On Optimal 4-Dimensional Metrics

Claude LeBrun* and Bernard Maskit

November 1, 2007

Abstract

We completely determine, up to homeomorphism, which simply connected compact oriented 4-manifolds admit scalar-flat, anti-self-dual Riemannian metrics. The key new ingredient is a proof that the connected sum $\overline{\mathbb{CP}}_2\#\overline{\mathbb{CP}}_2\#\overline{\mathbb{CP}}_2\#\overline{\mathbb{CP}}_2\#\overline{\mathbb{CP}}_2$ of five reverse-oriented complex projective planes admits such metrics.

1 Introduction

Marcel Berger [4] credits the late René Thom with the following vague, but fundamental, question:

Does every smooth compact manifold admit a best metric?

Berger eventually proposed a more precise version of the problem by asking which smooth compact n-manifoldsx M, $n \geq 3$, admit Riemannian metrics g which are *as flat as possible*, in the sense that they minimize the scale-invariant functional

$$g \longmapsto \mathcal{K}(g) = \int_M |\mathcal{R}_g|_g^{n/2} d\mu_g,$$

where $\mathcal{R}$ denotes the Riemann curvature tensor, $|\mathcal{R}|$ is its point-wise norm with respect to the metric, and $d\mu$ is the n-dimensional volume measure determined by the metric. The following terminology is then used to describe metrics which are "best" in this precise sense:

*Supported in part by NSF grant DMS-0604735.

Definition 1 *Let M be a smooth compact n-dimensional manifold, $n \geq 3$. A smooth Riemannian metric g on M is said to be an* optimal metric *if it is an absolute minimizer of the above-defined functional $\mathcal{K}$, in the sense that*

$$\mathcal{K}(\tilde{g}) \geq \mathcal{K}(g)$$

for every smooth Riemannian metric $\tilde{g}$ on M.

Berger's chief motivation for Definition 1 seems to have been that, as explained in §1.1 below, Einstein metrics on compact 4-manifolds are optimal in this sense. But this fact also shows that dimension 4 enjoys a peculiar, privileged status for the problem. Indeed, notice that, for any integer $m \geq 2$, there is a *non-optimal* Einstein metric g on $S^3 \times S^m$, given by the product of the standard 'round' metrics of radii $\sqrt{2}$ and $\sqrt{m-1}$, respectively. Since g is not flat, we obviously havee $\mathcal{K}(g) > 0$. However, there is a sequence of homogeneous metrics on $S^3 \times S^m$ with $\mathcal{K} \searrow 0$, as may be constructed by shrinking g along the fibers of the Hopf fibration $S^3 \times S^m \to \mathbb{CP}_1 \times S^m$. This shows that there are non-optimal Einstein metrics in any dimension ≥ 5. Since essentially the same argument also shows that the constant-curvature metric on S^3 isn't optimal either, the very special status of dimension four is now manifest.

However, while every 4-dimensional Einstein metrics is optimal, not every 4-dimensional optimal metric is Einstein. This fact, which is of fundamental importance for our purposes here, now merits a careful explanation.

1.1 Optimal Metrics in Dimension Four

Why, then, is dimension four so special? To a large extent, this is because the bundle of 2-forms on an oriented Riemannian 4-manifold (M, g) can be invariantly decomposed as a direct sum

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-, \tag{1}$$

of the (± 1)-eigenspaces $\Lambda^\pm$ of the Hodge $\star$ operator. Since the Riemann curvature tensor $\mathcal{R}$ may be thought of as a linear map $\Lambda^2 \to \Lambda^2$, the decomposition (1) therefore allows us to think of $\mathcal{R}$ as consisting of four blocks:

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \overset{\circ}{r} \\ \hline \overset{\circ}{r} & W_- + \frac{s}{12} \end{array} \right). \tag{2}$$

Here $W_\pm$ are the trace-free pieces of the appropriate blocks, and are called the self-dual and anti-self-dual Weyl curvatures, respectively; these pieces of the curvature tensor are *conformally invariant*, in the sense that they are unchanged if g replaced by $u^2 g$, where u is an arbitrary smooth positive function. The scalar curvature s is understood to act by scalar multiplication, whereas the trace-free Ricci curvature $\overset{\circ}{r} = r - \frac{s}{4} g$ acts on 2-forms by

$$\varphi_{ab} \longmapsto \frac{1}{2}\left[\overset{\circ}{r}_a^c \varphi_{cb} - \overset{\circ}{r}_b^c \varphi_{ca}\right].$$

In terms of this decomposition of the curvature tensor of an arbitrary metric g on a compact oriented 4-manifold M, the generalized Gauss-Bonnet theorem expresses the Euler characteristic of M as

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\overset{\circ}{r}|^2}{2} \right) d\mu_g, \tag{3}$$

while the Hirzebruch signature theorem allows us to express the signature of M as

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu_g \ . \tag{4}$$

Since our curvature functional becomes

$$\mathcal{K}(g) = \int_M |\mathcal{R}|^2 d\mu_g = \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 + \frac{|\overset{\circ}{r}|^2}{2} \right) d\mu_g \ ,$$

in dimension 4, we therefore have

$$\mathcal{K}(g) = 8\pi^2 \chi(M) + \int_M |\overset{\circ}{r}_g|^2 d\mu_g \ ,$$

implying Berger's observation that any Einstein metric g minimizes $\mathcal{K}$. However, *not every optimal metric is Einstein*, even in dimension four. For example, the Gauss-Bonnet and signature formulæ imply that

$$\mathcal{K}(g) = -8\pi^2(\chi + 3\tau)(M) + 2\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g$$

so, as was perhaps first observed by Lafontaine [27], another class of minimizers is given by the metrics for which both W_+ and s are identically zero. We will indicate the latter class of metrics by means of the following terminology:

Definition 2 *If M is a smooth oriented 4-manifold, a Riemannian metric g on M is said to be* anti-self-dual *(or, for brevity,* ASD*) if its self-dual Weyl curvature is identically zero:*

$$W_+ \equiv 0.$$

A metric g is called scalar-flat *(or, more briefly,* SF*) if it satisfies*

$$s \equiv 0.$$

Finally, we say that g is scalar-flat anti-self-dual *(or* SFASD*) if it satisfies both of these conditions.*

Notice that the ASD condition is conformally invariant; that is, if g is ASD, so is u^2g, for any positive function u. By contrast, if g has scalar curvature s, then $\hat{g} = u^2 g$ has scalar curvature $\hat{s}$ determined by

$$\hat{s}u^3 = (6\Delta + s)u,$$

where $\Delta = -\nabla \cdot \nabla$ is the Laplace-Beltrami operator of g. In particular, a conformal class on a compact manifold can contain at most one scalar-flat metric, up to overall constant rescaling.

An immediate corollary of the above ideas is that any locally conformally flat, scalar-flat metric on a compact 4-manifold is optimal. This fact alone provides many examples of non-Einstein optimal metrics on compact 4-manifolds with infinite fundamental group; for example, the product metric on $S^2 \times \Sigma$, where Σ is a compact surface of genus ≥ 2 equipped with a choice of hyperbolic (Gauss curvature -1) metric, and where the 2-sphere S^2 is taken to be equipped with its standard (Gauss curvature $+1$) metric. However, examples produced by this trick can never be simply connected [26]. Nonetheless, simply connected SFASD manifolds do exist in considerable profusion, and, as we shall explain in the next subsection, the entire purpose of this paper is to provide a complete topological classification of those simply connected 4-manifolds which admit optimal metrics of this special type.

1.2 The Main Result

The existence of an SFASD metric places tight constraints on the topology of a 4-manifold. Indeed, the following result was pointed out in [31]:

Proposition 1.1 *A smooth compact simply connected* 4*-manifold* M *admits scalar-flat anti-self-dual metrics only if*

- M *is homeomorphic to* $k\overline{\mathbb{CP}}_2$ *for some* $k \geq 5$*; or*
- M *is diffeomorphic to* $\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$ *for some* $k \geq 10$*; or else*
- M *is diffeomorphic to* $K3$*.*

The key point [28] is that a self-dual harmonic 2-form on a compact SFASD 4-manifold must be parallel; thus, either the intersection form on H^2 must be negative-definite, or else the metric must be Kähler. Moreover, as observed by Lafontaine [27], our Gauss-Bonnet-type formulæ imply that any SFASD manifold must satisfy $2\chi + 3\tau \leq 0$, with equality only if the metric is locally hyper-Kähler. The Proposition therefore follows from the celebrated results of Donaldson [11] and Freedman [15] on the topology of smooth 4-manifolds, together with an elegant plurigenus vanishing argument due to Yau [45].

The purpose of the present paper is to prove a near-converse of the above Proposition:

Theorem A *A smooth compact simply connected* 4*-manifold* M *admits scalar-flat anti-self-dual metrics if*

- M *is diffeomorphic to* $k\overline{\mathbb{CP}}_2$ *for some* $k \geq 5$*; or*
- M *is diffeomorphic to* $\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$ *for some* $k \geq 10$*; or*
- M *is diffeomorphic to* $K3$*.*

In particular, each of these manifolds carries an optimal metric.

Combining Proposition 1.1 and Theorem A, we thus have:

Corollary 1.2 *A compact simply connected topological 4-manifold M carries a smooth structure for which there is a compatible SFASD metric g iff M is homeomorphic to $K3$, $\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$, $k \geq 10$, or $k\overline{\mathbb{CP}}_2$, $k \geq 5$.*

It is still unknown whether any $k\overline{\mathbb{CP}}_2$ admits non-standard smooth structures, so the degree to which Theorem A represents a true converse of Proposition 1.1 remains poorly understood. Note, however, that each of the topological manifolds $K3$ and $\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$, $k \geq 10$, admits infinitely many exotic smooth structures, and Proposition 1.1 asserts that there can never be an SFASD metric compatible with any of these. Moreover, for large classes of such exotic smooth structures, related arguments even show [31, Theorem B] that these manifolds do not admit compatible optimal metrics of *any* kind.

Now, as indicated above, the proof of Proposition 1.1 hinges on the fact that any self-dual harmonic 2-form on an SFASD 4-manifold must be parallel. For this reason, the only such metrics on $K3$ are hyper-Kähler, and the corresponding existence assertion in Theorem A therefore follows from Yau's solution [46] of the Calabi conjecture. Similarly, the SFASD metrics on $\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$ are precisely the scalar-flat Kähler metrics on these spaces; for $k \geq 14$, the existence of such metrics was first shown by Kim, Pontecorvo, and the first author [22], a result which was later improved to $k \geq 10$ by Rollin and Singer [41]. On the other hand, the existence of SFASD metrics on $k\mathbb{CP}_2$, $k \geq 6$, was first shown in [31]. The main goal of the present paper is therefore to improve the last assertion in order to include the case of $k = 5$; however, we will in fact obtain a simple, unified construction of such metrics for all $k \geq 5$ at no added cost. We now begin by providing a brief overview of the entire construction.

1.3 Strategy of the Proof

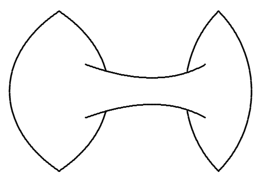
In order to prove Theorem A, we begin by choosing some integer $\ell \geq 3$, and then consider the two oriented conformally-flat orbifolds given by $S^4/\mathbb{Z}_2$ and $S^4/\mathbb{Z}_\ell$, where the relevant actions of $\mathbb{Z}_2$ and $\mathbb{Z}_\ell$ on the quaternionic projective line $\mathbb{HP}_1 = S^4$ are respectively generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/\ell} \end{pmatrix}.$$

Both of these orbifolds have two singular points, corresponding to $[1:0]$ and $[0:1] \in \mathbb{HP}_1$. However, while the two singularities of $S^4/\mathbb{Z}_2$ are on an equal footing, the two singularities of $S^4/\mathbb{Z}_\ell$ are instead mirror reflections of each other, corresponding to local actions on the quaternions $\mathbb{H} = \mathbb{R}^4$ respectively generated by

$$q \longmapsto e^{2\pi i/\ell} q \qquad \text{and} \qquad q \longmapsto q e^{-2\pi i/\ell}.$$

We now form the connected sum of these two orbifolds by removing a small round ball from the non-singular region of each, and then identifying the resulting S^3-boundaries via a reflection:



The resulting orbifold $V = (S^4/\mathbb{Z}_2)\#(S^4/\mathbb{Z}_\ell)$ is thereby endowed with an anti-self-dual orbifold conformal structure — indeed, with a locally conformally flat one.

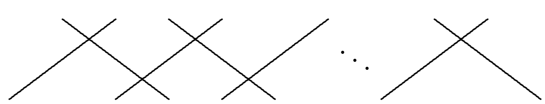
Now each singularity of the orbifold V looks like one familiar from the theory of complex algebraic surfaces [3], and each therefore has a so-called minimal resolution. We may thus desingularize V to obtain a smooth 4-manifold. Strictly on the level of smooth topology, this process amounts to replacing a neighborhood B^4/Γ of each singular point with a standard 4-manifold plug bounded by the same Lens space S^3/Γ. For the two $\Gamma = \mathbb{Z}_2$ singularities, the interior of our plug is just the degree -2 complex line bundle over $S^2 = \mathbb{CP}_1$. Similarly, for the singularity arising from the action of $\Gamma = \mathbb{Z}_\ell$ generated by

$$q \mapsto e^{2\pi i/\ell} q,$$

the interior of our plug is just the degree $-\ell$ complex line bundle over S^2. Finally, for the singularity arising from the action of $\Gamma = \mathbb{Z}_\ell$ generated by

$$q \mapsto q e^{-2\pi i/\ell},$$

the interior of our plug is a union of $\ell - 1$ copies of the degree -2 complex line bundle over S^2, plumbed together



in the manner dually indicated by the Dynkin diagram $A_{\ell-1}$:

$$\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\cdots\bullet\!-\!\bullet$$

Now it is known [30] that if this desingularization process is separately applied to $S^4/\mathbb{Z}_2$ and $S^4/\mathbb{Z}_\ell$, the resulting manifolds are respectively $2\overline{\mathbb{CP}}_2$ and $\ell\overline{\mathbb{CP}}_2$. Thus, the 4-manifold we obtain from $V = (S^4/\mathbb{Z}_2)\#(S^4/\mathbb{Z}_\ell)$ by this process will exactly be $M = (\ell+2)\overline{\mathbb{CP}}_2$. In particular, the $\ell = 3$ case will gives us the manifold $M = 5\overline{\mathbb{CP}}_2$, where the existence of optimal metrics is the key issue at stake.

Now each plug we have used to replace a singularity admits an asymptotically locally Euclidean (ALE) scalar-flat anti-self-dual (SFASD) metric [13, 17, 19, 29]. The ALE property means that the complement of a compact set is diffeomorphic to $(\mathbb{R}^4 - B^4)/\Gamma$ in such a manner that the metric takes the form

$$g_{jk} = \delta_{jk} + O(\varrho^{-2}),$$

where ϱ is the Euclidean radius. Since multiplying such a metric by a tiny positive number therefore yields a space whose geometry is macroscopically indistinguishable from the flat orbifold $\mathbb{R}^4/\Gamma$, one might thus hope to find anti-self-dual metrics on $M = (\ell+2)\overline{\mathbb{CP}}_2$ by grafting these ALE metrics onto the conformally flat conformal structure of the non-singular region of V, and then perturbing the grafted metric so as to once again solve the anti-self-duality equation. As we shall see, this does indeed work. Our approach to this aspect follows the direct analytic approach first proposed by Floer [14, 24, 41, 43]. We remark in passing that the alternative of a twistor approach [2, 39] to this problem might seem particularly tempting, but that the requisite stack-theoretic generalization of the Donaldson-Friedmann construction [10] has yet to be put on a solid footing; however, see [22, 34, 47] for evidence that such an approach should indeed be tractable. Note that this gluing procedure depends on a general vanishing result (Theorem 4.2) which had previously been asserted in [24, Theorem 8.4]; here, our contribution is to point out and repair a gap in the earlier proof.

Now, we have already noted that V carries a conformally flat structure. However, this conformally flat structure is far from unique. To the contrary, as we will see in §3 below, the theory of Kleinian groups [37] provides us with a connected family of conformally flat structures for which the corresponding limit sets have Hausdorf dimension varying from nearly zero to nearly two. By a result of Schoen-Yau [42], this implies that some of these conformal structures are represented by metrics of positive scalar curvature, while others are represented by metrics of negative scalar curvature. The punch line of our story is that the above gluing construction can be carried out uniformly in these additional parameters, and, setting $k = \ell + 2$, a gluing argument suggested by the work of Dominic Joyce [20] allows us to show the existence of 1-parameter family of ASD metrics on $M = k\overline{\mathbb{CP}}_2$ for which the scalar curvature changes sign in an analogous manner:

Theorem B *For any integer $k \geq 5$, the smooth oriented 4-manifold*

$$k\overline{\mathbb{CP}}_2 = \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{k}$$

admits a smooth 1-parameter family of anti-self-dual conformal metrics $[g_t]$, $t \in [-1, 1]$, such that $[g_1]$ contains a metric of positive scalar curvature, while $[g_{-1}]$ contains a metric of negative scalar curvature.

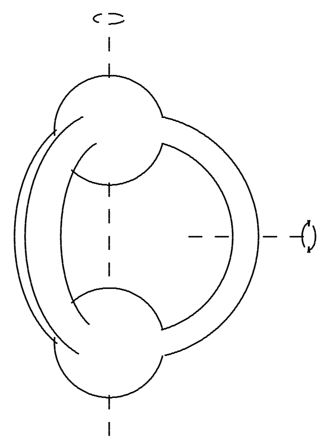
Now, because the lowest eigenvalue of the Yamabe Laplacian depends continuously on t, it is then easy to show that there must be some $t \in [-1, 1]$ such that $[g_t]$ contains a metric with scalar curvature $s \equiv 0$. In particular, this then shows that, for any $k \geq 5$, the 4-manifold $M = k\overline{\mathbb{CP}}_2$ carries a scalar-flat anti-self-dual metric. In light of our previous discussion, Theorem A is therefore an immediate consequence.

2 Uniformizing the Orbifold V

The orbifold V of §1.3 is actually expressible as a global quotient X/D_ℓ, where the locally conformally flat 4-manifold

$$X = \underbrace{(S^3 \times S^1) \# \cdots \# (S^3 \times S^1)}_{\ell - 1}$$

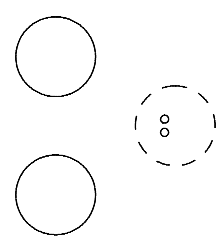
may be constructed by connecting two 4-spheres with ℓ tubes



and where the action of the dihedral group D_ℓ on X is then generated by a cyclic permutation of the handles and by an interchange of the two 4-spheres. This can be seen most directly by first recognizing $V = (S^4/\mathbb{Z}_2)\#(S^4/\mathbb{Z}_\ell)$ as a $\mathbb{Z}_\ell$-quotient of $S^4\#\ell(S^4/\mathbb{Z}_2)$, and then observing that this latter space can in turn be thought of as $X/\mathbb{Z}_2$.

2.1 The Kleinian Group Picture

Let us now explicitly realize X and V as quotients of regions of the 4-sphere by groups of Möbius transformations. Clearly, a fundamental region in the universal cover of X can be taken to be $S^4 = \mathbb{H} \cup \{\infty\}$ minus the ℓ open balls of radius ε centered at the ℓ^{th} roots of unity in $\mathbb{C} \subset \mathbb{H}$, together with the inversion of this complement into the ε-ball centered at 1:



The universal cover $\tilde{X}$ of X is thus realized as S^4 minus a Cantor set $\Lambda \subset \mathbb{H}$, where $\pi_1(X) = \mathbb{Z} * \cdots * \mathbb{Z}$ acts via a certain representation $\pi_1(X) \to PL(2, \mathbb{H})$. Similarly, we have $V = \tilde{X}/(\mathbb{Z}_2 * \mathbb{Z}_\ell)$, where $\mathbb{Z}_\ell$ is generated by the rotation

$$\beta(q) = e^{2\pi i/\ell} q,$$

and where $\mathbb{Z}_2$ is generated by the inversion

$$\alpha(q) = 1 + \varepsilon^2 (q-1)^{-1}.$$

Note that $\pi_1(X)$ is the subgroup of $\mathbb{Z}_2 * \mathbb{Z}_\ell$ generated by $(\alpha\beta)^2$ and its conjugates, and so is exactly the kernel of the obvious surjective homomorphism $\mathbb{Z}_2 * \mathbb{Z}_\ell \to D_\ell$. Since α and β actually actually lie in $PSL(2,\mathbb{C}) < PL(2,\mathbb{H})$, the relevant representations of $\mathbb{Z}_2 * \mathbb{Z}_\ell$ and $\pi_1(X) \triangleleft \mathbb{Z}_2 * \mathbb{Z}_\ell$ are actually $PSL(2,\mathbb{C})$-valued, and the limit set Λ is thus a subset of the plane $\mathbb{C} \subset \mathbb{H}$.

2.2 The Limit Set Λ

The limit set Λ comes, by construction, with a hierarchy of coverings by disks in $\mathbb{C}$. At the crudest level, it is contained in a union of ℓ disks of radius ε, where we are allowed to choose any positive $\varepsilon < \sin(\pi/\ell)$. We now express this choice as

$$\varepsilon = \frac{\sin(\pi/\ell)}{C+1}$$

for some real number $C > 0$. Each disk in the successive layers of the heirarchy contains $\ell - 1$ times as many disks as in the previous layers, and each disk in any layer has radius smaller than that of some disk in the preceding layer by a factor of better than C^{-2}. In particular, Λ is contained in a union of $\ell(\ell-1)^N$ disks of radius less than C^{-2N}, and so has d-dimensional Hausdorff measure $< \ell[(\ell-1)/C^{2d}]^N$. Taking $N \to \infty$, we thus see that the Hausdorff measure of Λ is zero in any dimension $> [\log(\ell-1)]/2\log C$. For fixed ℓ, it follows that that the Hausdorff dimension $\dim \Lambda$ can be taken arbitrarily close to 0 by taking ε to be sufficiently small.

This now implies that some of the constructed conformal structures on V are represented by conformally flat orbifold metrics of positive scalar curvature. Indeed, let us recall a remarkable result of Schoen and Yau [42], together with a slight refinement due to Nayatani [38]:

Lemma 2.1 (Schoen-Yau, Nayatani) *Let (M,[g]) be a compact, locally conformally flat n-manifold, $n \geq 3$, which can be uniformized as*

$$M = \Omega/G,$$

where $G \subset SO^{\uparrow}(n+1,1)$ is a Kleinian group and where $\Omega \subset S^n$ is the region of discontinuity of G. Let $g \in [g]$ be a metric on M in the fixed conformal *class for which the scalar curvature s does not change sign. Assume that the limit set Λ of G is infinite, and let* $\dim(\Lambda) > 0$ *denotes its Hausdorff dimension. Then*

$$\begin{aligned} s > 0 &\iff \dim(\Lambda) < \frac{n}{2} - 1 \\ s = 0 &\iff \dim(\Lambda) = \frac{n}{2} - 1 \\ s < 0 &\iff \dim(\Lambda) > \frac{n}{2} - 1. \end{aligned}$$

Now any conformal class on any compact manifold contains metrics for which the scalar curvature does not change sign; moreover, as observed by Trudinger [44], this sign coincides, for any background metric g, with that of the lowest eigenvalue λ of the Yamabe Laplacian, given in dimension 4 by

$$\Delta_g + \frac{s_g}{6} \ ;$$

indeed, if u is an eigenfunction with eigenvalue λ, then u is nowhere zero by the minimum principle, and $\hat{g} = u^2 g$ then has scalar curvature

$$\hat{s} = u^{-3} 6(\Delta_g + \frac{s_g}{6})u = 6\lambda u^{-2}$$

which everywhere has the same sign as λ. Let us apply this to

$$X = \underbrace{(S^3 \times S^1)\# \cdots \#(S^3 \times S^1)}_{\ell-1} \ .$$

By averaging, we may begin by choosing our background metric g on X to be invariant under the action of the finite group D_ℓ. Because the minimum principle implies that the lowest eigenvalue λ of the Yamabe Laplacian has multiplicity one, the corresponding lowest eigenfunction u is unique up to an overall multiplicative constant, and so must be invariant under the finite group of isometries D_ℓ. Thus $\hat{g} = u^2 g$ may either be viewed as a D_ℓ-invariant conformally flat metric on X, or else as a conformally flat orbifold metric on $V = X/D_\ell$. By Lemma 2.1, the sign of the scalar curvature of $\hat{g}$ will therefore be positive iff the Hausdorff dimension of the limit set Λ of our $\mathbb{Z}_2 * \mathbb{Z}_\ell$ action is less than 1. Since we have just seen that this can be accomplished by taking the parameter ε to be sufficiently small, it follows that some of our conformal classes are indeed represented by conformally flat orbifold metrics on V which have positive scalar curvature, as would also be predicted by more elementary considerations.

Our real goal is to show that the above metrics on V can be continuously deformed through locally conformally flat metrics in order to yield metrics of negative scalar curvature. We will do this by deforming the Kleinian groups $\mathbb{Z}_2 * \mathbb{Z}_\ell \hookrightarrow PSL(2, \mathbb{C})$ considered above into ones whose limit sets have Hausdorff dimension greater than 1. This problem is intimately tied to the theory of non-classical Schottky groups, and its solution will require the entire next section of the paper.

3 Deformations with large limit sets

The main point of this section is to show that there are quasi-conformal deformations of the Kleinian group G of §2.1 whose limit sets have Hausdorff dimension arbitrarily close to 2. The proof, which uses ideas going back to Bers [5] and Maskit [35], can be easily generalized to a larger class of Kleinian groups. Rather than work out a detailed formulation of this more general theorem, we content ourselves with assuming that G is a combination theorem free product of an elliptic cyclic group of order m, and an elliptic cyclic group of order $\ell \geq m$, where $\ell \geq 3$. That is, $G \cong \mathbb{Z}_m * \mathbb{Z}_\ell$ is the free product of an element α of order m, and an element β of order ℓ. For our application, we of course only need the case of $m = 2$, and it is the case of $\ell = 3$ which is of primary interest here.

3.1 Holomorphic coordinates

We note that, since $\ell \geq 3$, the fixed points of β can be distinguished. That is, there is no Möbius transformation conjugating β into itself and interchanging its fixed points. We label one of these fixed points as positive. (For example, if we conjugate β so that its positive fixed point is at 0, while its negative fixed point is at ∞, then this conjugate has the form $z \mapsto e^{2\pi i/\ell} z$.)

We normalize G so that α has fixed points at 0 and ∞, and so that the positive fixed point of β is at 1; if the order m of α is greater than 1, then we put the positive fixed point of α at 0. Let z_0 be the negative fixed point of β. For every $z \neq 1$, we define the homomorphism $\phi_z : G \to PSL(2, \mathbb{C})$ given by $\phi_z(\alpha) = \alpha$ and $\phi_z(\beta)$ is the Mobius transformation of order ℓ with positive fixed point at 1 and negative fixed point at z.

We observe that if $z = 0$, or $z = \infty$, then $\phi_z(G)$ contains parabolic elements and has a fixed point z. We also note that if z is real and negative, then the axes of $\phi_z(\alpha)$ and $\phi_z(\beta)$ intersect at a point in the upper half space; this point is necessarily a fixed point of $\phi_z(G)$. It was observed by Chuckrow [9] that no point on the algebraic convergence boundary of the deformation spaace of G has a fixed point in $\mathbb{H}^3$ or on the sphere at infinity.

We denote the complex plane, punctured at 1, and with the negative real axis deleted, by $\mathbb{C}^\bullet$. The homomorphism ϕ_z defines an embedding of $\mathbb{C}^\bullet$ into the space of homomorphisms of G into $PSL(2, \mathbb{C})$.

Let $\mathcal{T} = \mathcal{T}(G)$ denote the (quasi-conformal) deformation space of G; where the quasi-conformal mappings are as usual normalized so as to fix $(0, 1, \infty)$. With this normalization, every quasi-conformal mapping representing an element of $\mathcal{T}$ conjugates α into itself and preserves the positive fixed point of β. Since no element of the closure of $\mathcal{T}$ can conjugate G into a group with a fixed point, we can regard the closure of $\mathcal{T}$ as a subset of $\mathbb{C}^\bullet$.

It was shown by Ahlfors and Bers [1] that the image of a point under a family of quasi-conformal mappings is a holomorphic function of parameters for the family; it follows that if we regard $\mathcal{T}$ as being endowed with the complex structure defined by the projection from its universal covering space, the Teichmüller space of Riemann surfaces of genus 0 with four punctures, then this embedding of $\mathcal{T}$ into $\mathbb{C}^\bullet$ is holomorphic.

The parameter z appears to depend on our normalization. However, if we write the cross-ratio of four points as

$$(a, b; c, d) = \frac{(a - c)(b - d)}{(a - d)(b - c)}, \tag{5}$$

then $z = (z, 1; 0, \infty)$.

3.2 Geometric coordinates

Let $z \in \mathbb{C}^\bullet$, and let $\alpha_z = \phi_z(\alpha)$, and $\beta_z = \phi_z(\beta)$; since z does not lie on the closed negative real axis, there is a positive hyperbolic distance d between the axes of α_z and β_z.

We renormalize α_z and β_z, by sending 0 to $+1$ and by requiring that the common perpendicular between the axes of α_z and β_z have its endpoints at 0 and ∞, where 0 is closer to the axis of α_z. After this renormalization, α_z has fixed points at ± 1, with $+1$ the positive fixed point if $m > 2$; β_z has its positive fixed point at some point $t = |t|e^{i\theta}$, and its negative fixed point at $-t$. One sees at once that $|t| = e^d$, and that θ is the angle between the axes of α_z and β_z obtained by parallel transport along the common perpendicular. (More precisely, θ is the angle from the ray pointing to $+1$ along the axis of α_z to the ray pointing to t along the axis of β_z.) Then $t = e^{d+i\theta}$ is a geometric parameter for the subspace $\mathbb{C}^\bullet \subset Hom(G, PSL(2,\mathbb{C}))$.

Using the invariance of the cross-ratio, we obtain

$$z = z(t) = (-t, +t; +1, -1) = \frac{(t+1)^2}{(t-1)^2}. \tag{6}$$

It was shown by Gehring, Marshall and Martin [16] that for $z \in \mathcal{T}$ there is a positive lower bound to the distance d. As stated above, this is also easy to see using the result of Chuckrow [9]. It follows that $\sqrt{z}$, which is positive for z real and positive, is a holomorphic coordinate on T that extends to the boundary, from which it follows that

$$t = t(z) = e^{d+i\theta} = \frac{\sqrt{z}+1}{\sqrt{z}-1} \tag{7}$$

is also a holomorphic coordinate on $\mathcal{T}$ that extends to the boundary.

We note that each point t in the closure of $\mathcal{T}$ defines a homomorphism $\psi_t : G \to PSL(2,\mathbb{C})$, where $\psi_t = \phi_{z(t)}$. This homomorphism is in fact defined for all t in the exterior of the unit disc.

3.3 A boundary point of the first kind

There is a positive real number L such that if $|t| > L$, then $t \in \mathcal{T}$. For example, one can choose L to be the distance in the hyperbolic plane between the finite vertices of the triangle with angles π/m, π/ℓ and 0; equivalently, this is the distance between the fixed points of the elliptic generators of orders m and ℓ in the (m, ℓ, ∞)-triangle group. Easy computation shows that

$$\cosh L = \frac{1 + \cos(\pi/m)\cos(\pi/\ell)}{\sin(\pi/m)\sin(\pi/\ell)}. \tag{8}$$

For the case that especially interests us here, $m = 2$ and $\ell = 3$, this triangle group is the elliptic modular group, and $L = \log\sqrt{3}$.

For each fixed angle, θ, the ray $\arg t = \theta$ lies in $\mathcal{T}$ for $|t|$ sufficiently large, and does not lie even in the closure of $\mathcal{T}$ for $|t|$ sufficiently small. Hence there is some largest t_θ on this ray with t_θ on the boundary of $\mathcal{T}$.

For each fixed $g \in G$, the entries in the matrix representing $\psi_t(g)$ are holomorphic functions of t. We note that $\psi_t(g)$ is parabolic only if the square of its trace is equal to 4. Hence there are at most countably many points in the closure of $\mathcal{T}$ (in fact, in the exterior of the unit disc), for which $\psi_t(g)$ can be parabolic. Since G is countable, there are at most countably many points in the exterior of the unit disc for which some element of $\psi_t(G)$ is parabolic.

We now choose the direction θ_0 so that, for every point on the ray, $\arg t = \theta_0$, no element of the group $\psi_t(G)$ is parabolic. Let t_0 be the largest point on this ray, $\arg t = \theta_0$, where t_0 lies on the boundary of $\mathcal{T}$, and let $K = \psi_{t_0}(G)$. We will need below that there is a sequence of groups in $\mathcal{T}$ converging algebraically to K.

It follows from Chuckrow's theorem [9] that ψ_{t_0} is an isomorphism onto K; in particular, K is an algebraic free product of cyclic groups of orders m and ℓ; we also know that no element of K is parabolic.

Proposition 3.1 *Every point of $\mathbb{CP}_1$ is a limit point of K. In other words, K is a Kleinian group of the first kind.*

Proof. Suppose that K is of the second kind; that is, suppose that it acts discontinuously at some point on the sphere at infinity. Let Δ be a connected component of the set of discontinuity of K, and let H be the stability subgroup of Δ. Since K is finitely generated, it follows from Ahlfors' finiteness theorem that H is also finitely generated. Since K is an algebraic free product of cyclic groups of orders m and ℓ, it follows that H is an algebraic free product of a finite number of cyclic groups of orders m, ℓ, and/or infinity. The function group H can be decomposed, using combination theorem amalgamated free products and HNN extensions, into *basic* groups; these are finitely generated subgroups of H, each containing no accidental parabolic element, and each having, as a Kleinian group in its own right, a simply connected invariant component of its set of discontinuity [37]. It is also shown in [37] that every basic group is either Fuchsian or quasifuchsian of the first kind, degenerate, Euclidean or finite; it is also well known that every such quasifucshian or degenerate group is isomorphic to a Fuchsian group of the first kind, where the isomorphism preserves parabolic elements in both directions.

Suppose J is such a basic group. Since K is a free product of cyclic groups, so is J; since K contains no parabolic elements, neither does J. We conclude that J cannot be Fuchsian, quasi-Fuchsian or degenerate, for every finitely-generated Fuchsian group of the first kind that is a free product of cyclic groups necessarily contains a parabolic element. We also conclude that J is not Euclidean, for every Euclidean group contains parabolic elements. Hence J is finite.

Since no basic subgroup of H is either Fuchsian or quasi-Fuchsian, the set of discontinuity of H is connected [37], implying both that $H = K$ and that Δ is the full set of discontinuity of K. We now have that K is a function group isomorphic to G. By [36], it therefore follows that there is a quasi-conformal homeomorphism w realizing the isomorphism ψ_{t_0}. But this contradicts our assumption that t_0 lies on the boundary of $\mathcal{T}$ and not in its interior. Thus K cannot be of the second kind, and so must be of the first kind, as claimed. ∎

Theorem 3.2 *For every $\epsilon > 0$, there is some $t_\epsilon \in \mathcal{T}$ such that the limit set of $\psi_{t_\epsilon}(G)$ has Hausdorff dimension greater than $2 - \epsilon$.*

Proof. Proposition 3.1 tells us that the limit set of K has Hausdorff dimension 2. However, a result of Bishop and Jones [6] asserts that if $K_n \to K$ algebraically, then $\liminf \dim \Lambda(K_n) \geq \dim \Lambda(K)$, where $\dim \Lambda$ denotes the Hausdorff dimension of the limit set of the relevant group. Our result therefore follows from the fact that, by construction, K is the algebraic limit of groups of the form $\psi_t(G)$. ∎

3.4 Scalar-Flat Orbifold Metrics

We now specialize these conclusions to our original case of $m = 2$, and apply them to our geometric setting. Doing so immediately gives us an orbifold analogue of Theorem B:

Proposition 3.3 *For each $\ell \geq 3$, there is a smooth family h_t of metrics on $X = (\ell - 1)(S^1 \times S^3)$, $t \in [-1, 1]$, such that*

- *for each t, the metric h_t is locally conformally flat and D_ℓ-invariant;*
- *the metric h_1 is conformally related to a metric with $s > 0$; and*
- *the metric h_{-1} is conformally related to a metric with $s < 0$.*

Proof. Choose two points in T, one corresponding to a group of Hausdorff dimension > 1, the other corresponding to a group with Hausdorff dimension < 1. Since the manifold T is connected, these two points can be joined by a smooth arc. This arc then corresponds to a family of manifolds diffeomorphic to $X = (\ell - 1)(S^1 \times S^3)$, together with a smooth family of flat conformal structures on them and a smooth family of actions of the dihedral group D_ℓ compatible with these conformal structures. The total space of this family is then diffeomorphic to $X \times I$ such a manner that the D_ℓ action is independent of the parameter. We now represent our conformal structures by a smooth family of metrics. By adding all the pull-backs of these metrics with respect to the fixed D_ℓ-action, we then obtain representatives h_t which are D_ℓ-invariant. By Lemma 2.1, the metric at one end-point, say h_{-1}, is then conformal to a metric with negative scalar curvature, while the metric at the other end-point, say h_1, is conformal to a metric with positive scalar curvature. ∎

Now, for each t, let λ_t be the lowest eigenvalue of the Yamabe Laplacian

$$\Delta_{h_t} + \frac{s_{h_t}}{6}$$

and recall that, by the minimum principle, any corresponding eigenfunction u_t must be everywhere non-zero. Hence λ_t has multiplicity 1, and so varies continuously with t. However, we know that $\lambda_{-1} < 0$, and $\lambda_{+1} > 0$, so the intermediate value theorem predicts the existence of some $t_0 \in [-1, 1]$ such that $\lambda_{t_0} = 0$. Letting u_{t_0} be the corresponding positive eigenfunction, the conformally flat metric $h = u_{t_0}^2 h_{t_0}$ is then D_ℓ-invariant, and has scalar curvature $s \equiv 0$. Thus:

Proposition 3.4 *For each ℓ, the orbifold $V = [(\ell - 1)(S^1 \times S^3)]/D_\ell$ admits locally conformally flat, scalar-flat orbifold metrics. Such metrics are optimal, in the the sense of the natural orbifold extension of Definition 1.*

3.5 Deformation Theory

In order to carry out our gluing construction, we will want to know that the deformation theory of anti-self-dual conformal structures is formally unobstructed for each of the constructed flat conformal structures on V. We will deduce this from an analogous statement about flat conformal structures on the connected sum $X = (\ell - 1)(S^1 \times S^3)$.

To make this precise, recall that the deformation theory of anti-self-dual conformal structures on any 4-orbifold Y is governed by the elliptic complex [10]

$$0 \to \Gamma(TY) \xrightarrow{\mathcal{L}} \Gamma(\odot_0^2 \Lambda^1) \xrightarrow{DW_+} \Gamma(\odot_0^2 \Lambda^+) \to 0$$

where $\odot_0^2$ denotes the trace-free symmetric product. Here $\mathcal{L}$ computes the Lie derivative of the conformal metric along vector fields, while DW_+ is the linearization of the self-dual Weyl tensor at g. The deformation theory is unobstructed at a given conformal metric iff H^2 of this complex vanishes. (We remark in passing that this amounts [10, 12] to saying that the Kodaira-Spencer deformation theory of the corresponding twistor space is unobstructed.)

Proposition 3.5 *For any flat conformal structure $[g]$ on the orbifold $V = (S^4/\mathbb{Z}_2)\#(S^4/\mathbb{Z}_\ell)$, the differential operator*

$$DW_+ : C^\infty(\odot_0^2 \Lambda^1) \to C^\infty(\odot_0^2 \Lambda^+)$$

is surjective. In other words, the deformation theory of ASD conformal structures on V is unobstructed at $[g]$.

Proof. By ellipticity, it is enough to show that $\ker(DW_+)^* = 0$. But we saw in §2 that V can be expressed as a global quotient X/D_ℓ, where

$$X = \underbrace{(S^3 \times S^1)\# \cdots \#(S^3 \times S^1)}_{\ell-1} .$$

Thus any flat conformal structure on V pulls back to a flat conformal structure on X, and any $\varphi \in \ker(DW_+)^*$ pulls back to an element of the cokernel of the corresponding operator DW_+ on X. However, a Mayer-Vietoris argument due to Eastwood and Singer shows [31, Theorem 8.2] that the deformation of ASD conformal structures is unobstructed at every flat conformal structure on X. Since the pull-back of φ to X therefore vanishes, so does φ itself, and DW_+ is therefore surjective, as claimed. ∎

4 ALE Metrics

In addition to the conformally flat orbifolds considered in the previous section, our construction will crucially involve the use of two classes of examples of complete, non-compact anti-self-dual 4-manifolds which are asymptotically locally Euclidean (ALE). The examples we will use are in fact all scalar-flat and Kähler. After an explicit description of the metrics we will actually need, we then prove a general result (Theorem 4.2, §4.3) concerning the deformation theory of ALE scalar-flat Kähler manifolds.

4.1 The Gibbons-Hawking Metrics

The first class of building blocks we will need consists of the Gibbons-Hawking gravitational instantons [17], which may be understood [19] as Ricci-flat Kähler metrics on the minimal resolutions of the singular complex surfaces

$$xy = z^\ell .$$

Such a singular complex surface can explicitly be identified with $\mathbb{C}^2/\mathbb{Z}_\ell$, where $\mathbb{Z}_\ell \subset SU(2)$ is generated by

$$\begin{pmatrix} e^{2\pi i \ell} & 0 \\ 0 & e^{-2\pi i \ell} \end{pmatrix},$$

via the map

$$\begin{array}{rcl} \mathbb{C}^2/\mathbb{Z}_\ell & \longrightarrow & \{\ (x, y, z) \quad | \quad xy = z^\ell\ \} \\ \mathbb{Z}_\ell \cdot (u, v) & \longmapsto & (u^\ell, v^\ell, uv). \end{array}$$

The resolution of the singularity is accomplished by replacing the origin with a string of $\ell - 1$ copies $\mathbb{CP}_1$, each of self-intersection -2, and each only intersecting its succesor and/or predecessor, so that the pattern of intersections is dual to that indicated by the Dynkin diagram $A_{\ell-1}$:

$$\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\bullet$$

Metrics in this family can be written in closed form by means of the *Gibbons-Hawking ansatz* [17]. Choose ℓ distinct point $p_1, \ldots, p_\ell$ in $\mathbb{R}^3$, and set $\mathcal{U} = \{p_1, \ldots, p_\ell\}$. Let $\rho_j : \mathbb{R}^3 \to \mathbb{R}$ be the Euclidean distance to p_j, and define $V : \mathcal{U} \to \mathbb{R}^+$ by

$$V = \sum_{j=1}^{\ell} \frac{1}{2\rho_j}.$$

Then $\Delta V = 0$ on $\mathcal{U}$, so the 2-form

$$\star dV = V_x dy \wedge dz + V_y dz \wedge dx + V_z dy \wedge dz$$

is closed. Moreover, $[(\star dV)/2\pi] \in H^2(\mathcal{U}, \mathbb{Z})$, since the integral of $\star dV$ on a small 2-sphere centered at p_j is -2π, and such spheres generate $H_2(\mathcal{U}, \mathbb{Z})$. Let $\varpi : P \to \mathcal{U}$ be the circle bundle with first Chern class $[(\star dV)/2\pi]$. By the Chern-Weil theorem, there is a connection 1-form θ on P with $d\theta = \varpi^* \star dV$. The Gibbons-Hawking metric on P is then given by

$$g_{GH} = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2.$$

The metric-space completion Y_ℓ of (P, g) is then a smooth Riemannian manifold, and is obtained by merely adding one point $\hat{p}_j$ for each each of the chosen points p_j, and we then have a smooth projection

$$\begin{array}{ccccc} Y_\ell & = & P & \cup & \{\hat{p}_1, \ldots, \hat{p}_\ell\} \\ \downarrow & & \downarrow \varpi & & \downarrow \\ \mathbb{R}^3 & = & \mathcal{U} & \cup & \{p_1, \ldots, p_\ell\} \end{array}$$

which, near each critical point $\hat{p}_j$, is modelled on the map

$$\begin{array}{rcl} \mathbb{R}^4 = \mathbb{H} & \longrightarrow & \Im m\ \mathbb{H} = \mathbb{R}^3 \\ q & \longmapsto & qi\bar{q}\ . \end{array}$$

The Riemannian 4-manifold (Y_ℓ, g) is then ALE, with $\Gamma = \mathbb{Z}_\ell$, and is *hyper-Kahler*, in the sense that Λ^+ is trivialized by three linearly independent parallel sections. It is also asymptotically locally Euclidean (ALE); indeed, the region $\rho_1 > \max(|p_j - p_1|)$ in Y_ℓ may be identified with the complement of a ball in $\mathbb{R}^4/\mathbb{Z}_\ell$ in such a manner such that $\varrho = \sqrt{2\ell\rho_1}$ becomes the Euclidean radius, and such that

$$g = g_{\text{Eucl}} + O(\varrho^{-4}), \quad \partial^k g = O(\varrho^{-4-k}).$$

4.2 Line-Bundle Metrics

The second family of ALE metrics we will need for our construction consists of the so-called LeBrun metrics [29] on the total spaces of negative-degree complex line bundles $L \to \mathbb{CP}_1$ over the 2-sphere. For each integer $\ell \geq 1$, such a metric on the $c_1 = -\ell$ line bundle may be obtained by taking the metric-space completion of the metric

$$g_{LB} = \frac{d\varrho^2}{1 + \frac{\ell-2}{\varrho^2} - \frac{\ell-1}{\varrho^4}} + \varrho^2 \left(\sigma_1^2 + \sigma_2^2 + \left[1 + \frac{\ell-2}{\varrho^2} - \frac{\ell-1}{\varrho^4} \right] \sigma_3^2 \right) \qquad (9)$$

on $(\mathbb{C}^2 - B)/\mathbb{Z}_\ell$, where $B \subset \mathbb{C}^2$ is the closed unit ball, and where the $\mathbb{Z}_\ell$ action is generated by scalar multiplication by $e^{2\pi i/\ell}$ on $\mathbb{C}^2$. Here σ_1, σ_2, σ_3 is the standard $SU(2)$-invariant orthonormal co-frame on the the unit sphere $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$, and it is therefore easy to show that g is $U(2)$-invariant. When a 2-sphere (corresponding to the zero section of the line bundle) is added along $\varrho = 1$, the metric extends smoothly and becomes complete. For our purposes, we will need only the cases $\ell \geq 2$. Note that when $\ell = 2$, g is the celebrated Eguchi-Hanson metric, and is also given by the $\ell = 2$ case of the Gibbons-Hawking construction described in §4.1. When $\ell \neq 2$, however, the metric is no longer Ricci-flat; moreover, it is then just ALE of order 2, rather than of order 4:

$$g_{LB} = g_{\text{Eucl}} + O(\varrho^{-2}), \quad \partial^k g = O(\varrho^{-2-k}).$$

4.3 A Vanishing Theorem

The examples described in the two previous sub-section are all ALE scalar-flat Kähler manifolds. However, that these examples by no means constitute an exhaustive list; there are many others [7, 25]. The purpose of the present section is to prove a general vanishing result pertaining to all such ALE spaces. We begin with a lemma which provides useful information on the asymptotic structure of such manifolds.

Lemma 4.1 *Let (Y, J, g) be a complete scalar-flat Kähler manifold of real dimension 4. Suppose, moreover, that g is ALE in the weak sense that*

$$g = g_{Eucl} + O(\varrho^{-\frac{3}{2}-\delta}), \quad \partial g = O(\varrho^{-\frac{5}{2}-\delta})$$

for some $\delta > 0$ in some real asymptotic coordinate system. Let $U \subset Y$ be the domain of such an asymptotic coordinate system, and let $\tilde{U} \approx S^3 \times \mathbb{R}$ be its universal cover. Then there is a non-singular complex surface (S, J) obtained by adding a $\mathbb{CP}_1$ of self-intersection $+1$ to $\tilde{U}$ at infinity. Moreover, small deformations of this holomorphic curve pass through every point of $\tilde{U}$ in a neighborhood of infinity. The complex surface (Y, J) therefore has only one end, and is obtained from a non-singular, rational complex surface by removing a (typically singular) rational curve. Moreover, g is strongly ALE, of order ≥ 2, in the sense that in a (perhaps better) asymptotic coordinate chart,

$$g = g_{Eucl} + O(\varrho^{-2}), \quad \partial^k g = O(\varrho^{-2-k})$$

for every positive integer k.

Proof. The fact that the weak form of the ALE condition implies the stronger form for a scalar-flat Kähler surface is proved in [8, Proposition 12] by first showing that adding an extra twistor line P at infinity to the twistor space of $\tilde{U}$ still yields a complex manifold Z. Let S_0 be the image of the section of the twistor projection $(Z - P) \to \tilde{U}$ given by J, and let $\bar{S}_0$ similarly corresponding to $-J$. Then $S_0 \cap \bar{S}_0 = \varnothing$, and, because g is scalar-flat Kähler, Pontecorvo's theorem [40] tells us that $S_0 \cup \bar{S}_0$ is the non-degenerate zero locus of a holomorphic section of $\kappa_Z^{-1/2}$. However, since $P \subset Z$ has complex codimension 2, this section uniquely extends to all of Z by Hartog's theorem. Since S_0 and $\bar{S}_0$ are interchanged by the real structure of Z, this section is real, and its restriction to P, which is essentially a section of $\mathcal{O}(2)$ on $\mathbb{CP}_1$, must either vanish identically, or vanish only at an antipodal pair of points. In the latter case, however, Pontecorvo's theorem would say that g extended to $\tilde{U} \cup \{\infty\}$ as a scalar-flat Kähler metric, contradicting the fact that g is complete. The extension therefore vanishes on all of P, and $S_0 \cup \bar{S}_0 \cup P$ is therefore a (singular) complex hypersurface in Z. Let S be the irreducible component of this surface containing S_0. Since every point of P is either in the closure of S_0 or in the closure of $\bar{S}_0$, P must meet S in infinitely points. Hence $P \subset S$, and $S = S_0 \cup P$. Since a generic complex twistor line therefore meets S in exactly one point, and since we can foliate any small region of Z with a holomorphic family of complex twistor lines which are not contained in S, the nullstellensatz [18] now implies that S is actually non-singular. Thus S is a smooth complex surface obtained by adding $P \cong \mathbb{CP}_1$ to $S_0 = (\tilde{U}, J)$. Moreover, since $P \subset Z$ has normal bundle

$\mathcal{O}(1) \oplus \mathcal{O}(1)$, whereas $P \cdot S = 1$, the adjunction formula implies that the normal bundle of $P \subset S$ has degree one.

The normal bundle of the projective line $P \subset S$ is therefore $\mathcal{O}(1)$. Since $H^1(\mathbb{CP}_1, \mathcal{O}(1)) = 0$, Kodaira's theorem on deformations of compact complex submanifolds [23] guarantees that every element of $H^0(\mathbb{CP}_1, \mathcal{O}(1))$ arises from a variation of $P \subset S$ through holomorphic curves, and the union of these curves must therefore fill out some open neighborhood $V \subset S$ of P.

The fundamental group Γ of the end U acts on $\tilde{U}$ by isometries preserving J, and these extend to $\tilde{U} \cup \{\infty\}$ as conformal isometries. These in turn lift to the twistor space Z as biholomorphisms sending S to S and P to P. In particular, Γ acts holomorphically on S in a manner sending P to itself. Since the automorphism group of the first infinitesimal neighborhood of $P \subset S$ is $GL(2, \mathbb{C})$, the action of the finite group Γ acts on $TS|_P$ is realized by that of a finite subgroup of the maximal compact subgroup $U(2) \subset GL(2, \mathbb{C})$. If $\Gamma_0 \subset \Gamma$ is the normal subgroup corresponding to the center $U(1) \subset U(2)$, the quotient $\check{S} = S/\Gamma_0$ is then a smooth complex surface, and $S \to \check{S}$ is then a branched cyclic cover ramified along P. The induced action of $\check{\Gamma} = \Gamma/\Gamma_0$ on $\check{S}$ then has only cyclic isotropy groups with isolated fixed points, so we can resolve all the singularities of $S/\Gamma = \check{S}/\check{\Gamma}$ by replacing them with Hirzebruch-Jung strings [3, Theorem 5.4]. Notice that the generic deformation of P in S will descend to $\check{S}$ as a $\mathbb{CP}_1$ of self-intersection $|\Gamma_0| \geq 1$, and that any such curve which avoids the finite fixed-point set of $\check{\Gamma}$ will then become an immersed $\mathbb{CP}_1$ with positive normal bundle in the resolution of S/Γ.

Applying this picture, we may thus compactify Y as a non-singular compact complex surface $\hat{Y}$ by adding a tree of $\mathbb{CP}_1$'s to each end. But the compactification $\hat{Y}$ then in particular contains holomorphically immersed $\mathbb{CP}_1$'s with positive normal bundle passing through each point of in a region at each end of Y. Since the pluricanonical line-bundle K^m and the cotangent bundle Ω^1 pull back to any such curve as negative bundles, all holomorphic sections of K^m or Ω^1 must vanish along all such curves, and since these curves sweep out an open subset of $\hat{Y}$, uniqueness of analytic continuation implies that $H^0(\hat{Y}, \Omega^1) = 0$ and $H^0(\hat{Y}, \mathcal{O}(K^m)) = 0$ for all $m > 0$. Moreover, the homology class of any such curve has positive self-intersection, as may be seen by moving a given $\mathbb{CP}_1$ within the appropriate Kodaira family; hence $b_+(\hat{Y}) \neq 0$. The Kodaira-Enriques classification [3, 18] therefore tells us that $\hat{Y}$ is a blow-up of either $\mathbb{CP}_2$ or a Hirzebruch surface. In particular, $b_+(\hat{Y}) = 1$. It follows that Y can only one end; otherwise, we would get two disjoint curves of positive self-intersction from any pair of ends, forcing

b_+ to be larger. Hence Y is obtained from a rational complex surface $\hat{Y}$ by removing a (possibly singular) embedded rational curve, as claimed. $\blacksquare$

With the ground now properly prepared, we are now ready to prove our main vanishing result. Actually, this very result was previously stated in [24, Theorem 8.4], but the proof given there unfortunately assumes that the complex structure is asymptotically biholomorphic to $\mathbb{C}^2/\Gamma$, and this, alas, is simply not true in general. However, we shall now see that much the same proof can be made to work in full generality with a modicum of extra care.

Theorem 4.2 *Let (Y, g) be any ALE scalar-flat Kahler manifold of real dimension 4, and let $(\hat{Y}, [g])$ be its 1-point conformal compactification, considered as a compact ASD orbifold. Then the deformation theory of ASD conformal structures on $\hat{Y}$ is unobstructed at $[g]$. That is, the linearized self-dual Weyl curvature*

$$DW_+ : C^\infty(\odot_0^2 T^*Y) \to C^\infty(\odot_0^2 \Lambda^+)$$

is surjective at $[g]$.

Proof. Consider the adjoint equation

$$(DW_+)^* \varphi = 0 \tag{10}$$

for an element $\varphi \in \Gamma(\odot_0^2 \Lambda^+)$ of the cokernel on the orbifold $\hat{Y}$. If ${\varphi^a}_{bcd}$ is simply considered as a totally trace-free tensor field with the same symmetries as W_+, then (10) takes the explicit form

$$\left(\nabla^b \nabla^d + \frac{1}{2} r^{bd}\right) {\varphi^a}_{bcd} = 0, \tag{11}$$

so $(DW_+)(DW_+)^*$ is elliptic, and any distributional solution φ on $\hat{Y}$ is necessarily smooth in the orbifold sense. Moreover, DW_+ belongs to an elliptic complex, so that DW_+ is surjective iff (10) has no non-trivial solutions. Now the geometric nature of the construction of DW_+ implies that it is conformally invariant; and since the L^2 inner product on such tensor fields ${\varphi^a}_{bcd}$ is conformally invariant, equation (10) is conformally invariant without any added conformal weight. Thus the pull-back of φ to Y satisfies both (10) and the asymptotic fall-off conditions

$$|\varphi| = O(\varrho^{-4}), \quad |\nabla^k \varphi| = O(\varrho^{-4-k})$$

at infinity, where ϱ denotes the Euclidean radius function in the asymptotic chart, and where the norms and derivatives are all with respect to g. Working henceforth only on (Y, g), our objective is therefore to show that (10) and these boundary conditions suffice to imply that φ vanishes.

To this end, think of φ as belonging to End (Λ^+), and consider the self-dual 2-form ϕ obtained by applying this endomorphism to the Kähler form ω; that is, let

$$\phi = \varphi(\omega) \in \Gamma(\Lambda^+)$$

be the contraction of φ with ω. We then recall that

$$\Lambda^+ = \mathbb{R}\omega \oplus \Re e(K)$$

on any Kähler surface, where $K = \Lambda^{2,0}$ is the canonical line bundle. Thus

$$\phi = f\omega + \alpha$$

for some function f and the real part α of some $(2,0)$-form. Then (10) implies [33, Corollary 2.3] that

$$d^-\delta(f\omega + \alpha) - f\rho = 0,$$

where ρ is the Ricci form of g. Using the Kähler identities to rewrite this as

$$(d\Lambda d^c + d^c\Lambda d)\alpha - 2dd^c f - \omega\Delta f - 2f\rho = 0$$

and then applying dd^c, we thus obtain the fourth order scalar equation

$$(\Delta^2 + 2r \cdot \nabla\nabla)f = 0.$$

However, this so-called Lichnerowicz equation can be re-written [32] as

$$\nabla_\nu \nabla^{\bar{\mu}} \nabla_{\bar{\mu}} \nabla^\nu f = 0,$$

or, in other words, as

$$(\bar{\partial}\partial^\#)^*(\bar{\partial}\partial^\#)f = 0,$$

where $\partial^\#$ means the $(1,0)$ component of the gradient. But since

$$|\nabla^k f| = O(\varrho^{-4-k})$$

two integrations by parts give us

$$\int_Y |(\bar{\partial}\partial^{\#})f|^2 d\mu = \int_Y f(\bar{\partial}\partial^{\#})^*(\bar{\partial}\partial^{\#})f d\mu = 0.$$

Hence $\bar{\partial}\partial^{\#}f = 0$, and $\partial^{\#}f = (\nabla f - iJ\nabla f)/2$ is thus a holomorphic vector field. But this means that the Hamiltonian vector field $J\nabla f$ preserves both the Kähler form ω and the complex structure J. Therefore $\xi = J\nabla f$ also preserves g, and so is a Killing vector field. In particular, ξ may be viewed as a conformal Killing field on $(\hat{Y}, [g])$. But our fall-off conditions say that $\xi = O(\varrho^{-5})$, and in inverted coordinates about the point at infinity, ξ therefore has a zero of order 6. Since a conformal Killing field is completely determined by its 2-jet at any point, it follows that $\xi = J\nabla f$ vanishes identically. This shows that f is constant, and our fall-off therefore implies that $f \equiv 0$.

It follows that the self-dual 2-form $\phi = \alpha$ satisfies

$$d^-\delta\phi = 0.$$

Hence $d\delta\phi$ is self-dual, and $|d\delta\phi|^2 d\mu = d\delta\phi \wedge d\delta\phi$. Integration therefore gives

$$\int_Y |d\delta\phi|^2 d\mu = \int_Y d\,(\delta\phi \wedge d\delta\phi) = 0$$

by Stokes' theorem and the fact that $|\delta\phi \wedge d\delta\phi| = O(\varrho^{-11})$. Thus $d\delta\phi = 0$, and the self-dual 2-form ϕ is therefore killed by the Hodge Laplacian. But since our manifold has $s = 0$ and $W_+ = 0$, the Weitzenböck formula for the Hodge Laplacian just says

$$(d+\delta)^2\phi = \nabla^*\nabla\phi = 0.$$

Since $\phi \cdot \nabla\phi = O(\varrho^{-9})$, integration by parts therefore says that

$$\int_Y |\nabla\phi|^2 d\mu = \int_Y \langle\phi, \nabla^*\nabla\phi\rangle d\mu = 0$$

so that $\nabla\phi = 0$ is parallel, and so in particular, $|\phi|$ is constant. But $|\phi| \to 0$ at infinity, so it follows that $\phi \equiv 0$.

We must therefore have $\varphi = \beta + \overline{\beta}$ for some $\beta \in \Gamma(\kappa^{\otimes 2})$, where $\kappa = \Lambda^{2,0}$ is the canonical line bundle. Because the Ricci tensor belongs to $\Lambda^{1,1}$, equation (11) now simplifies to become

$$\nabla^\kappa\nabla^\mu\beta_{\kappa\lambda\mu\nu} = 0.$$

Letting $\psi \in \Gamma(\Omega^1 \otimes \kappa)$ be defined by

$$\psi_{\kappa\lambda\nu} = \nabla^\mu \beta_{\kappa\lambda\mu\nu},$$

we then have $\partial^* \psi = 0$, so $\overline{\partial}\psi = 0$, and ψ is holomorphic.

Now let $U \subset Y$ be an asymptotic region of Y, and let $\tilde{U} \approx S^3 \times \mathbb{R}$ be its universal cover. We have shown, in Lemma 4.1, that we can form a complex surface (S, J) by adding a holomorphic curve $P \cong \mathbb{CP}_1$ of self-intersection $+1$ at infinity; moreover, deformations of P then gives us rational holomorphic curves of self-intersection $+1$ sweeping out an open neighborhood $V \subset S$ of P. The asymptotic fall-off of ϕ now implies that the pull-back of ψ to $\tilde{U}$ extends continuously, and hence holomorphically, to S. But V is swept out by $\mathbb{CP}_1$'s with normal bundle $\mathcal{O}(1)$. Since the restriction of $\Omega^1 \otimes \kappa$ to such a curve is isomorphic to $\mathcal{O}(-5) \oplus \mathcal{O}(-4)$, it follows that ψ vanishes identically on V, hence on its image in Y, and thus on all of Y by the uniqueness of analytic continuation.

The vanishing of ψ now tells us that

$$g^{\kappa\bar{\pi}} \nabla_{\bar{\pi}} \beta_{\kappa\mu\nu\lambda} = \nabla^\kappa \beta_{\kappa\mu\nu\lambda} = 0$$

so we now have $\overline{\partial}\beta = 0$, and β is itself holomorphic. But once again, our fall-off conditions imply that the pull-back of β to $\tilde{U}$ extends holomorphically to S. Since the restriction of $\kappa^{\otimes 2}$ to a $\mathbb{CP}_1$ of self-intersection $+1$ is isomorphic to $\mathcal{O}(-6)$, we thus see that β must vanish identically on the open set $V \subset S$ swept out by such curves, and hence on all of Y by analytic continuation. Thus $\varphi = \beta + \bar{\beta} \equiv 0$, and the cokernel of DW_+ is trivial, as claimed. ∎

5 The Gluing Construction

We are now in a position to construct the desired family of anti-self-dual metrics on $k\overline{\mathbb{CP}}_2$, $k \geq 5$. The basic tool we will need is the following gluing result [24, Theorem C]; cf. [14, 43].

Proposition 5.1 (Floer *et al.*) *Let $(V_j, [g_j])$ be a finite collection of smooth compact oriented 4-dimensional orbifolds with anti-self-dual conformal structure, and suppose that these conformal structures are* deformation-unobstructed *in the sense that*

$$DW_+ : C^\infty(\odot_0^2 T^*Y) \to C^\infty(\odot_0^2 \Lambda^+)$$

is surjective for each of these spaces. Further, suppose we are given gluing data*, consisting of a finite collection of pairs* $(p_\alpha, q_\alpha) \in \coprod V_j$ *of distinct points, together with a collection of linear maps*

$$\Lambda_\alpha : T_{p_\alpha} \to T_{q_\alpha}$$

between the orbifold tangent space of $\coprod V_j$ *at these points, such that* Λ_α *is an orientation-reversing conformal isometry, and such that, for each* α*, either*

- p_α *and* q_α *are both manifold (non-singular) points of* $\coprod V_j$*; or else*
- p_α *and* q_α *are both orbifold (singular) points, the relevant orbifolds are locally modelled on* $T_{p_\alpha}/\Gamma_\alpha$ *and* $T_{q_\alpha}/\Gamma_\alpha$ *for the same finite group* Γ_α*, and the relevant representations of* Γ_α *are intertwined by* Λ_α*.*

Let M *be the new orbifold obtained from* $\coprod V_j$ *by first removing a small ball around each* p_α *and* q_α*, and then identifying the resulting boundaries via the orientation-reversing diffeomorphism indicated by* Λ_α*. Then* V *admits a family of anti-self-dual metrics* $[g_{\mathrm{u}}]$*, smoothly (but perhaps redundantly) parameterized by deformation-unobstructed ASD conformal structures on* $\coprod V_j$ *and choices of gluing data. Moreover, these conformal structures resemble those specified on each* V_j *in the following sense: there are sequences of these* g_{u_I} *of these metrics and diffeomorphisms* $\Phi_I : (V_j - \{p_\alpha, q_\alpha\}) \hookrightarrow M$ *such that the conformal metrics* $[\Phi_I^* g_{u_I}]$ *converge to* $[g_j]$ *in the* C^2 *topology on every compact subset* $\mathbf{K} \subset (V_j - \{p_\alpha, q_\alpha\})$*.*

We remark that the redundancy of parameterization alluded to above only occurs if some of the V_j carry conformal Killing vector fields, and can be eliminated by adding marked points to the picture. Note that this parameterization by no means describes the entire moduli space of anti-self-dual metrics on V, about which still know all too little; rather, it simply says that this moduli space has a preferred end, where we know quite a bit.

As sketched in §1.3, we now consider the orbifolds $V_1 = S^4/\mathbb{Z}_2$, and $V_2 = S^4/\mathbb{Z}_\ell$, $\ell \geq 3$, together with two copies V_3 and V_4 of the one-point compactification of the Eguchi-Hanson space Y_2, one copy V_5 of the compactified $A_{\ell-1}$ Gibbons-Hawking space Y_ℓ, and one copy V_6 of the compactification of the $c_1 = -\ell$ line bundle over $\mathbb{CP}_1$ equipped with the LeBrun metric. We consider the ordinary connect sum $V = V_1 \# V_2$ centered at manifold points, and then take the generalized connect sum

$$M_\ell = V \#_{\mathbb{Z}_2} V_3 \#_{\mathbb{Z}_2} V_4 \#_{\mathbb{Z}_\ell} V_5 \#_{\mathbb{Z}_\ell} V_6$$

centered at appropriate pairs of singular points. Because we have eliminated all the singular points, this gives a smooth compact 4-manifold, and it is moreover easy to see by Seifert-van Kampen and Mayer-Vietoris that M_ℓ is simply connected, and has $b_2 = \ell + 2$. By Proposition 3.5 and Theorem 4.2, the hypotheses of Proposition 5.1 are fulfilled, and we thus have

Corollary 5.2 *The gluing strategy proposed in §1.3 does actually give rise, via Proposition 5.1, to a connected family of anti-self-dual metrics on the smooth compact simply connected manifold M_ℓ with $b_2 = \ell + 2$, $\ell \geq 3$. Moreover, for any compact subset $\mathbf{K}$ of the complement of the four singular points of V, there are embeddings $\mathbf{K} \hookrightarrow M$ on which some of these metrics are arbitrarily close, in the C^2 sense, to any one of the conformally flat metrics h of §3.4. On the other hand, one can also construct some metrics of this family by instead applying Proposition 5.1 to the orbifolds $V_1, \ldots, V_6$, without forming the connect sum $V = V_1 \# V_2$ as an intermediate step.*

Note that, in the complex coordinate on $\mathcal{T}$ described in §3.1 , direct construction from $V_1, \ldots, V_6$ corresponds to the region $z \approx 1$, where $V = V_1 \# V_2$ has an extremely long neck separating the two orbifold points of type $\mathbb{Z}_\ell$ from the two singularities of type $\mathbb{Z}_2$.

5.1 Negative Scalar Curvature

Let us now show that some of the conformal structures of Corollary 5.2 have representatives of negative scalar curvature.

To see this, recall that if the lowest eigenvalue of the Yamabe Laplacian of a compact Riemannian manifold is negative, the metric can then be conformally rescaled to yield a metric of negative scalar curvature simply by multiplying it by the square of the corresponding eigenfunction. In dimension 4, it therefore suffices to show that

$$\inf_{u \in L^2_1} \frac{\int [6|\nabla u|^2 + su^2] d\mu}{\int u^2 d\mu}$$

is negative; in particular, it suffices to display some smooth test function u such that

$$\int_M [6|\nabla u|^2 + su^2] d\mu_g < 0.$$

Note that this test function need not be positive everywhere.

We saw in §3.4 that $V = V_1 \# V_2$ admits conformally flat orbifold metrics of negative scalar curvature. By rescaling , we can therefore find such metrics with scalar curvature $s < -1$. Let u be a function V which is identically zero on the $(\varepsilon/2)$-balls about the four oribifold singularities, equal to 1 outside the corresponding (ε)-balls, and has $|\nabla u| < 3/\varepsilon$ in the transition annuli. We further take $\varepsilon \in (0,1)$ to be small enough so that the total volume of these four (orbifold) ε-balls is less than $C\varepsilon^4$ where e.g. $C = 2\pi^2$. With respect to the background metric h, we then have

$$\int_Y [6|\nabla u|^2 + su^2] d\mu_h < 54C\varepsilon^2 - [\mathrm{Vol}\ (Y) - C\varepsilon^4] < 55C\varepsilon^2 - \mathrm{Vol}\ (Y)$$

which is negative for ε is sufficiently small. For such an ε, let $\mathbf{K} \subset V$ be the complement of the four $(\varepsilon/2)$-balls about the orbifold points. Corollary 5.2 then tells us that there exists a family of representative metrics g_j for some of our conformal structures on M_ℓ, together with a sequence of diffeomorphisms $\Phi_j : \mathbf{K} \hookrightarrow M_\ell$, such that $\Phi_j^* g_j \to h$ in C^2 on $\mathbf{K}$. In particular,

$$\int_{\mathbf{K}} [6|\nabla u|^2 + su^2] d\mu_{\Phi_j^* g_j} < 0$$

for sufficiently large j. For some such j, we can thus construct a test function $\hat{u}$ on M_ℓ by extending the push-forward $u \circ \Phi_j^{-1}$ by zero, and this test function then satisfies

$$\int_{M_\ell} [6|\nabla \hat{u}|^2 + s\hat{u}^2] d\mu_{g_j} < 0.$$

Thus $(M, [g_j])$ can be conformally rescaled to yield a metric of negative scalar curvature, and we have:

Lemma 5.3 *For each $\ell \geq 3$, some of the anti-self-dual conformal classes on M_ℓ constructed in Corollary 5.2 can be represented by Riemannian metrics of negative scalar curvature.*

Note that the gist of this argument is quite soft, and widely applicable. The moral is that negative scalar curvature on one summand can be used to generate negative scalar curvature on a generalized connected sum; cf. [20].

5.2 Positive Scalar Curvature

We next show that other conformal classes in our connected family have representatives of positive scalar curvature. Results of Joyce [20] indicate that this can be done by a direct argument in the spirit of the proof of Lemma 5.3; however, the positive case in this setting is technically quite delicate. Instead, we will proceed here via an entirely different route, exploiting the considerable body of information concerning the spaces in question.

To this end, let us instead consider the manifolds

$$N = V_3 \#_{\mathbb{Z}_2} V_1 \#_{\mathbb{Z}_2} V_4 = V_3 \#_{\mathbb{Z}_2} V_4$$

and

$$N_\ell = V_5 \#_{\mathbb{Z}_\ell} V_2 \#_{\mathbb{Z}_\ell} V_6 = V_5 \#_{\mathbb{Z}_\ell} V_6.$$

In both these cases, Proposition 5.1 allows us to deduce the existence of ASD conformal structures built out of the given building blocks. However, one knows families of explicit ASD metrics of with the required degenerations on these manifolds; some of these metrics admit semi-free circle actions, and arise from the hyperbolic ansatz [30, pp. 235–237], and the relevant degenerations involve bringing the centers together until they collide. In particular, this explicitly identifies N as the smooth manifold $2\overline{\mathbb{CP}}_2$, and N_ℓ as $\ell\overline{\mathbb{CP}}_2$. Moreover, all of these explicit conformal classes are known [30, p. 235] to admit representatives of positive scalar curvature. Some of the metrics of our family are therefore obtained by directly applying Proposition 5.1 to glue together some of these positive-scalar-curvature ASD metrics on N and N_ℓ. For such ordinary connect sums, however, the gluing procedure can be realized by the Donaldson-Freedman construction [10], and in this context, a theorem of Atiyah on the the Penrose Transforms of Yamabe Green's functions allows one to show [31, 21] that connected sums of positive-scalar-curvature ASD manifolds with small gluing parameters again have representatives of positive scalar curvature. This shows the following:

Lemma 5.4 *For each $\ell \geq 3$, M_ℓ is diffeomorphic to $(\ell+2)\overline{\mathbb{CP}}_2$, and some of the anti-self-dual conformal classes on M_ℓ constructed in Corollary 5.2 can be represented by Riemannian metrics of positive scalar curvature.*

5.3 The Main Theorems

Combining Lemmas 5.3 and 5.4, we now have a smooth connected family of anti-self-dual metrics on the connected sum $(\ell+2)\overline{\mathbb{CP}}_2$, $\ell \geq 3$, such that some of the conformal classes are represented by metrics of positive scalar curvature, and others are represented by metrics of negative scalar curvature. Choosing a smooth path in our parameter space which joins a conformal class of the first type to one of the second, and setting $k = \ell+2$, we have therefore proved

Theorem B *For any integer $k \geq 5$, the smooth oriented 4-manifold $k\overline{\mathbb{CP}}_2$ admits a smooth 1-parameter family of anti-self-dual Riemannian metrics g_t, $t \in [-1, 1]$, such that $[g_1]$ contains a metric of positive scalar curvature, while $[g_{-1}]$ contains a metric of negative scalar curvature.*

Now let λ_t be the smallest eigenvalue of the Yamabe Laplacian

$$\Delta + \frac{s}{6}$$

for the metric g_t. By the minimum principle, any corresponding eigenfunction u_t must be everywhere non-zero. Hence λ_t has multiplicity 1, and so varies continuously with t. However, we know that $\lambda_{-1} < 0$, and $\lambda_{+1} > 0$, so the intermediate value theorem predicts the existence of some $t_0 \in [-1, 1]$ such that $\lambda_{t_0} = 0$. Letting u_{t_0} be the corresponding positive eigenfunction, the ASD metric $g = u_{t_0}^2 g_{t_0}$ then has scalar curvature $s \equiv 0$. This proves

Theorem 5.5 *For each $k \geq 5$, the smooth compact oriented 4-manifold*

$$k\overline{\mathbb{CP}}_2 = \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{k}$$

admits scalar-flat anti-self-dual Riemannian metrics. In particular, each of these spaces carries optimal metrics, in the sense of Definition 1.

Combining this with the results of Yau [46] on $K3$ and Rollin-Singer [41] on $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, we have therefore proved Theorem A, as promised.

Authors' address: Mathematics Department, SUNY, Stony Brook, NY 11794-3651.
Keywords: 4-manifold, anti-self-dual metric, scalar curvature, Kleinian group.
Mathematics Subject Classification: 53C25 (primary), 30F40 (secondary).

References

[1] L. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. (2), 72 (1960), pp. 385–404.

[2] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A, 362 (1978), pp. 425–461.

[3] W. Barth, C. Peters, and A. V. de Ven, *Compact Complex Surfaces*, Springer-Verlag, 1984.

[4] M. Berger, *Riemannian Geometry During the Second Half of the Twentieth Century*, American Mathematical Society, Providence, RI, 2000. Reprint of the 1998 original.

[5] L. Bers, *On boundaries of Teichmüller spaces and on Kleinian groups. I*, Ann. of Math. (2), 91 (1970), pp. 570–600.

[6] C. J. Bishop and P. W. Jones, *Hausdorff dimension and Kleinian groups*, Acta Math., 179 (1997), pp. 1–39.

[7] D. M. J. Calderbank and M. A. Singer, *Einstein metrics and complex singularities*, Invent. Math., 156 (2004), pp. 405–443.

[8] X. X. Chen, C. LeBrun, and B. Weber, *On conformally Kähler, Einstein manifolds*. e-print arXiv:0705.0710, 2007.

[9] V. Chuckrow, *On Schottky groups with applications to Kleinian groups*, Ann. of Math. (2), 88 (1968), pp. 47–61.

[10] S. Donaldson and R. Friedman, *Connected sums of self-dual manifolds and deformations of singular spaces*, Nonlinearity, 2 (1989), pp. 197–239.

[11] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom., 18 (1983), pp. 279–315.

[12] M. G. Eastwood and M. A. Singer, *The Fröhlicher [Frölicher] spectral sequence on a twistor space*, J. Differential Geom., 38 (1993), pp. 653–669.

[13] T. Eguchi and A. J. Hanson, *Self-dual solutions to Euclidean gravity*, Ann. Physics, 120 (1979), pp. 82–106.

[14] A. Floer, *Self-dual conformal structures on $\ell\mathbf{C}\mathrm{P}^2$*, J. Differential Geom., 33 (1991), pp. 551–573.

[15] M. Freedman, *On the topology of 4-manifolds*, J. Differential Geom., 17 (1982), pp. 357–454.

[16] F. W. Gehring, T. H. Marshall, and G. J. Martin, *The spectrum of elliptic axial distances in Kleinian groups*, Indiana Univ. Math. J., 47 (1998), pp. 1–10.

[17] G. W. Gibbons and S. W. Hawking, *Classification of gravitational instanton symmetries*, Comm. Math. Phys., 66 (1979), pp. 291–310.

[18] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[19] N. J. Hitchin, *Polygons and gravitons*, Math. Proc. Cambridge Philos. Soc., 85 (1979), pp. 465–476.

[20] D. Joyce, *Constant scalar curvature metrics on connected sums*, Int. J. Math. Math. Sci., (2003), pp. 405–450.

[21] M. Kalafat, *Scalar curvature and connected sums of self-dual 4-manifolds.* e-print arXiv:math/0611769, 2006.

[22] J. Kim, C. LeBrun, and M. Pontecorvo, *Scalar-flat Kähler surfaces of all genera*, J. Reine Angew. Math., 486 (1997), pp. 69–95.

[23] K. Kodaira, *A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds*, Ann. of Math. (2), 75 (1962), pp. 146–162.

[24] A. Kovalev and M. Singer, *Gluing theorems for complete anti-self-dual spaces*, Geom. Funct. Anal., 11 (2001), pp. 1229–1281.

[25] P. B. Kronheimer, *Instantons gravitationnels et singularités de Klein*, C. R. Acad. Sci. Paris Sér. I Math., 303 (1986), pp. 53–55.

[26] N. H. Kuiper, *On conformally-flat spaces in the large*, Ann. of Math. (2), 50 (1949), pp. 916–924.

[27] J. Lafontaine, *Remarques sur les variétés conformément plates*, Math. Ann., 259 (1982), pp. 313–319.

[28] C. LeBrun, *On the topology of self-dual* 4*-manifolds*, Proc. Amer. Math. Soc., 98 (1986), pp. 637–640.

[29] ——, *Counter-examples to the generalized positive action conjecture*, Comm. Math. Phys., 118 (1988), pp. 591–596.

[30] ——, *Explicit self-dual metrics on* $\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2$, J. Differential Geom., 34 (1991), pp. 223–253.

[31] ——, *Curvature functionals, optimal metrics, and the differential topology of four-manifolds*, in Different Faces of Geometry, Int. Math. Ser. (N. Y.), Kluwer/Plenum, New York, 2004, pp. 199–256.

[32] C. LeBrun and S. R. Simanca, *Extremal Kähler metrics and complex deformation theory*, Geom. Funct. Anal., 4 (1994), pp. 298–336.

[33] C. LeBrun and M. Singer, *Existence and deformation theory for scalar-flat Kähler metrics on compact complex surfaces*, Invent. Math., 112 (1993), pp. 273–313.

[34] ——, *A Kummer-type construction of self-dual* 4*-manifolds*, Math. Ann., 300 (1994), pp. 165–180.

[35] B. Maskit, *On boundaries of Teichmüller spaces and on Kleinian groups. II*, Ann. of Math. (2), 91 (1970), pp. 607–639.

[36] ——, *Isomorphisms of function groups*, J. Analyse Math., 32 (1977), pp. 63–82.

[37] ——, *Kleinian Groups*, Springer-Verlag, Berlin, 1988.

[38] S. Nayatani, *Patterson-Sullivan measure and conformally flat metrics*, Math. Z., 225 (1997), pp. 115–131.

[39] R. Penrose, *Nonlinear gravitons and curved twistor theory*, General Relativity and Gravitation, 7 (1976), pp. 31–52.

[40] M. Pontecorvo, *On twistor spaces of anti-self-dual Hermitian surfaces*, Trans. Amer. Math. Soc., 331 (1992), pp. 653–661.

[41] Y. Rollin and M. Singer, *Non-minimal scalar-flat Kähler surfaces and parabolic stability*, Invent. Math., 162 (2005), pp. 235–270.

[42] R. Schoen and S.-T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math., 92 (1988), pp. 47–71.

[43] C. H. Taubes, *The existence of anti-self-dual conformal structures*, J. Differential Geom., 36 (1992), pp. 163–253.

[44] N. Trudinger, *Remarks concerning the conformal deformation of metrics to constant scalar curvature*, Ann. Scuola Norm. Sup. Pisa, 22 (1968), pp. 265–274.

[45] S. T. Yau, *On the curvature of compact Hermitian manifolds*, Invent. Math., 25 (1974), pp. 213–239.

[46] ——, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math., 31 (1978), pp. 339–411.

[47] J. Zhou, *Connected Sums of Self-Dual Orbifolds*, PhD thesis, State University of New York at Stony Brook, 1995.