Geometry of 4-Manifolds:

Curvature in the Balance

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$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

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Proposition. *Assume* $n \geq 4$. *Then* (M^n, g) *locally conformally flat* $\Longleftrightarrow W \equiv 0$.

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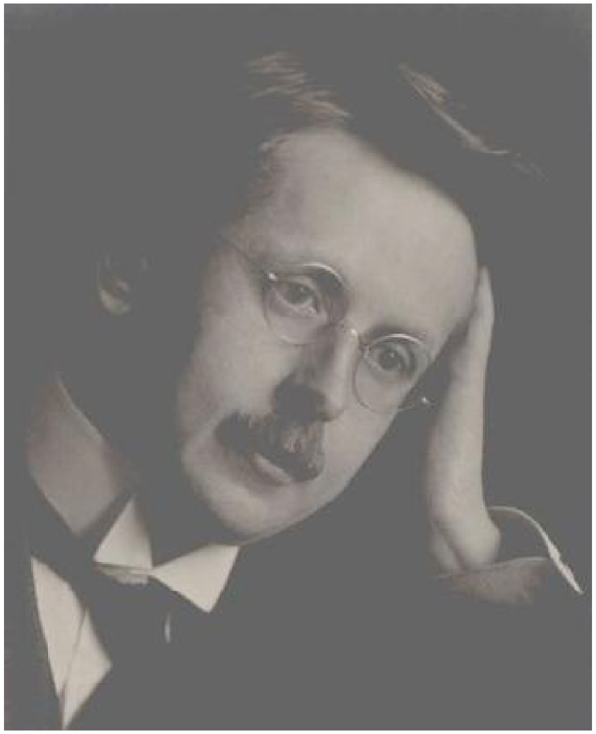

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- Do there exist minimizers?

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Of course, conformally Einstein good enough!

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Ricci-flat $\Longrightarrow$ $W = \mathcal{R}$.

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since, for fixed CY on $K3$, $\quad \mathscr{W}(g) \propto \text{Vol}(\mathbb{T}^{m-4})$.

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Integrals give four scale-invariant functionals.

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However, these are not independent!

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Euler characteristic

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu$$

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$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

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e.g. critical for Weyl functional

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So $\int |W_+|^2 d\mu$ equivalent to Weyl functional.

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Today's theme: How do these compare in size,

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Today's theme: How do these compare in size,

for specific classes of metrics on interesting 4-manifolds?

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More general Riemannian metrics?

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Excluded: Round S^4, Fubini-Study $\overline{\mathbb{CP}}_2$.

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with equality $\Leftrightarrow$ g *is locally Kähler-Einstein.*

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Excluded: Del Pezzo Surfaces (10 diffeotypes)

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How are these results proved?

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Any complex surface M with b_1 even carries both metrics with $>$ and with $<$.

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Equivalent to

$$\frac{1}{4\pi^2} \int_M |W_+|^2 d\mu_g \overset{?}{\geq} \frac{1}{3}(2\chi + 3\tau)(M).$$

Since

$$\mathscr{W}([g]) = -12\pi^2\tau(M) + 2\int_M |W_+|^2 d\mu_g$$

this is really a question about $\inf \mathscr{W}$.

For (M^4, g) compact oriented Riemannian,

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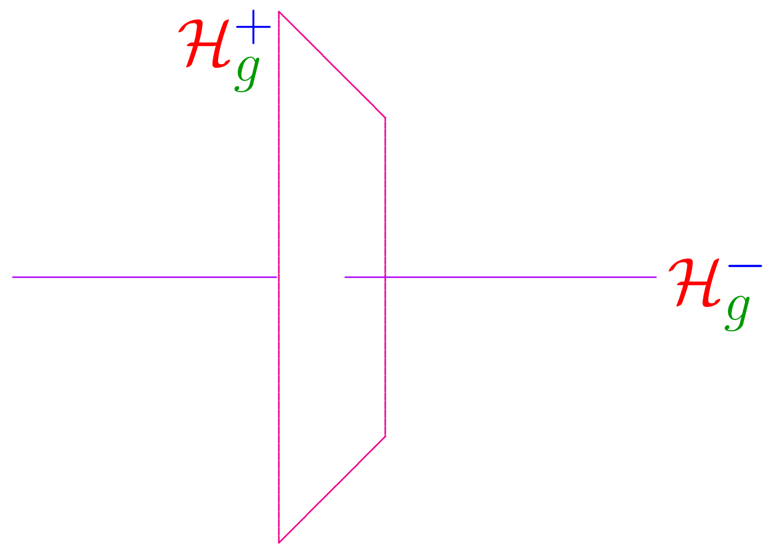
However, they are genuinely metric-dependent as soon as we allow for more general changes of g.



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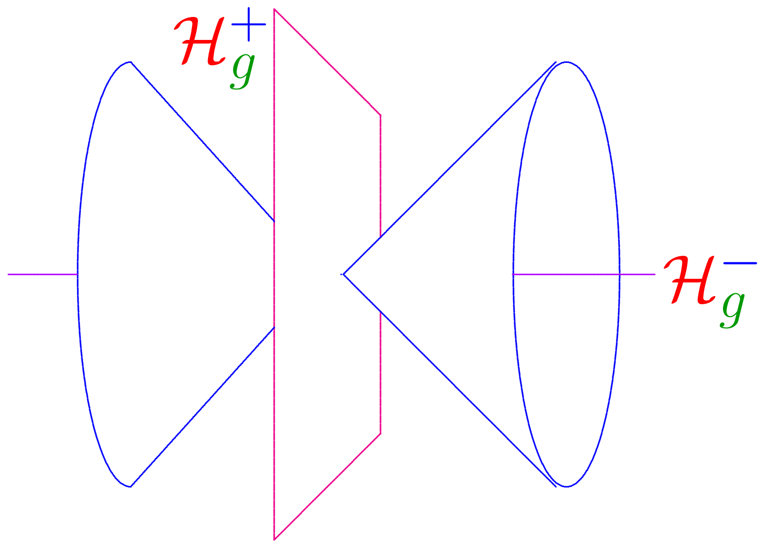

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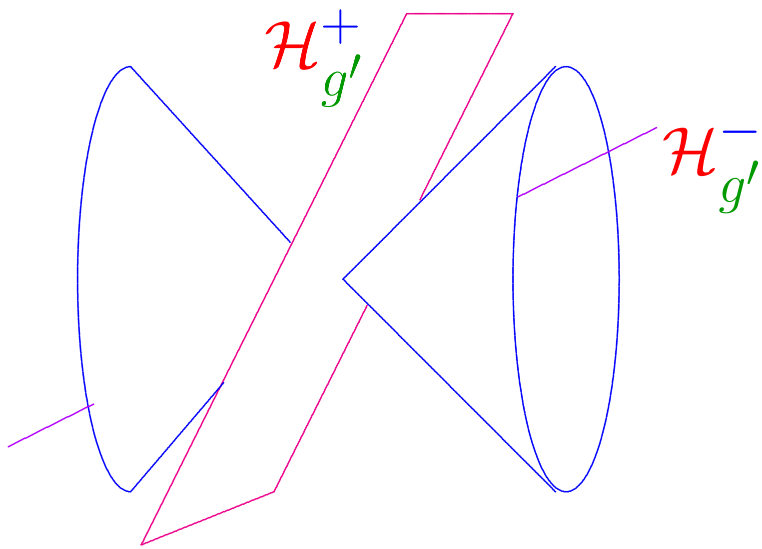

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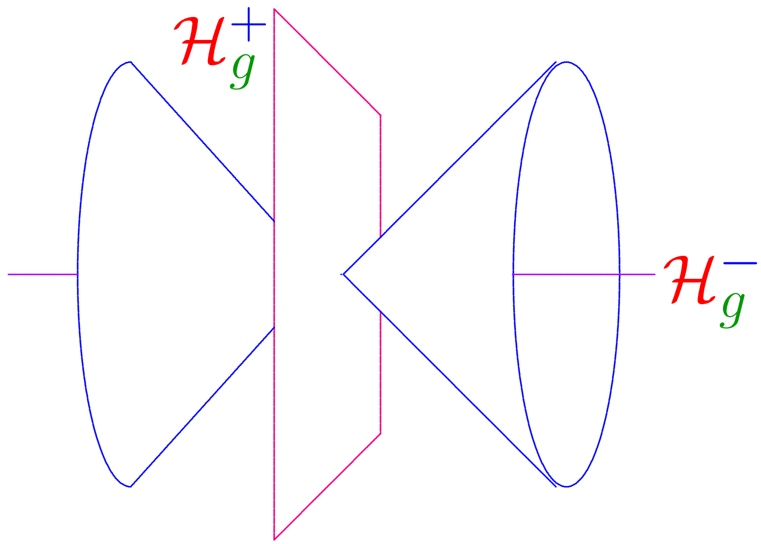

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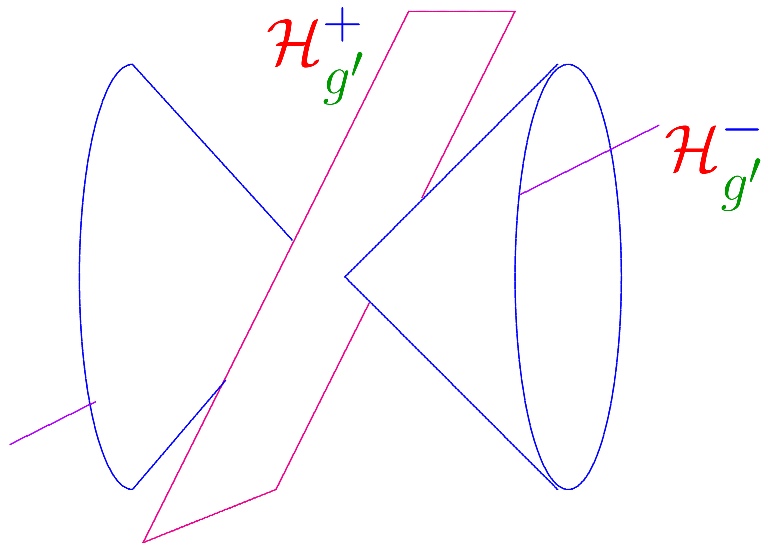

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Often using complex geometry, via twistor spaces. . .

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Context: 1978 paper building on Penrose '76.

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If g has s of fixed sign, agrees with sign of $Y_{[g]}$.

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Proposition (Atiyah-Hitchin-Singer '78). *The Fubini-Study metric on $\mathbb{CP}_2$ is self-dual. Consequently, minimizes Weyl functional.*

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Kuiper '49: $\therefore$ Round S^4! $\Rightarrow\Leftarrow$

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Commonality between $\mathbb{CP}_2$ and $S^2 \times S^2$?

Kähler-Einstein, with $\lambda > 0$.

Natural Generalization:

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(M^4, J) for which c_1 is a Kähler class $[\omega]$.

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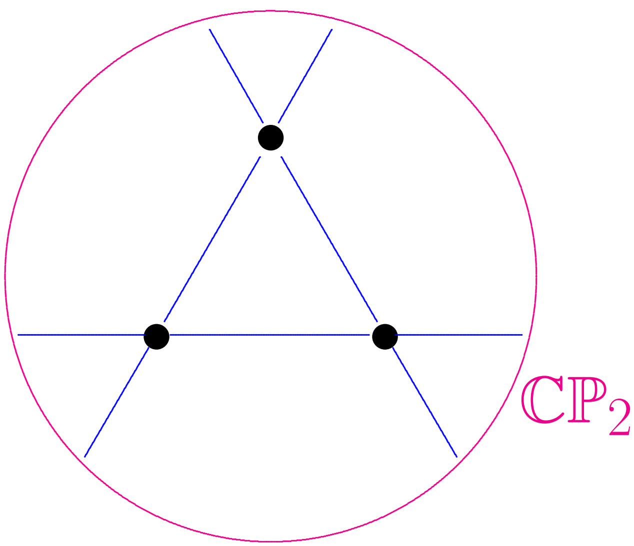

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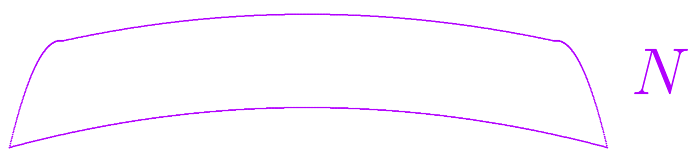

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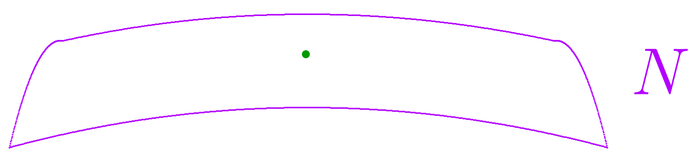

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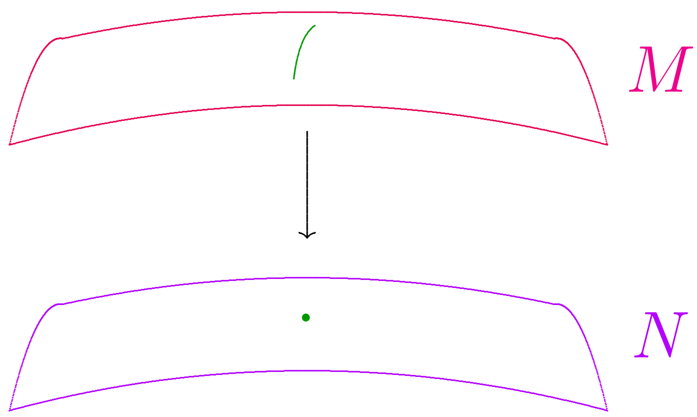

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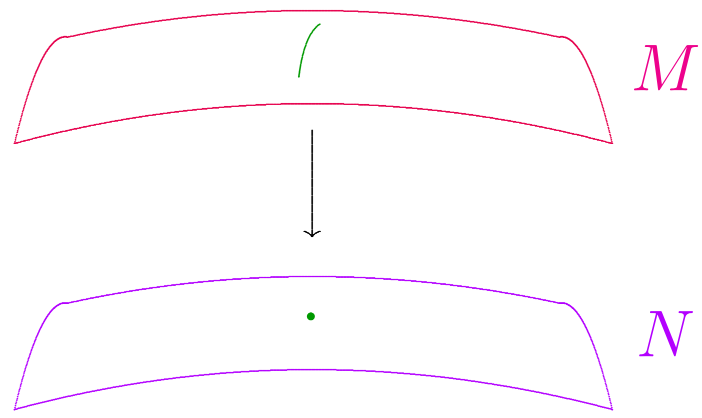

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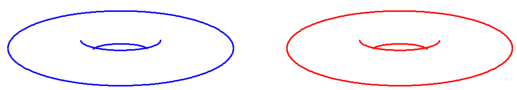

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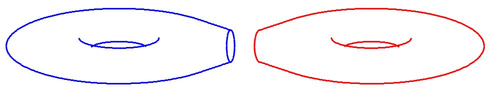

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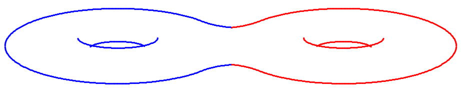

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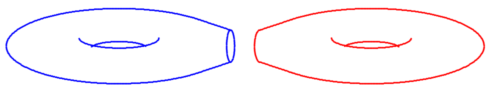

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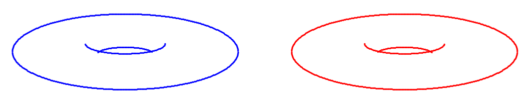

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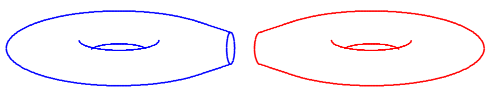

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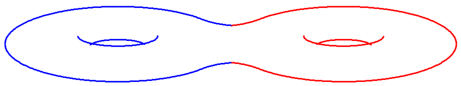

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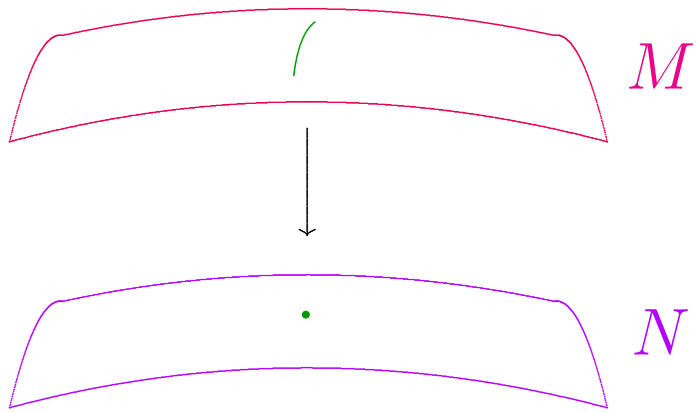

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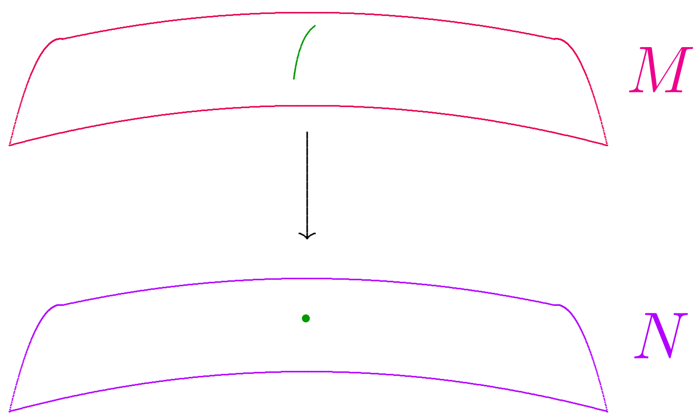

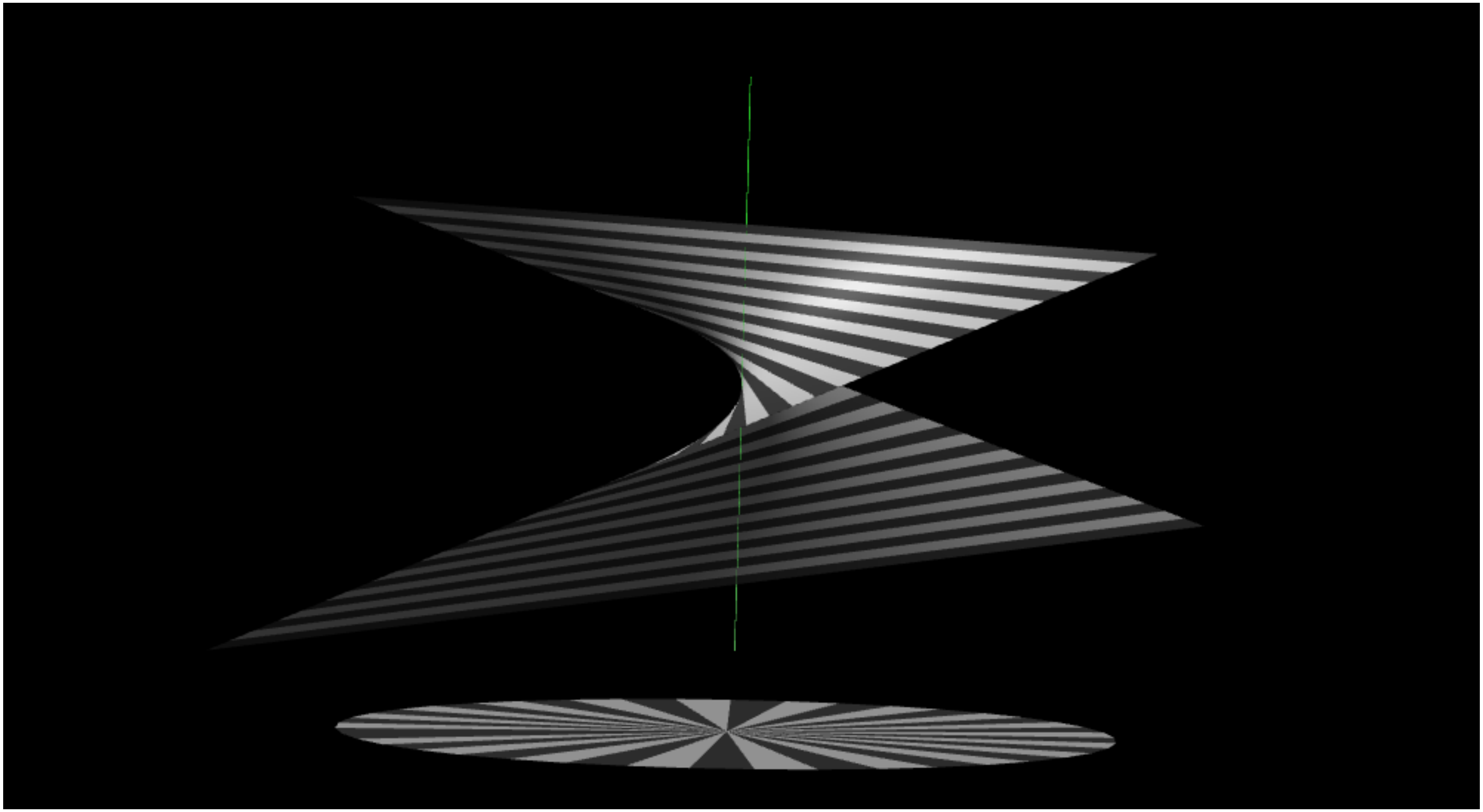

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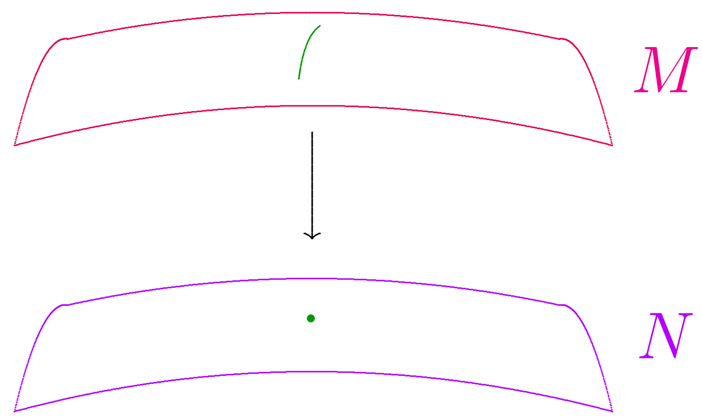

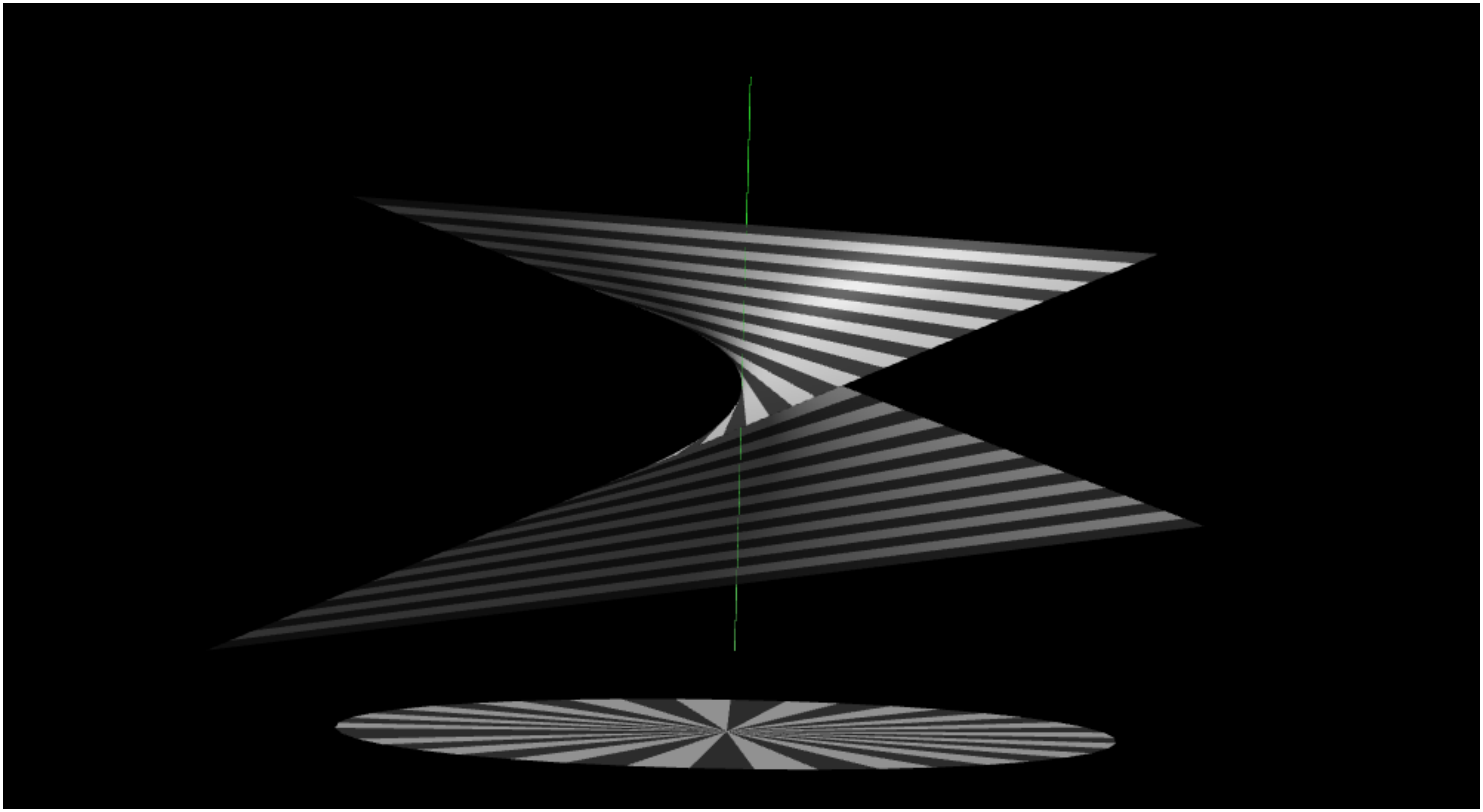

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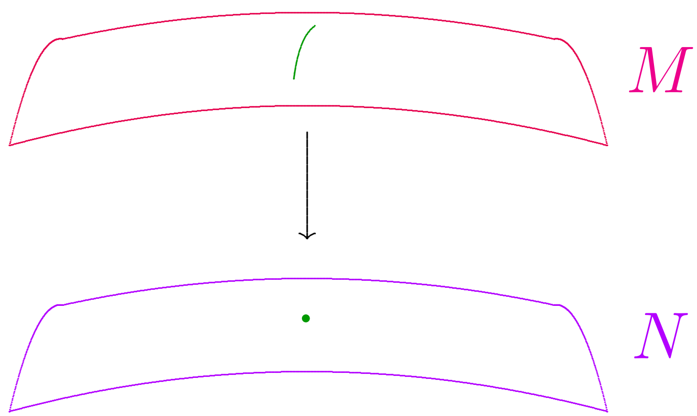

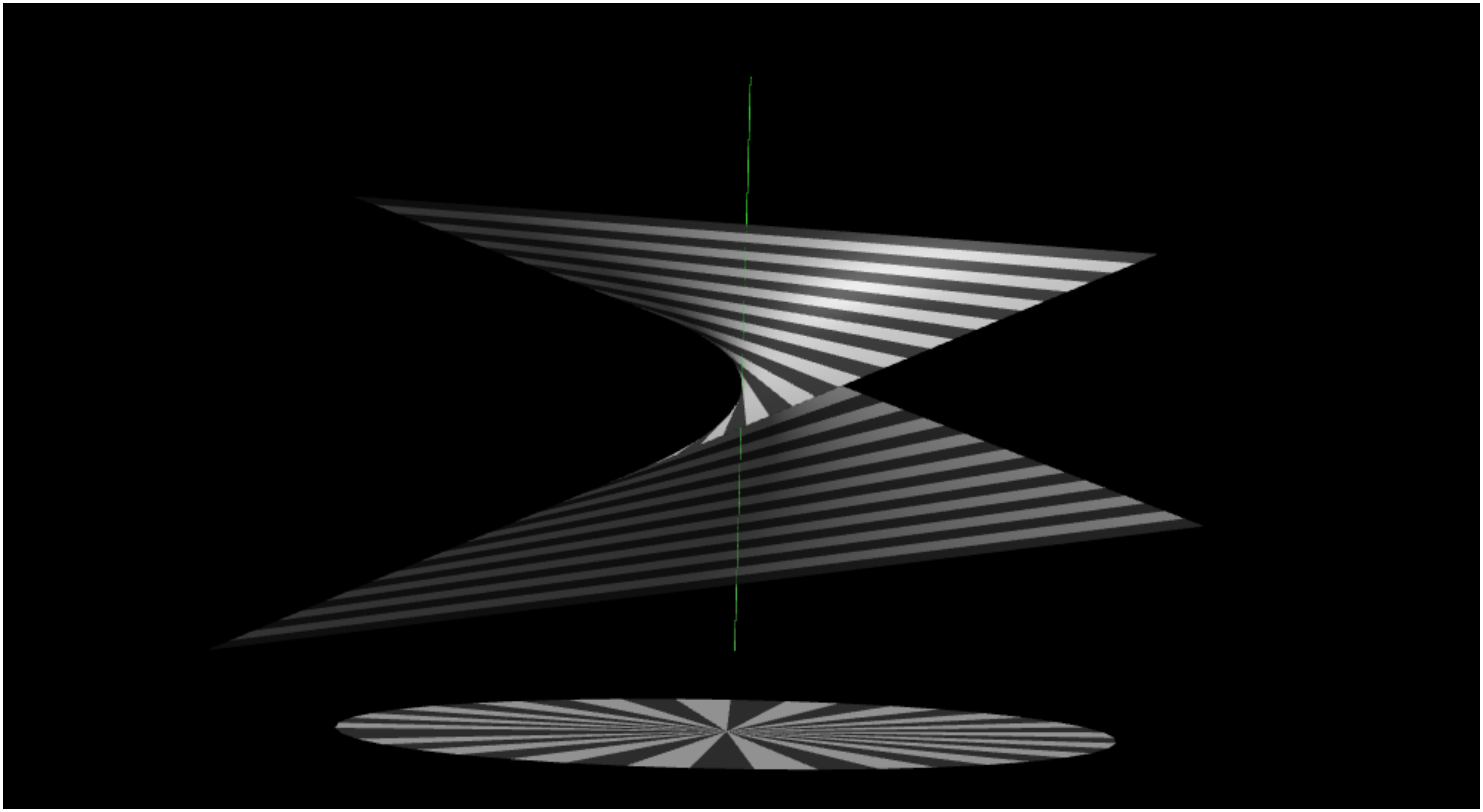

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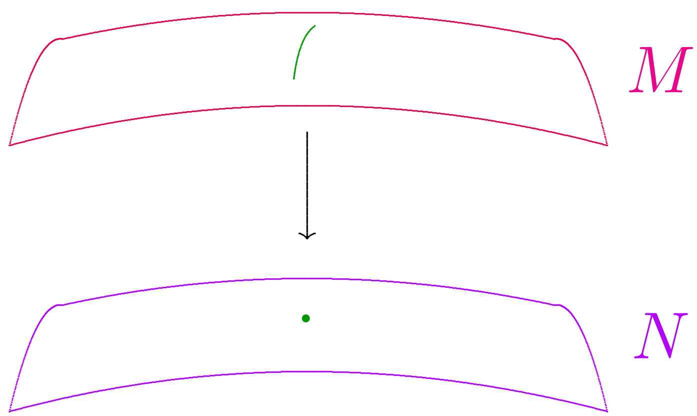

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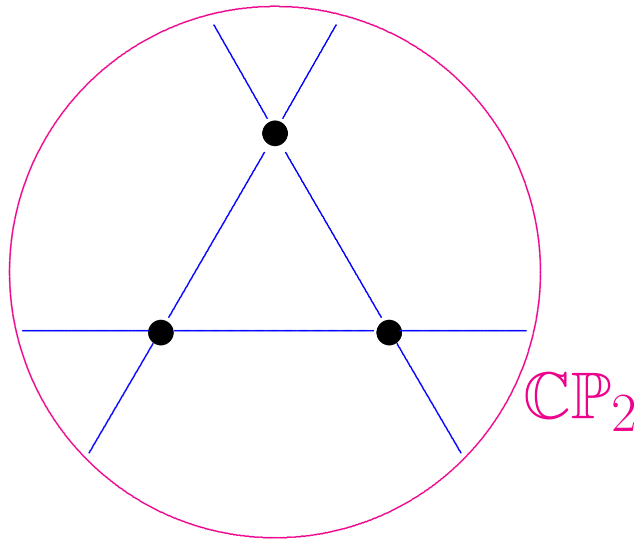

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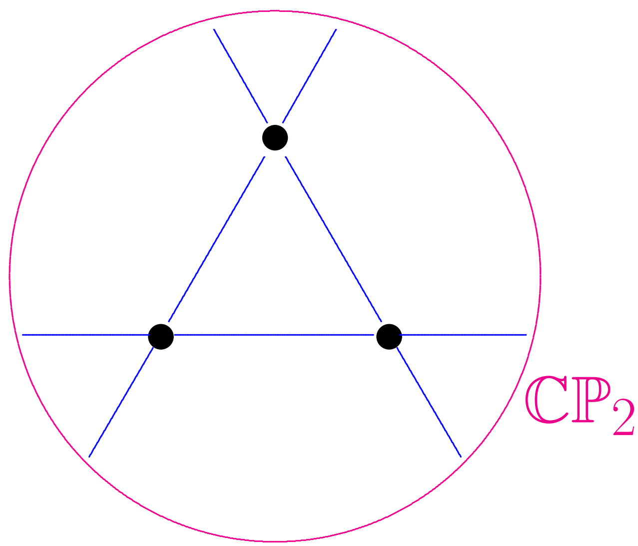

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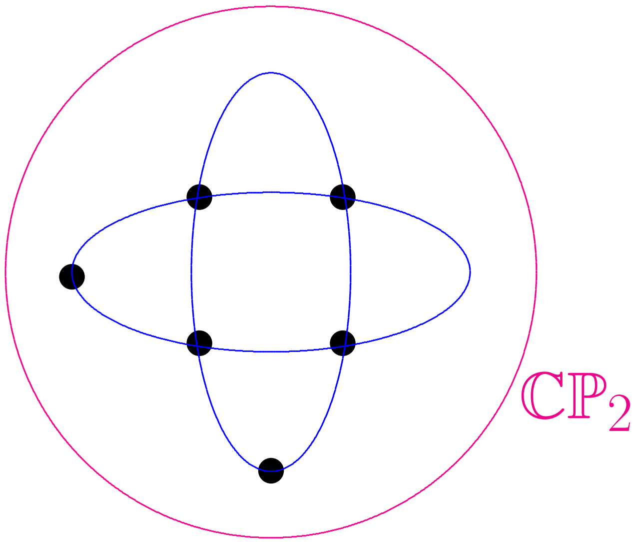

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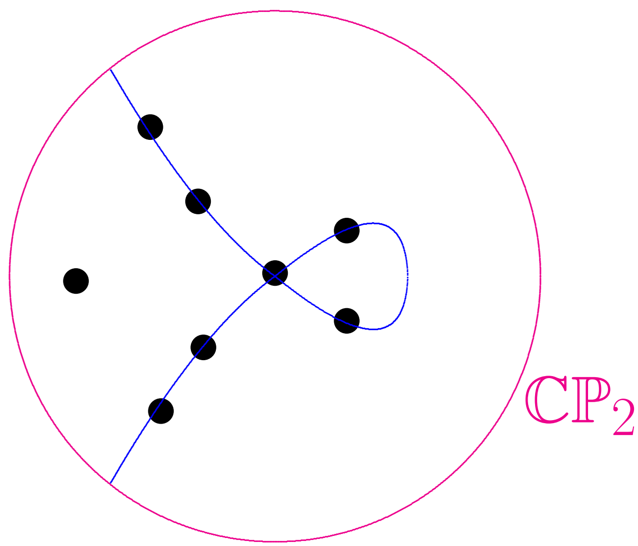

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Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page

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One reason this seems satisfying...

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But this is not needed in above result.

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What about $S^2 \times S^2$?

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Applies in much greater generality.

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But says nothing about $Y([g]) < 0$ realm.

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Method: Weitzenböck formula

$$0 = \frac{1}{2}\Delta|\omega|^2 + |\nabla\omega|^2 - 2W_+(\omega, \omega) + \frac{s}{3}|\omega|^2$$

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$$\Longrightarrow \exists \widehat{g} = u^2 g \quad \text{s.t.} \quad \widehat{\mathfrak{s}} := \widehat{s} - 2\sqrt{6}\widehat{|W_+|} \leq 0.$$

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Open condition in C^2 topology on metrics.

(Harmonic forms depend continuously on metric.)

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This recovers Gursky's inequality — but for a different open set of conformal classes!

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Inequality not limited to the positive Yamabe realm!

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Same technique covers conformally Kähler, Einstein cases among classes with fixed T^2 symmetry.

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Method: Almost-Kähler geometry:

$$\int_M \left[\frac{2s}{3} + W_+(\omega, \omega)\right] d\mu = 4\pi c_1 \bullet [\omega]$$

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This is apparently not an accident!

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What happens there in the Yamabe-negative realm?

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$\to$ Miyaoka-Yau line! Can choose spin or non-spin!

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But. . .

a general understanding of $\inf \mathscr{W}$ still eludes us!

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Finally, let me just say. . .

Happy Birthday, David!

Happy Birthday, David!

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And Best Wishes For Your Retirement!

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Thanks for the invitation!

It's a pleasure being here!

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