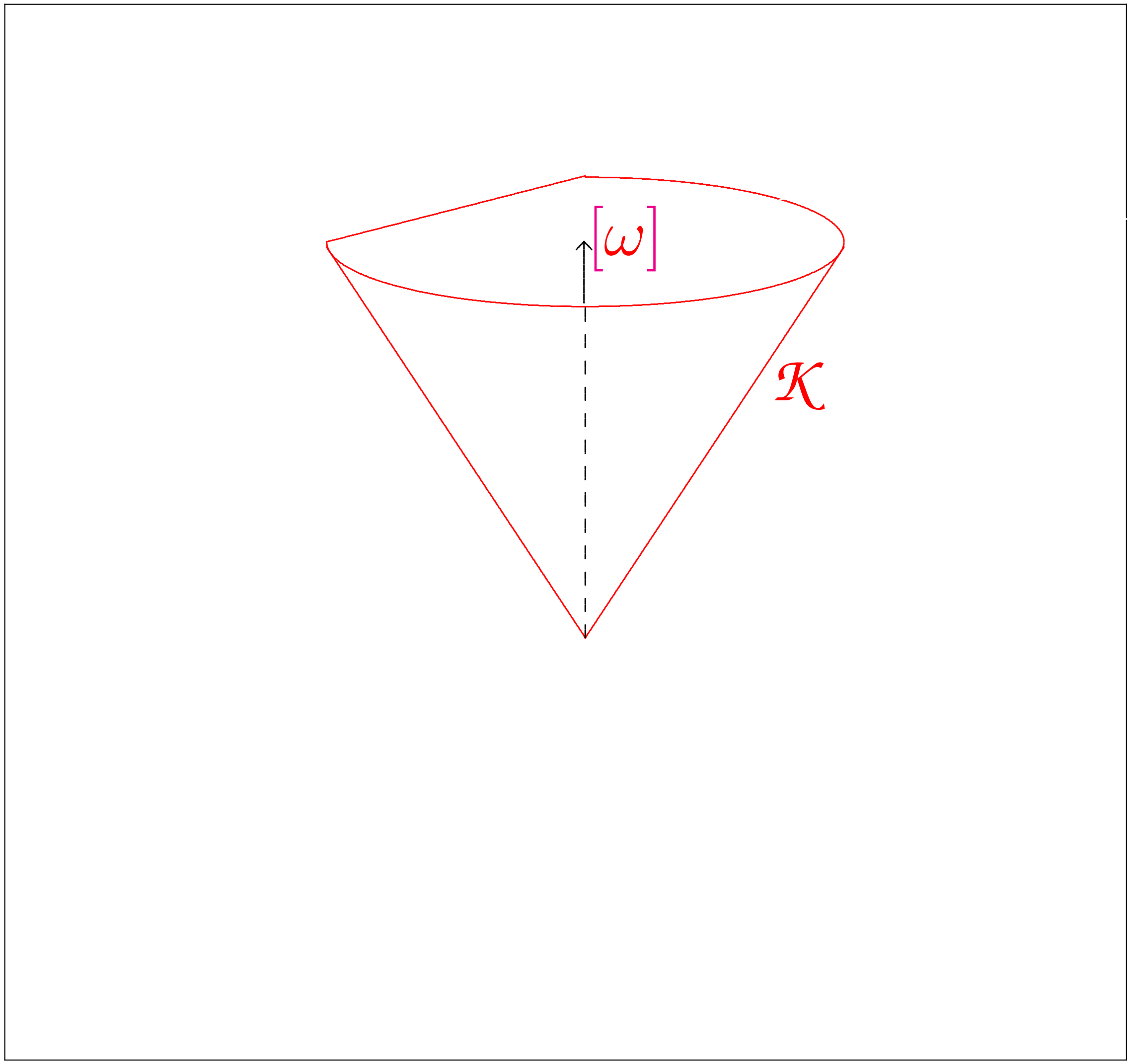
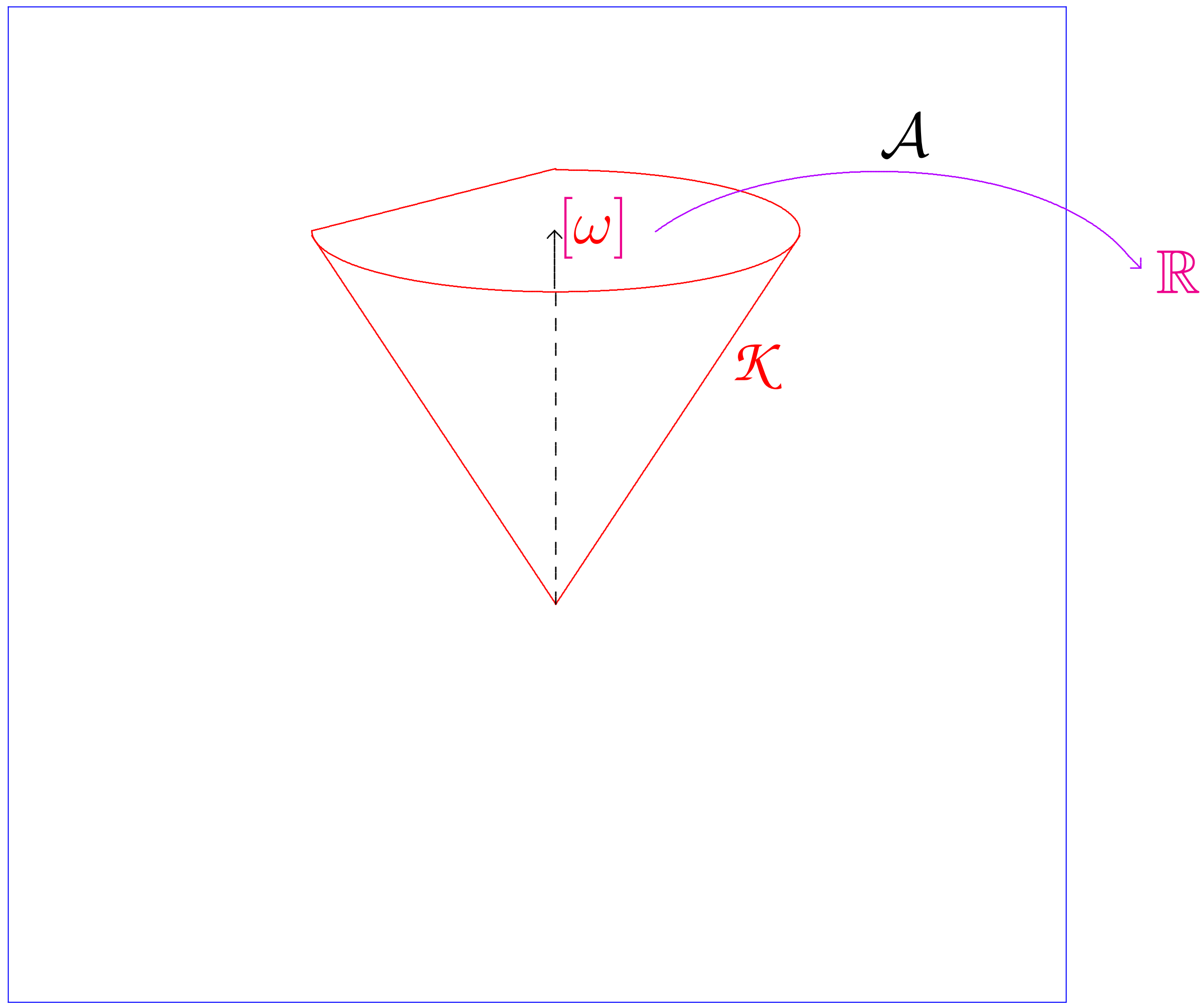
$\mathcal{A}$ is function on Kähler cone $\mathcal{K} \subset H^2(M, \mathbb{R})$.



$$\mathcal{K} \subset H^{1,1}(M, \mathbb{R}) \subset H^2(M, \mathbb{R})$$



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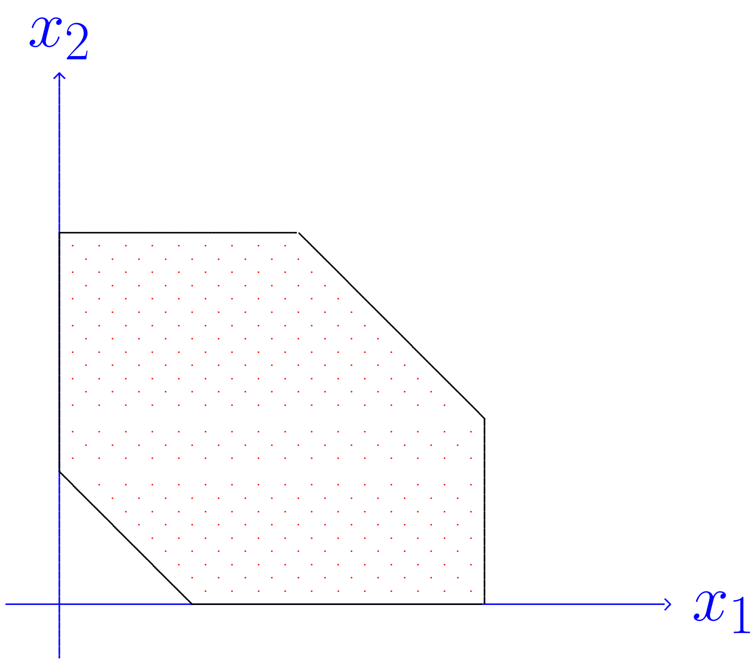
Most important cases are toric, and the action $\mathcal{A}$ can be directly computed from moment polygon.



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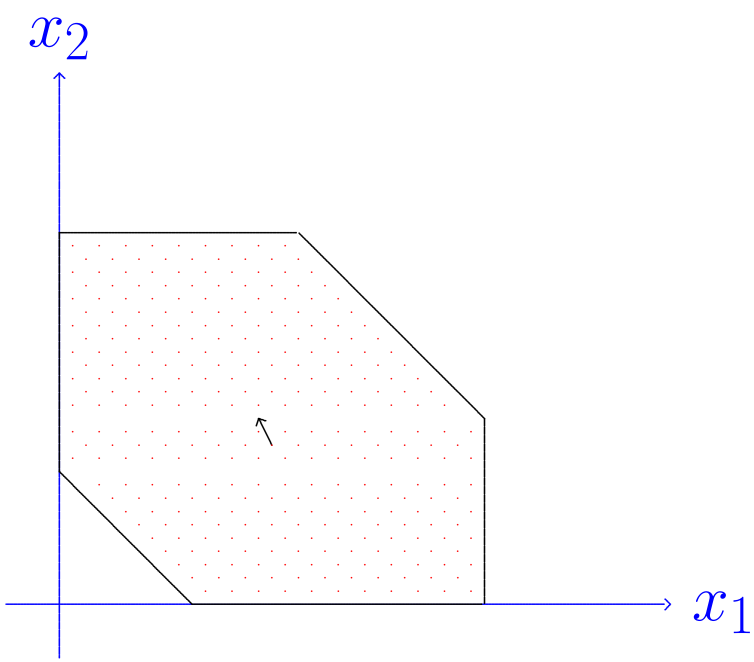

$$\mathcal{A}([\omega]) = \frac{|\partial P|^2}{2}\left(\frac{1}{|P|} + \vec{\mathfrak{D}} \cdot \Pi^{-1}\vec{\mathfrak{D}}\right)$$

Action Function on Kähler Cone

For any extremal Kähler (M^4, g, J),

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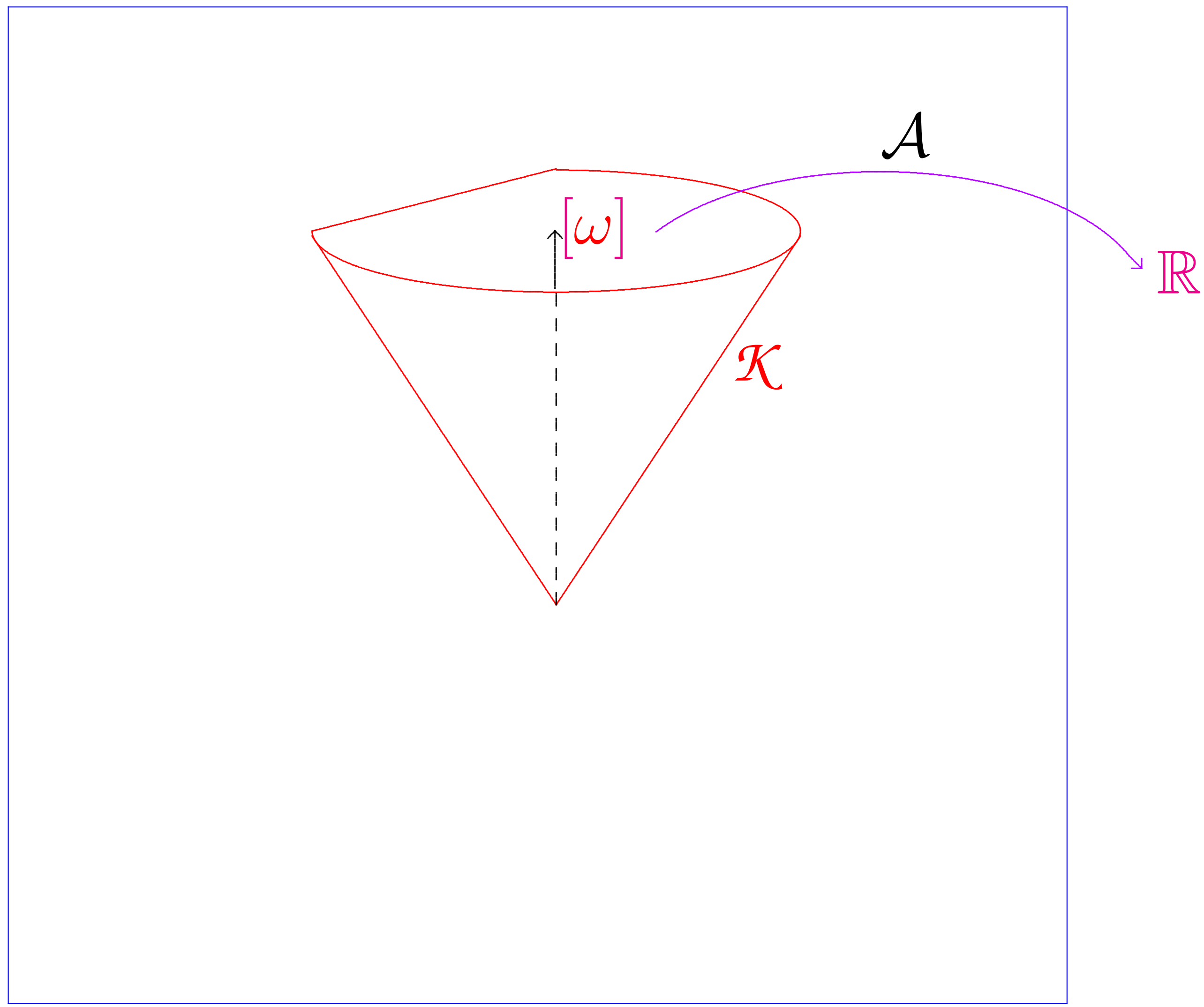

$$\mathcal{K} \subset H^{1,1}(M, \mathbb{R}) \subset H^2(M, \mathbb{R})$$

Restriction of $\mathscr{W}_+$ to Kähler metrics?

On Kähler metrics,

$$\int |W_+|^2 d\mu = \int \frac{s^2}{24} d\mu$$

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Andrzej Derdziński : For Kähler metrics g,

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Global implications?

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Moreover, each case actually occurs.

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If not Kähler-Einstein:

I. s *is positive. Then*

$(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. s *is zero. Then*

(M, g, J) *SFK, but not Ricci-flat.*

III. s *changes sign. Then*

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This happens $\Longleftrightarrow$ $c_1 > 0$.

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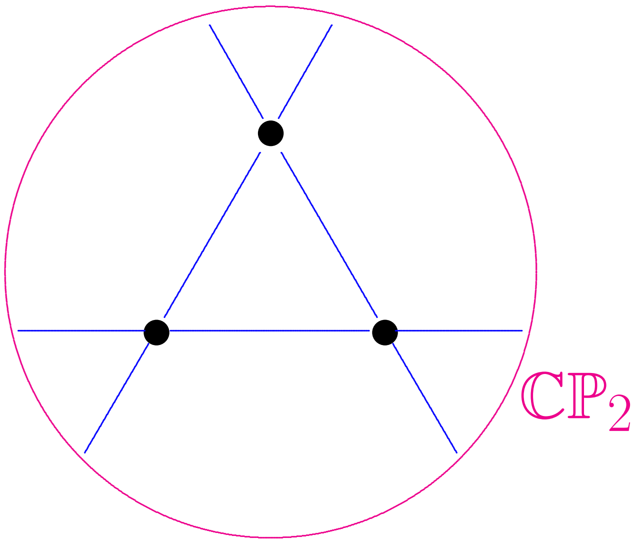

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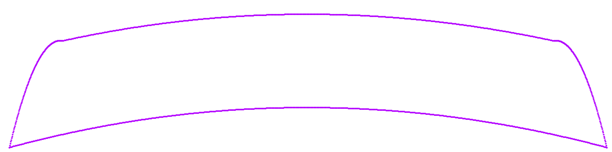

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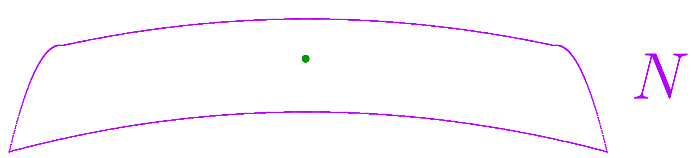

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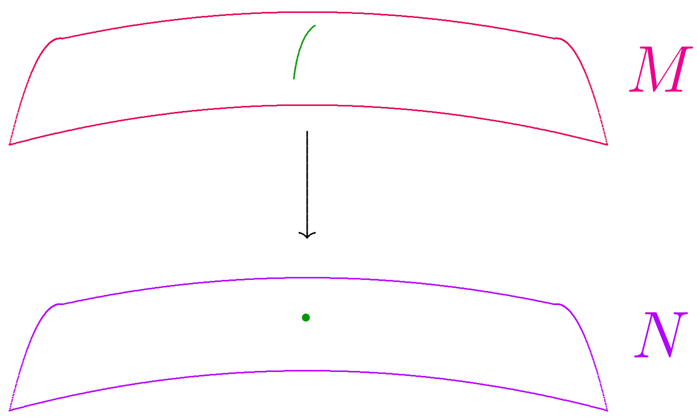

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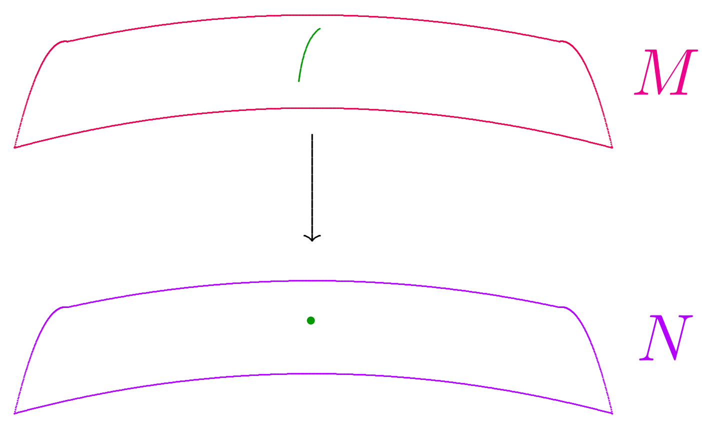
If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$



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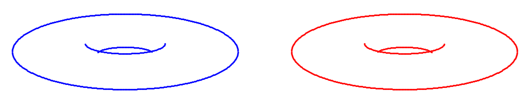

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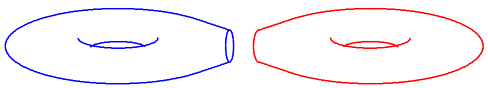

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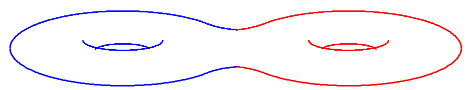

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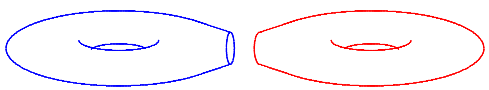

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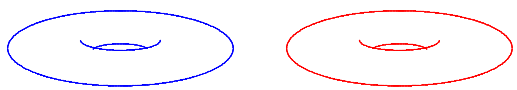

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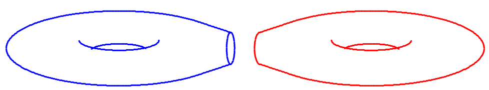

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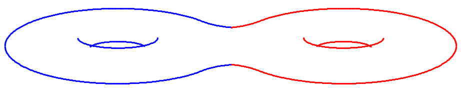

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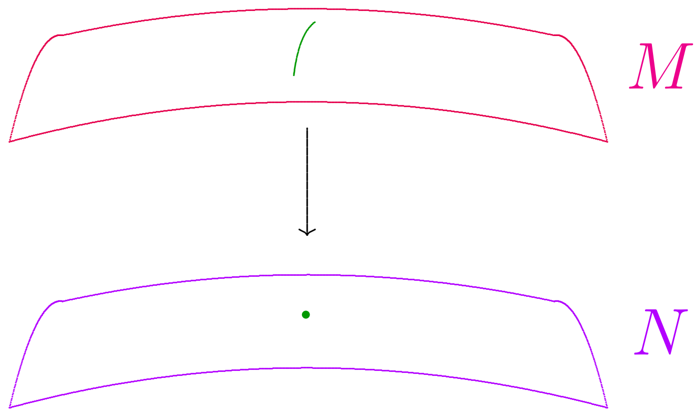

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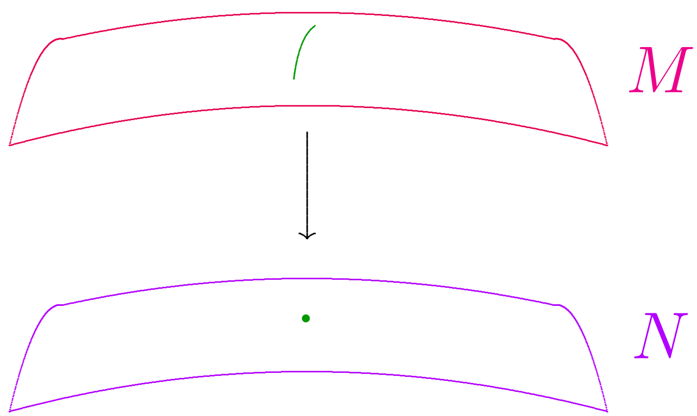

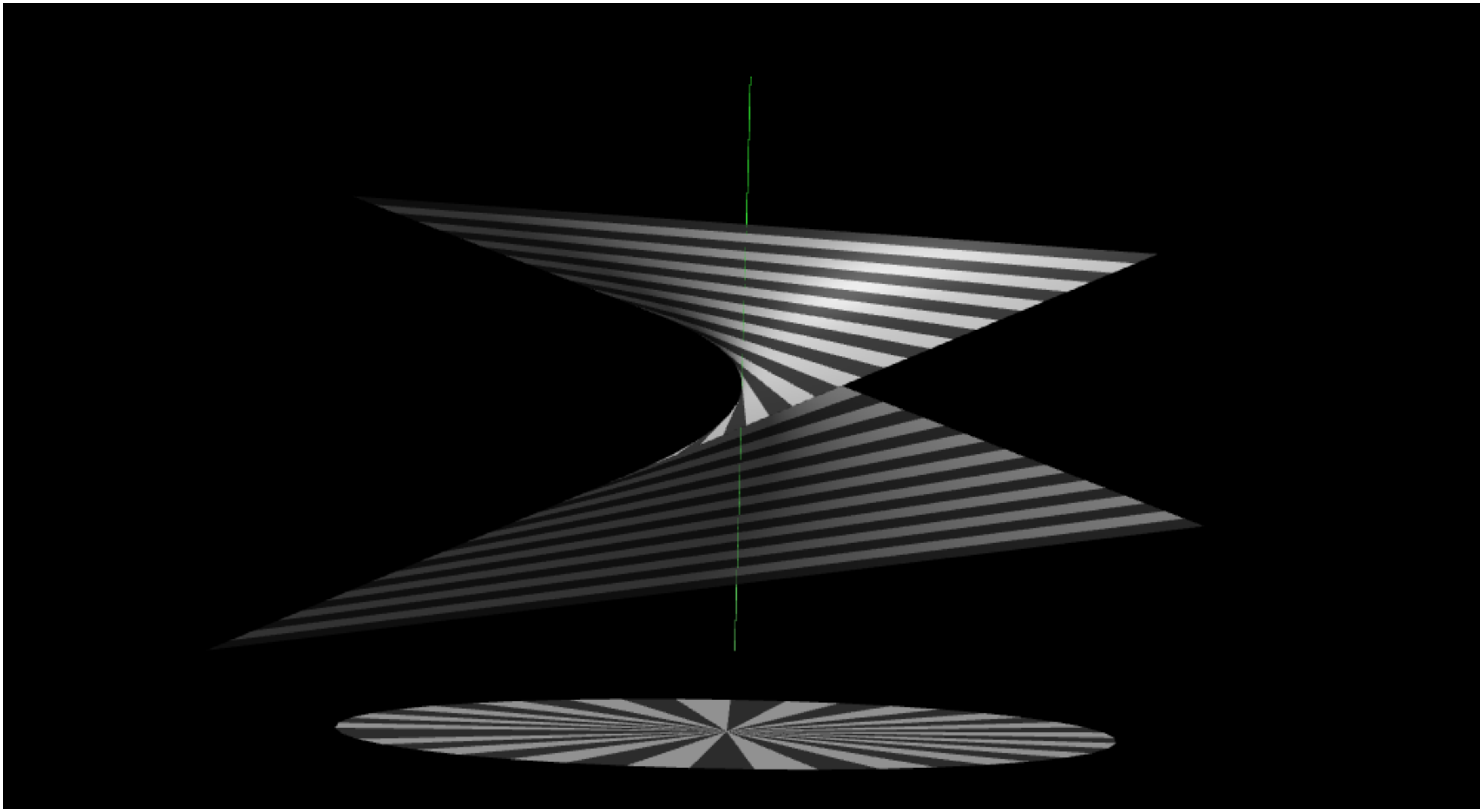

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If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

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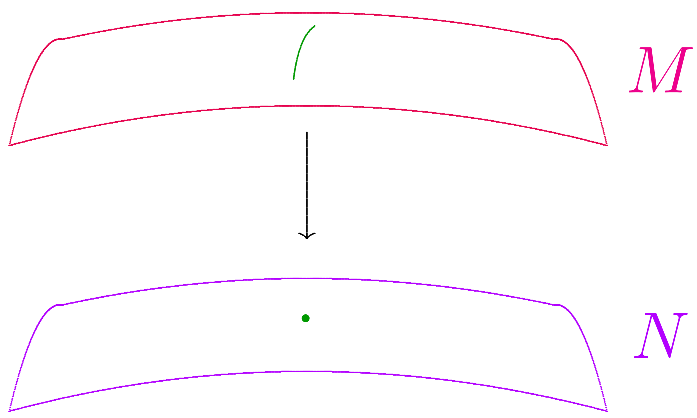

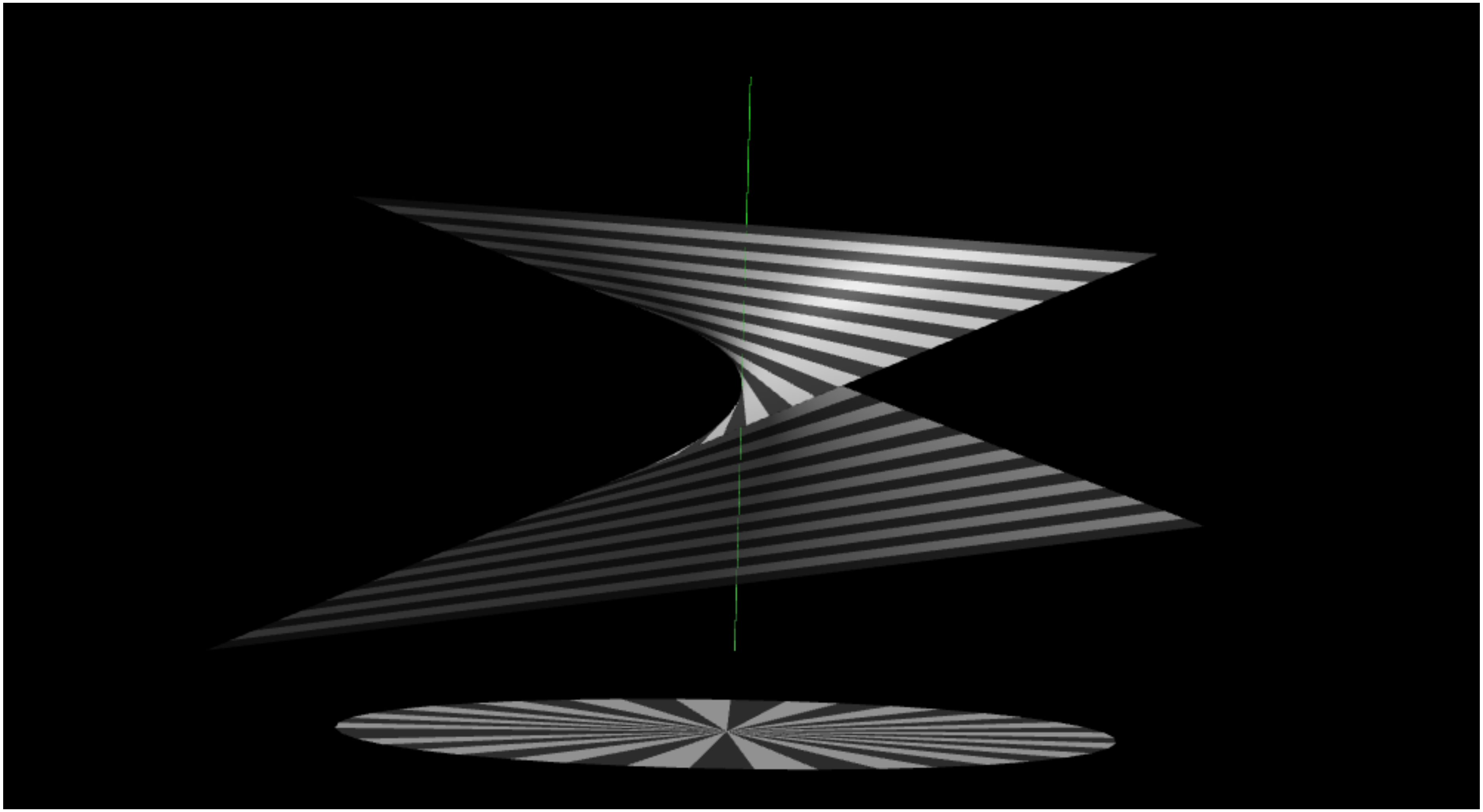

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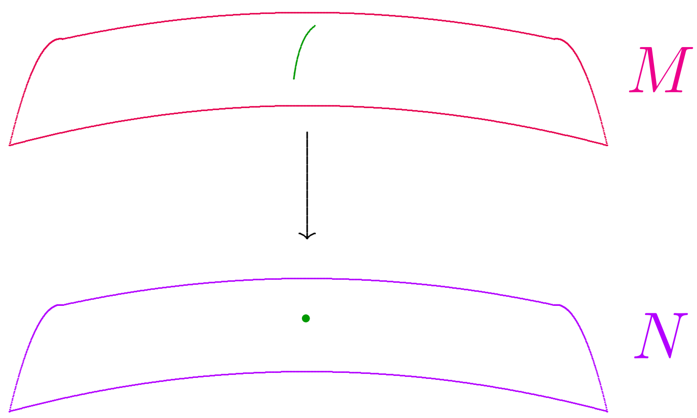

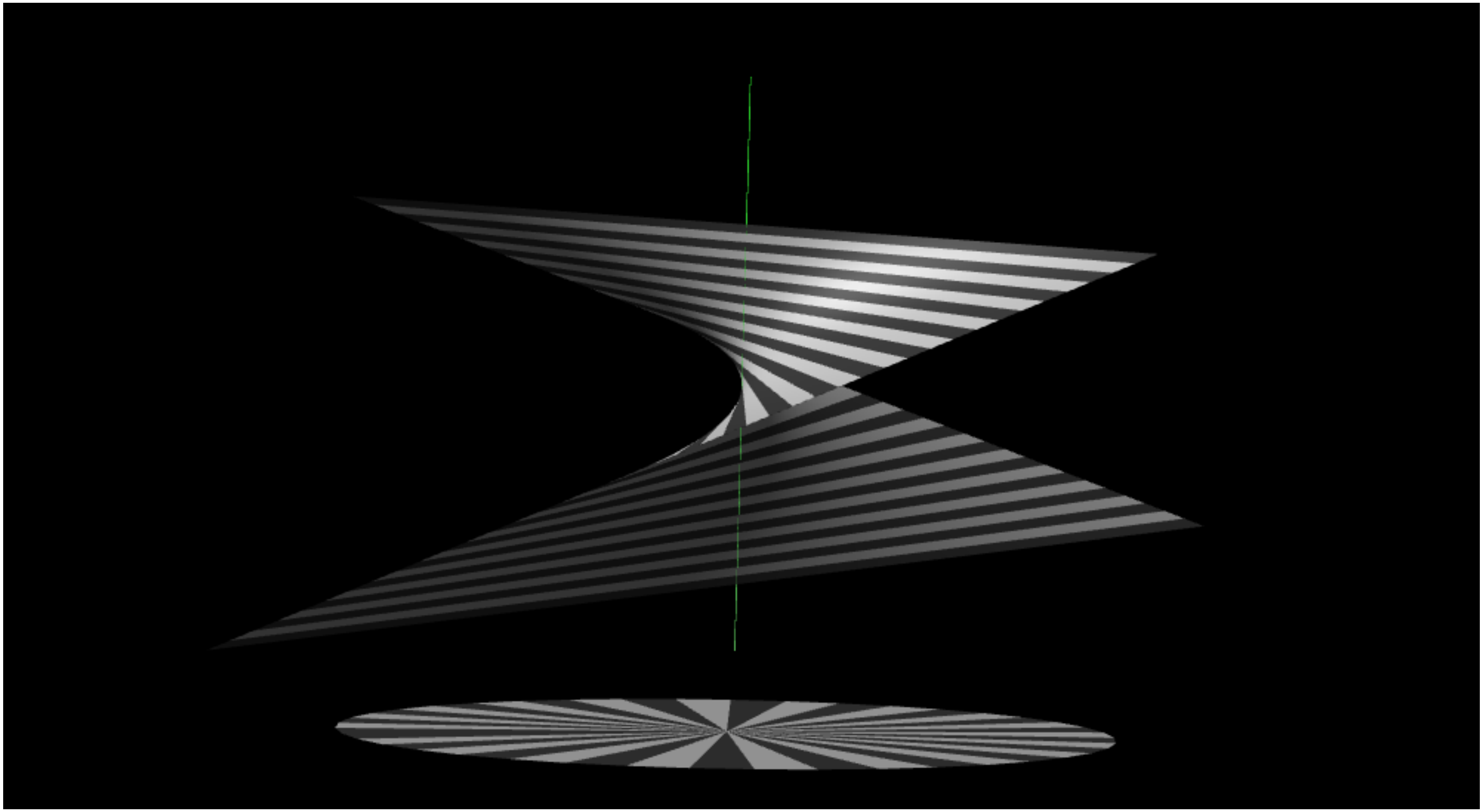

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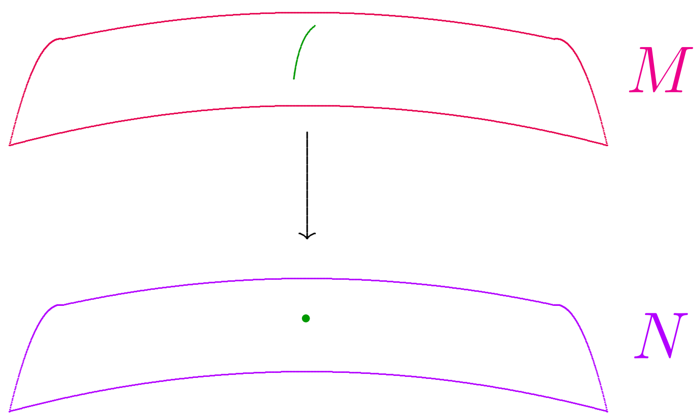
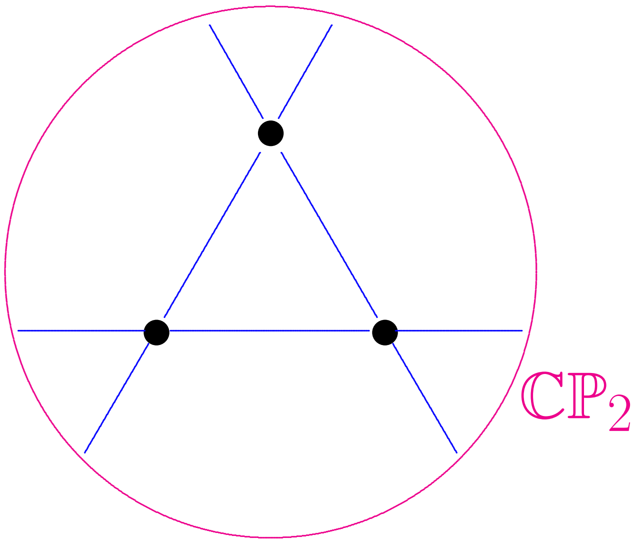


Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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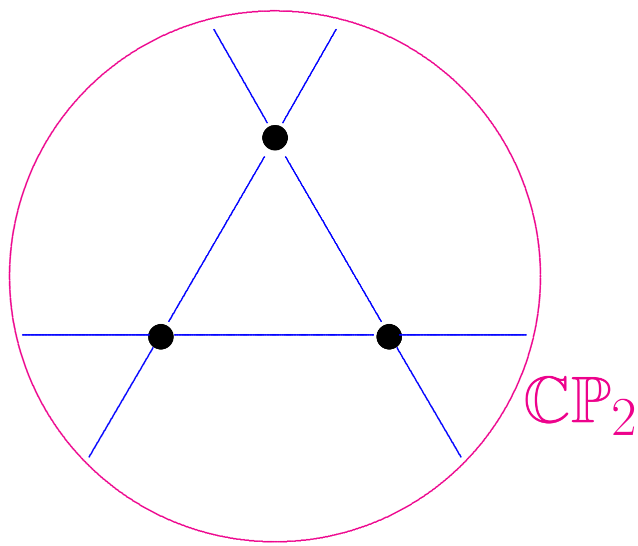

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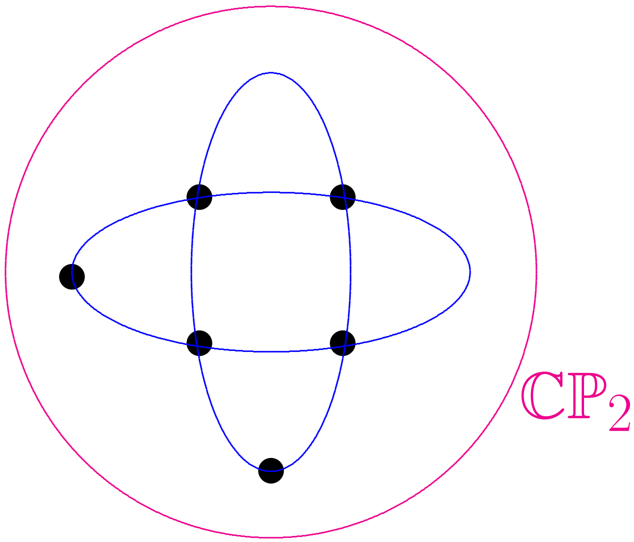

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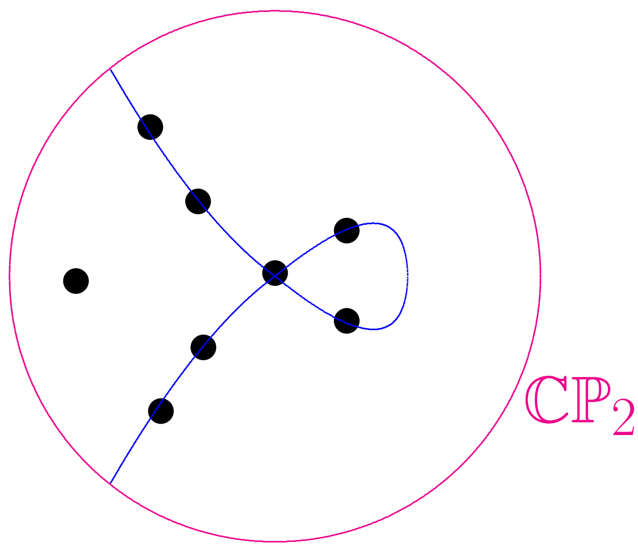

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Existence: Page

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One reason this seems satisfying. . .

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But this is not needed in above result.

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Understand all Einstein metrics on del Pezzos.

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Exactly one connected component of moduli space!

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Kähler $\Longrightarrow \Lambda^+ = \mathbb{R}\omega \oplus \Re e \Lambda^{2,0}$

$$W^+ = \text{trace-free part of } \begin{bmatrix} 0 & & \\ & 0 & \\ & & \frac{s}{4} \end{bmatrix}$$

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$$g \rightsquigarrow h = f^2 g \quad \Longrightarrow \quad \det(W^+) \rightsquigarrow f^{-6}\det(W^+).$$

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method also proves more general results.

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Theorem B.

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Claim: (M, h) compact Einstein $\Longrightarrow$ J integrable.

Theorem B. *Let* (M, h) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W^{+} : \Lambda^{+} \to \Lambda^{+}$$

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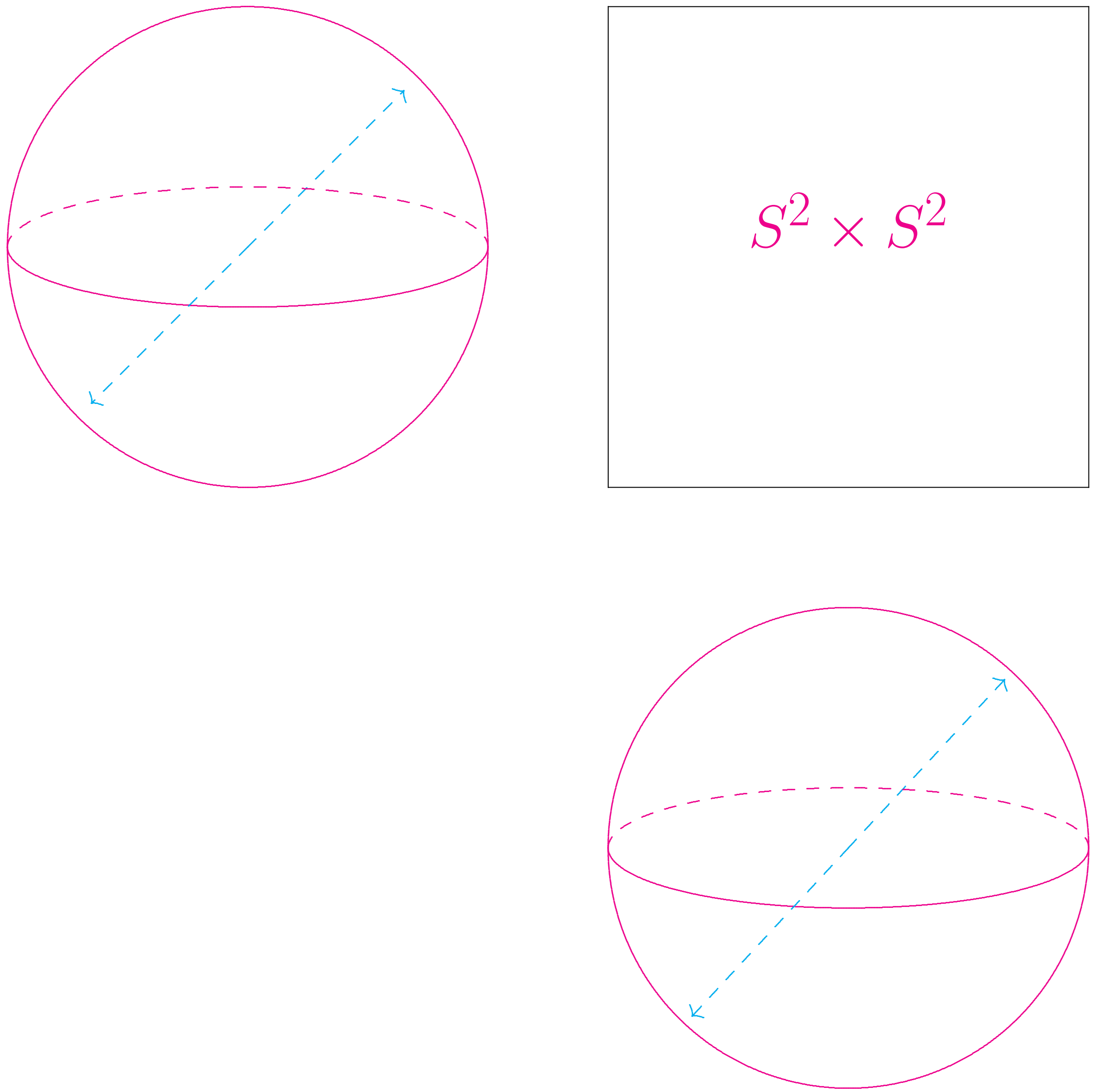
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Correctly understand equation $\delta W^+ = 0$.

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$$0 = \nabla^*\nabla W^+ + \frac{s}{2}W^+ - 6W^+ \circ W^+ + 2|W^+|^2 I$$

for $W^+ \in \mathrm{End}(\Lambda^+)$, with respect to h.

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This identity has many applications.

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Example. If $\exists$ harmonic ω with $W^+(\omega, \omega) > 0$,

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thus showing that g must actually be Kähler.

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To analyze Wu's criterion:

Let $\alpha \geq \beta \geq \gamma$ be eigenvalues of W^+:

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So $\alpha = \alpha_h : M \to \mathbb{R}^+$ a smooth function. Set

$$f = {\alpha_h}^{-1/3}, \qquad g = f^{-2}h = {\alpha_h}^{2/3}h.$$

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Now choose $\omega \in \Gamma\Lambda^+$ so that

$$W_g^+(\omega) = \alpha\,\omega, \quad |\omega|_g \equiv \sqrt{2},$$

after at worst passing to double cover $\hat{M} \to M$.

$$0 = \int_{\hat{M}} \Big[\langle W^+, \nabla^*\nabla(\omega \otimes \omega) \rangle + \frac{s}{2} W^+(\omega, \omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2 \Big] f \, d\mu$$

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$$0 = \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) - 2W^+(\omega, \nabla^e \nabla_e \omega) + \frac{s}{2} W^+(\omega, \omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2 \Big] f \, d\mu$$

$$0 = \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) - 2\alpha\langle \omega, \nabla^e \nabla_e \omega \rangle$$
$$+\frac{s}{2}\alpha|\omega|^2 - 6\alpha^2|\omega|^2 + 2|W^+|^2|\omega|^2\Big] f \; d\mu$$

because

$$W_g{}^+(\omega) = \alpha\omega$$

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$$0 \geq \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha\langle \omega, \nabla^*\nabla\omega\rangle$$
$$+\frac{s}{2}\alpha|\omega|^2 - 6\alpha^2|\omega|^2 + 3\alpha^2|\omega|^2\Big] f \; d\mu$$

because

$$|W_g{}^+|^2 \geq \frac{3}{2}\alpha^2$$

$$0 \geq \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha \langle \omega, \nabla^* \nabla \omega \rangle$$

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$$|\omega|_g^2 = 2 \quad \Longrightarrow \quad (\nabla_e \omega) \perp \omega$$

$$\det(W^+) > 0 \quad \Longrightarrow \quad W^+(\nabla_e \omega, \nabla^e \omega) \leq 0$$

$$0 \geq \int_M \Big[\quad 2\alpha\langle \omega, \nabla^*\nabla\omega\rangle + \frac{s}{2}\alpha|\omega|^2 - 3\alpha^2|\omega|^2\Big] f \; d\mu$$

$$|\omega|_g^2 = 2 \quad \Longrightarrow \quad (\nabla_e\omega) \perp \omega$$

$$\det(W^+) > 0 \quad \Longrightarrow \quad -W^+(\nabla_e\omega, \nabla^e\omega) \geq 0$$

$$0 \geq \int_M \left[2\alpha \langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2} \alpha |\omega|^2 - 3\alpha^2 |\omega|^2 \right] f \, d\mu$$

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But

$$\alpha f \equiv 1$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2} |\omega|^2 - 3|\omega|^2 \alpha \right] d\mu$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle - 3W^+(\omega, \omega) + \frac{s}{2}|\omega|^2 \right] d\mu$$

$$0 \geq \int_M \left[\frac{1}{2}|\nabla\omega|^2 + \frac{3}{2}\langle\omega, \left(\nabla^*\nabla - 2W^+ + \frac{s}{3}\right)\omega\rangle\right] d\mu$$

$$0 \geq \int_M \left[\tfrac{1}{2}|\nabla\omega|^2 + \tfrac{3}{2}\langle\omega, \quad (d+d^*)^2 \quad \omega\rangle\right] d\mu$$

Because

$$(d+d^*)^2 = \nabla^*\nabla - 2W^+ + \frac{s}{3}$$

on $\Gamma\Lambda^+$.

$$0 \geq \frac{1}{2} \int_M |\nabla \omega|^2 \, d\mu + 3 \int_M |d\omega|^2 \, d\mu$$

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So $\nabla \omega \equiv 0$, and g is Kähler!

Theorem B. *Let* (M, h) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M*. Then* h *is conformally Kähler, and* M *is a Del Pezzo surface.*

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Produces harmonic ω with $W^+(\omega, \omega) > 0$.

Now use my earlier result!

Theorem C. *Let* (M, h) *be a compact oriented Riemannian* 4*-manifold with* $\delta W^+ = 0$. *If*

$$W^+ \neq 0 \quad \textit{and} \quad \det(W^+) \geq -\frac{5\sqrt{2}}{21\sqrt{21}}|W^+|^3$$

everywhere on M, *then actually* $\det(W^+) > 0$. *In particular, if* (M, h) *is a simply-connected Einstein manifold, then* h *is conformally Kähler, and* M *is a Del Pezzo surface.*

Obrigado por me convidar!

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