Einstein Manifolds,

Conformal Curvature, &

Anti-Holomorphic Involutions

Claude LeBrun
Stony Brook University

Twistor Theory and Beyond
September 27, 2021

For my friend and collaborator

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In belated celebration of his 60th birthday

For my friend and collaborator
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and many remarkable research achievements.

Main references:

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Bach-Flat Kähler Surfaces

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Journal of Geometric Analysis
30 (2020) 2491–2514

Einstein Manifolds, Self-Dual Weyl Curvature, and Conformally Kähler Geometry

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Mathematical Research Letters
28 (2021) 127–144

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Annales Mathématiques du Québec
45(2) (2021) 391–405

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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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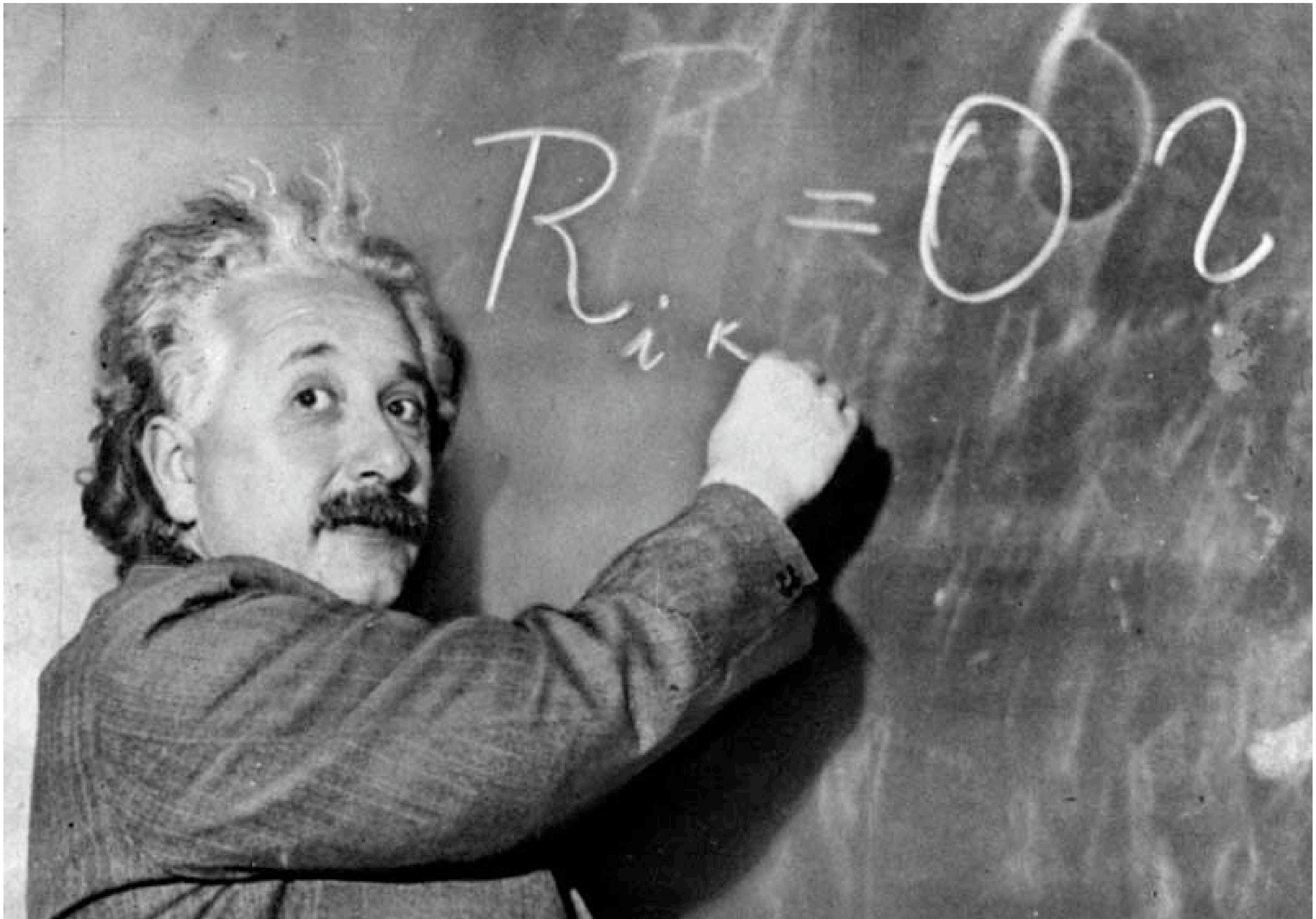

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My early ambitwistor approach to the problem hinged on correctly understanding of the Einstein condition from a conformal point of view.

Many of the important ideas originated in the work of Lionel Mason, Paul Tod, and their collaborators.

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Conformal gravity, the Einstein equations and spaces of complex null geodesics

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Received 13 October 1986

Abstract. The aim of this work is to give a twistorial characterisation of the field equations of conformal gravity and of Einstein spacetimes. We provide strong evidence for a particularly concise characterisation of these equations in terms of 'formal neighbourhoods' of the space of complex null geodesics.

We consider second-order perturbations of the metric of complexified Minkowski space. These correspond to certain infinitesimal deformations of its space of complex null geodesics, $P\mathbb{N}$.

$P\mathbb{N}$ has a natural codimension one embedding into a larger space (the product of twistor space and its dual). We show that deformations extend automatically to the fourth-order embedding (that is, the fourth formal neighbourhood). They extend to the fifth formal neighbourhood if and only if the corresponding perturbation in the metric has vanishing Bach tensor (these are the equations of conformal gravity). Finally, deformations which extend to the sixth formal neighbourhood correspond to perturbations in the metric that are conformally related to ones satisfying the Einstein equations, at least when the Weyl curvature is sufficiently algebraically general.

One can attempt to construct such formal neighbourhoods in the fully curved case. We present arguments which suggest that our results will also hold when spacetime is fully curved.

1. Introduction

Penrose's non-linear graviton construction (Penrose 1976) provides sufficient new mathematical insight into Einstein's equations that one would expect that it should also have wide ranging physical applications. However the construction only produces gravitons in helicity eigenstates, that is spacetimes with either pure ASD (anti-self-dual) or pure SD (self-dual) Weyl curvature. In order to make contact with physics one should be able to have general Weyl curvature (subject only to field equations).

In this paper we present evidence for a particular type of generalisation of the non-linear graviton construction. This uses the space of complex null geodesics instead of twistor space.

LeBrun (1983) has proved that a complex spacetime together with its conformal structure, M, can be reconstructed from its space of complex null geodesics, PN (see § 2 for a statement of the theorem). This generalises the part of Penrose's non-linear graviton construction in which it is shown that an ASD spacetime can be reconstructed from its twistor space.

It remains to characterise field equations on M in terms of holomorphic structures on PN. In this paper we present a second-order analysis which suggests a particularly

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Conformal Einstein Spaces[1]

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and

K. P. TOD[2]

Mathematical Institute, Oxford University, Oxford, England

Received March 16, 1984

Abstract

We study conformal transformations in four-dimensional manifolds. In particular, we present a new set of two necessary and sufficient conditions for a space to be conformal to an Einstein space. The first condition defines the class of spaces conformal to C spaces, whereas the last one (the vanishing of the Bach tensor) gives the particular subclass of *C* spaces which are conformally related to Einstein spaces.

§(1): *Introduction and Mathematical Preliminaries*

The study of Riemannian spaces conformally related to Einstein spaces is a problem which has been addressed since the 1920s.

The first work on this subject was that of H. W. Brinkman [1]. He studied the necessary and sufficient conditions for spaces to be conformally related to Einstein spaces in n dimensions with a particular example in four-dimensions. However, since his arguments involved existence and compatibility of differential equations [1, 2], a constructive set of necessary and sufficient conditions is very difficult to infer. Later, other authors also contributed to a further understanding of the problem [3], but owing to the vast variety of algebraically distinct types of Weyl tensors, no general set of conditions have yet been found in n dimensions.

[1] This work has been partially supported by a grant from the National Science Foundation.
[2] S.E.R.C. Advanced Fellow.

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Communications in Mathematical Physics

Thickenings and Conformal Gravity

Claude LeBrun*

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Received June 6, 1989

Abstract. A twistor correspondence is given for complex conformal space-times with vanishing Bach and Eastwood–Dighton tensors; when the Weyl curvature is algebraically general, these equations are precisely the conformal version of Einstein's vacuum equations with cosmological constant. This gives a fully curved version of the linearized correspondence of Baston and Mason [B–M].

0. Introduction

In this paper we provide a twistor correspondence for conformal gravity, meaning roughly a reformulation of the conformally invariant aspects of Einstein's vacuum equations in terms of deformations of complex analytic spaces. This correspondence was conjectured by Baston and Mason [B–M] on the basis of some insightful (albeit heuristic) arguments concerning the linearized theory, and the chief new idea that will be explored here, the rôle of Poisson structures (cf. [W]) in the relevant extension problem, arose directly from the efforts of the present author to give the calculations of Baston and Mason precise meaning.

We work throughout in the context of conformal classes of complex-Riemannian 4-manifolds. Recall that a complex-Riemannian manifold is a complex manifold equipped with a non-degenerate holomorphic symmetric 2-tensor, so that each tangent space is endowed with a complex quadratic form; two such complex-Riemannian metrics are called conformally equivalent if one is obtained from the other by multiplication by a non-zero holomorphic function. Such structures arise naturally from the analytic continuation of real-analytic pseudo-Riemannian metrics and their conformal classes into the complex domain, and one may return to the realm of pseudo-Riemannian geometry by restricting to the fixed-point set of an anti-holomorphic involution ("complex conjugation") respecting the structure. While it is possible to reformulate some of our results without this foray into the complex domain, we will avoid so doing here for the sake of brevity.

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* Research supported in part by NSF grant DMS-8704401

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Conversely, these conditions $\Rightarrow$ $\exists$ Einstein $h = u^2 g$ near any $p \in M$ where $W^\pm : \Lambda^\pm \to \Lambda^\pm$ max rank.

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$\therefore$ On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

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(M^4, h) also compact, but not Kähler-Einstein $\Longrightarrow$

$$s > 0 \quad \text{and} \quad h = \text{ const } s^{-2} g$$

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Moreover, each case actually occurs.

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I. $s > 0$ *everywhere. Then*

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III. $s < 0$ *somewhere. Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda < 0$*; or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$*, and* $M - \mathcal{Z}$ *has exactly two components.*

If not Kähler-Einstein:

I. s *is positive. Then*

$(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. s *is zero. Then*

(M, g, J) *SFK, but not Ricci-flat.*

III. s *changes sign. Then*

$(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$, *and* $M - \mathcal{Z}$ *has exactly two components.*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$; *or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$, *and* $M - \mathcal{Z}$ *has exactly two components.*

Main interest today:

I. $\min s > 0$. *Then*

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This happens $\iff$ $c_1 > 0$.

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$\iff$ (M^4, J) is a Del Pezzo surface.

Del Pezzo surfaces:

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(M^4, J) for which c_1 is a Kähler class $[\omega]$.

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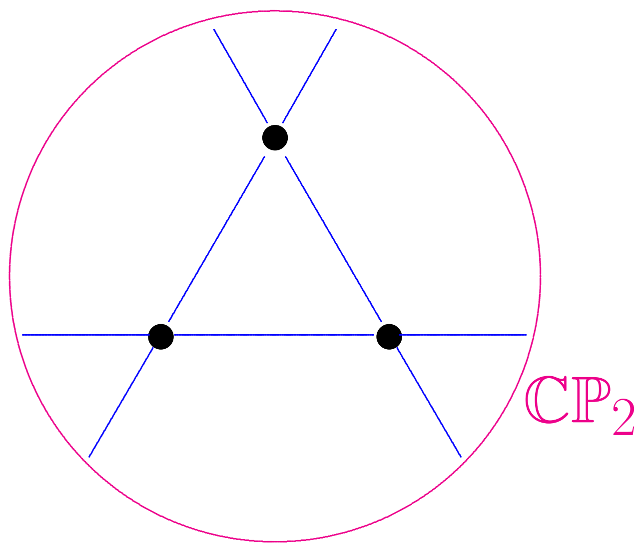

Blowing up:

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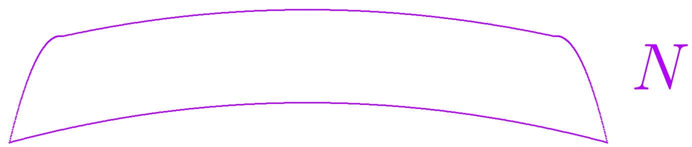
If N is a complex surface,



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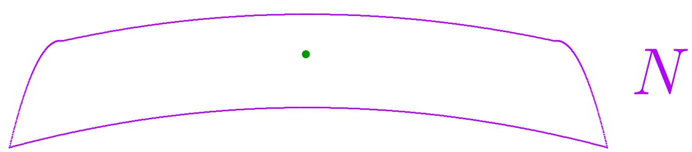
If N is a complex surface, may replace $p \in N$



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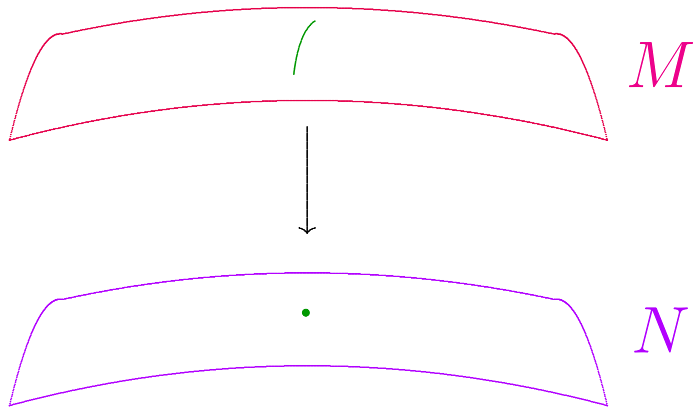

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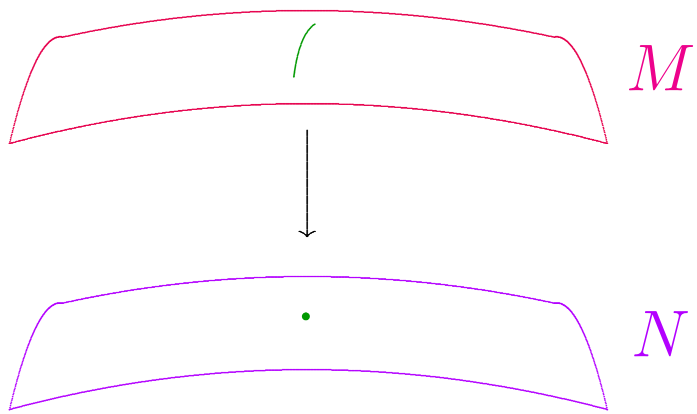
If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$



Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

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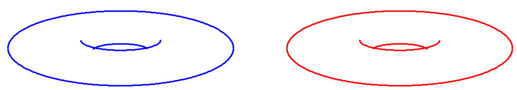

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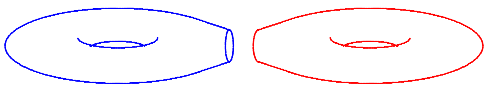

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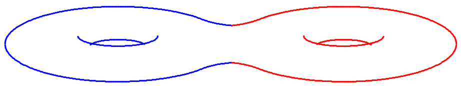

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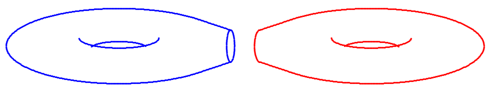

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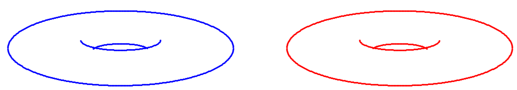

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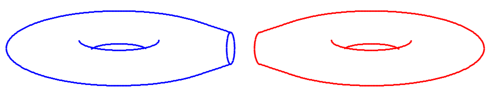

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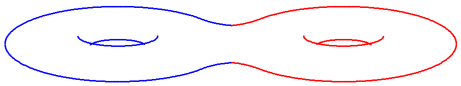

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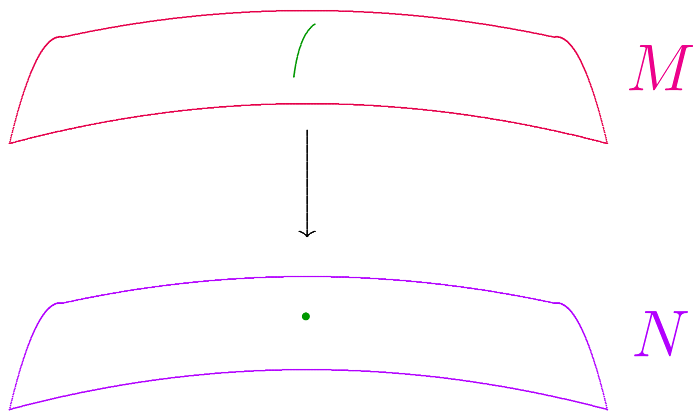

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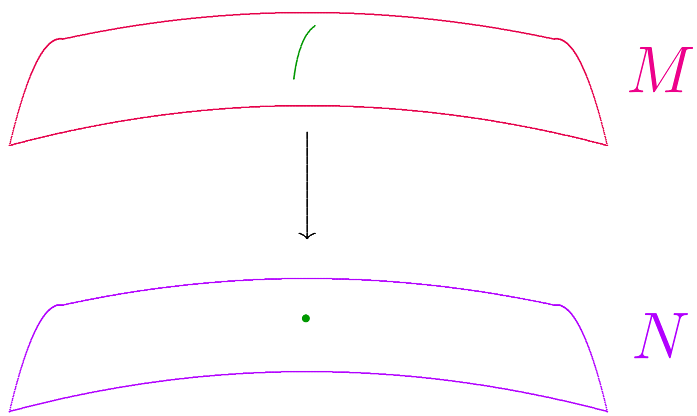

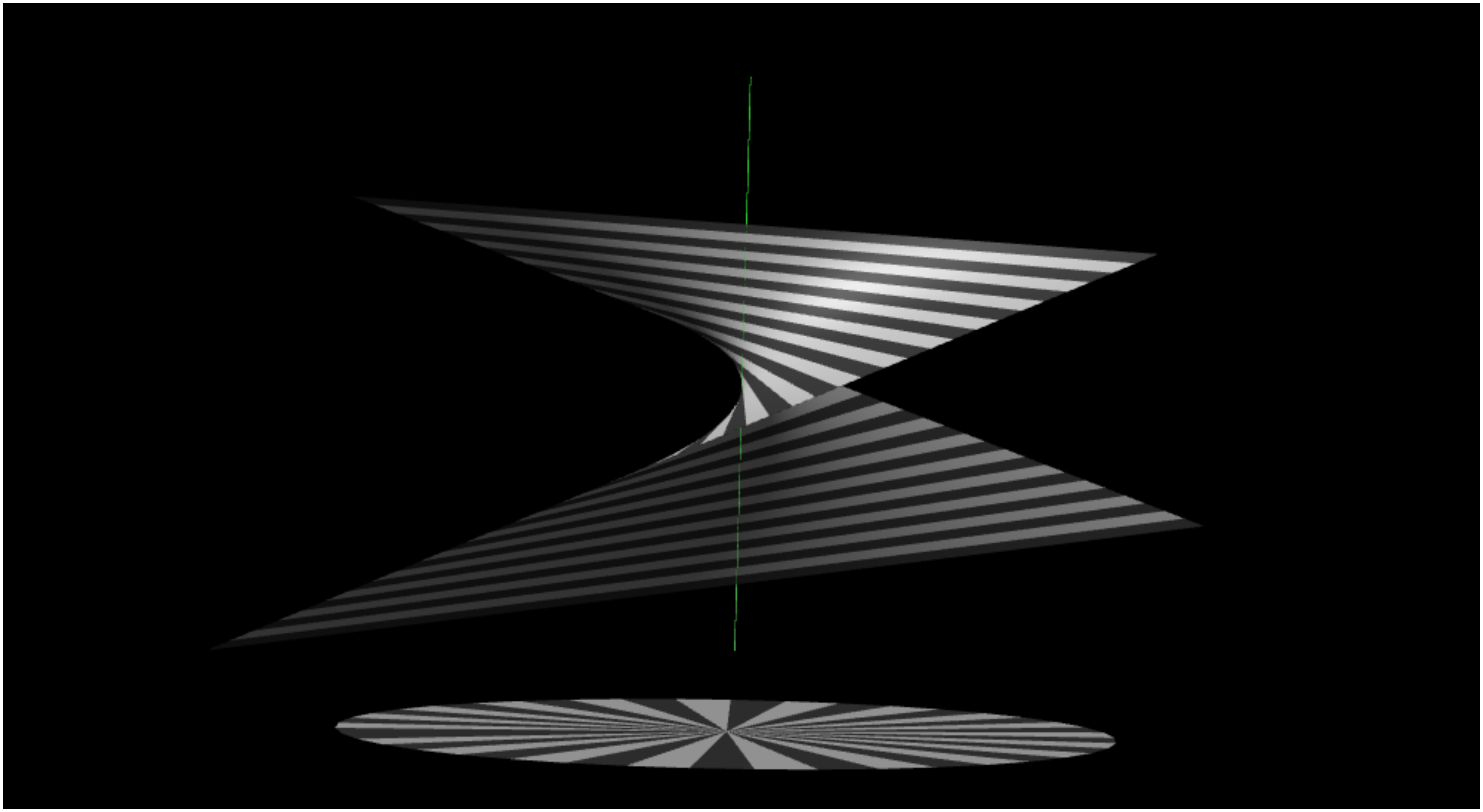

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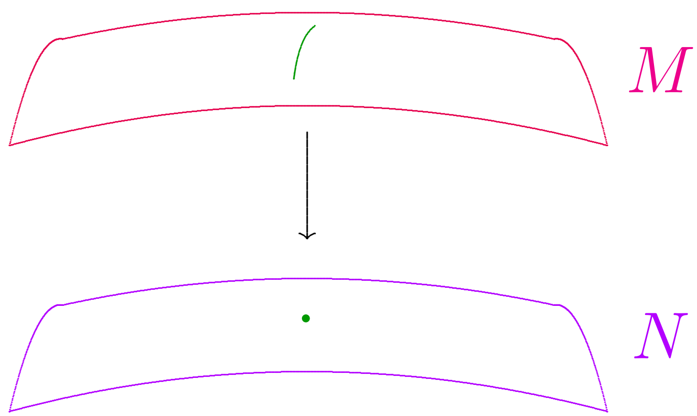

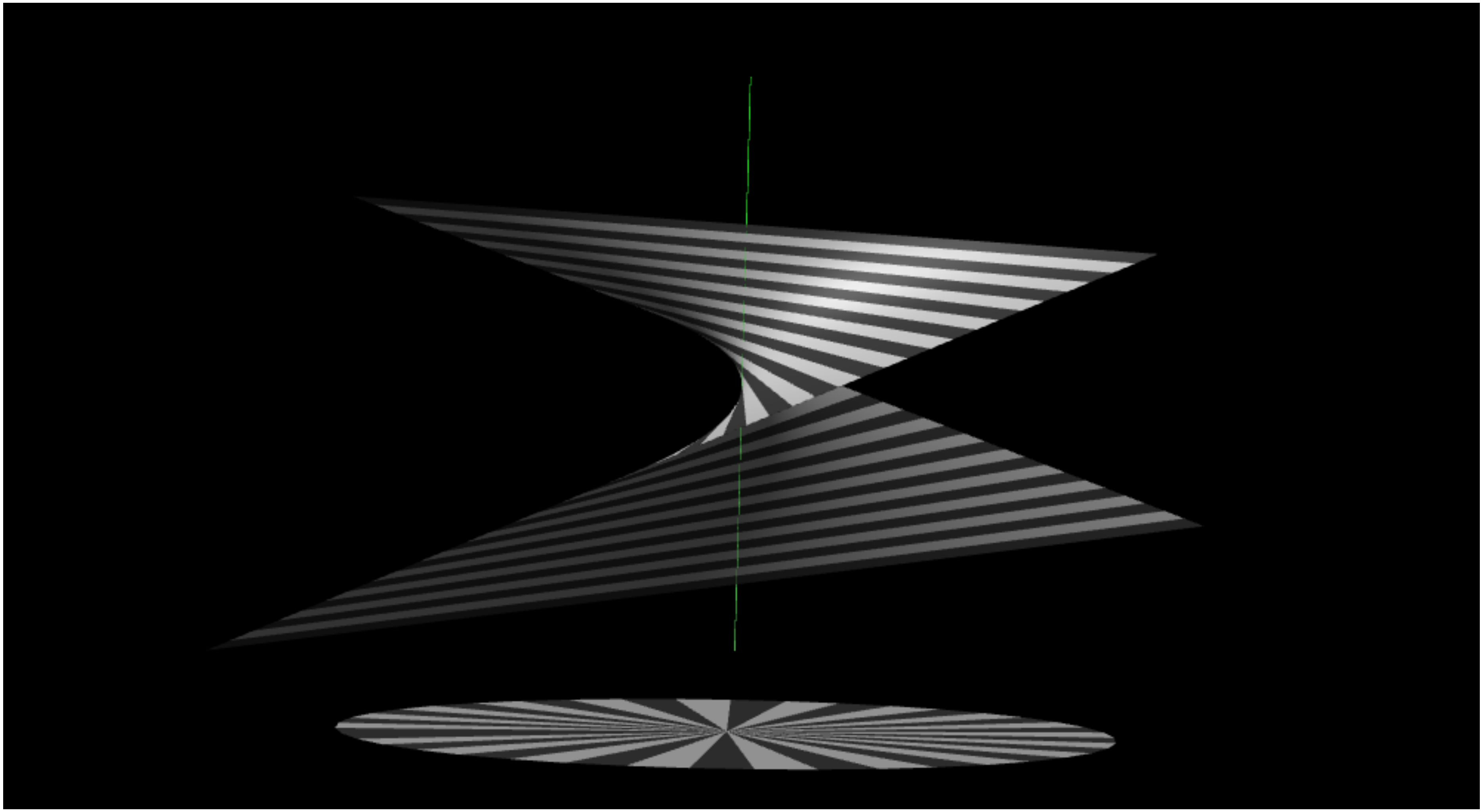

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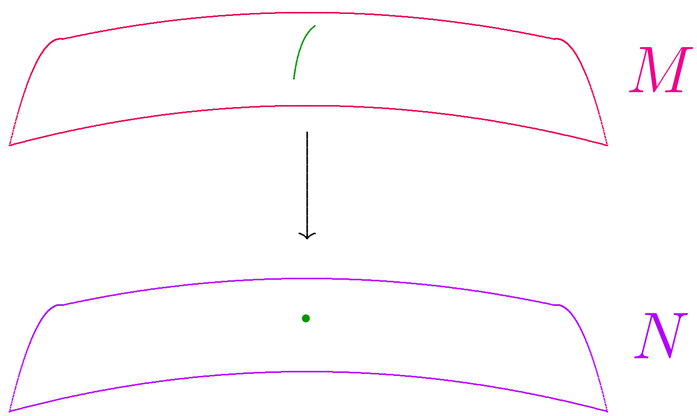

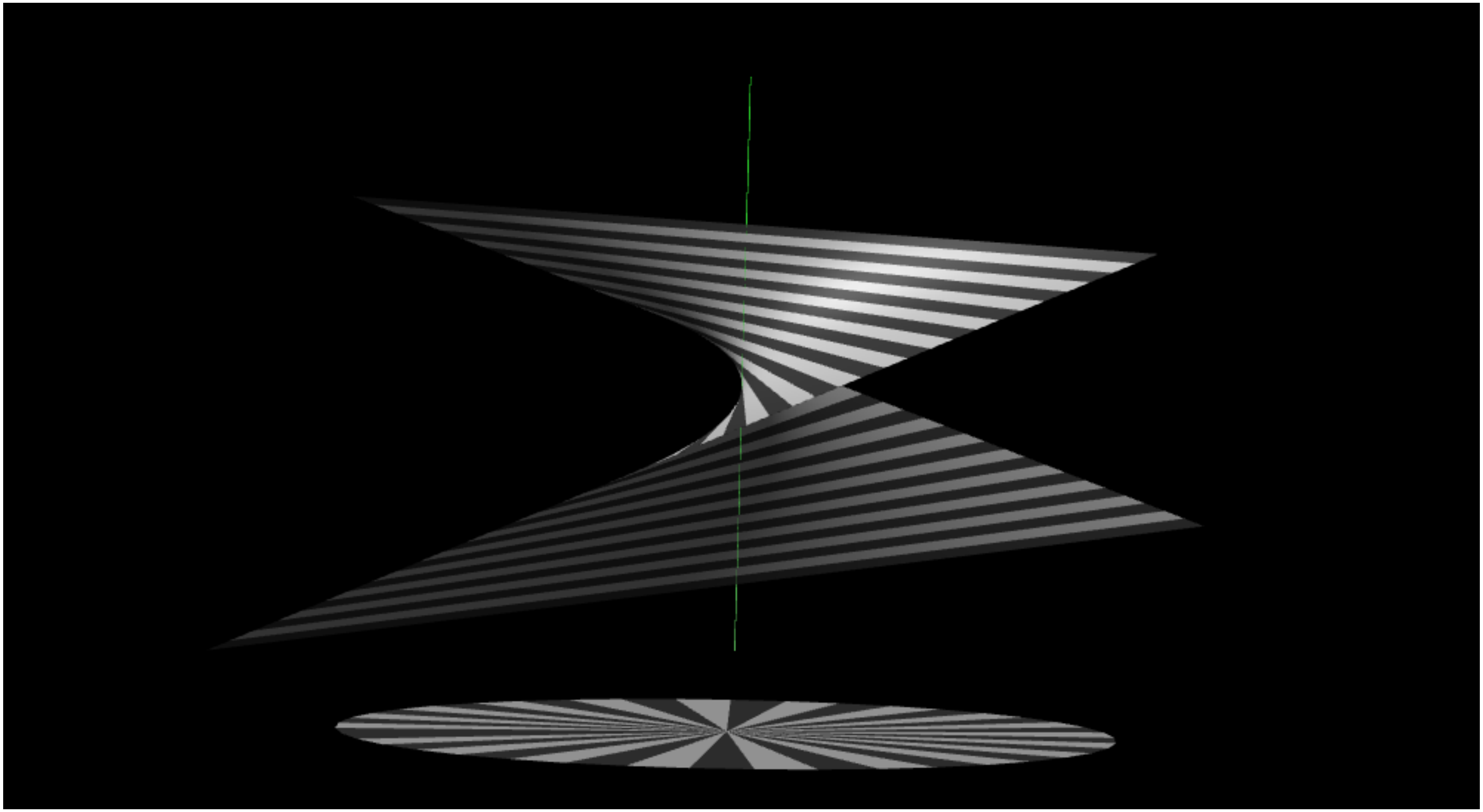

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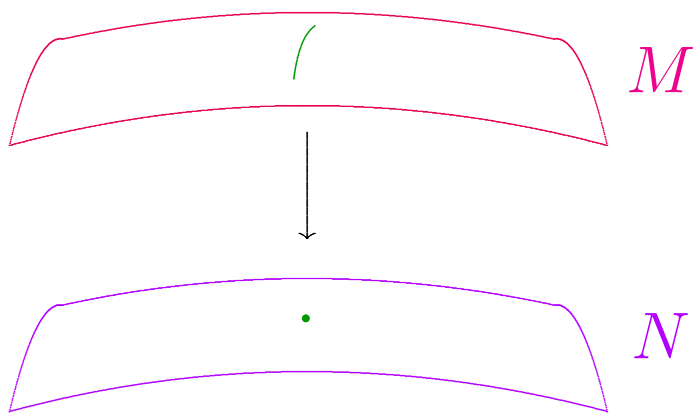
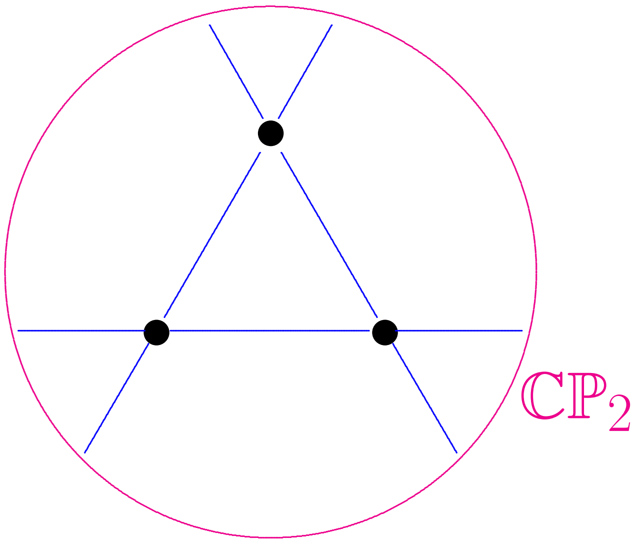


Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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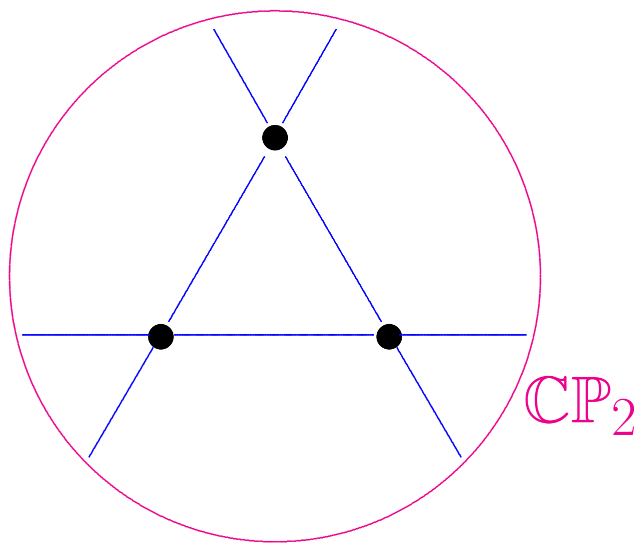

No 3 on a line,

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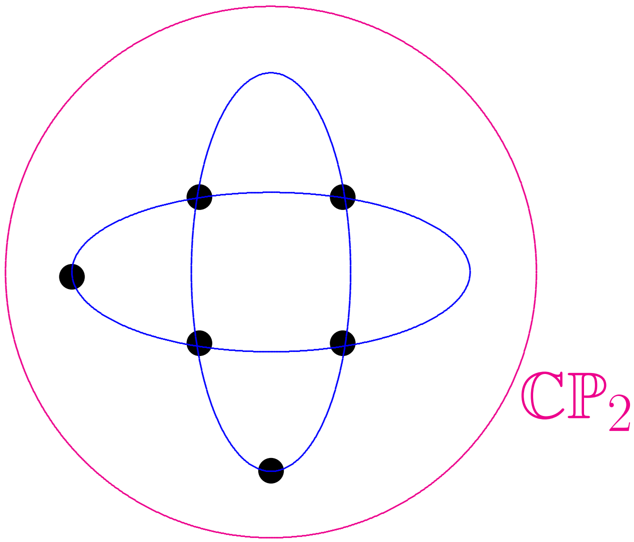

No 3 on a line, no 6 on conic,

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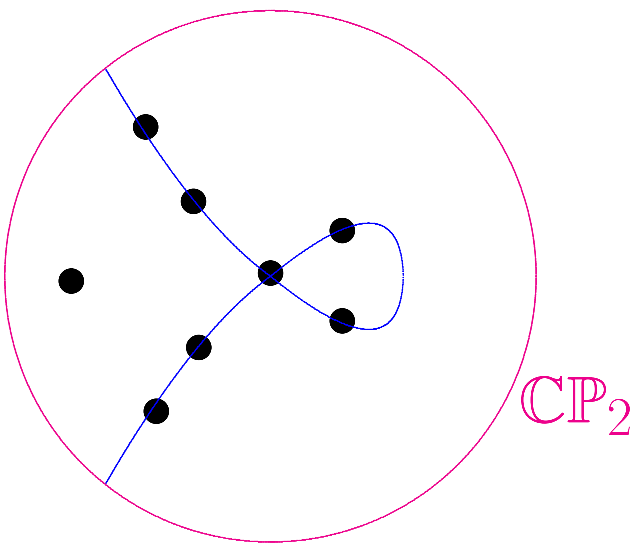

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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Theorem.

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Theorem. *Each del Pezzo* (M^4, J) *admits a* J*-compatible conformally Kähler, Einstein metric, and this metric is unique*

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Existence: Page

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Existence: Page-Derdziński,

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun,

2016

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Understand all Einstein metrics on del Pezzos.

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Is Einstein moduli space connected?

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Completely understood for certain 4-manifolds:

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Known to be connected for certain 4-manifolds:

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Berger,

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Berger, Hitchin, Besson-Courtois-Gallot, L.

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Progress to date:

Nice characterizations of known Einstein metrics.

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Understand all Einstein metrics on del Pezzos.

Is Einstein moduli space connected?

Progress to date:

Nice characterizations of known Einstein metrics.

Exactly one connected component of moduli space!

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Corollary. *These known Einstein metrics on any del Pezzo M^4 sweep out exactly one connected component of the Einstein moduli space $\mathscr{E}(M)$.*

Reasonably satisfying result.

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$$W^+ = \text{trace-free part of } \begin{bmatrix} 0 & & \\ & 0 & \\ & & \frac{s}{4} \end{bmatrix}$$

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for these metrics & conformal rescalings:

$$g \rightsquigarrow h = f^2 g \quad \Longrightarrow \quad \det(W^+) \rightsquigarrow f^{-6} \det(W^+).$$

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method also proves more general results.

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L (2020): related classification result.

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either on M or double cover $\widetilde{M}$.

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Claim: (M, h) compact Einstein $\Longrightarrow$ J integrable.

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Simply connected hypothesis is essential!

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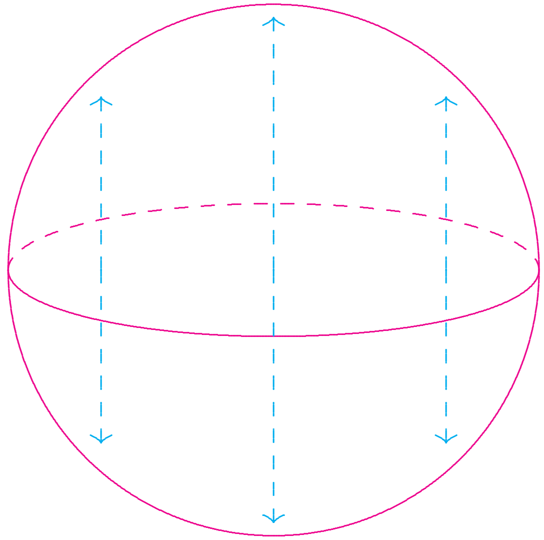
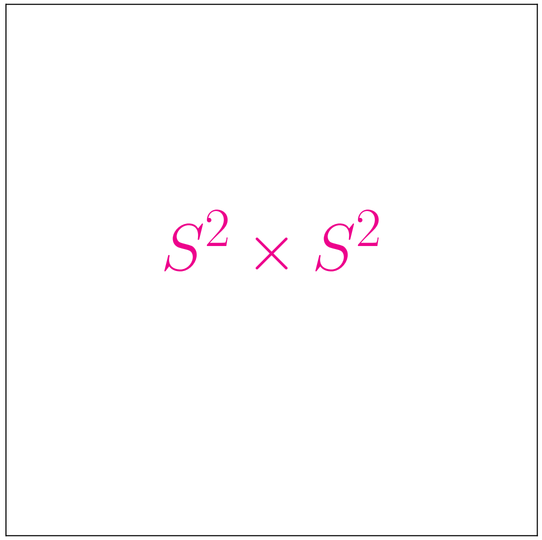
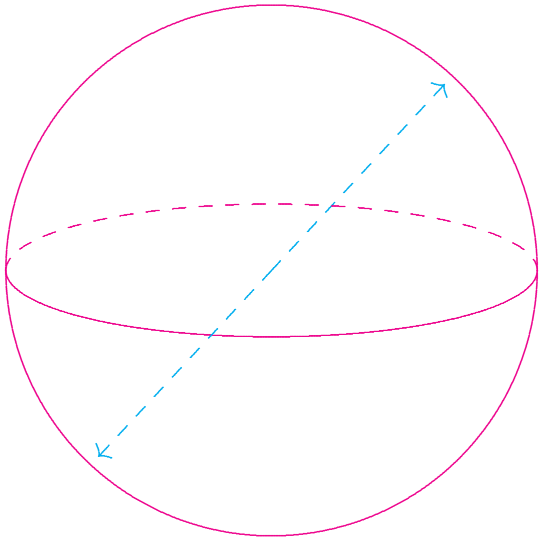




Oriented spin 4-manifold

$\mathscr{P} := (S^2 \times S^2)/\langle \mathfrak{a} \times \mathfrak{r} \rangle$



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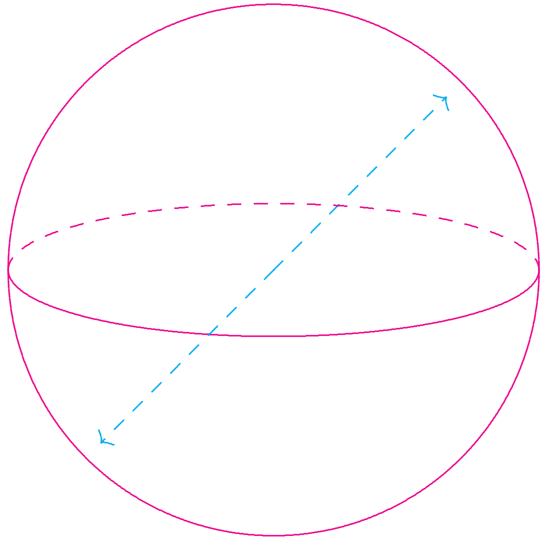
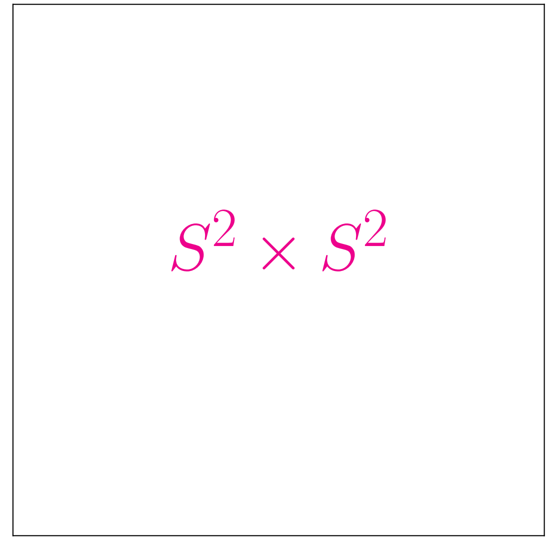
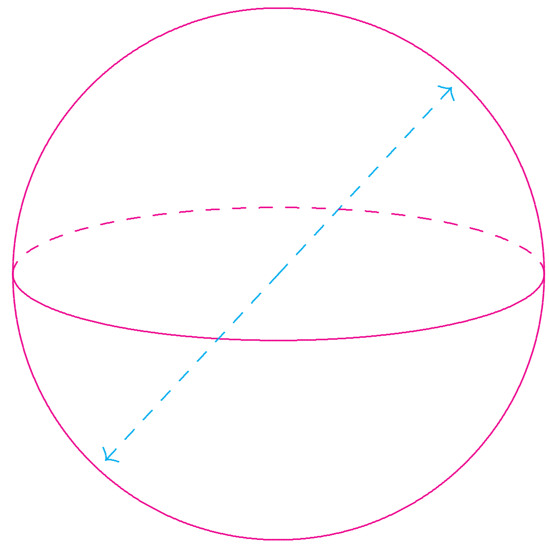




Non-spin 4-manifold

$\mathscr{Q} := (S^2 \times S^2)/\langle \mathfrak{a} \times \mathfrak{a} \rangle$



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Theorem C. *Let M be smooth compact oriented 4-manifold with $\pi_1 \neq 0$. Then, M admits an Einstein metric h with $\det(W^+) > 0 \iff$*

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Theorem D. *There are exactly 15 diffeotypes of compact oriented* 4*-manifolds* M *that carry Einstein metrics* h *with* $\det(W^+) > 0$ *everywhere.*

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Why is $\mathscr{E}_{\det}(M) \subset \mathscr{E}(M)$ open and closed?

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Theorem E. *Let* (M, h) *be a compact oriented Einstein* 4*-manifold. If*

$$\det(W^+) > -\frac{5\sqrt{2}}{21\sqrt{21}}|W^+|^3$$

everywhere on M*, then actually* $\det(W^+) > 0$*. Consequently, all the results described remain true if we merely impose this ostensibly weaker hypothesis.*

Some indication of the proof:

For clarity, let's just assume $\det(W^+) > 0 \ldots$

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for $fW^+ \in \mathrm{End}(\Lambda^+)$.

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with $\omega \otimes \omega$, and integrate by parts. This yields:

$$0 = \int_M \left[\langle W^+, \nabla^*\nabla(\omega\otimes\omega)\rangle + \frac{s}{2}W^+(\omega,\omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2\right] f \, d\mu$$

Let $\alpha \geq \beta \geq \gamma$ be eigenvalues of W^+:

$$W^+ = \begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix}$$

$$\alpha + \beta + \gamma = 0$$

$$\alpha > 0, \quad \gamma < 0, \qquad \text{if } W^+ \neq 0$$

$$\det(W^+) = \alpha\beta\gamma$$

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$$f = {\alpha_h}^{-1/3}, \qquad g = f^{-2}h = {\alpha_h}^{2/3}h.$$

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Now choose $\omega \in \Gamma\Lambda^+$ so that

$$W_g^+(\omega) = \alpha\,\omega, \quad |\omega|_g \equiv \sqrt{2},$$

after at worst passing to double cover $\hat{M} \to M$.

$$
\begin{aligned}
0 = \int_{\hat{M}} \Big[& \langle W^+, \nabla^*\nabla(\omega \otimes \omega)\rangle \\
& + \frac{s}{2} W^+(\omega, \omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2 \Big] f \; d\mu
\end{aligned}
$$

$$0 = \int_M \Big[\langle W^+, \nabla^* \nabla (\omega \otimes \omega) \rangle + \frac{s}{2} W^+(\omega, \omega) - 6 |W^+(\omega)|^2 + 2 |W^+|^2 |\omega|^2 \Big] f \, d\mu$$

$$0 = \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) - 2W^+(\omega, \nabla^e \nabla_e \omega) + \frac{s}{2} W^+(\omega, \omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2 \Big] f \, d\mu$$

$$0 = \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) - 2\alpha\langle \omega, \nabla^e \nabla_e \omega\rangle$$

$$+\frac{s}{2}\alpha|\omega|^2 - 6\alpha^2|\omega|^2 + 2|W^+|^2|\omega|^2\Big] f \; d\mu$$

because

$$W_g{}^+(\omega) = \alpha\omega$$

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$$0 \geq \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha\langle \omega, \nabla^*\nabla\omega\rangle$$
$$+\frac{s}{2}\alpha|\omega|^2 - 6\alpha^2|\omega|^2 + 3\alpha^2|\omega|^2\Big] f \; d\mu$$

because

$$|W_g{}^+|^2 \geq \frac{3}{2}\alpha^2$$

$$0 \geq \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha \langle \omega, \nabla^* \nabla \omega \rangle$$

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$$|\omega|_g^2 = 2 \quad \Longrightarrow \quad (\nabla_e \omega) \perp \omega$$

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$$|\omega|_g^2 = 2 \quad \Longrightarrow \quad (\nabla_e \omega) \perp \omega$$

$$\det(W^+) > 0 \quad \Longrightarrow \quad W^+(\nabla_e \omega, \nabla^e \omega) \leq 0$$

$$0 \geq \int_M \Big[\quad 2\alpha\langle\omega, \nabla^*\nabla\omega\rangle + \frac{s}{2}\alpha|\omega|^2 - 3\alpha^2|\omega|^2\Big] f \, d\mu$$

$$|\omega|^2_g = 2 \quad \Longrightarrow \quad (\nabla_e\omega) \perp \omega$$

$$\det(W^+) > 0 \quad \Longrightarrow \quad -W^+(\nabla_e\omega, \nabla^e\omega) \geq 0$$

$$0 \geq \int_M \left[2\alpha \langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2} \alpha |\omega|^2 - 3\alpha^2 |\omega|^2 \right] f \, d\mu$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^*\nabla\omega\rangle + \frac{s}{2}|\omega|^2 - 3\alpha|\omega|^2\right](\alpha f)\, d\mu$$

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But

$$\alpha f \equiv 1$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2} \alpha |\omega|^2 - 3|\omega|^2 \alpha \right] d\mu$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle - 3W^+(\omega, \omega) + \frac{s}{2}|\omega|^2 \right] d\mu$$

$$0 \geq \int_M \left[\frac{1}{2}|\nabla\omega|^2 + \frac{3}{2}\langle\omega, \left(\nabla^*\nabla - 2W^+ + \frac{s}{3}\right)\omega\rangle\right] d\mu$$

$$0 \geq \int_M \left[\frac{1}{2}|\nabla\omega|^2 + \frac{3}{2}\langle\omega, \quad (d+d^*)^2 \quad \omega\rangle\right] d\mu$$

Because

$$(d+d^*)^2 = \nabla^*\nabla - 2W^+ + \frac{s}{3}$$

on $\Gamma\Lambda^+$.

$$0 \geq \frac{1}{2} \int_M |\nabla \omega|^2 \, d\mu + 3 \int_M |d\omega|^2 \, d\mu$$

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So $\nabla\omega \equiv 0$, and g is Kähler!

Belated Birthday Greetings, Lionel!

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And Many Happy Returns!