Einstein Metrics,

Harmonic Forms, &

Conformally Kähler Geometry

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Stony Brook University

Seminário Geometria em Lisboa
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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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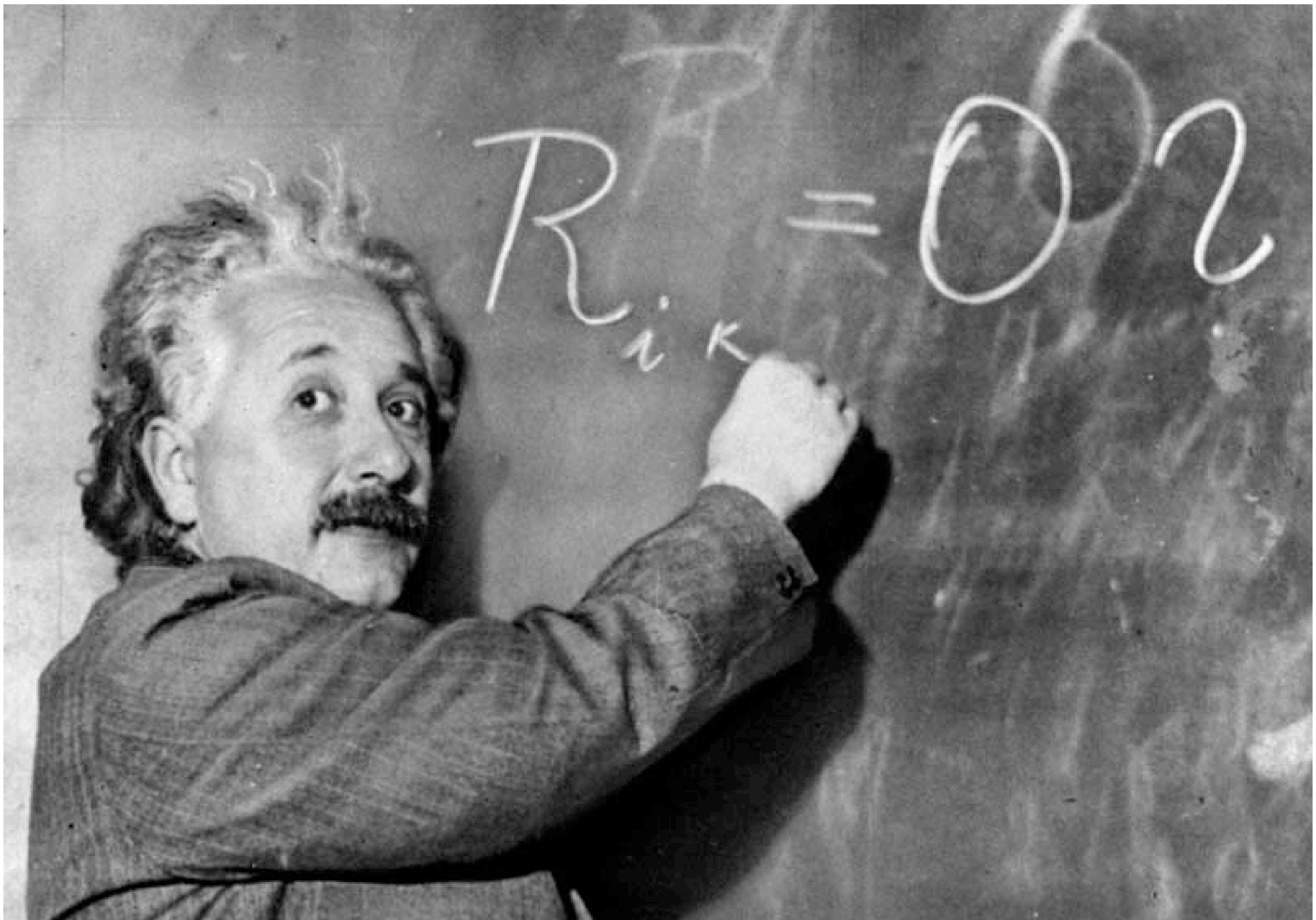

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$$s = r_j^{\ j} = \mathcal{R}^{ij}{}_{ij}.$$

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When $n = 4$, situation is more encouraging...

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One key question:

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One key question:

Does enough rigidity really hold in dimension four to make this a genuine geometrization?

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Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric h (unrelated to ω)? What if we also require $\lambda \geq 0$?*

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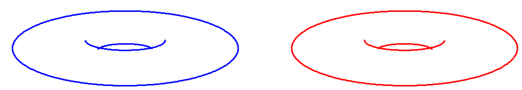

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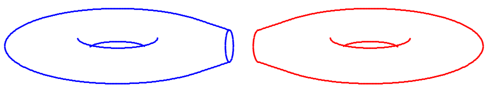

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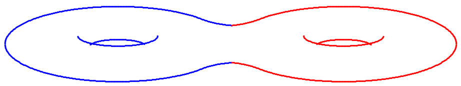

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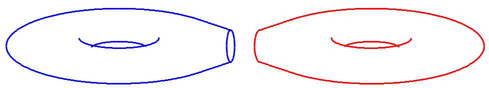

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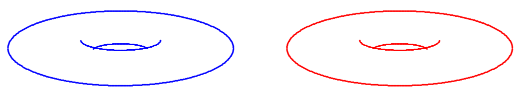

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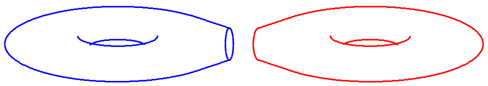

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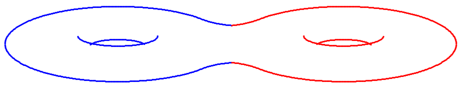

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Calabi/Yau: Admits Ricci-flat Kähler metrics.

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Definitive list . . .

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Know an Einstein metric on each manifold.

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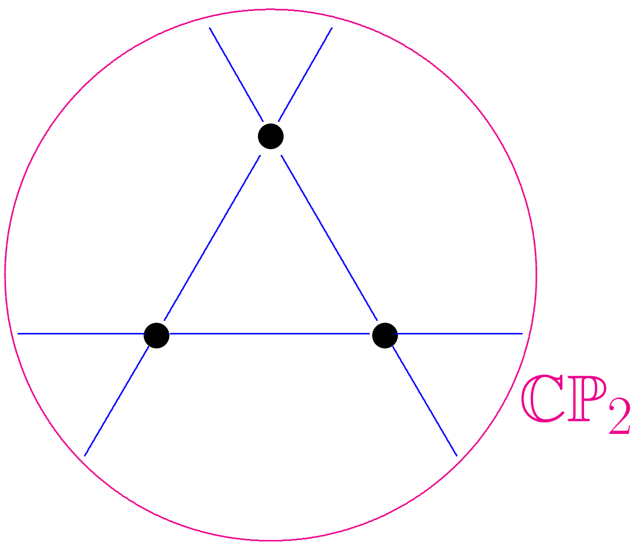

Blowing up:

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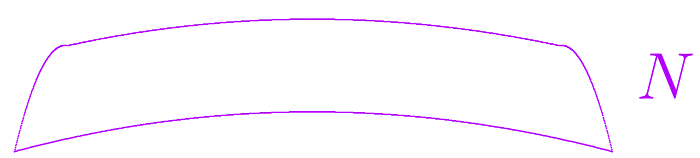

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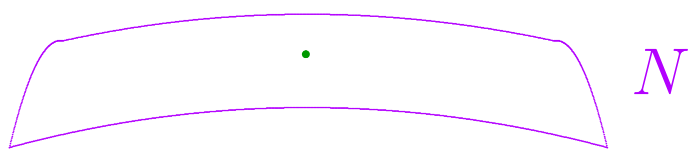

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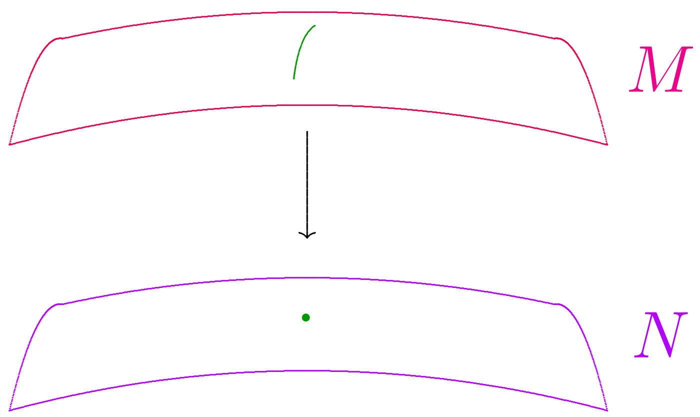

Blowing up:

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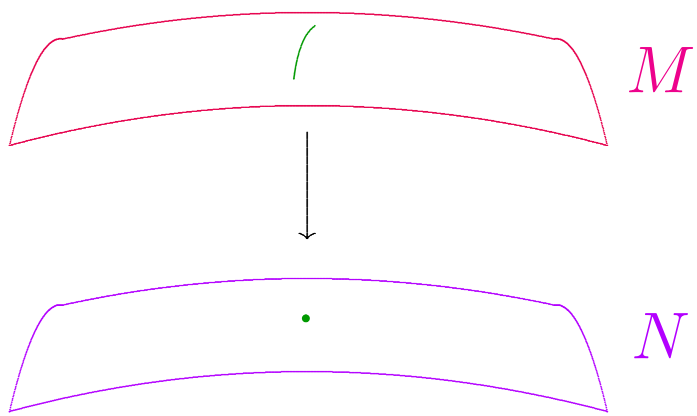
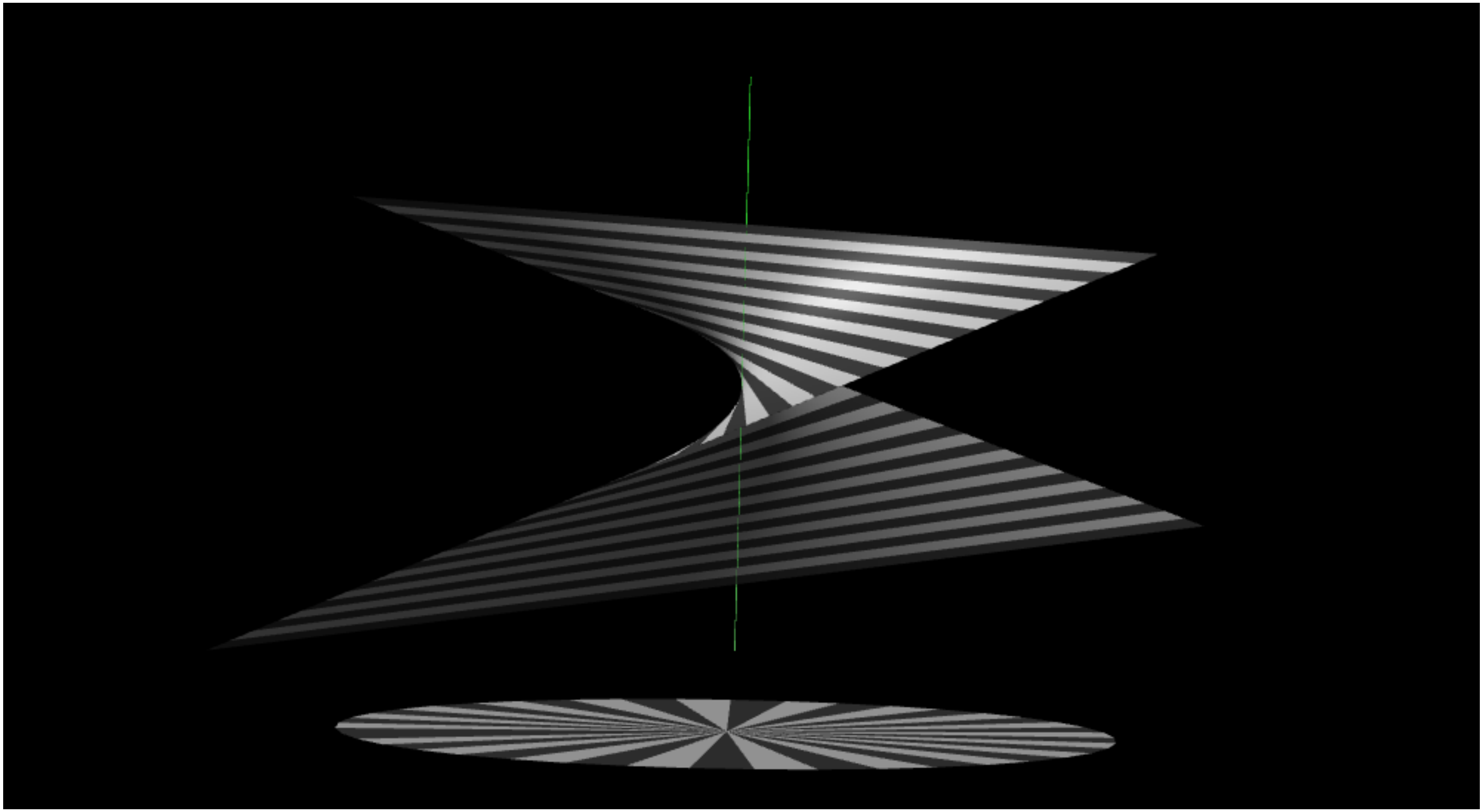
$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.





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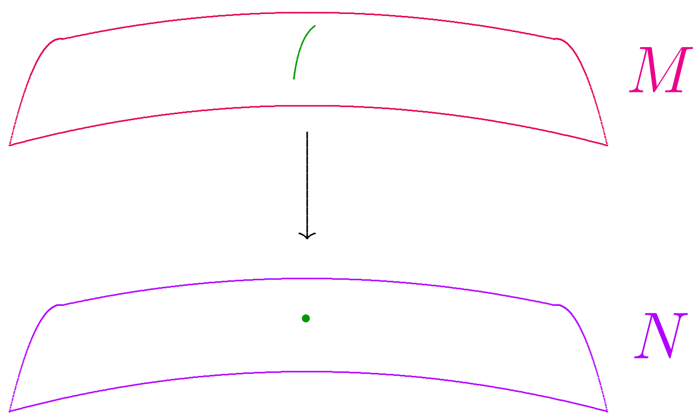

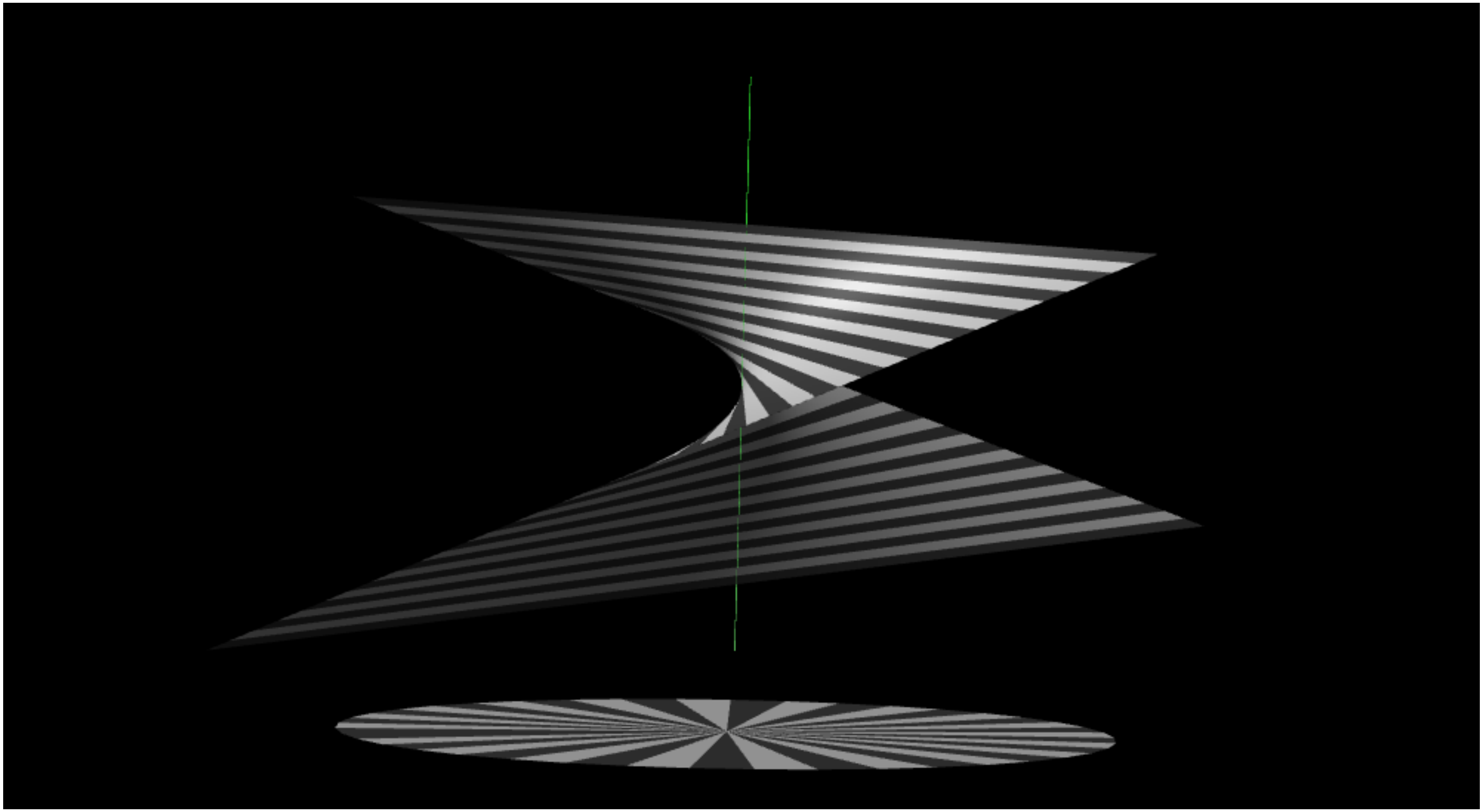

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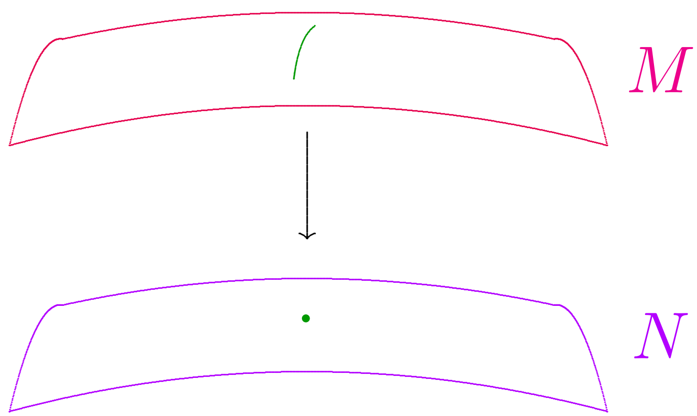

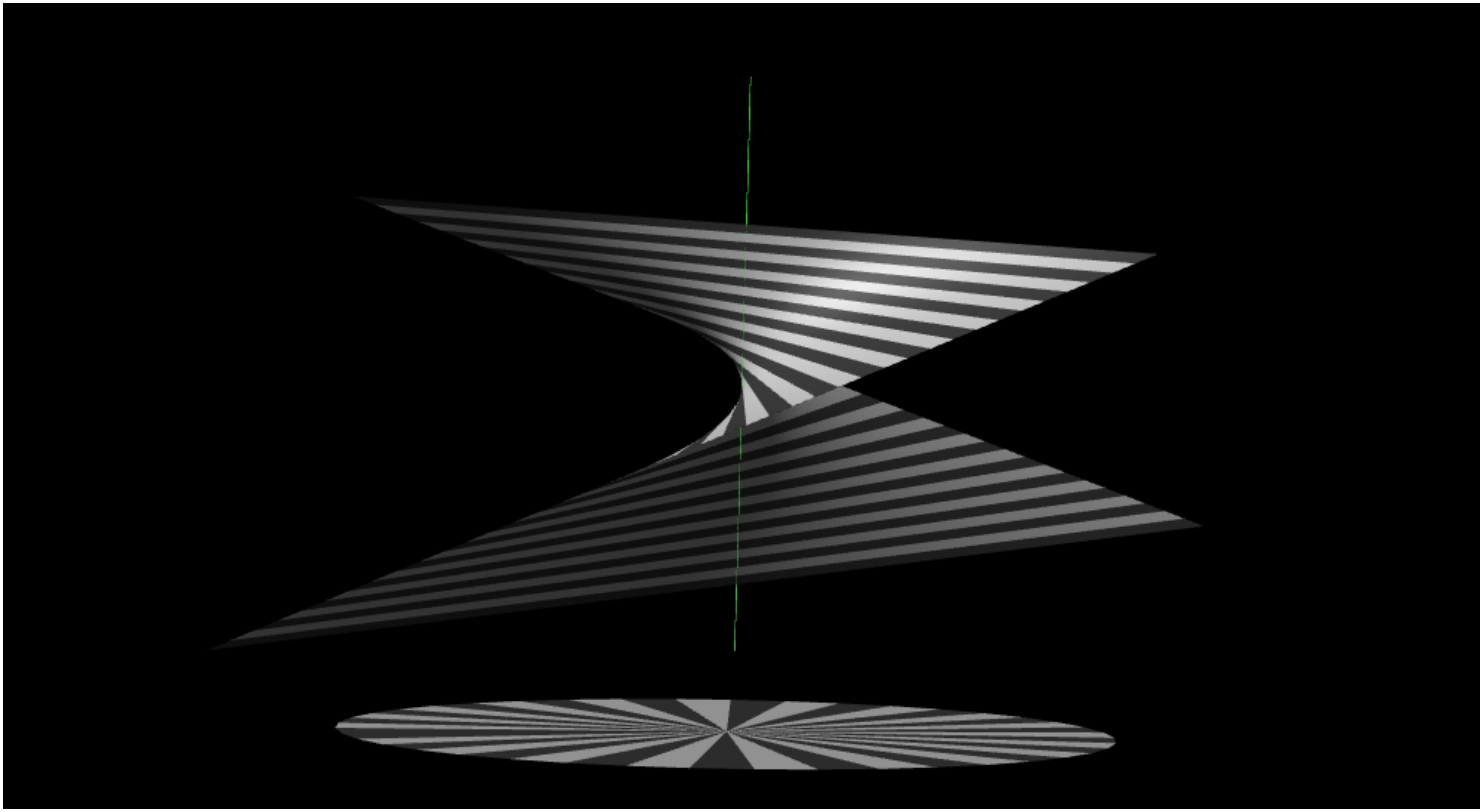

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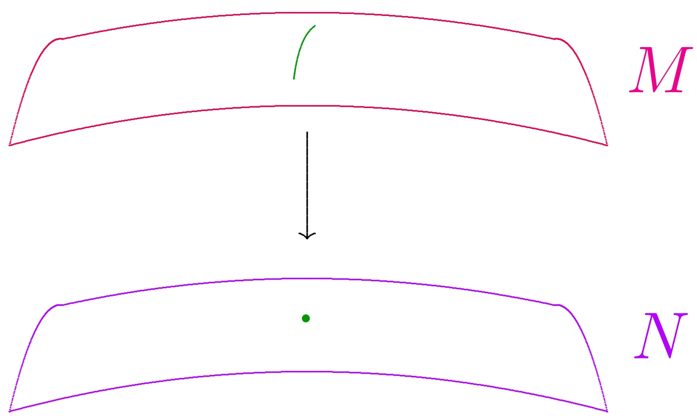

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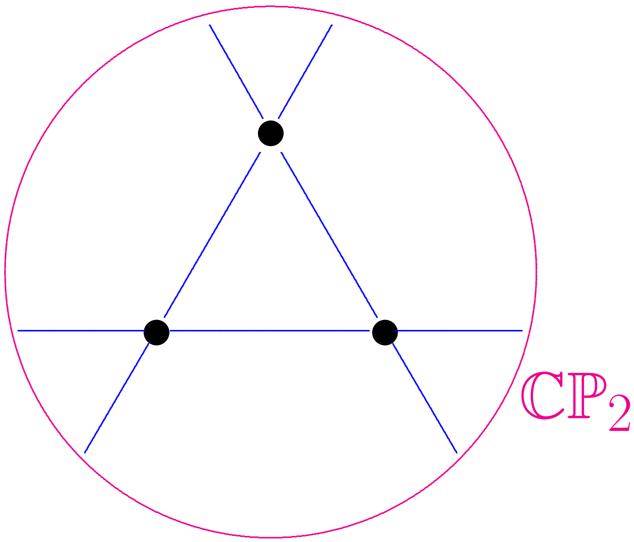

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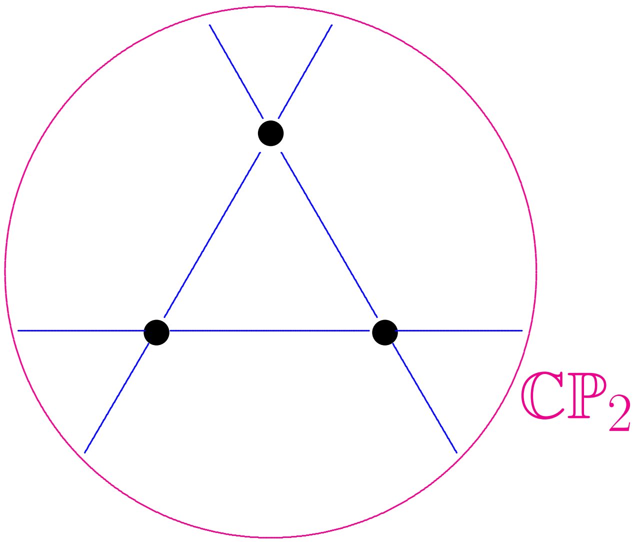

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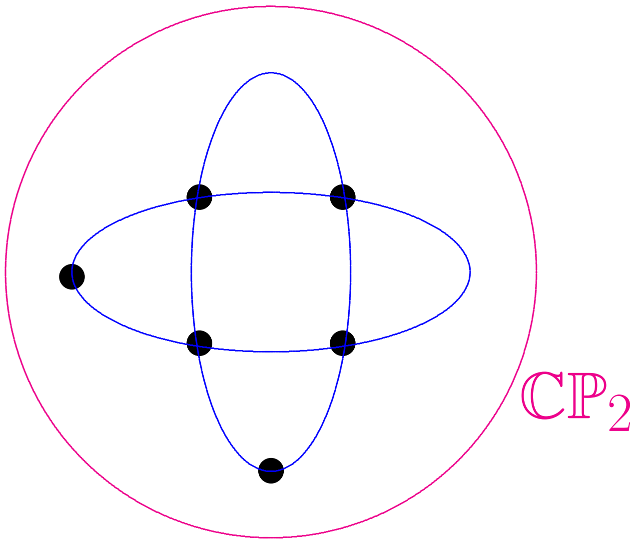

No 3 on a line, no 6 on conic,

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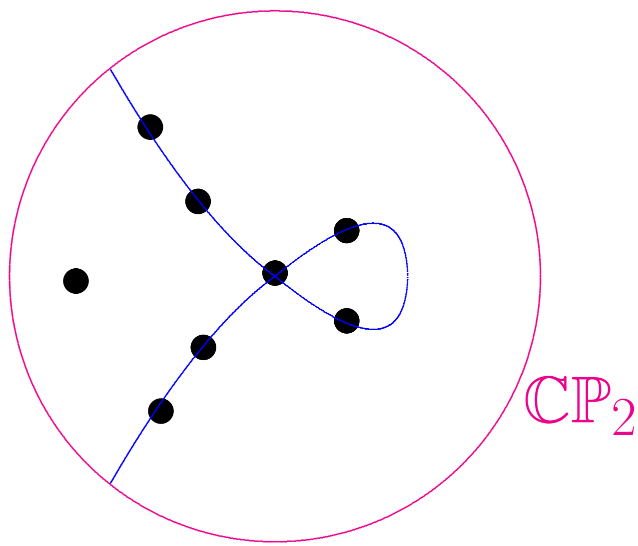

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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Theorem. *Each Del Pezzo (M^4, J) admits a compatible conformally Kähler Einstein metric, and this metric is unique up to automorphisms.*

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Existence: Tian, Odaka-Spotti-Sun, Chen-L-Weber.

Uniqueness: Bando-Mabuchi '87, L '12.

Above the line:

Moduli space $\mathscr{E}(M) \neq \varnothing$. But is it connected?

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Basic problem:

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Understand all Einstein metrics on Del Pezzos.

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Natural characterization of known Einstein metrics.

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Natural characterization of known Einstein metrics.

Exactly one connected component of moduli space.

Formulation depends on . . .

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$$h \rightsquigarrow u^2 h$$

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Because of this . . .

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Λ^{+}	$W^{+} + \frac{s}{12}$	$\mathring{r}$
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$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W^{+} &= \text{self-dual Weyl curvature} \\ W^{-} &= \text{anti-self-dual Weyl curvature} \end{aligned}$$

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self-dual & anti-self-dual harmonic forms.

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self-dual & anti-self-dual harmonic forms.

Notice these spaces are conformally invariant.

Hodge theory:

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Since $\star$ is involution of RHS, $\Longrightarrow$

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More generally, their dimensions

$$b_\pm(M) = \dim \mathcal{H}^\pm_h$$

are completely metric-independent, and
are oriented homotopy invariants of M.

Key background result:

Theorem (L '15)**.**

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Yes — with a reasonable extra hypothesis on ω...

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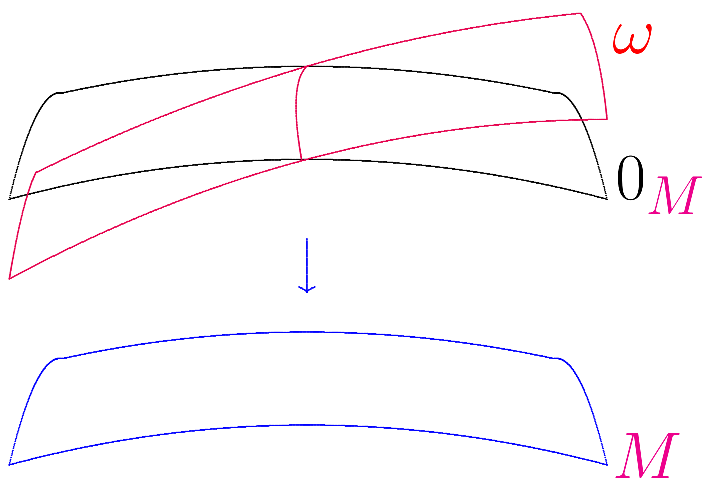

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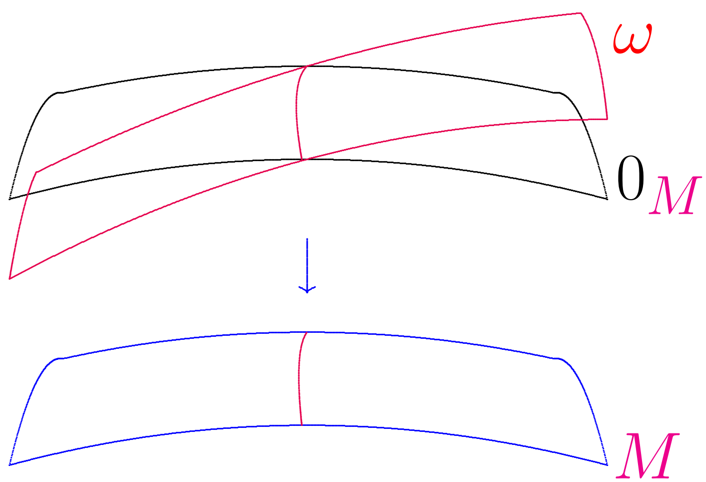

$\Longrightarrow$ Zero set Z of ω has codimension 3:

$$Z \approx \sqcup_{j=1}^{n} S^1.$$

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Theorem (Taubes, et al). *If $b_+(M) \neq 0$, such forms exist for an open dense set of metrics h on M.*

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Indeed, all known Einstein metrics on Del Pezzo surfaces arise this way!

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Conversely, these complex surfaces all admit $\lambda \geq 0$ *Einstein metrics* h *of the above type.*

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Moral: Taubes' genericity result does not guarantee genericity among metrics solving an equation!

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Kähler $\Longrightarrow$

$$W^+ = \begin{bmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{bmatrix}$$

Counter-examples promised by **Theorem C**:

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$$W^+(\omega, \omega) \geq 0, \qquad W^+(\omega, \omega) \not\equiv 0$$

Before discussing **Theorems A & B**,

consider simpler case when $W^+(\omega, \omega) > 0$.

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$\therefore h \propto s^{-2} g$ globally on M.

Obrigado por me convidar!