Einstein Manifolds,

Conformal Curvature, &

Anti-Holomorphic Involutions

Claude LeBrun
Stony Brook University

Irish Geometry Seminar
February 23, 2021

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"...the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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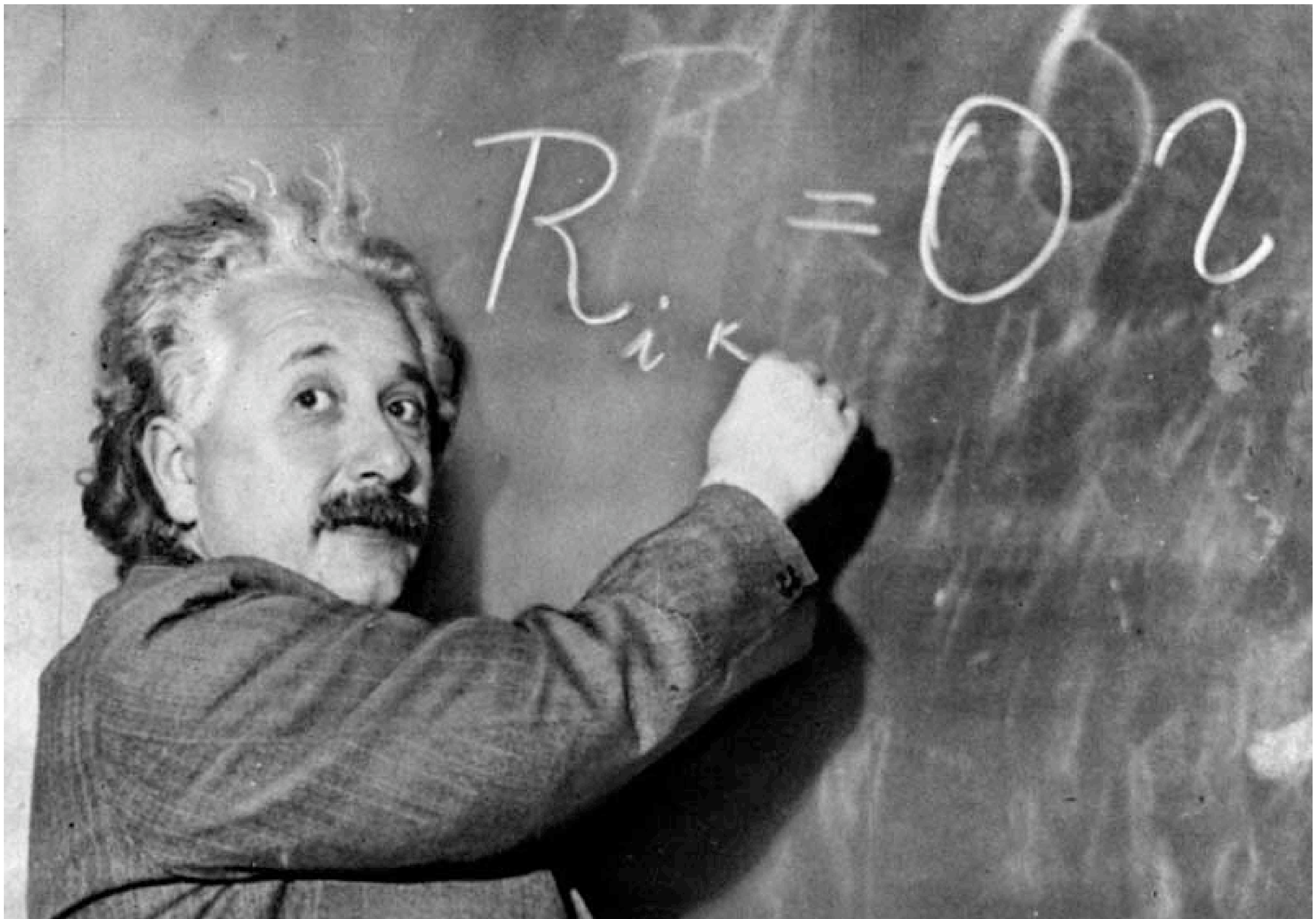

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Has same sign as the *scalar curvature*

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When $n = 4$, situation is more encouraging...

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Many 4-manifolds do not admit Einstein metrics!
Becomes more extreme if we demand $\lambda \geq 0 \ldots$

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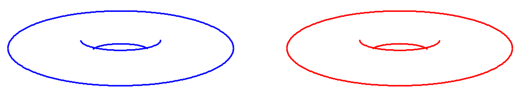

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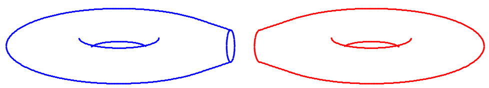

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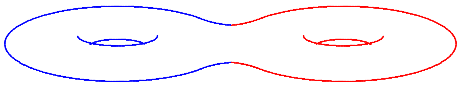

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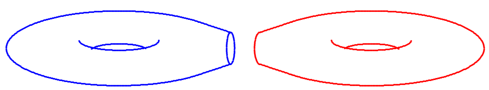

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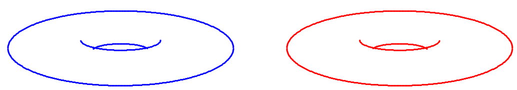

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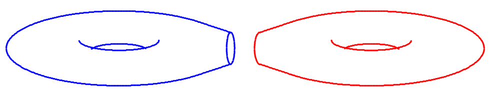

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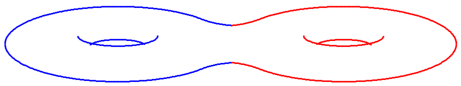

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Typical model: Smooth quartic in $\mathbb{CP}_3$.

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Same conclusion if we instead require $\exists$ complex structure J rather than ω.

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All such results depend on . . .

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Because of this . . .

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One standard tool:

For (M^4, h) compact oriented…

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where $b_\pm(M) = \max\dim$ subspaces $\subset H^2(M, \mathbb{R})$ on which intersection pairing

$$\cup : H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \longrightarrow H^4(M, \mathbb{R}) = \mathbb{R}$$

is positive (resp. negative) definite.

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with equality $\Longleftrightarrow$ Riemannian connection ∇ on $\Lambda^+ \to M$ is flat $\Longleftrightarrow$ (M, h) finitely covered by flat T^4 or Calabi-Yau $K3$.

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

del Pezzo surfaces,
K3 surface, Enriques surface,
Abelian surface, Hyper-elliptic surfaces.

$$
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Definitive list . . .

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But we understand some cases better than others!

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Moduli space $\mathscr{E}(M)$

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Moduli space $\mathscr{E}(M) = \{\text{Einstein } h\}/(\text{Diffeos} \times \mathbb{R}^+)$

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Moduli space $\mathscr{E}(M)$ completely understood.

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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

Above the line:

Know an Einstein metric on each manifold.

$$
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Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

Above the line:

Moduli space $\mathscr{E}(M) \neq \varnothing$.

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Moduli space $\mathscr{E}(M)$ connected!

Above the line:

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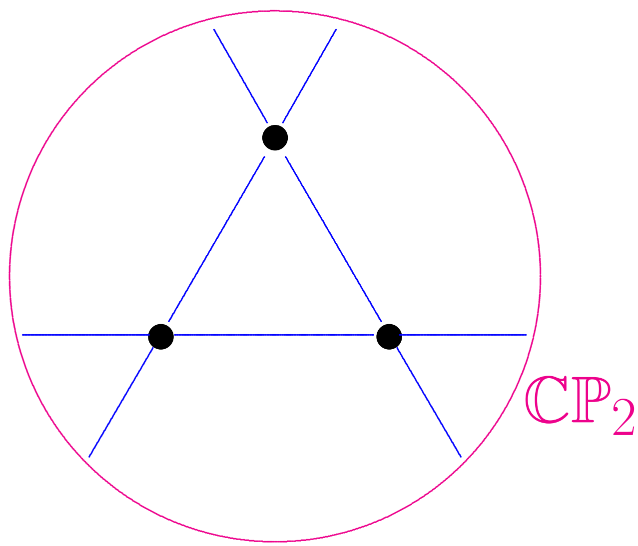

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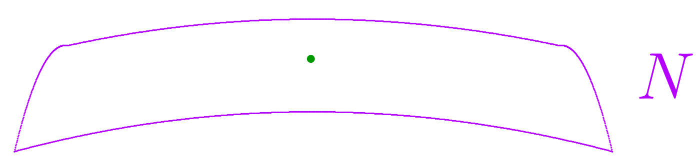
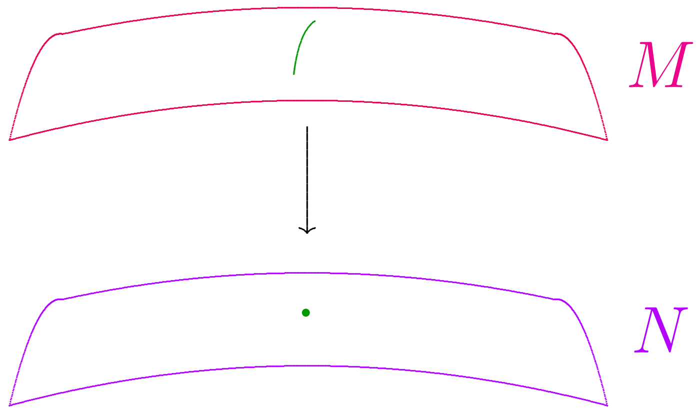
If N is a complex surface, may replace $p \in N$



Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$

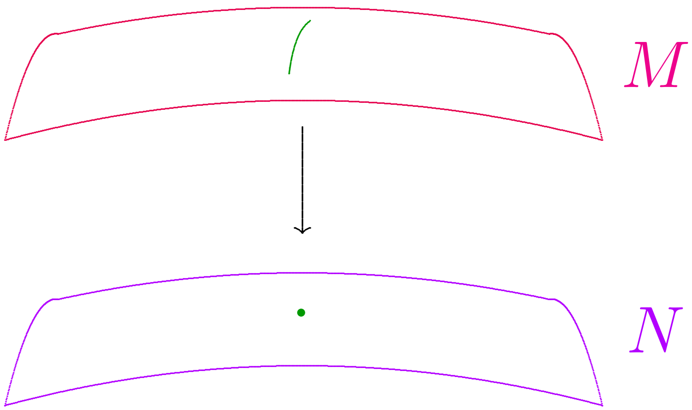


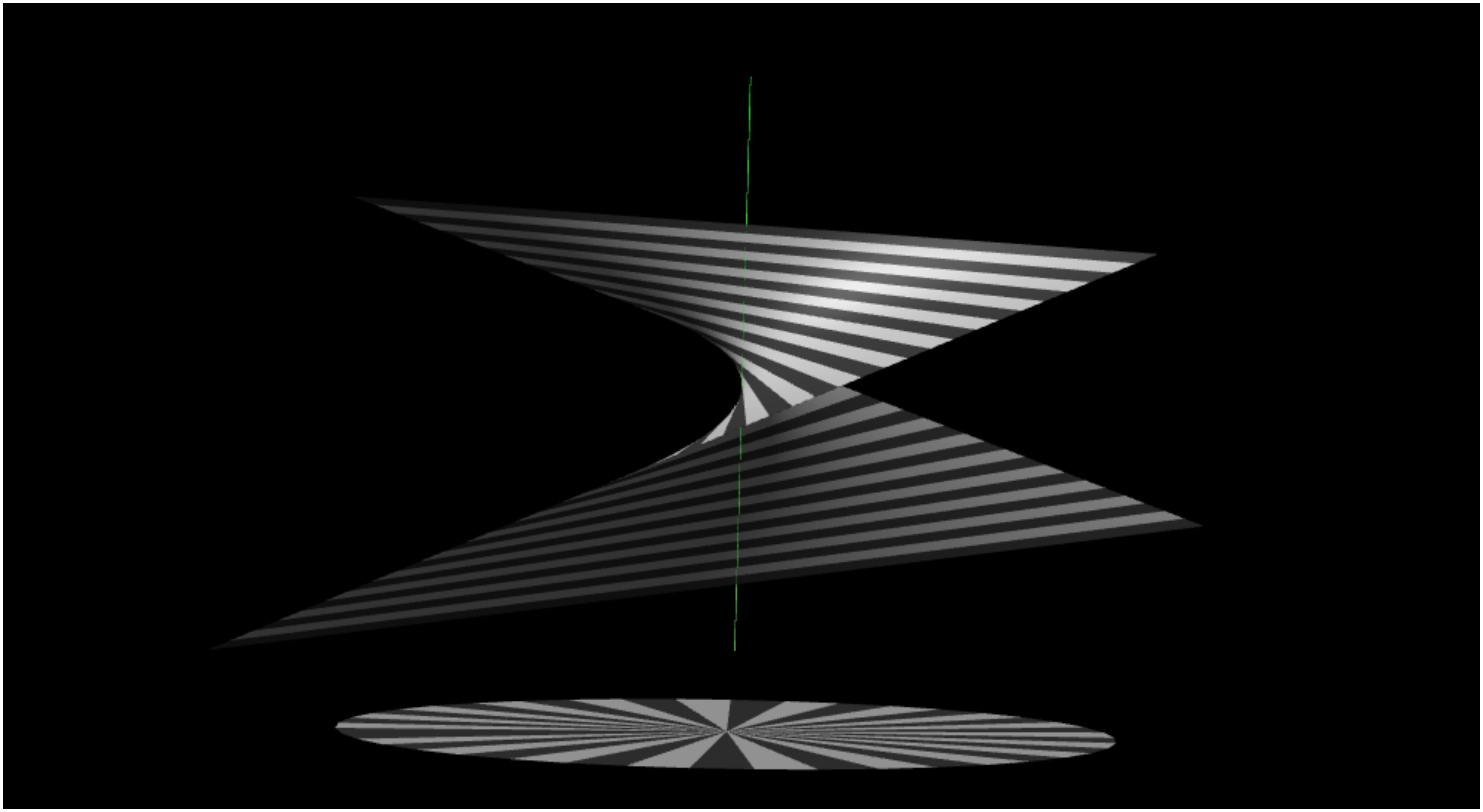
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.





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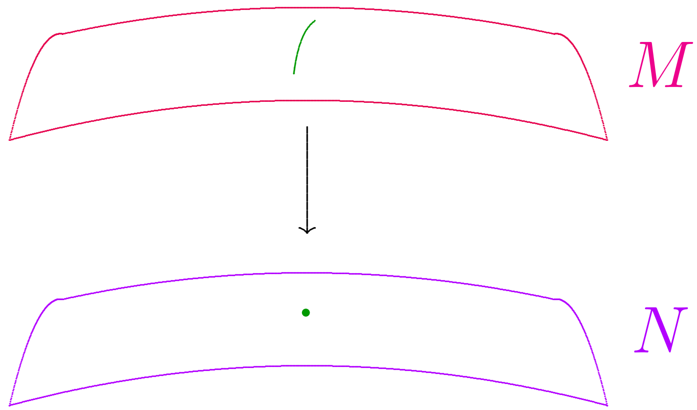

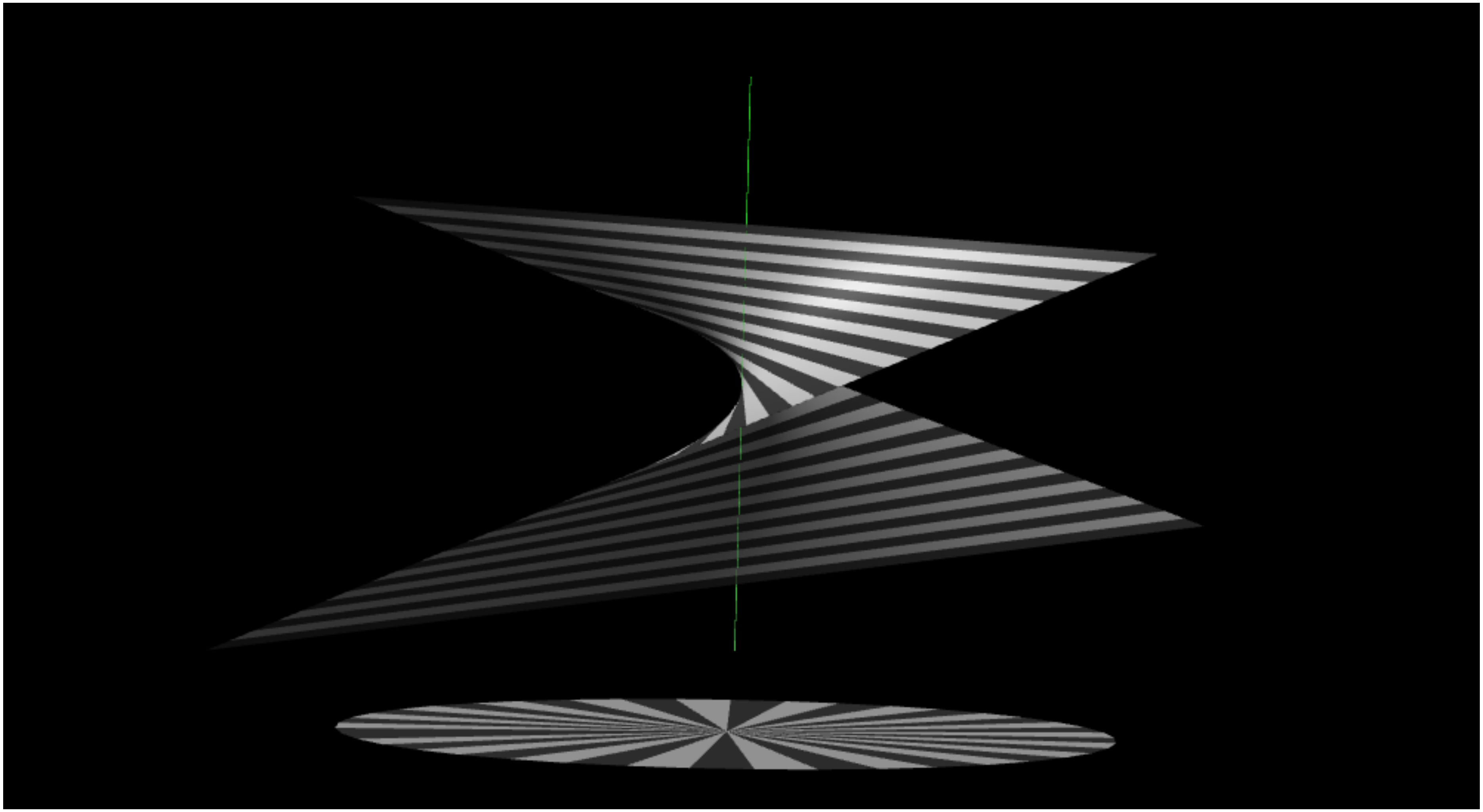

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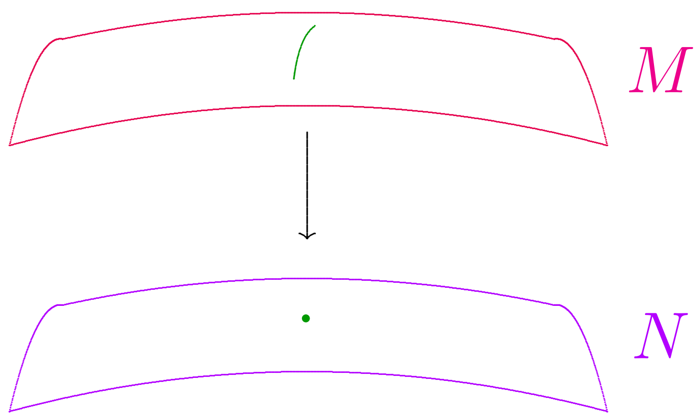

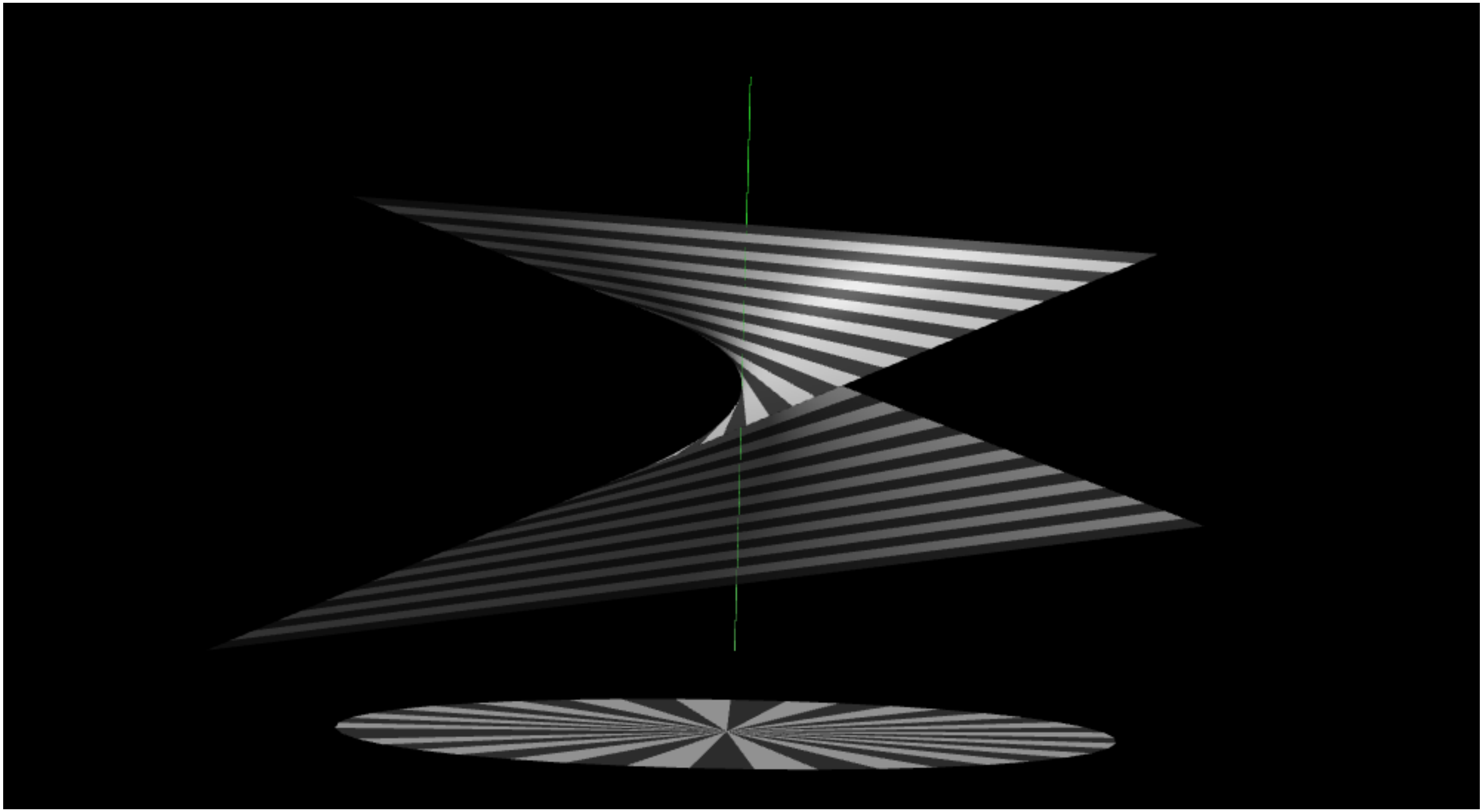

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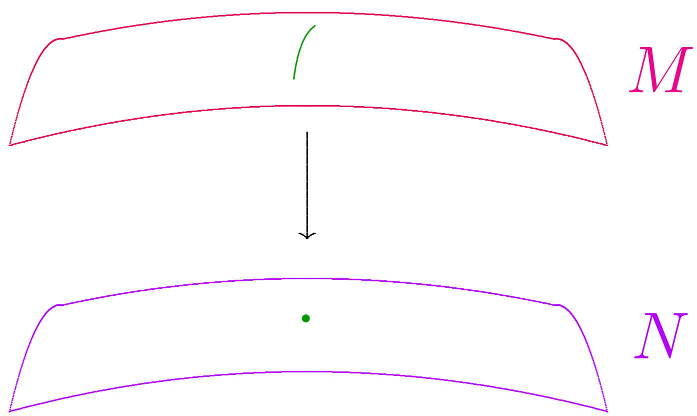
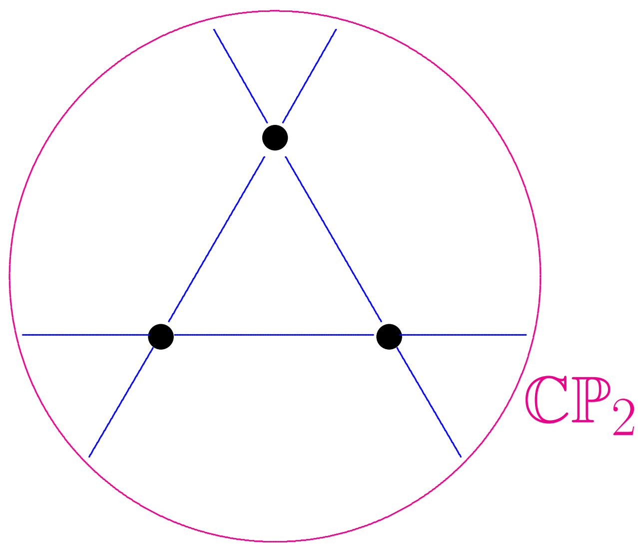


Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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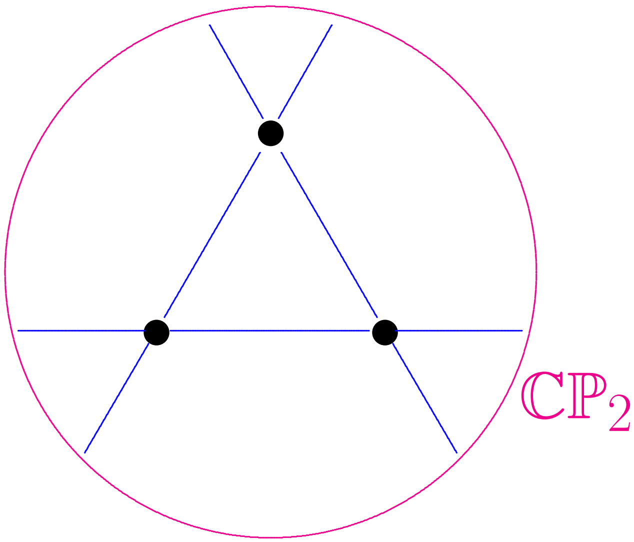

No 3 on a line,

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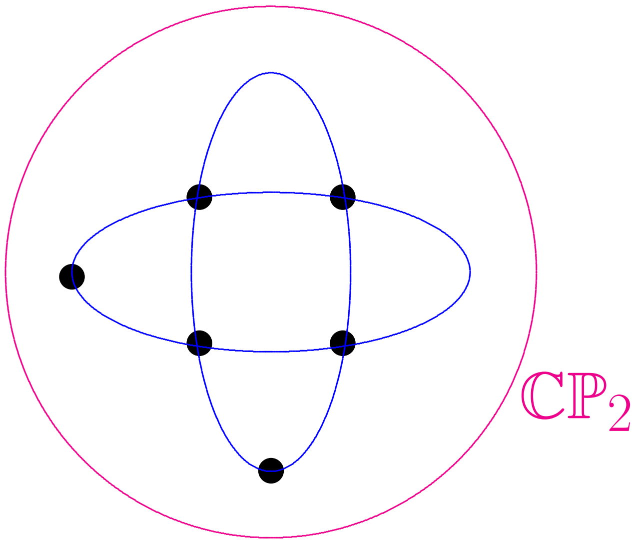

No 3 on a line, no 6 on conic,

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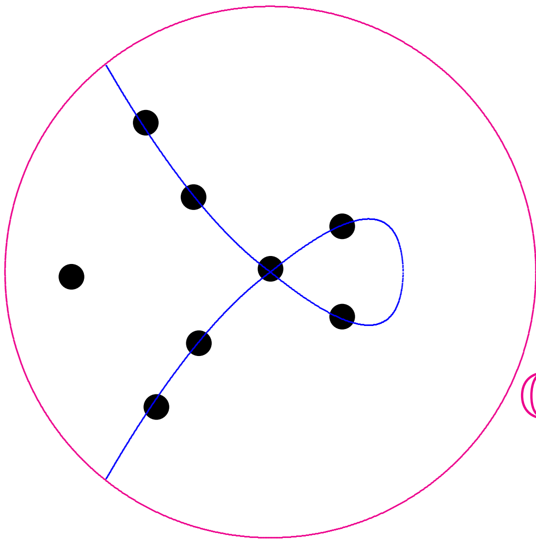

$\mathbb{CP}_2$

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

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Exactly one connected component of moduli space!

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Kähler $\Longrightarrow \Lambda^+ = \mathbb{R}\omega \oplus \Re e \Lambda^{2,0}$

$$W^+ = \text{trace-free part of } \begin{bmatrix} 0 & & \\ & 0 & \\ & & \frac{s}{4} \end{bmatrix}$$

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for these metrics & conformal rescalings:

$$g \rightsquigarrow h = f^2 g \quad \Longrightarrow \quad \det(W^+) \rightsquigarrow f^{-6} \det(W^+).$$

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method also proves more general results.

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L (2020): related classification result.

Theorem A.

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either on M or double cover $\widetilde{M}$.

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Get almost-complex structure J on M or $\widetilde{M}$ by

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Claim: (M, h) compact Einstein $\Longrightarrow$ J integrable.

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Corollary. *Every simply-connected compact oriented Einstein* (M^4, h) *with* $\det(W^+) > 0$ *is diffeomorphic to a del Pezzo surface. Conversely, every del Pezzo* M^4 *carries Einstein* h *with* $\det(W^+) > 0$*, and these sweep out exactly one connected component of moduli space* $\mathscr{E}(M)$.

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Theorem A. *Let* (M, h) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M*. Then* h *is conformal to an orientation-compatible Bach-flat extremal Kähler metric* g *with scalar curvature* $s > 0$ *on* M*.*

Simply connected hypothesis is essential!

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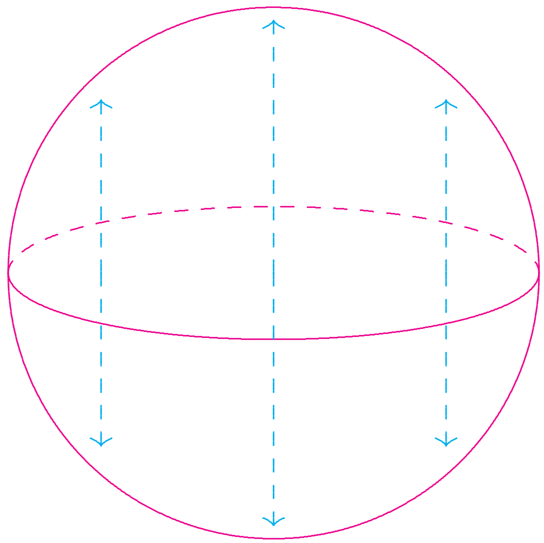
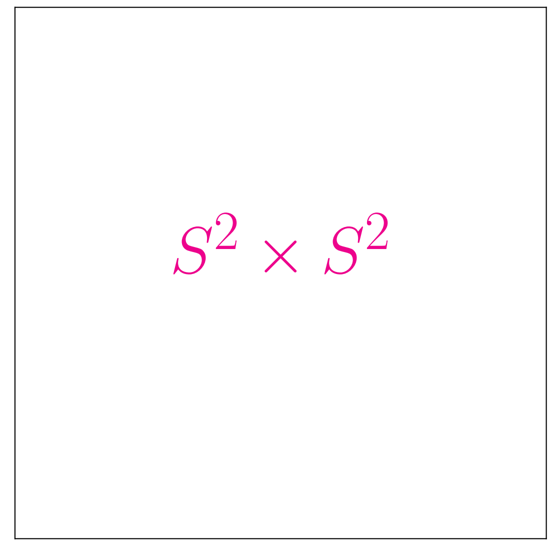
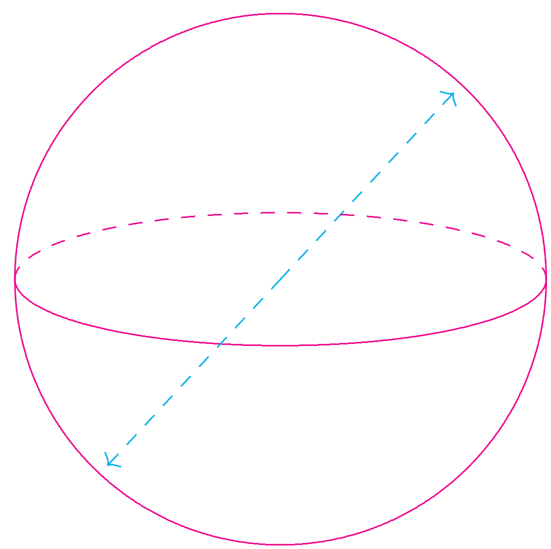




Oriented spin 4-manifold

$\mathscr{P} := (S^2 \times S^2)/\langle \mathfrak{a} \times \mathfrak{r} \rangle$



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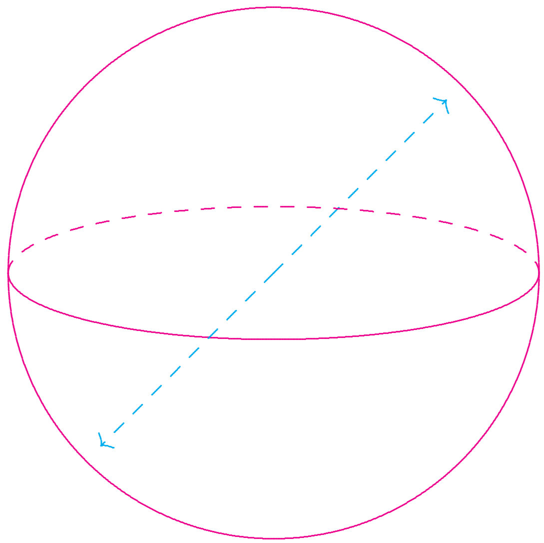
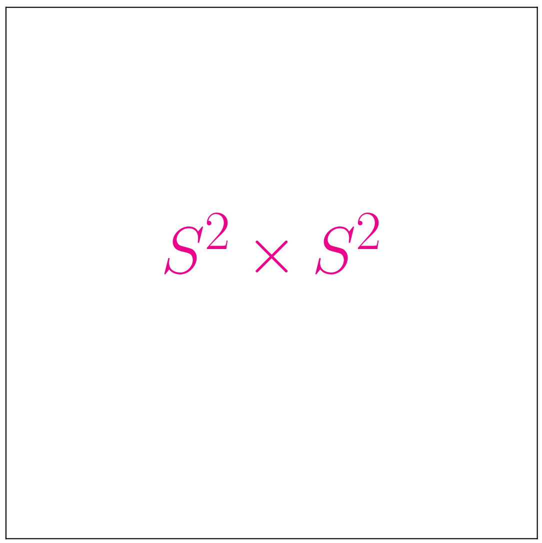
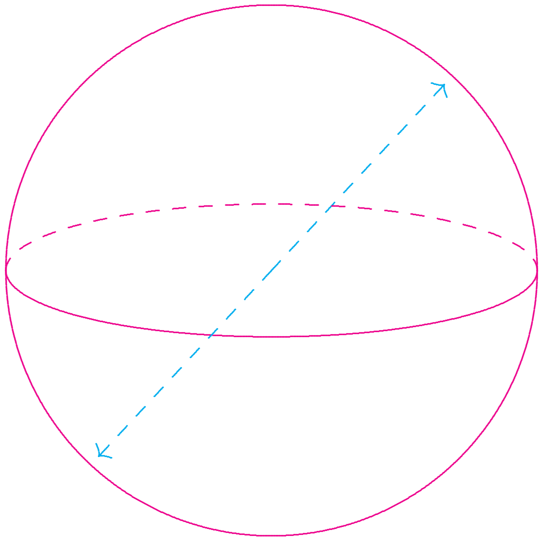




Non-spin 4-manifold

$\mathscr{Q} := (S^2 \times S^2)/\langle \mathfrak{a} \times \mathfrak{a} \rangle$



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But $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ isn't spin!

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Cute fact: No 4-manifold on this list admits an orientation-compatible almost-complex structure!

Indeed, they all have Todd genus

$$\mathbf{Td} = \frac{\chi + \tau}{4} = \frac{1 - b_1 + b_+}{2} = \frac{1}{2} \notin \mathbb{Z}.$$

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Why is $\mathscr{E}_{\det}(M) \subset \mathscr{E}(M)$ open and closed?

Open: $\det(W^+) > 0$.

Closed: $\det(W^+) = \frac{1}{3\sqrt{6}}|W^+|^3$ and $s \geq 0$.

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Theorem D. *Let* (M, h) *be a compact oriented Einstein* 4*-manifold. If*

$$W^+ \neq 0 \quad \textit{and} \quad \det(W^+) \geq -\frac{5\sqrt{2}}{21\sqrt{21}}|W^+|^3$$

everywhere on M*, then actually* $\det(W^+) > 0$*. Consequently, all the results described remain true if we merely impose this ostensibly weaker hypothesis.*

Some indication of the proof:

For clarity, let's just assume $\det(W^+) > 0\ldots$

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as proxy for Einstein equation.

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for $fW^+ \in \mathrm{End}(\Lambda^+)$.

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$$0 = \nabla^*\nabla(fW^+) + \frac{s}{2}fW^+ - 6fW^+ \circ W^+ + 2f|W^+|^2 I$$

with $\omega \otimes \omega$, and integrate by parts.

$$0 = \int_M \left[\langle \nabla^*\nabla(fW^+), \omega \otimes \omega\rangle + \cdots\right] d\mu$$

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$$0 = \int_M \left[\langle W^+, \nabla^*\nabla(\omega\otimes\omega)\rangle + \frac{s}{2}W^+(\omega,\omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2\right] f\, d\mu$$

Let $\alpha \geq \beta \geq \gamma$ be eigenvalues of W^+:

$$W^+ = \begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix}$$

$$\alpha + \beta + \gamma = 0$$

$$\alpha > 0, \quad \gamma < 0, \quad \text{if } W^+ \neq 0$$

$$\det(W^+) = \alpha\beta\gamma$$

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$$f = {\alpha_h}^{-1/3}, \qquad g = f^{-2}h = {\alpha_h}^{2/3}h.$$

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Now choose $\omega \in \Gamma\Lambda^+$ so that

$$W_g^+(\omega) = \alpha\,\omega, \quad |\omega|_g \equiv \sqrt{2},$$

after at worst passing to double cover $\hat{M} \to M$.

$$0 = \int_{\hat{M}} \Big[\langle W^+, \nabla^* \nabla (\omega \otimes \omega) \rangle + \frac{s}{2} W^+(\omega, \omega) - 6 |W^+(\omega)|^2 + 2 |W^+|^2 |\omega|^2 \Big] f \, d\mu$$

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$$0 = \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) - 2W^+(\omega, \nabla^e \nabla_e \omega) + \frac{s}{2} W^+(\omega, \omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2 \Big] f \, d\mu$$

$$0 = \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) - 2\alpha \langle \omega, \nabla^e \nabla_e \omega \rangle + \frac{s}{2}\alpha|\omega|^2 - 6\alpha^2|\omega|^2 + 2|W^+|^2|\omega|^2 \Big] f \, d\mu$$

because

$$W_g{}^+(\omega) = \alpha\omega$$

$$0 = \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha \langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2}\alpha |\omega|^2 - 6\alpha^2 |\omega|^2 + 2|W^+|^2 |\omega|^2 \Big] f \, d\mu$$

$$0 \geq \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha \langle \omega, \nabla^* \nabla \omega \rangle$$

$$+\frac{s}{2}\alpha|\omega|^2 - 6\alpha^2|\omega|^2 + 3\alpha^2|\omega|^2 \Big] f \, d\mu$$

because

$$|W_g{}^+|^2 \geq \frac{3}{2}\alpha^2$$

$$0 \geq \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha \langle \omega, \nabla^* \nabla \omega \rangle$$

$$+\frac{s}{2}\alpha|\omega|^2 - 3\alpha^2|\omega|^2 \Big] f \; d\mu$$

$$|\omega|_g^2 = 2 \quad \Longrightarrow \quad (\nabla_e \omega) \perp \omega$$

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$$|\omega|_g^2 = 2 \quad \Longrightarrow \quad (\nabla_e \omega) \perp \omega$$

$$\det(W^+) > 0 \quad \Longrightarrow \quad W^+ \sim \begin{bmatrix} + & & \\ & - & \\ & & - \end{bmatrix}$$

$$0 \geq \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha \langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2} \alpha |\omega|^2 - 3\alpha^2 |\omega|^2 \Big] f \, d\mu$$

$$|\omega|_g^2 = 2 \implies (\nabla_e \omega) \perp \omega$$

$$\det(W^+) > 0 \implies W^+(\nabla_e \omega, \nabla^e \omega) \leq 0$$

$$0 \geq \int_M \Big[\quad 2\alpha\langle \omega, \nabla^*\nabla\omega\rangle + \frac{s}{2}\alpha|\omega|^2 - 3\alpha^2|\omega|^2\Big] f \, d\mu$$

$$|\omega|^2_g = 2 \implies (\nabla_e \omega) \perp \omega$$

$$\det(W^+) > 0 \implies -W^+(\nabla_e \omega, \nabla^e \omega) \geq 0$$

$$0 \geq \int_M \left[2\alpha \langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2} \alpha |\omega|^2 - 3\alpha^2 |\omega|^2 \right] f \, d\mu$$

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But

$$\alpha f \equiv 1$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2} \alpha |\omega|^2 - 3|\omega|^2 \alpha \right] d\mu$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle - 3W^+(\omega, \omega) + \frac{s}{2}|\omega|^2 \right] d\mu$$

$$0 \geq \int_M \left[\frac{1}{2}|\nabla\omega|^2 + \frac{3}{2}\langle\omega, \left(\nabla^*\nabla - 2W^+ + \frac{s}{3}\right)\omega\rangle\right] d\mu$$

$$0 \geq \int_M \left[\frac{1}{2}|\nabla\omega|^2 + \frac{3}{2}\langle\omega, \quad (d+d^*)^2 \quad \omega\rangle\right] d\mu$$

Because

$$(d+d^*)^2 = \nabla^*\nabla - 2W^+ + \frac{s}{3}$$

on $\Gamma\Lambda^+$.

$$0 \geq \frac{1}{2} \int_M |\nabla \omega|^2 \, d\mu + 3 \int_M |d\omega|^2 \, d\mu$$

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So $\nabla\omega \equiv 0$, and g is Kähler!

Go Raibh Maith Agat!

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Thanks for the invitation!

Slán Agat!

Slán Agat!

Bye!