Einstein Metrics,

Four-Manifolds, &

Conformally Kähler Geometry

Claude LeBrun
Stony Brook University

Einstein Spaces and Special Geometry,
Institut Mittag-Leffler. 10 juli, 2023.

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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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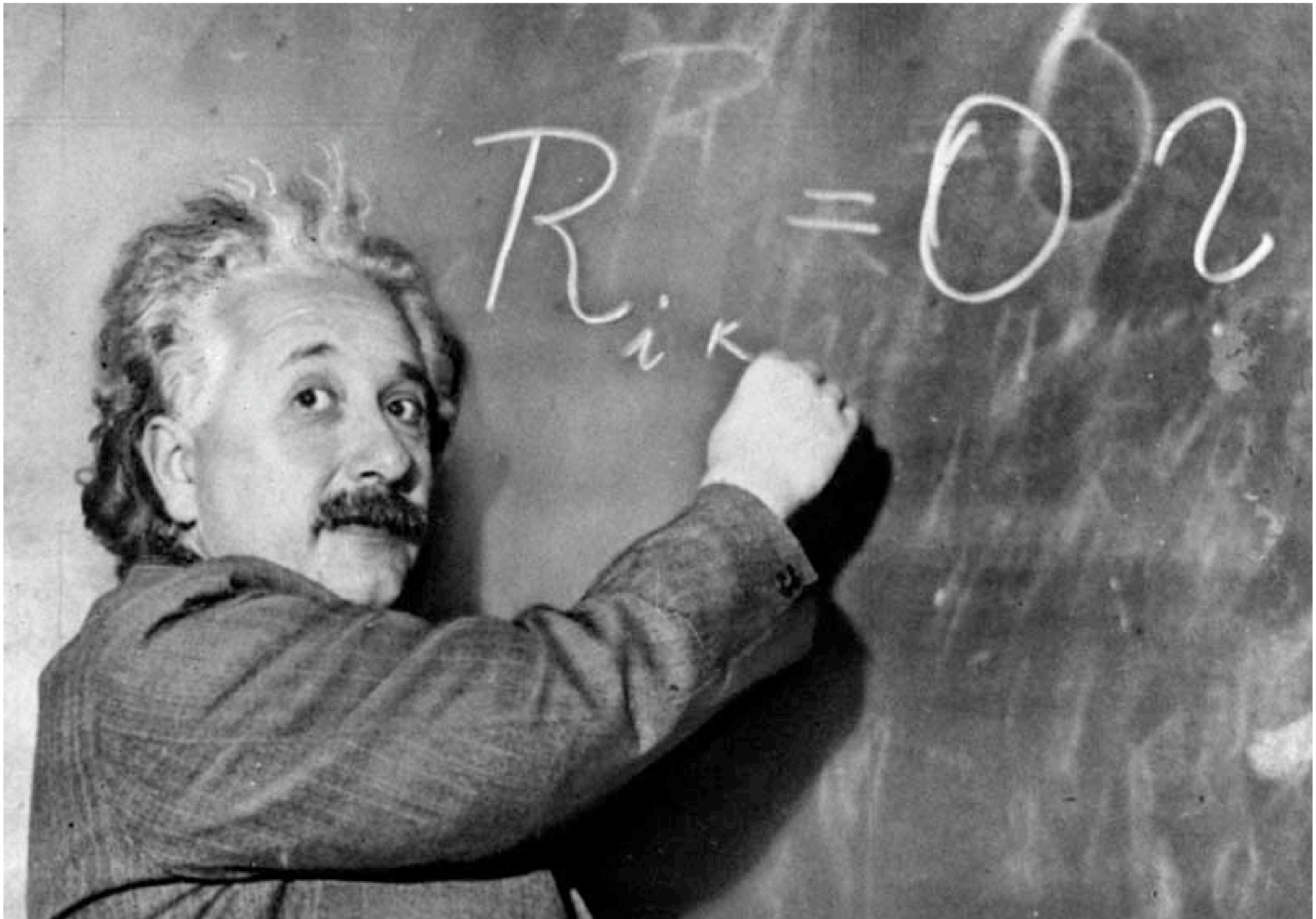

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Has same sign as the *scalar curvature*

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- When $n \geq 6$, wide open. Maybe???

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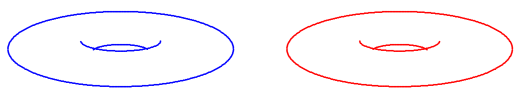

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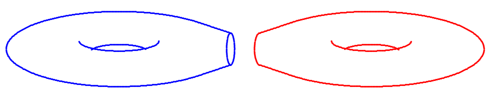

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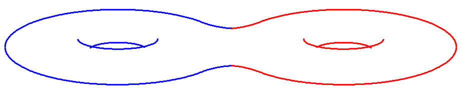

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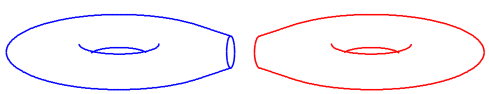

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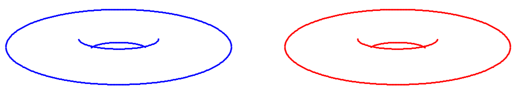

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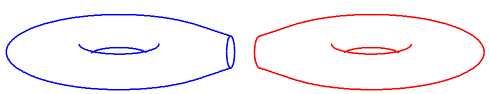

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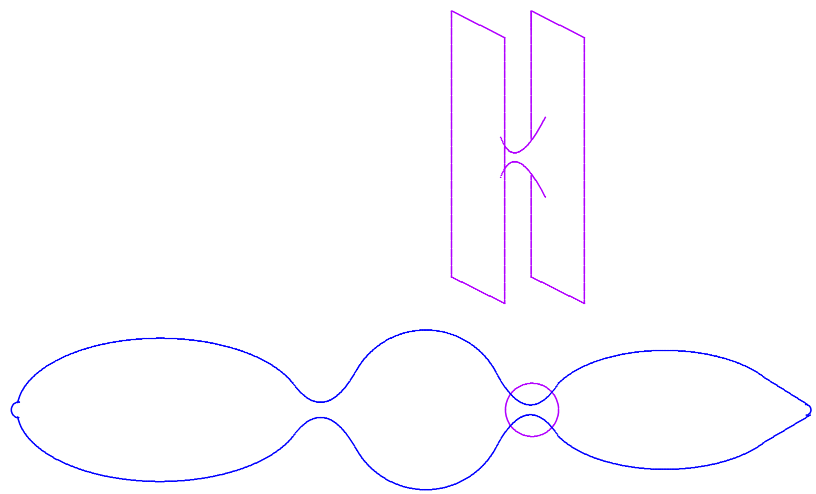

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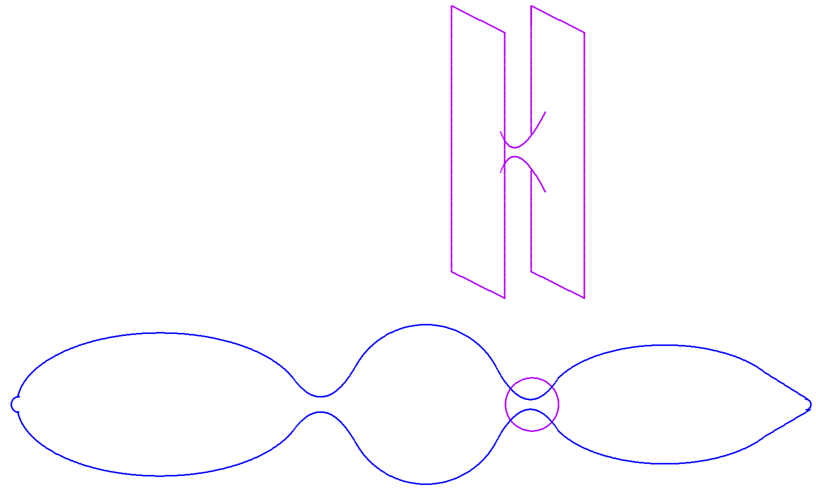

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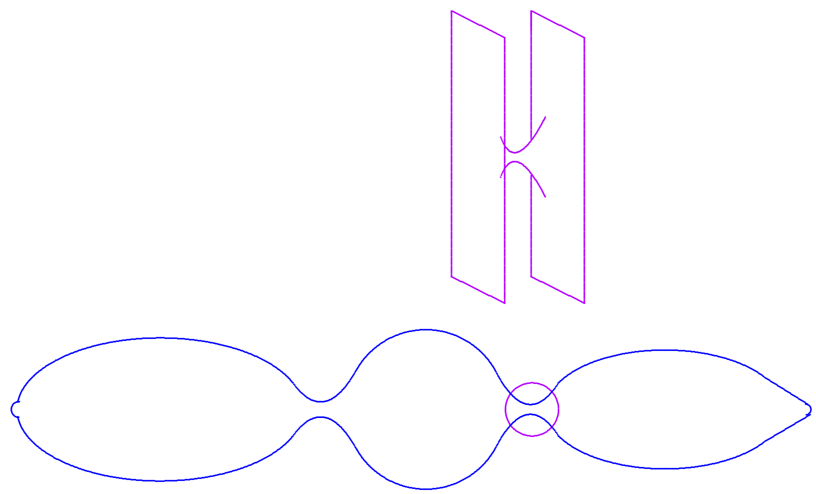

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By contrast, high-dimensional Einstein metrics too common; have little to do with geometrization.

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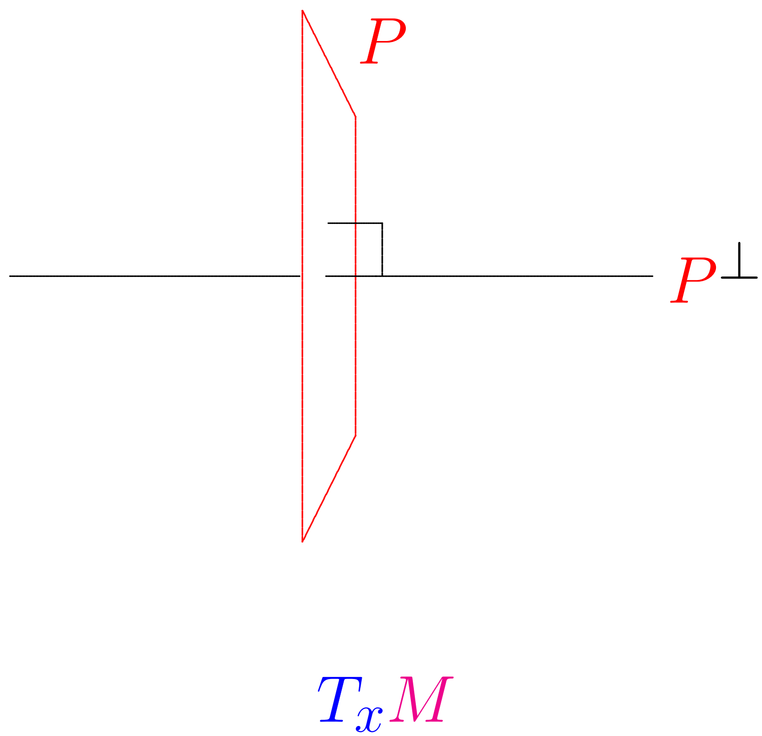

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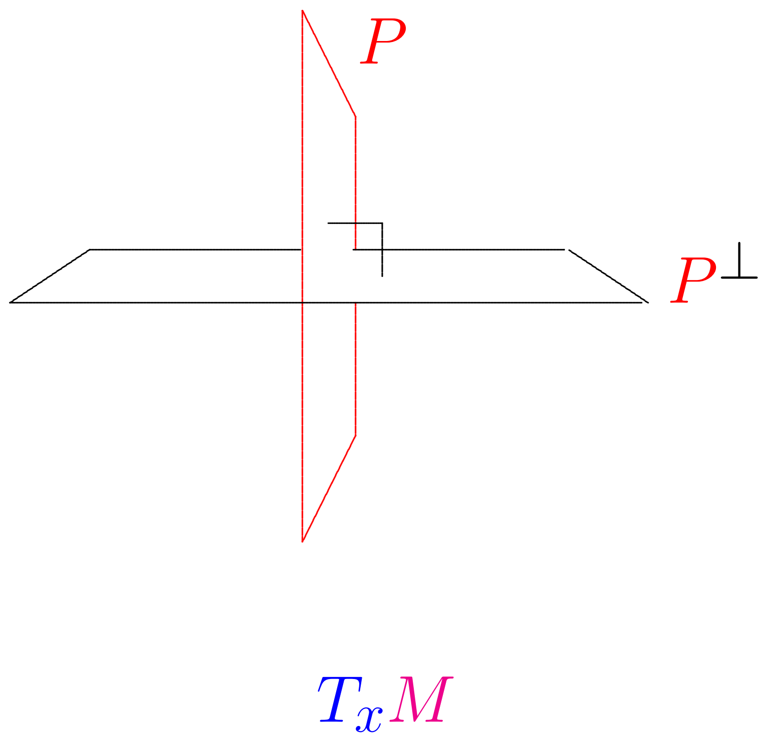

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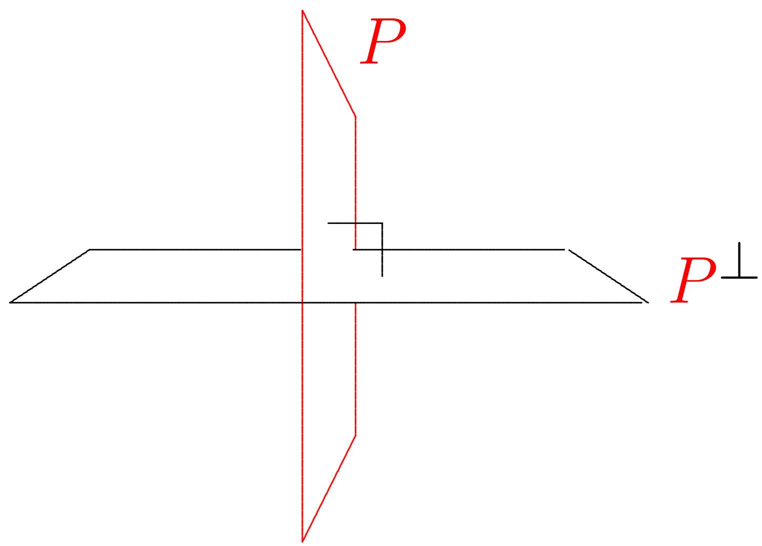

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The numbers

$$b_\pm(M) = \dim \mathcal{H}^\pm_g$$

are independent of g, and so are invariants of M.

$b_{\pm}(M)$?

Best understood in terms of intersection pairing

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Diagonalize:

$$\left[\begin{array}{cccccc} +1 & & & & & \\ & \ddots & & & & \\ & & +1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array}\right].$$

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$$b_2(M) = b_+(M) + b_-(M)$$

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"Signature" of M.

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Example. For any symplectic (M^4, ω),
$\exists$ "adapted" Riemannian g such that $\omega \in \mathcal{H}^+_g$.

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There is no higher-dimensional version of this story!

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In practice, this means that psc metrics are only obstructed on most, but not quite all, symplectic M^4 with $b_+ = 1$.

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Conformally Kähler:

$$g = u^2 h$$

$\exists$ some Kähler metric h & some smooth function u.

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"Goldberg-Sachs Theorem."

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False in higher dimensions!

Theorem (L '09)**.**

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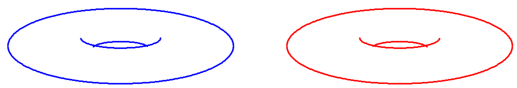
Connected sum $\#$:



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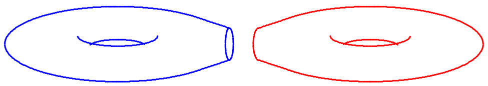

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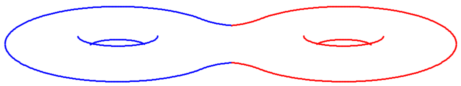

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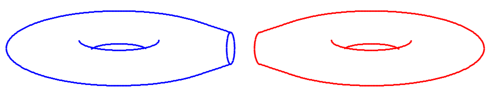

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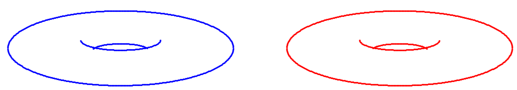

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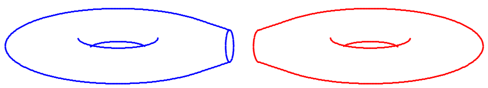

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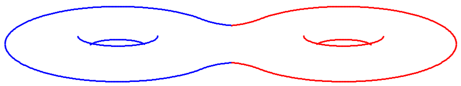

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Del Pezzo surfaces,
K3 surface, Enriques surface,
Abelian surface, Hyper-elliptic surfaces.

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There are also attractive results in the $\lambda < 0$ realm, where Seiberg-Witten really comes into play. But less definitive, and beyond the scope of this lecture.

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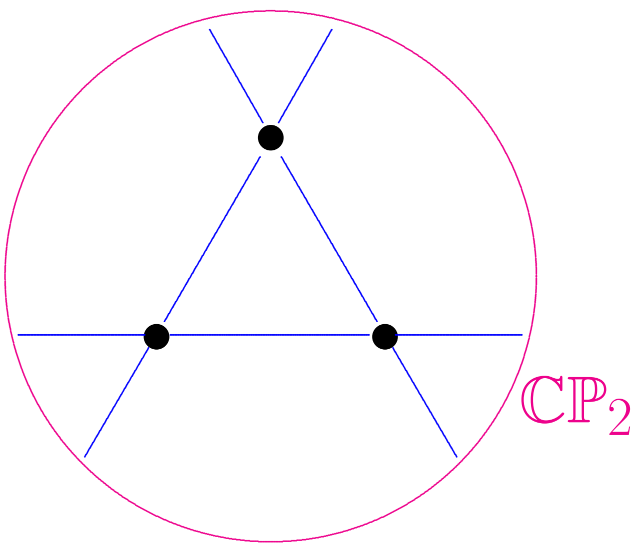

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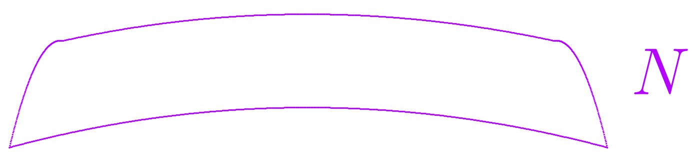

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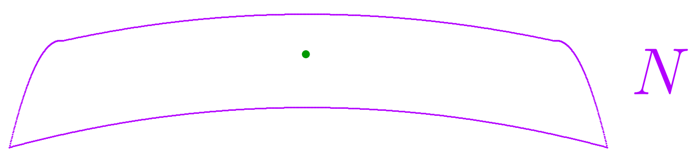

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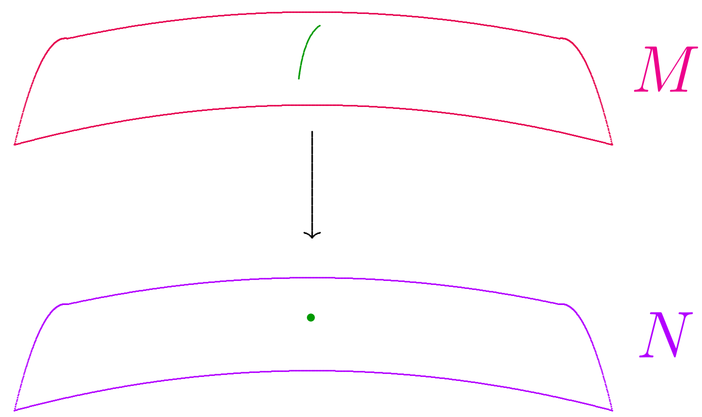

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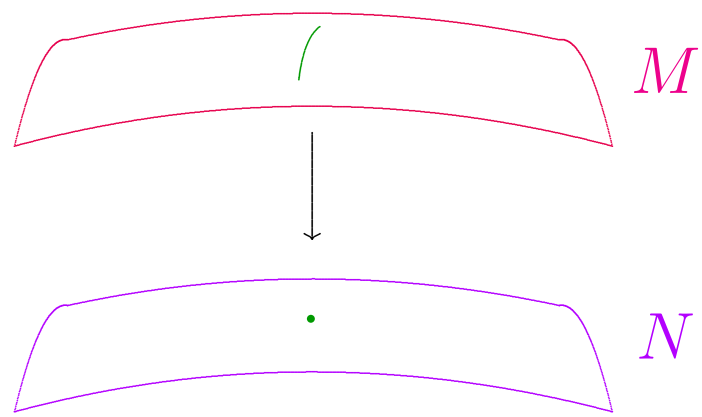

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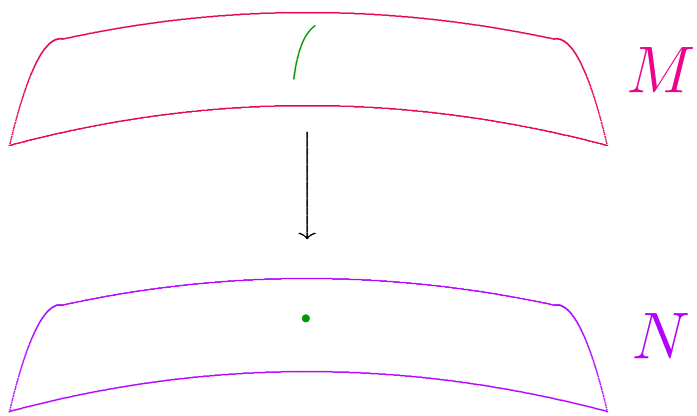

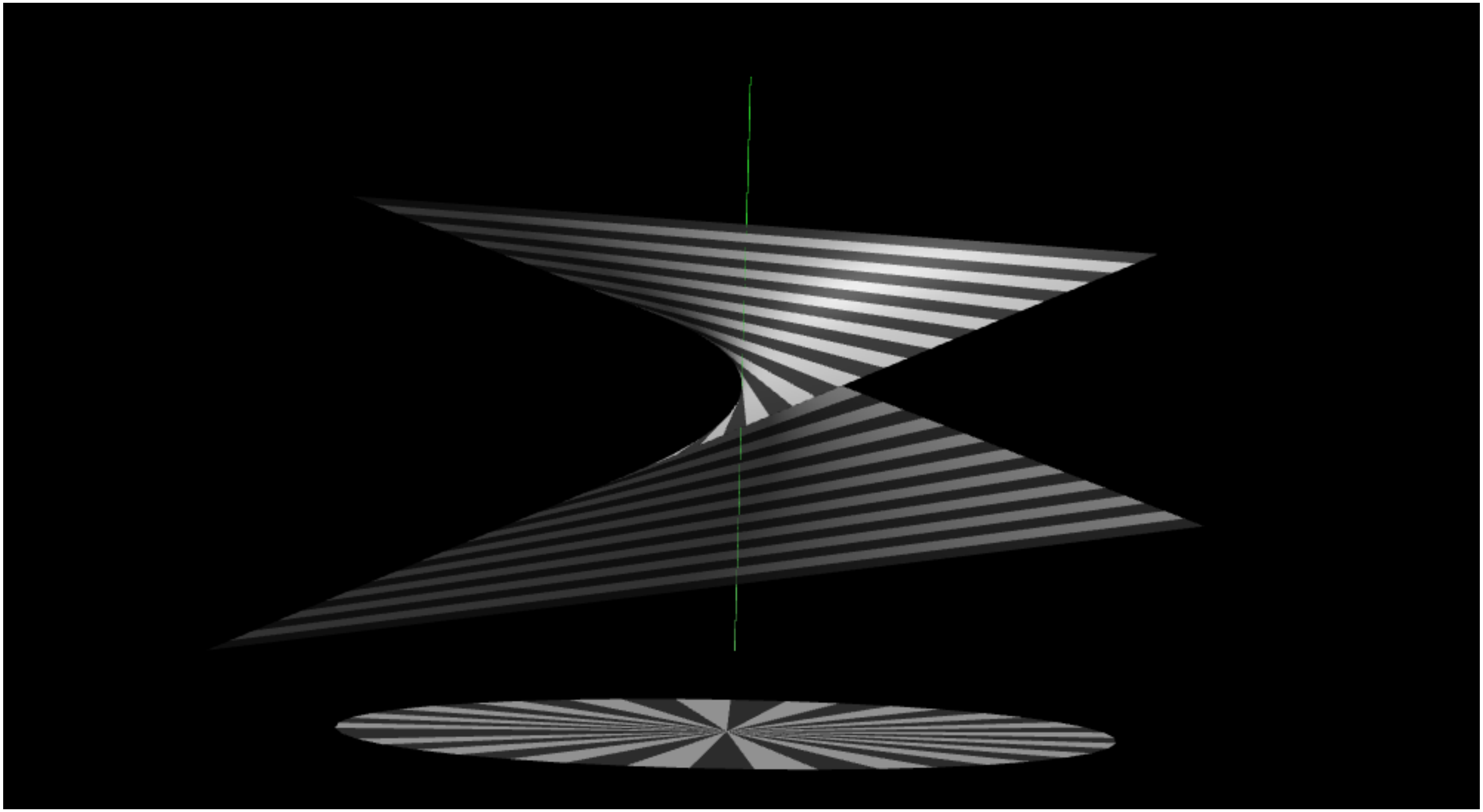

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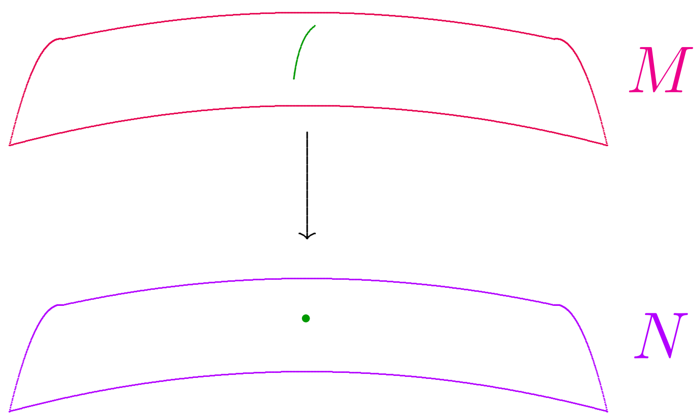

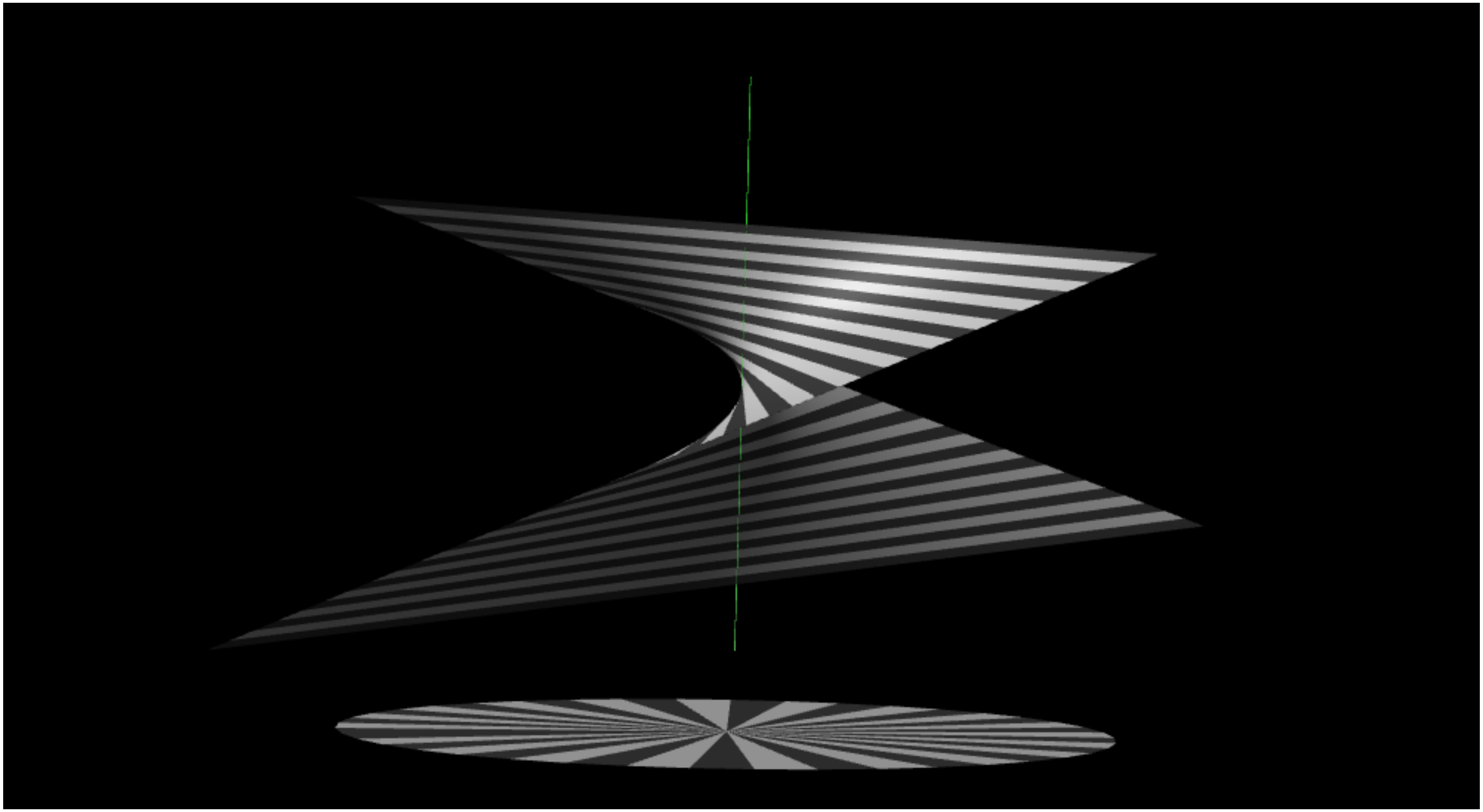

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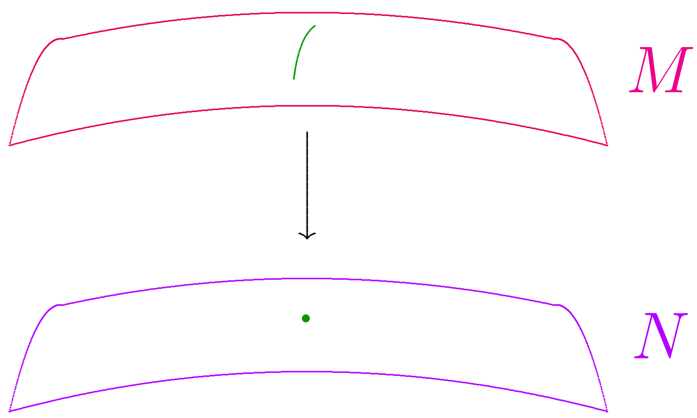

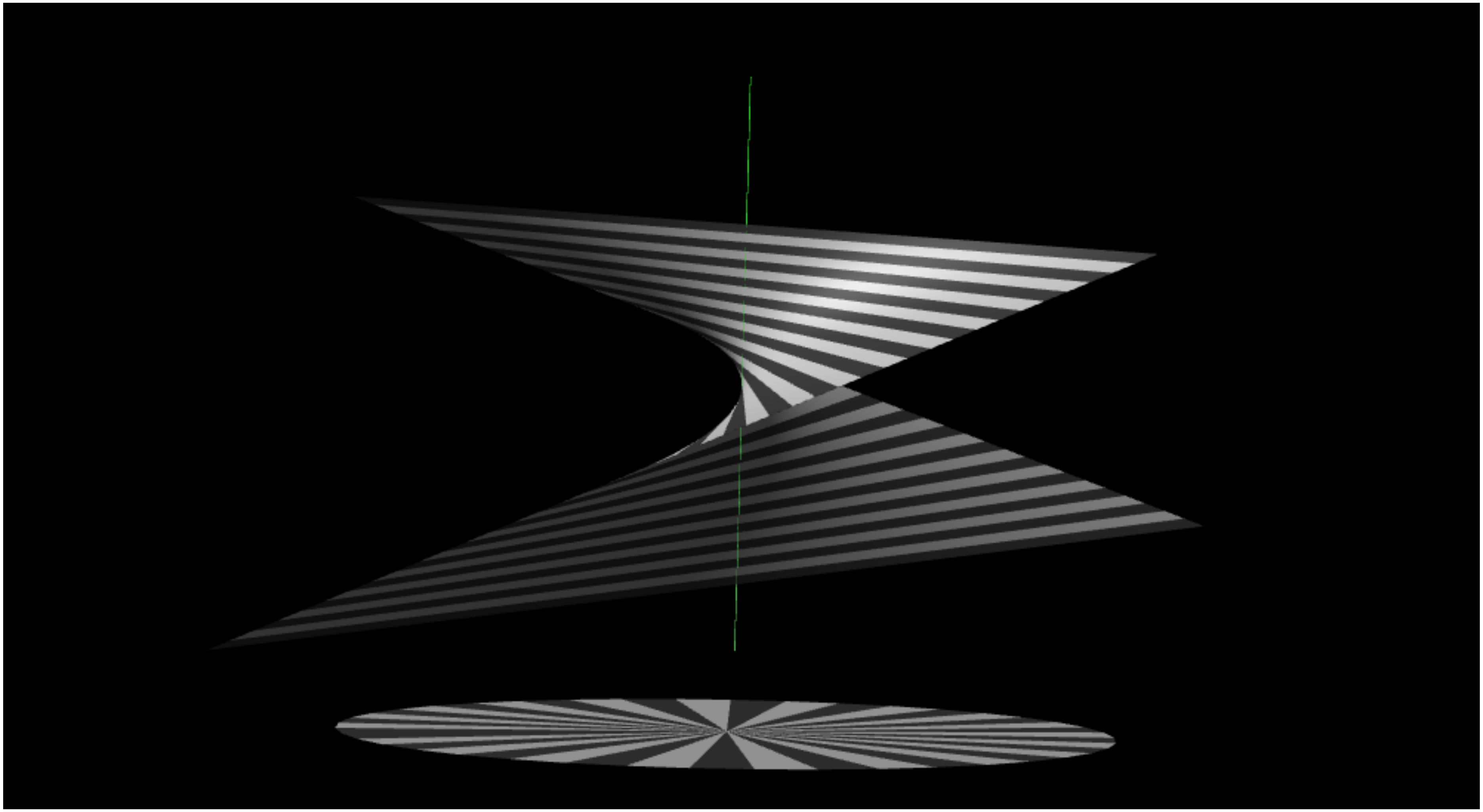

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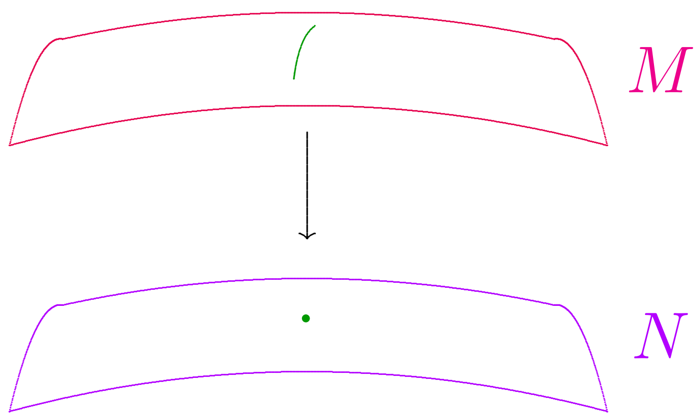

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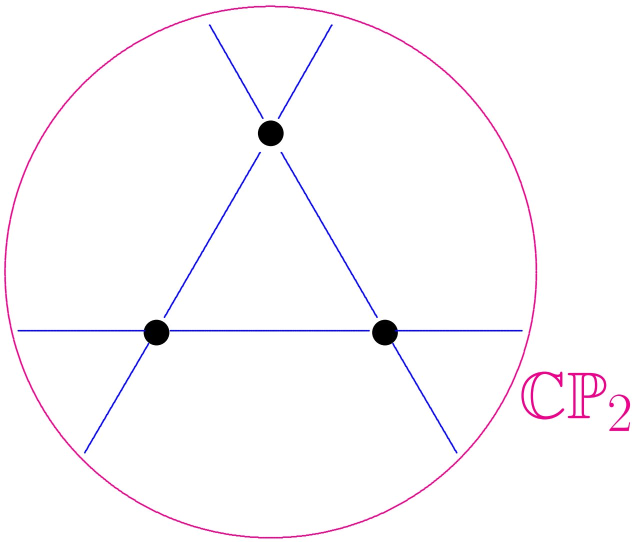

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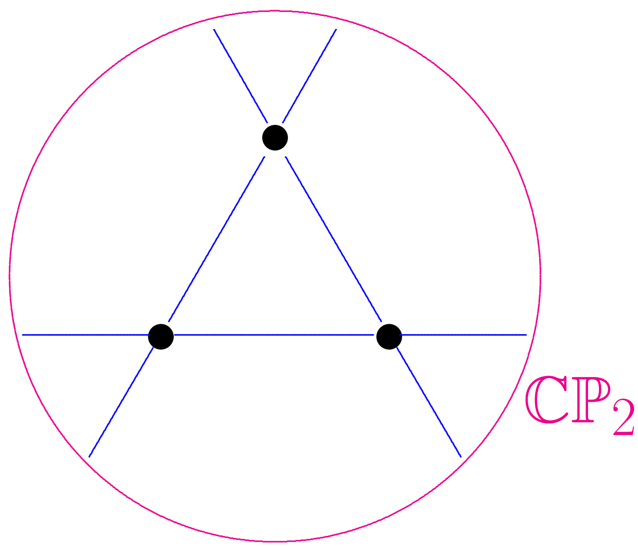

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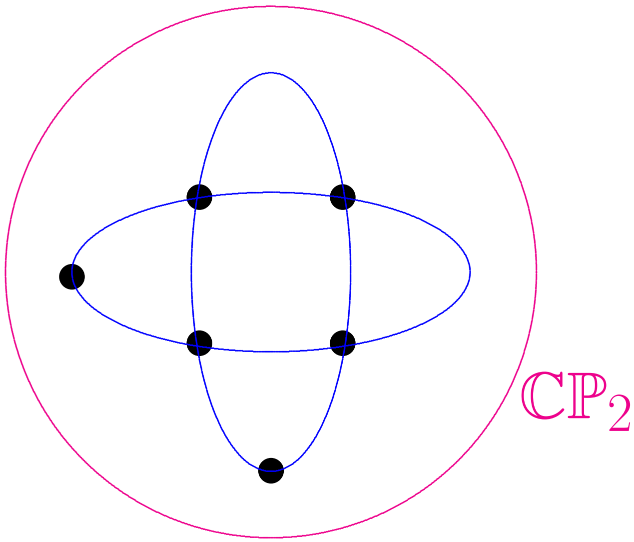

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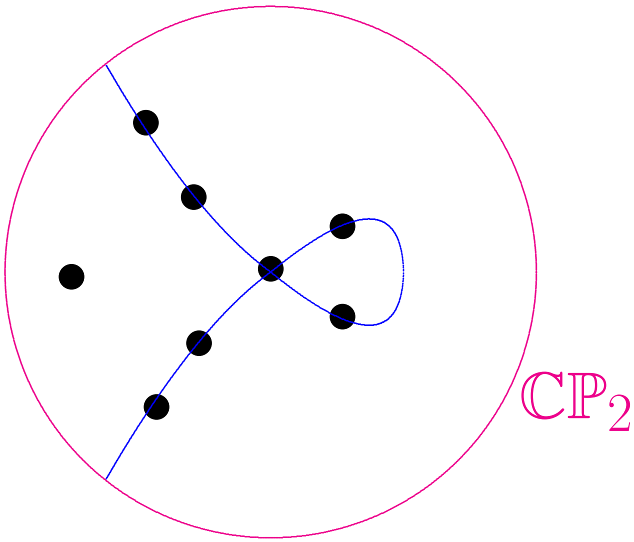

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Existence: Page

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Another 4-Dimensional Pecularity

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When $n = 4$, Einstein metrics satisfy a remarkable conformally-invariant equation.

On Riemannian n-manifold (M, g),

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$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2} \mathring{r}^{[a}{}_{[c} \delta^{b]}{}_{d]} + \frac{2}{n(n-1)} \, s \, \delta^{a}_{[c} \delta^{b}_{d]}$$

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where

$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W &= \text{Weyl curvature} \end{aligned}$$

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$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

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Proposition. *Assume* $n \geq 4$. *Then* (M^n, g) *locally conformally flat* $\Longleftrightarrow W \equiv 0$.

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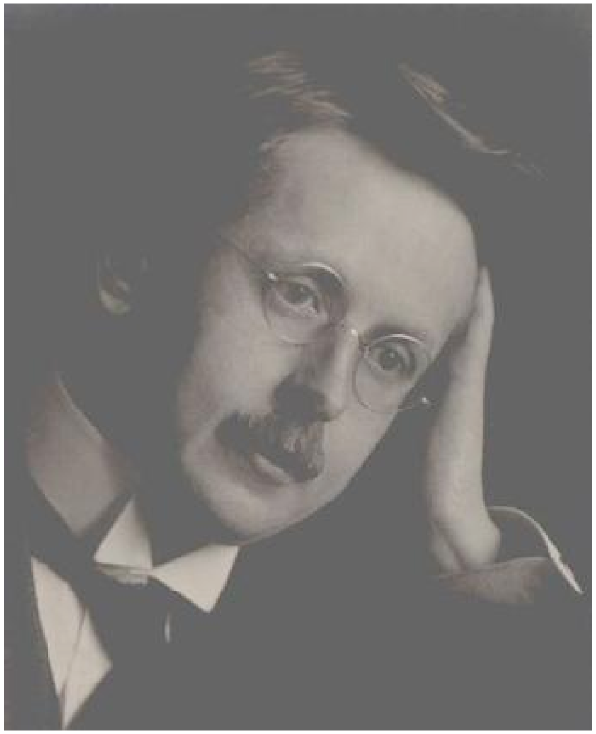

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Measures deviation $[g]$ from conformal flatness.

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Of course, conformally Einstein good enough!

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But when $n \neq 4$, Einstein $\not\Rightarrow$ critical point of $\mathscr{W}$!

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When $n = 4$, conf. Einstein $\Rightarrow$ critical for $\mathscr{W}$.

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In $n = 4$, $\exists$ conformally-invariant decomposition

$$W = W_+ + W_-$$

For (M^4, g) compact oriented Riemannian,

Euler characteristic

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu$$

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(M^4, g, J) Kähler.

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- *$g_t = g + tB$ is Kähler metric for small t.*

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Thus, a Kähler metric is Bach-flat $\Longleftrightarrow$ critical for restriction of $\mathscr{W}_+$ to Kähler metrics!

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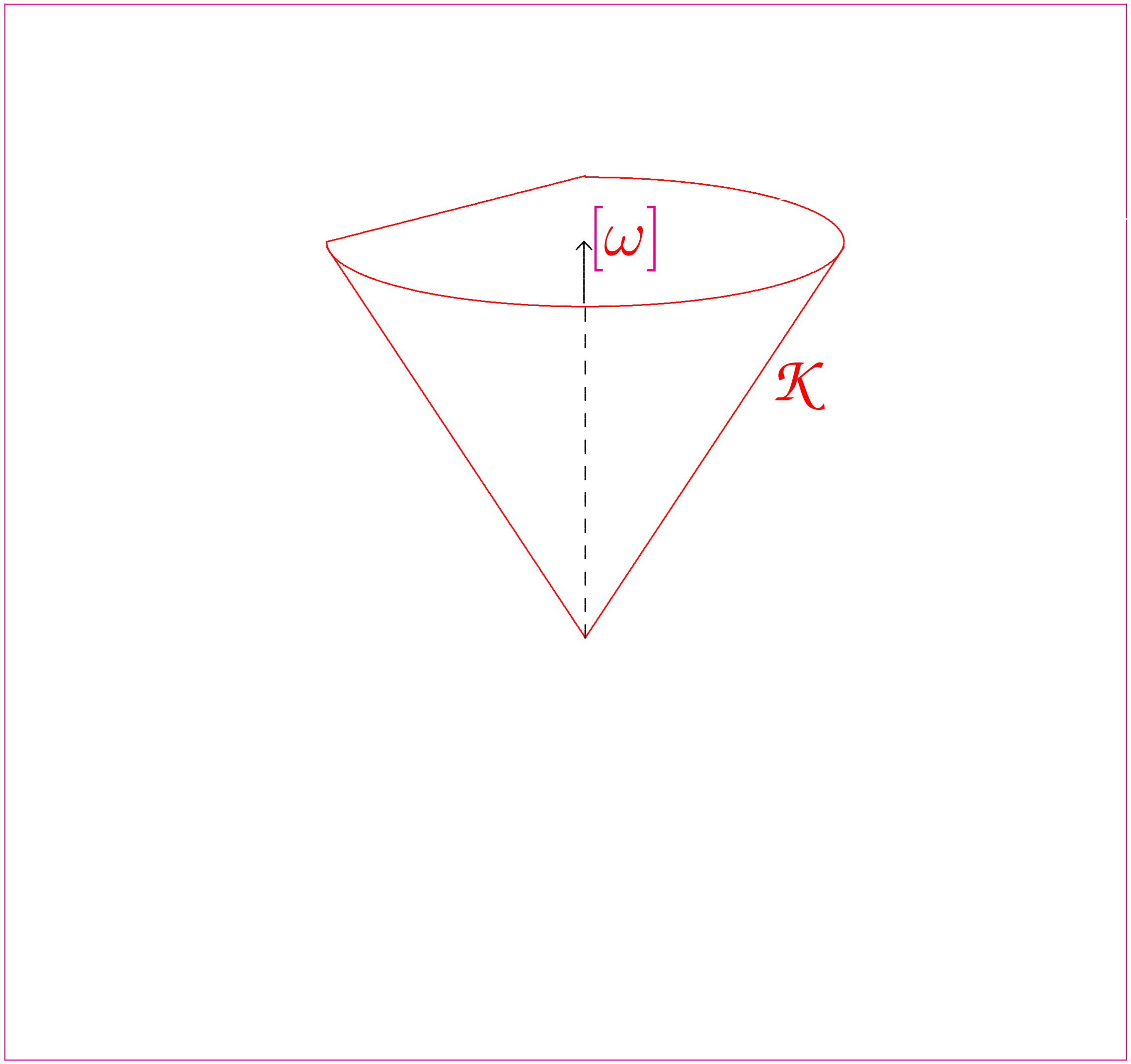

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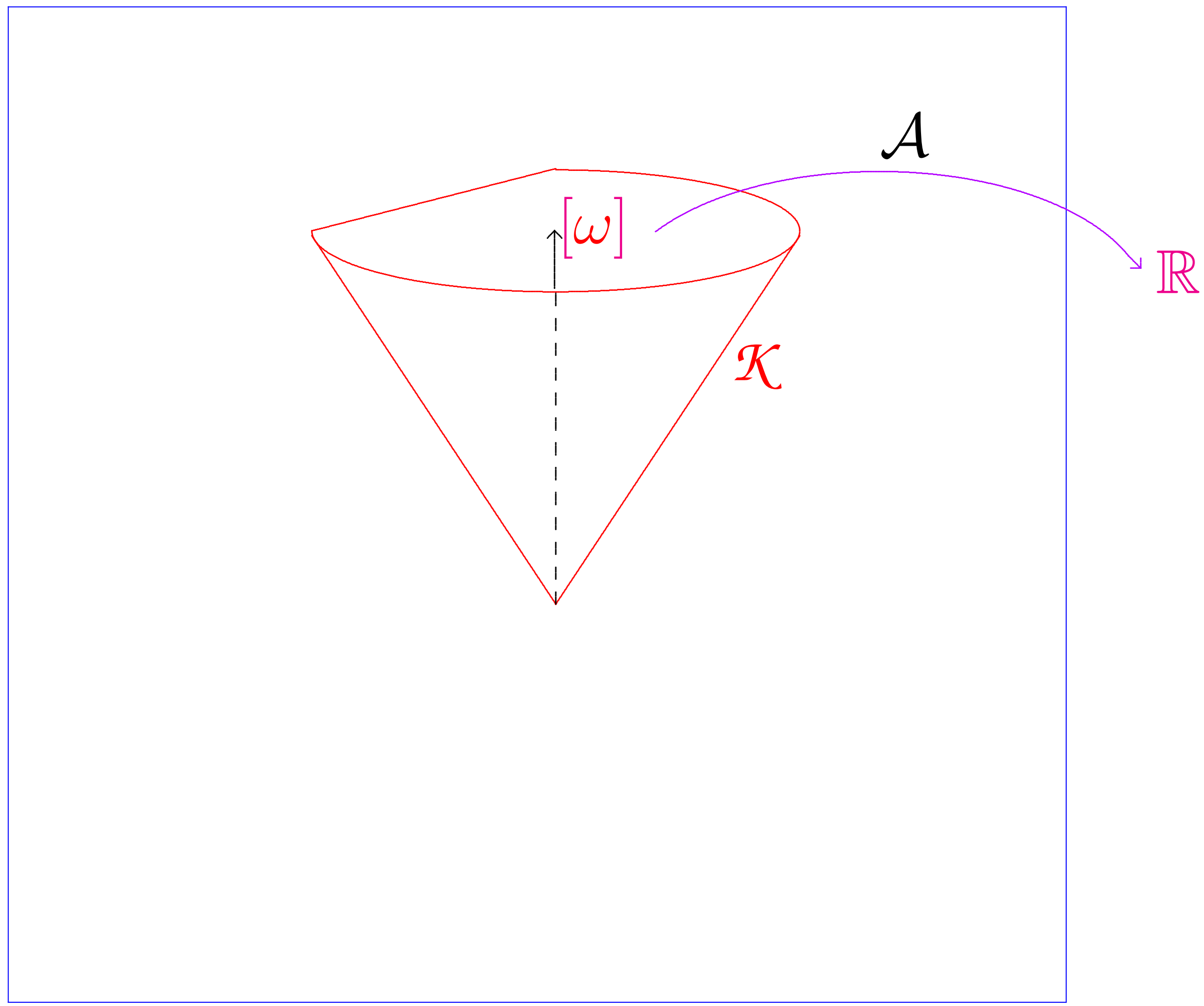

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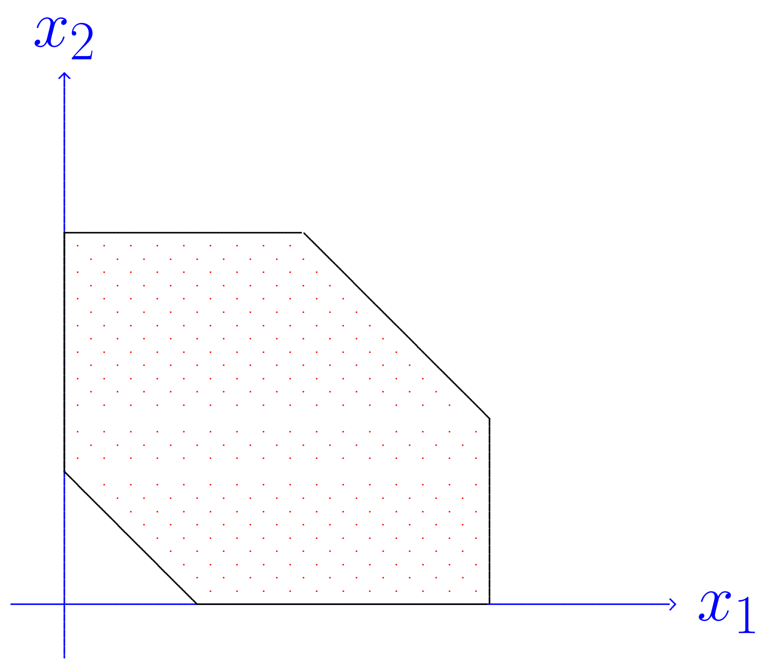

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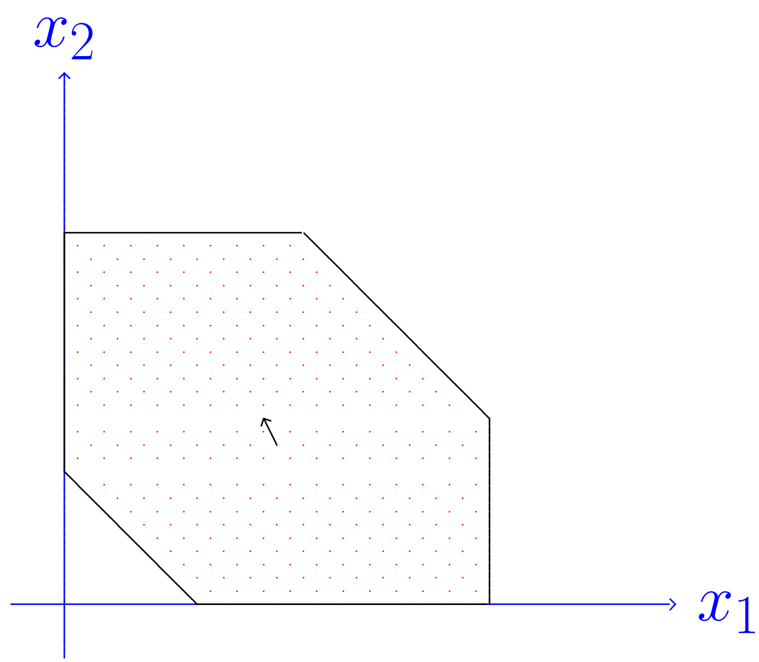

$$\mathcal{A}([\omega]) = \frac{|\partial P|^2}{2}\left(\frac{1}{|P|} + \vec{\mathfrak{D}} \cdot \Pi^{-1}\vec{\mathfrak{D}}\right)$$

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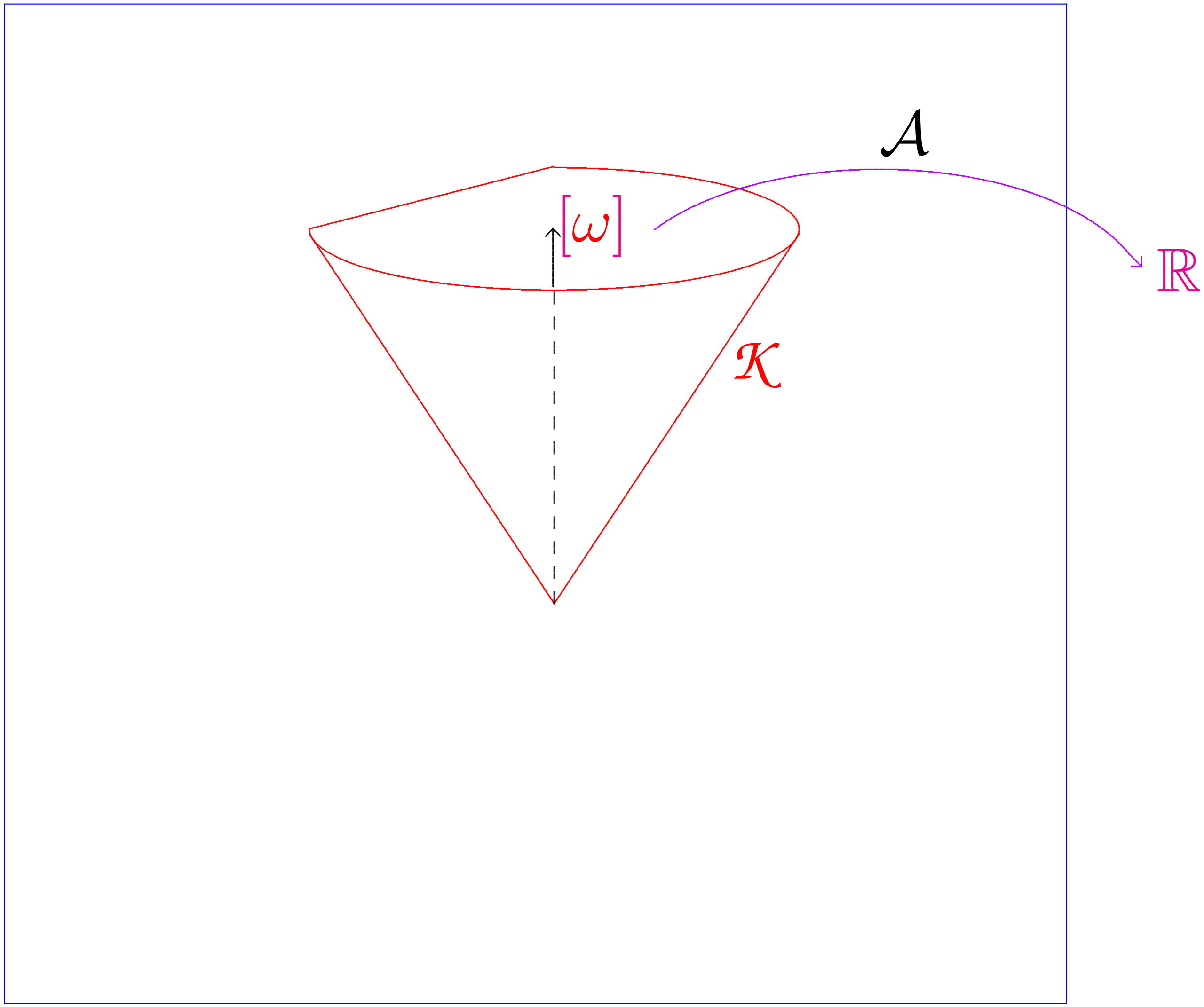

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Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

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Exactly one connected component of moduli space!

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for these metrics & conformal rescalings:

$$g \rightsquigarrow h = f^2 g \quad \Longrightarrow \quad \det(W_+) \rightsquigarrow f^{-6} \det(W_+).$$

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Claim: (M, g) compact Einstein $\Longrightarrow$ J integrable.

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Integrability proof based on Weitzenböck formula

$$\boxed{0 = \nabla^*\nabla W_+ + \frac{s}{2}W_+ - 6W_+ \circ W_+ + 2|W_+|^2 I}$$

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Corollary. *Every simply-connected compact oriented Einstein (M^4, h) with $\det(W_+) > 0$ is diffeomorphic to a del Pezzo surface. Conversely, every del Pezzo M^4 carries Einstein h with $\det(W_+) > 0$, and these sweep out exactly one connected component of moduli space $\mathscr{E}(M)$.*

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Techniques used extend today's results.

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