Mass, Scalar Curvature, &

Kähler Geometry, I

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Stony Brook University

Seminario de Geometría
ICMAT, October 29, 2018

Key results joint with

Key results joint with

Hans-Joachim Hein
Fordham University

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Fordham University

Mass in Kähler Geometry
Comm. Math. Phys. 347 (2016) 621–653.

Most recent results:

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Mass, Kähler Manifolds,
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`arXiv: 1810.11417 [math.DG]`

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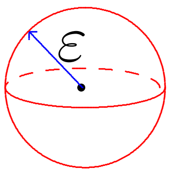

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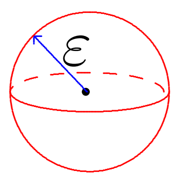

The metric g is called scalar-flat if it satisfies $s \equiv 0$.

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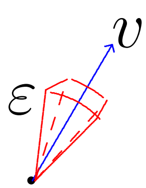

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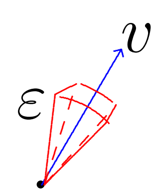

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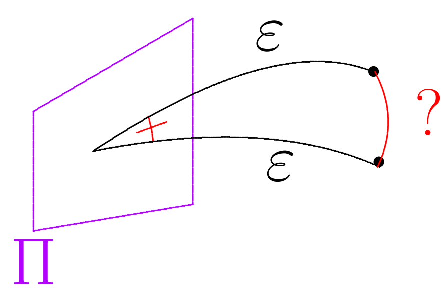

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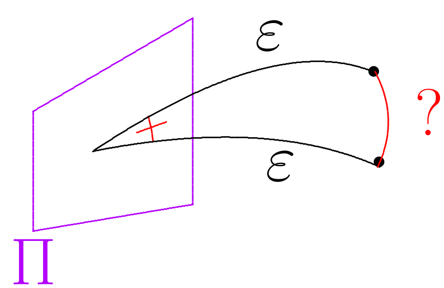

The metric g is called flat if it satisfies $K \equiv 0$.

Simple, natural problem:

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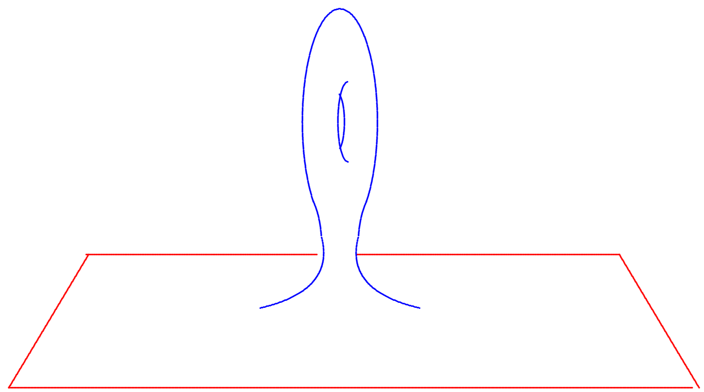
(M^n, g) complete non-compact Riemannian n-manifold.



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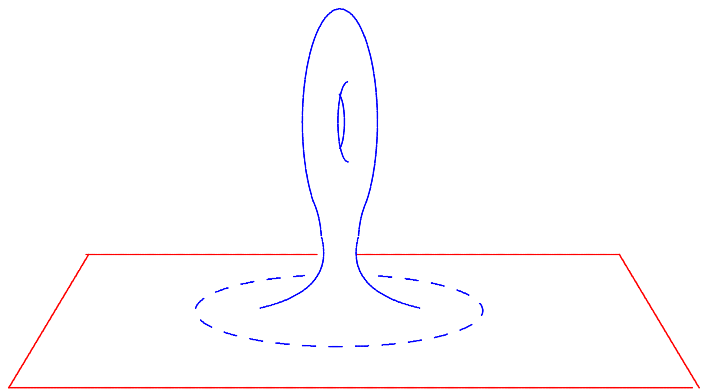
Suppose $\exists K \subset M$ compact



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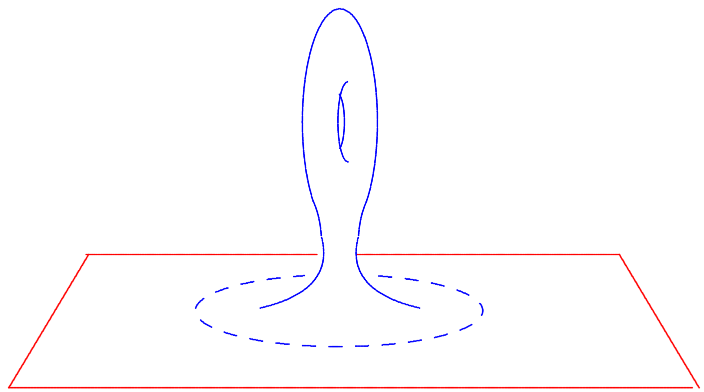

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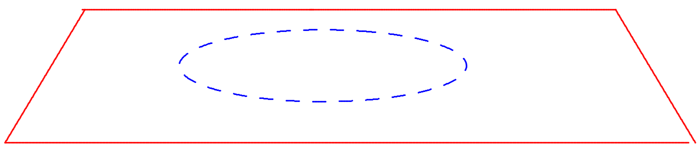

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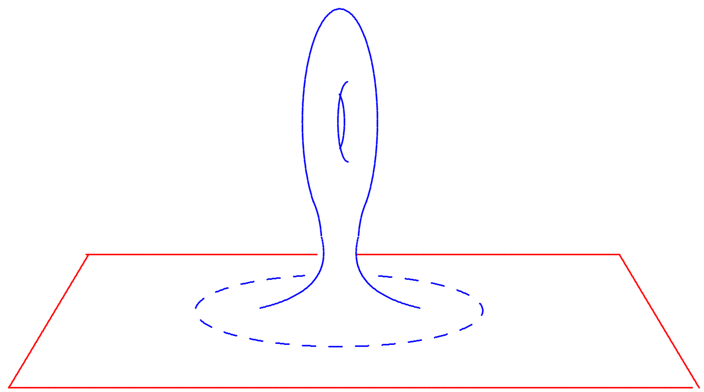

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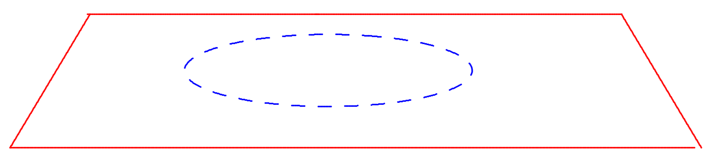

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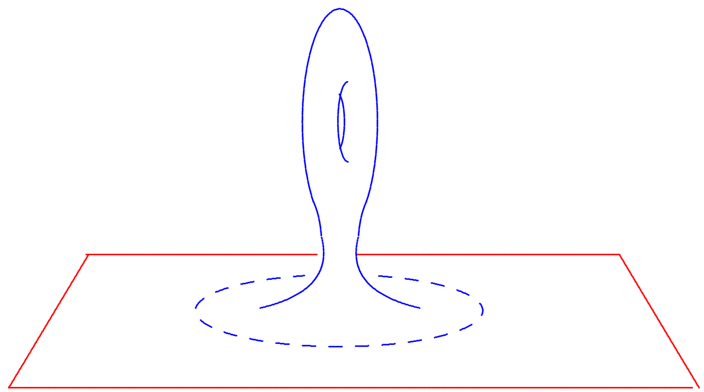

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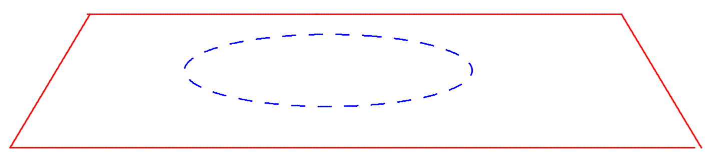

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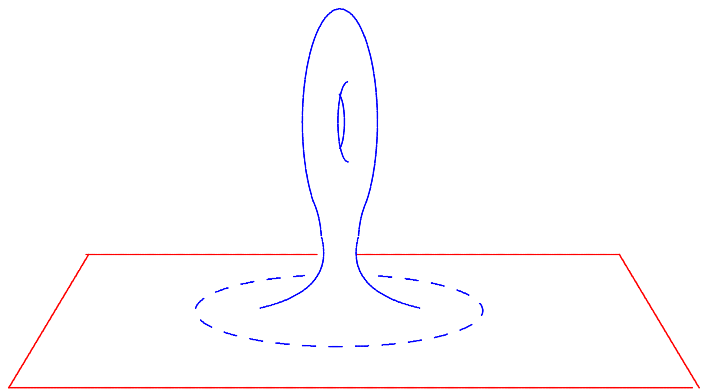

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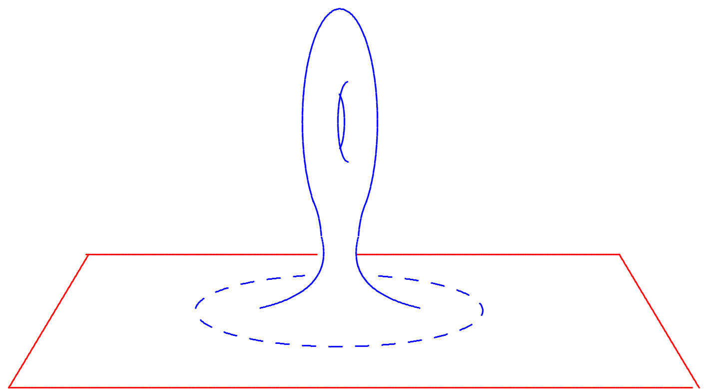

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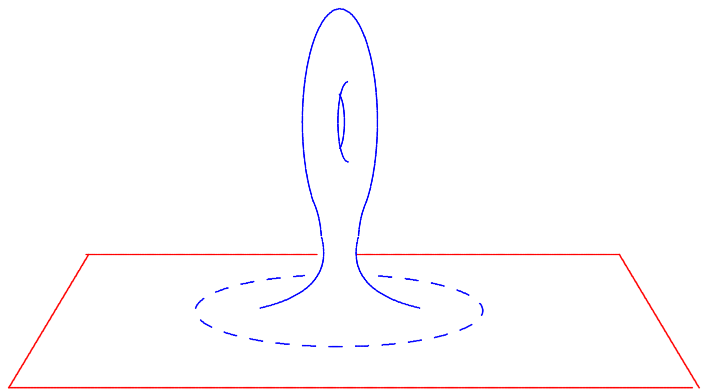

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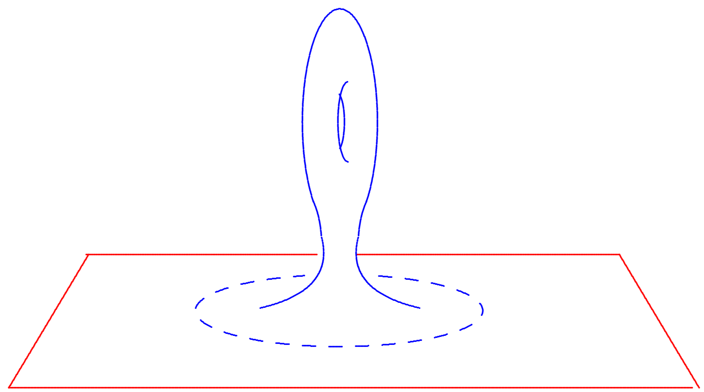

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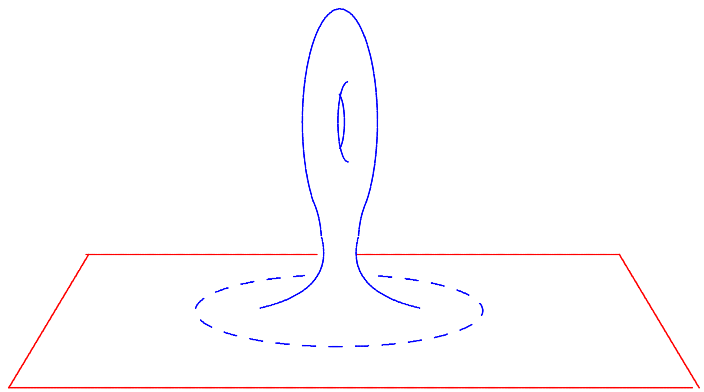

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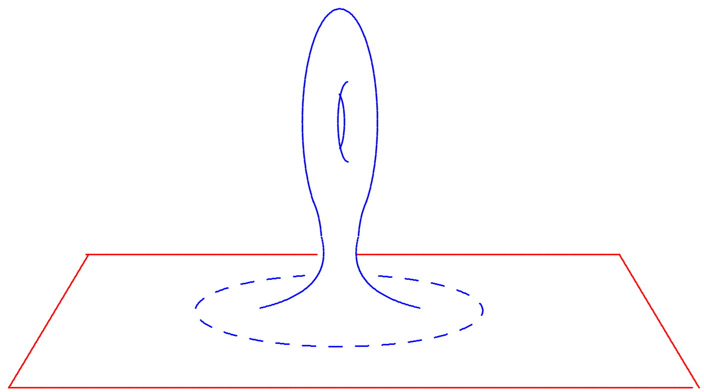

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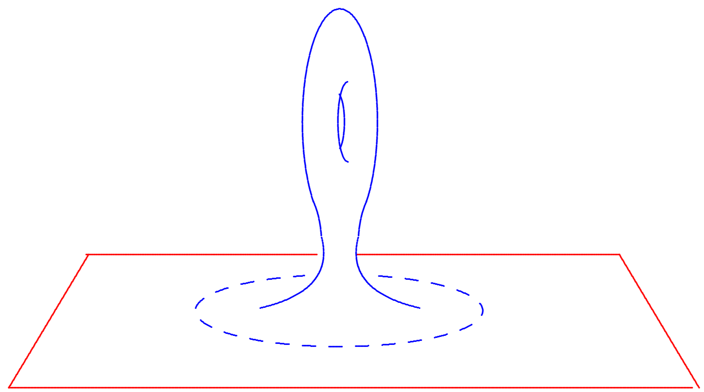

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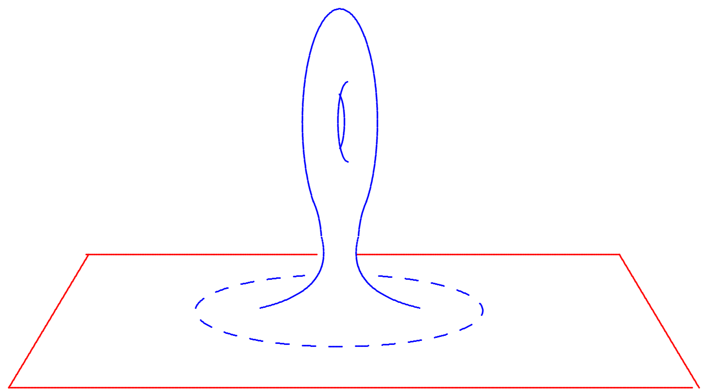

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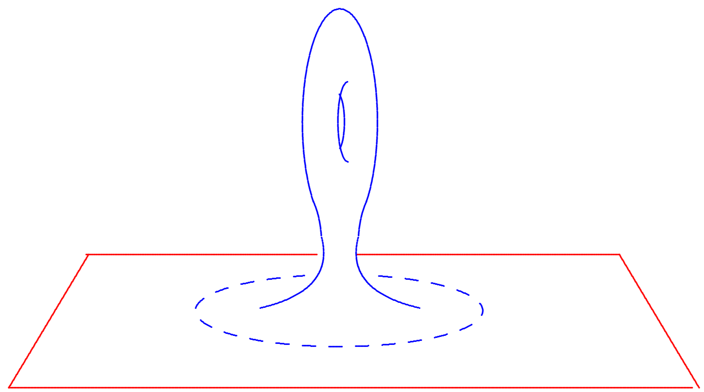

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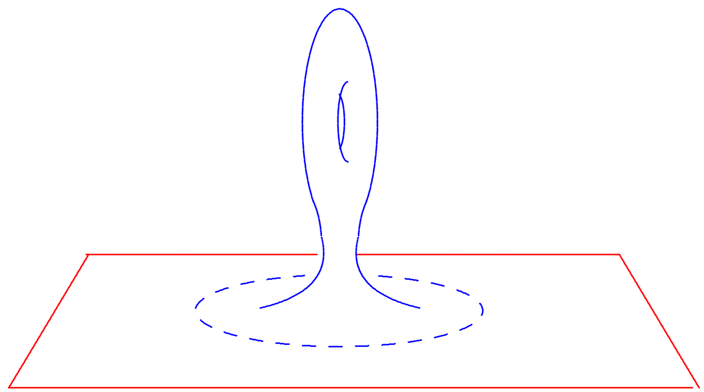

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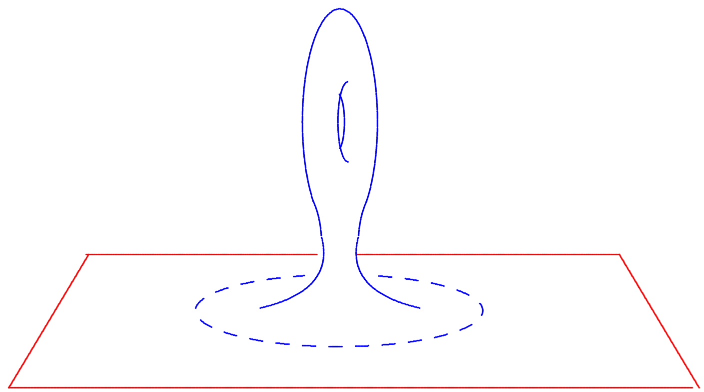

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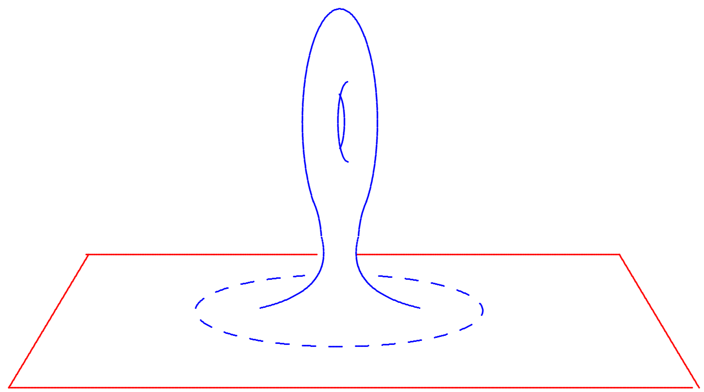

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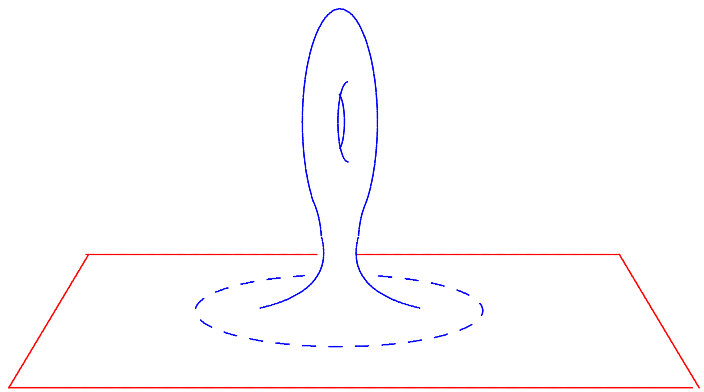

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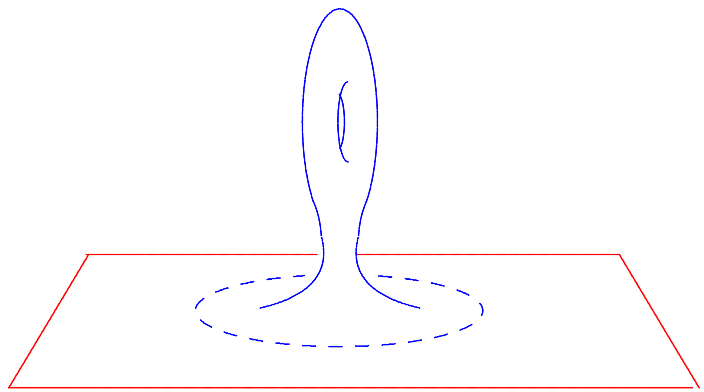

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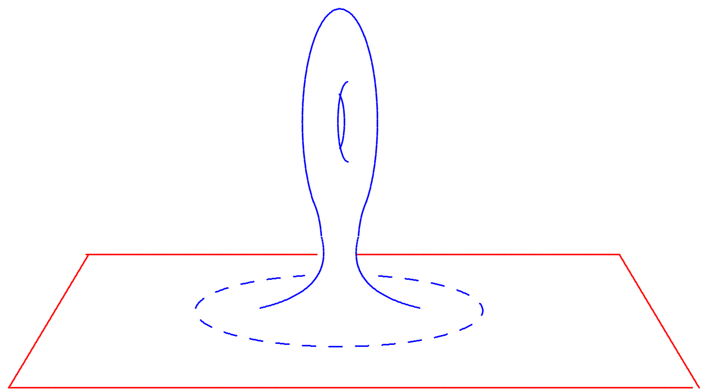

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Get result even with appropriate fall-off to Euclidean. . .

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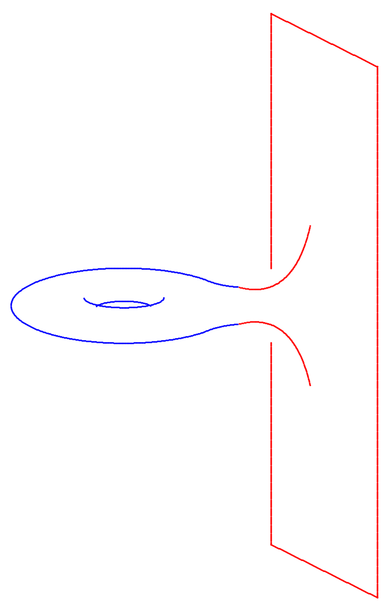

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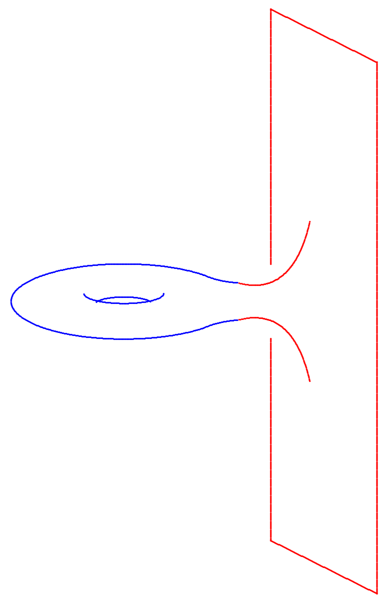

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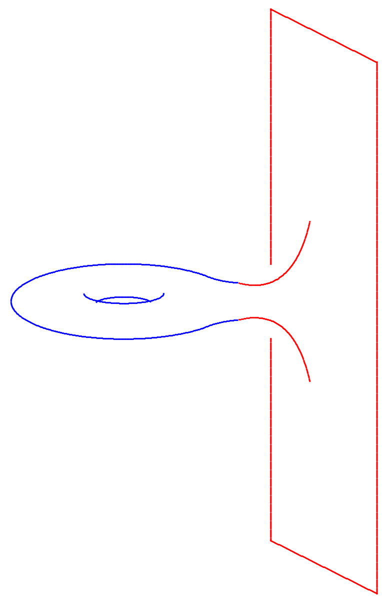

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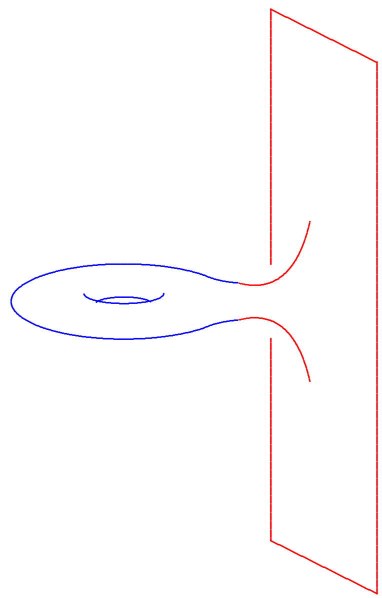

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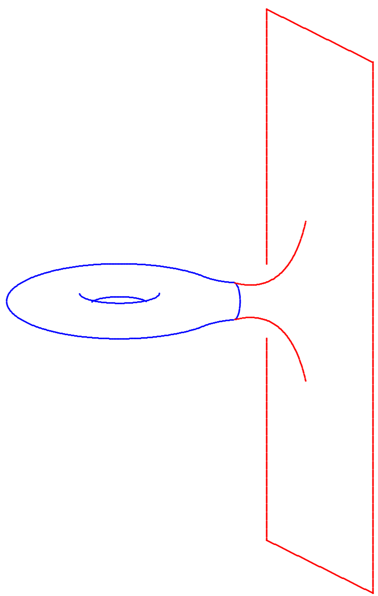

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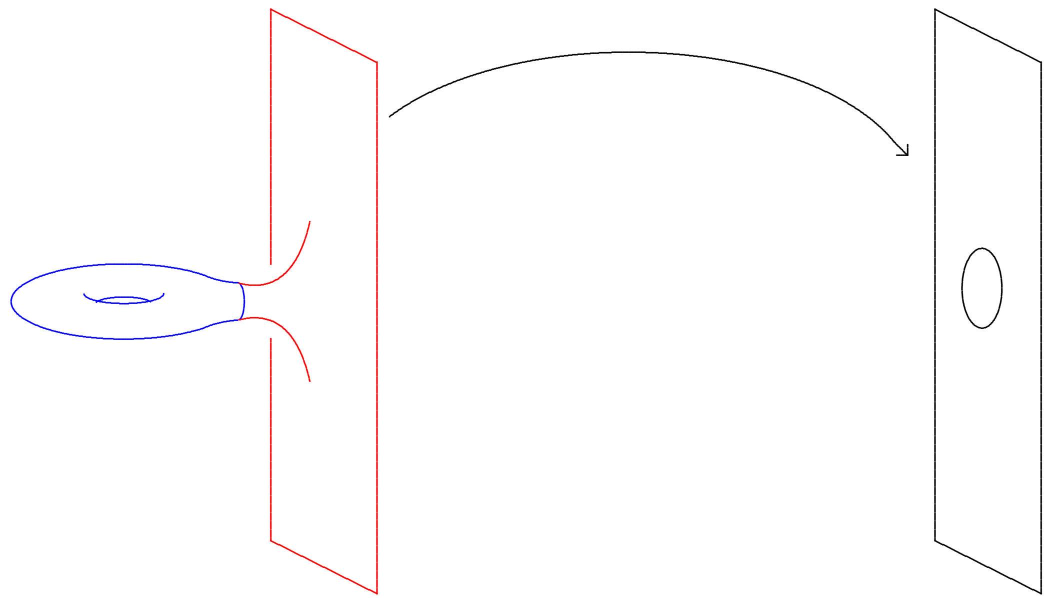

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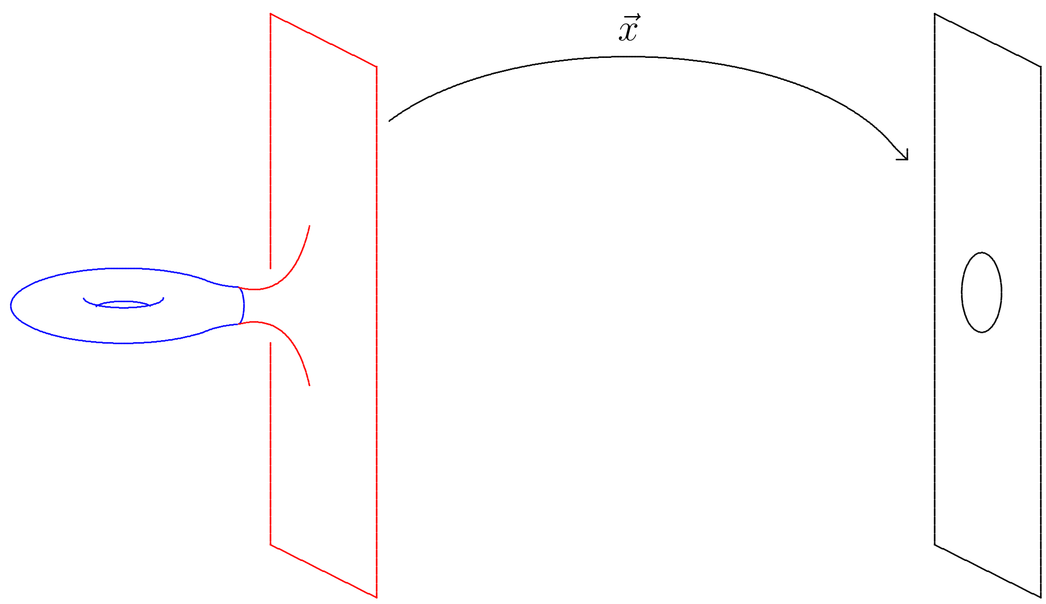

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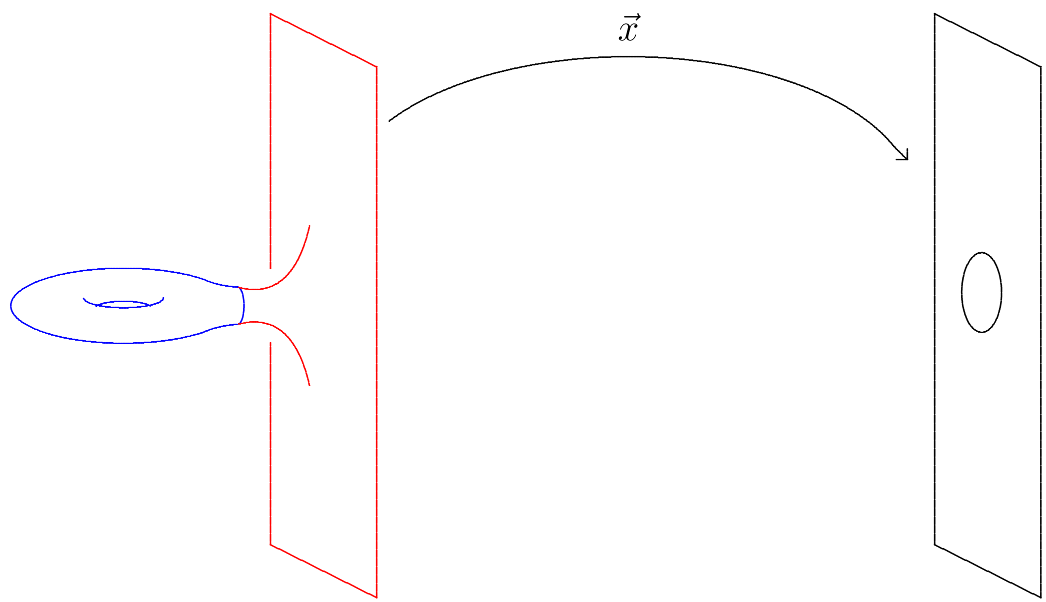

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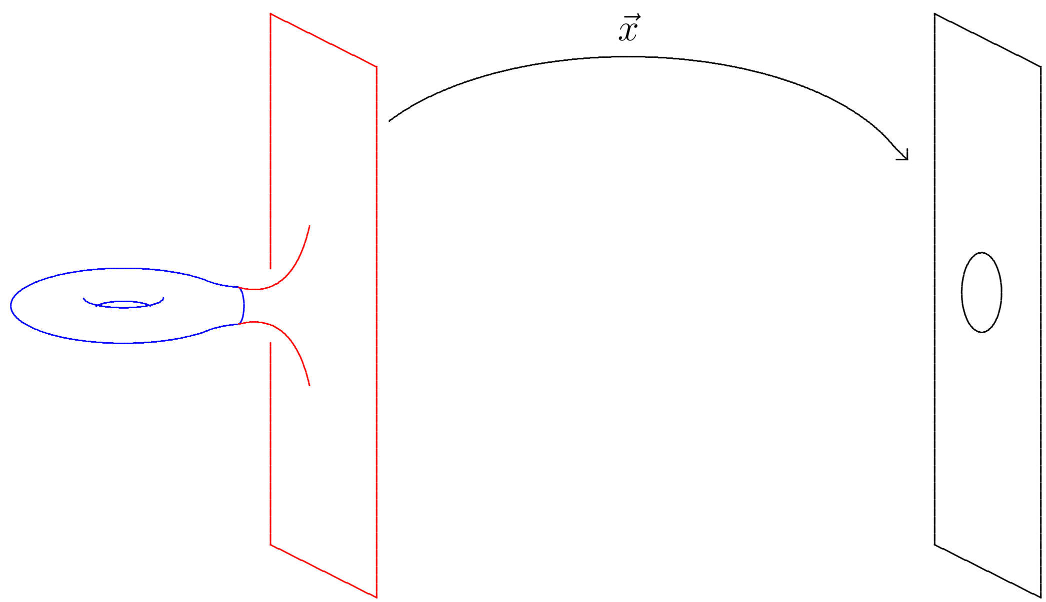

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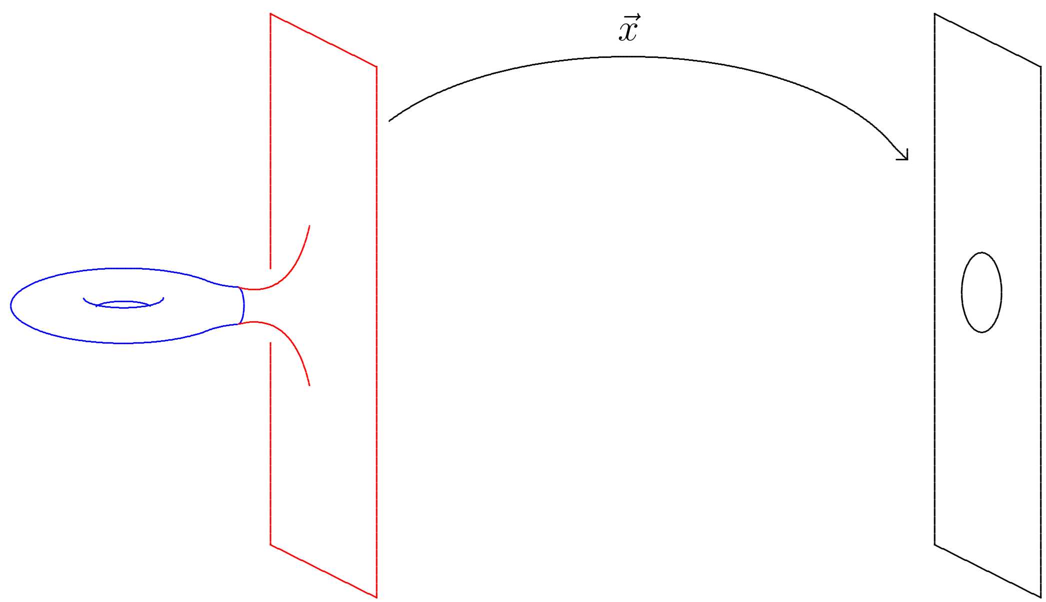
Definition. *A complete, non-compact Riemannian n-manifold (M^n, g) is called* asymptotically Euclidean *(AE) if there is a compact set $K \subset M$ such that*  *$M - K$ is diffeomorphic to $\mathbb{R}^n - D^n$ in such a manner that*

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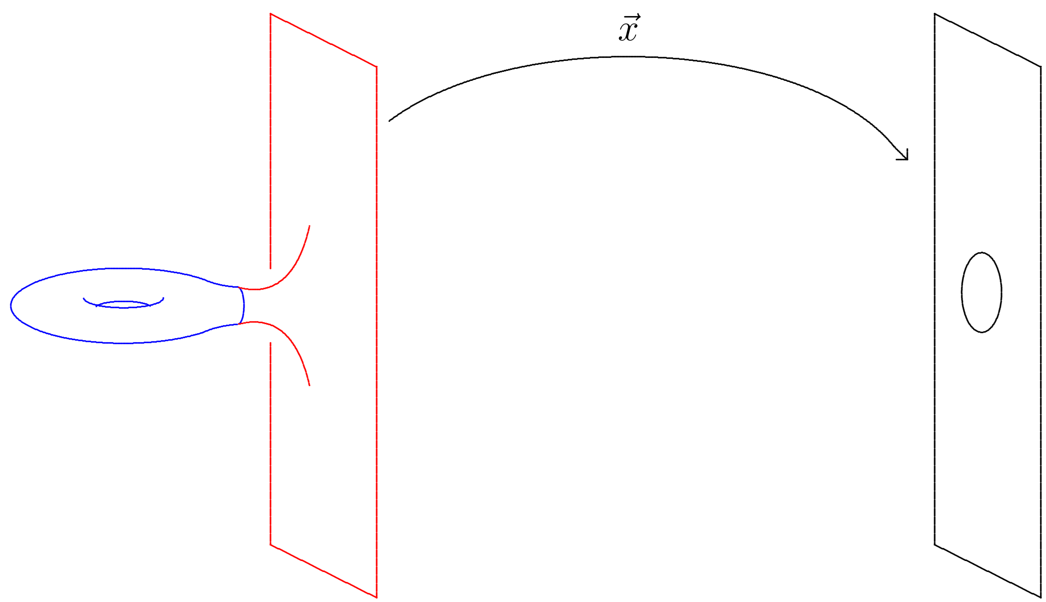

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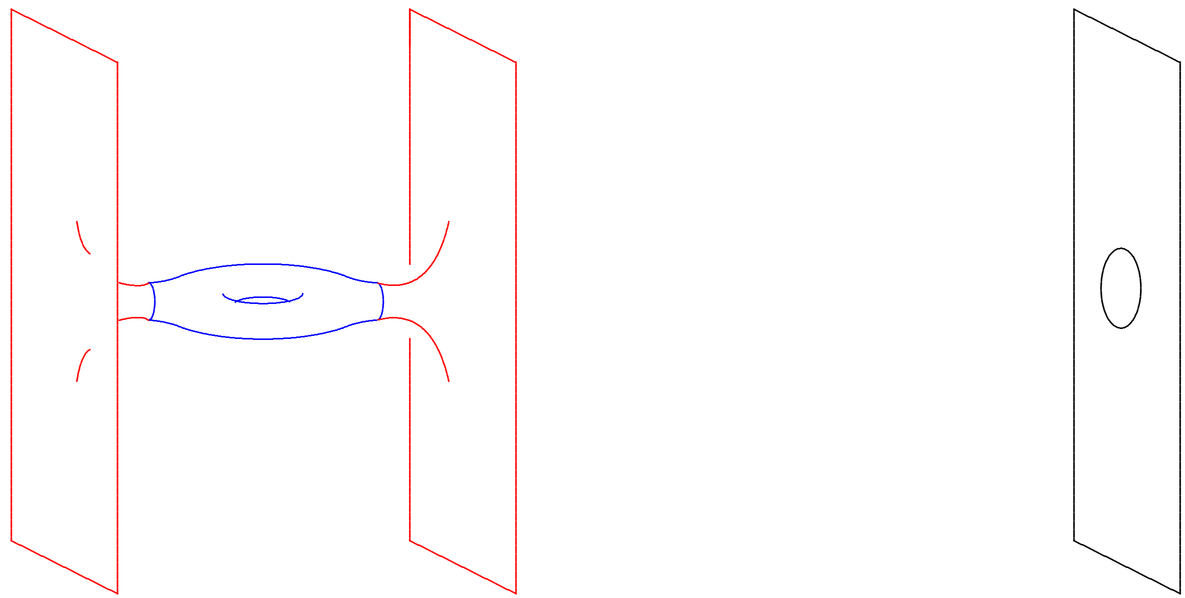

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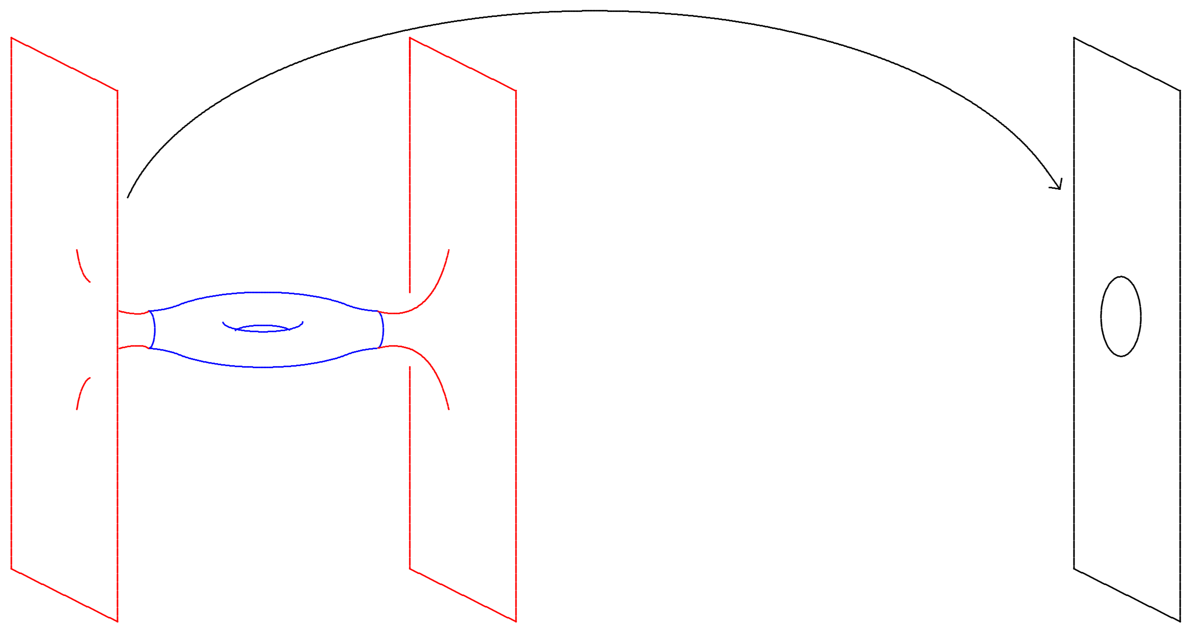

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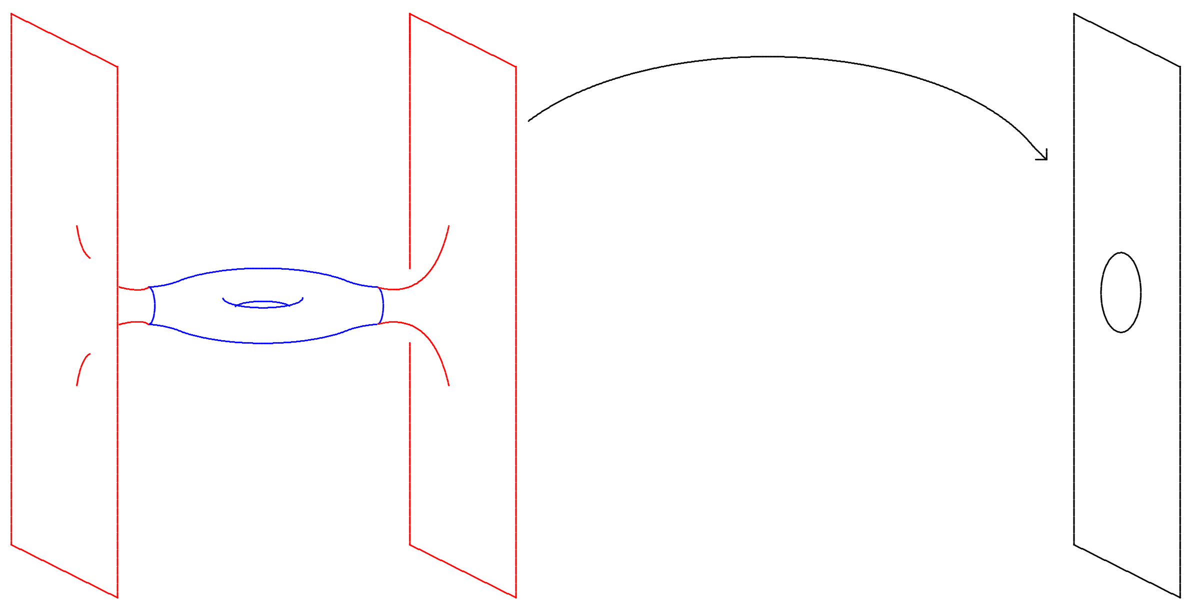

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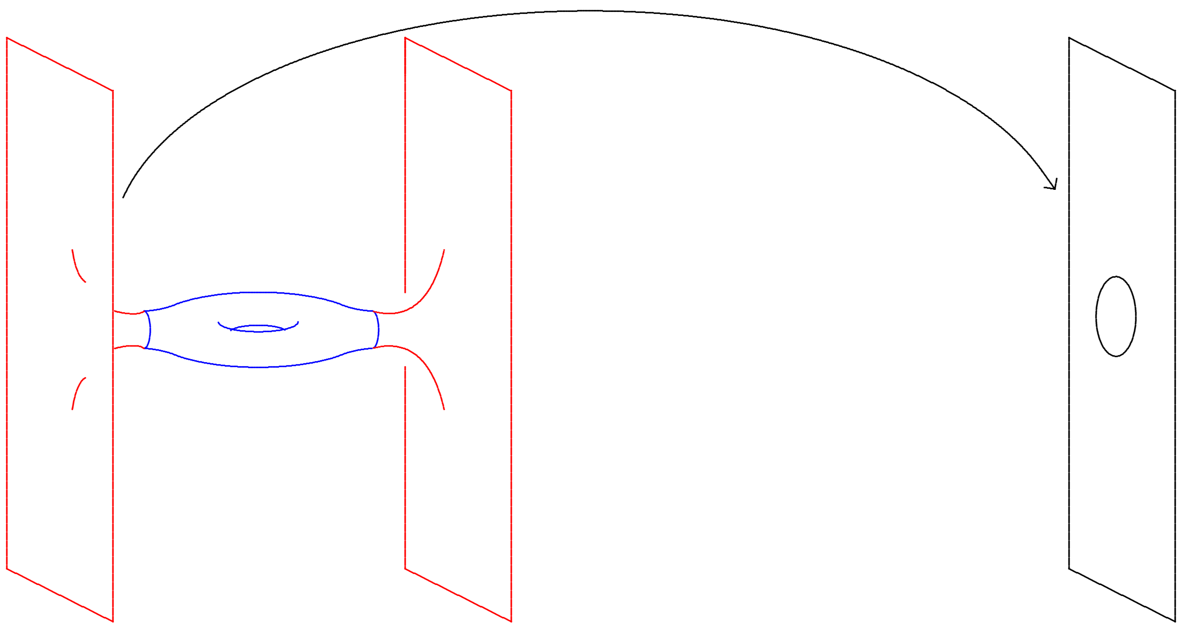

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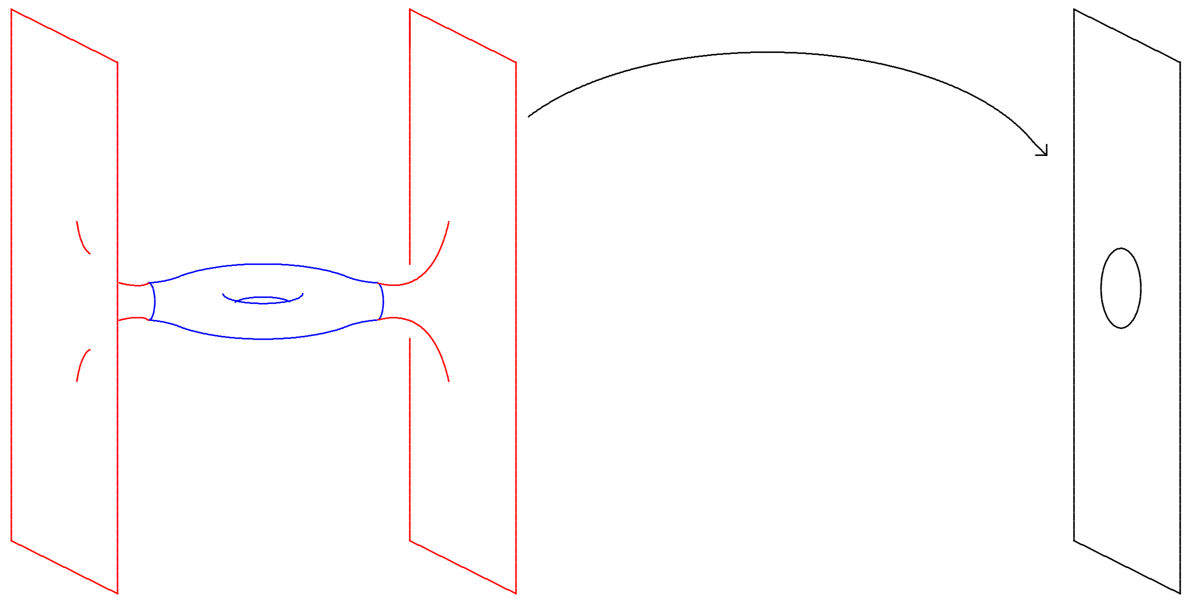

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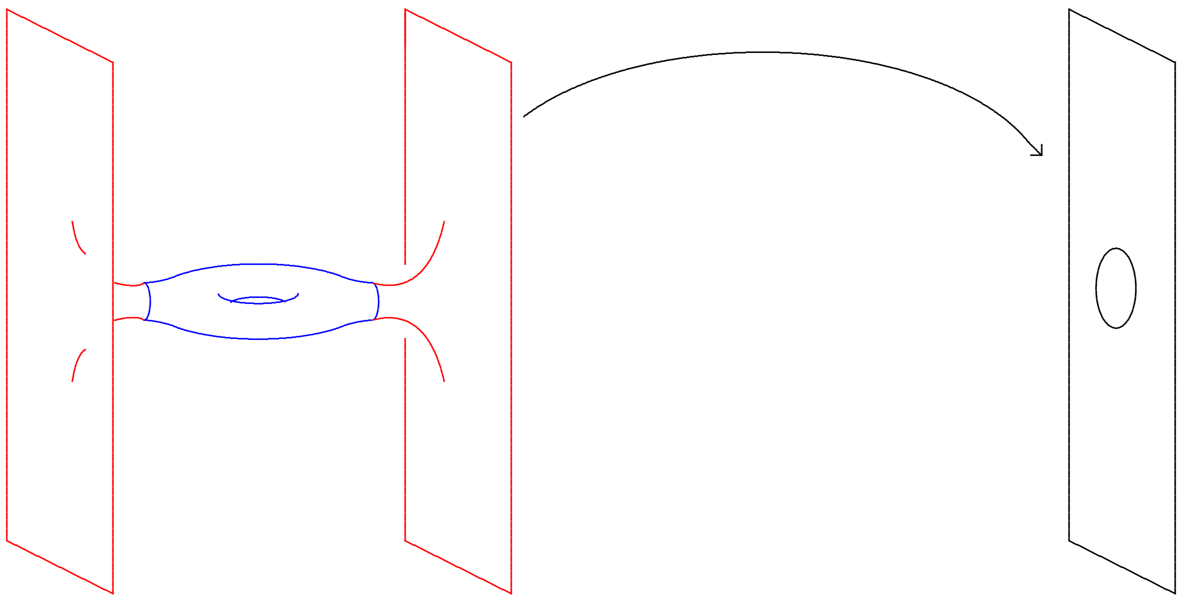

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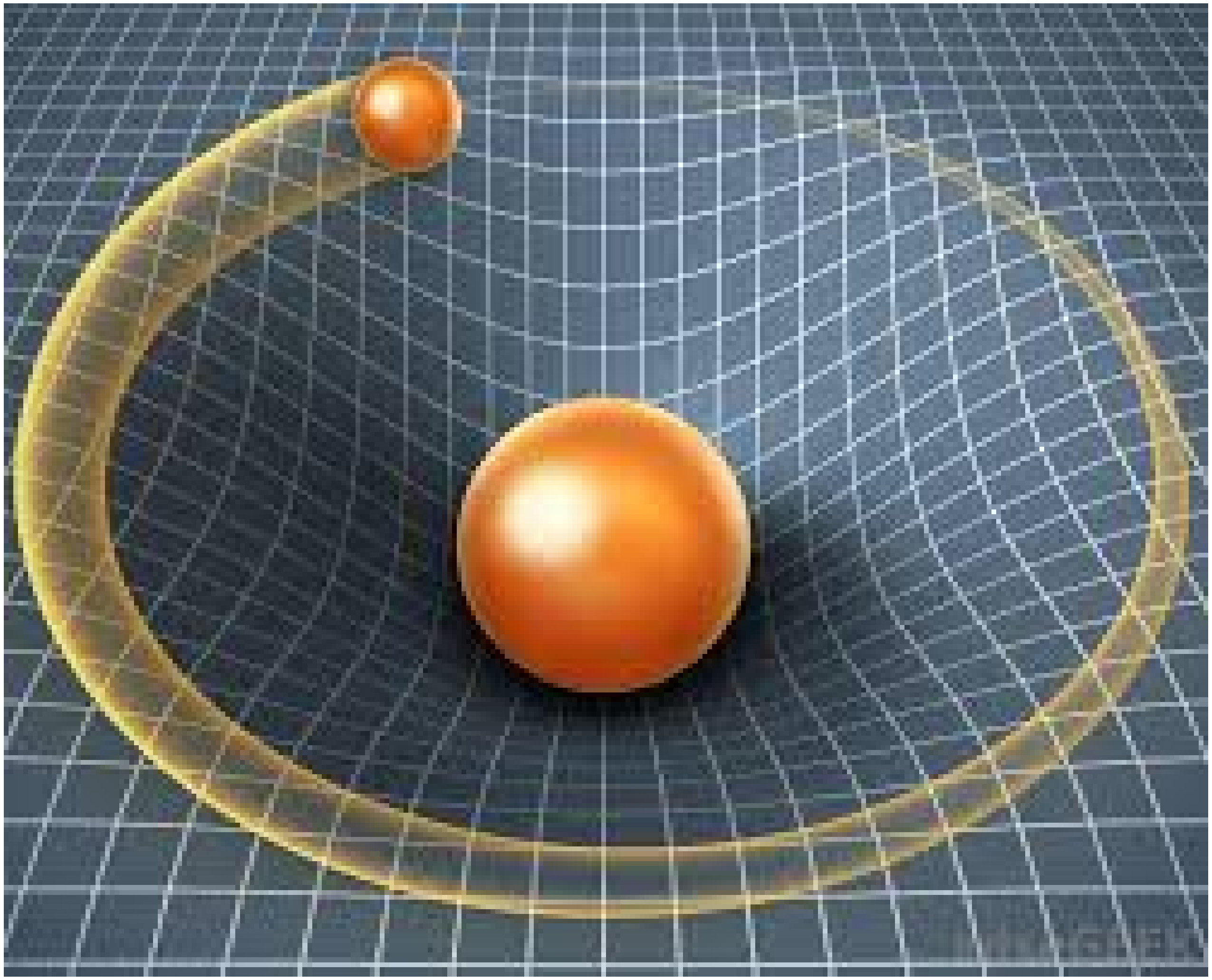

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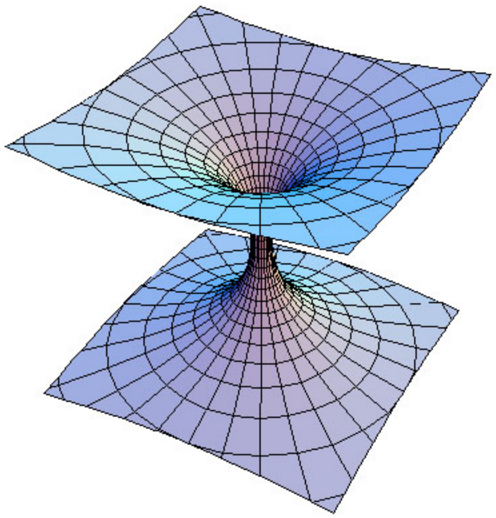

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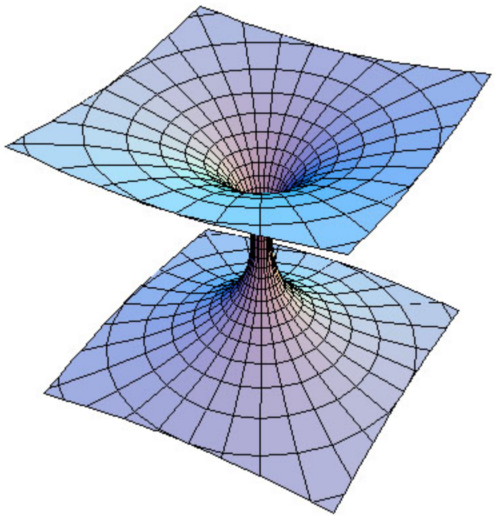

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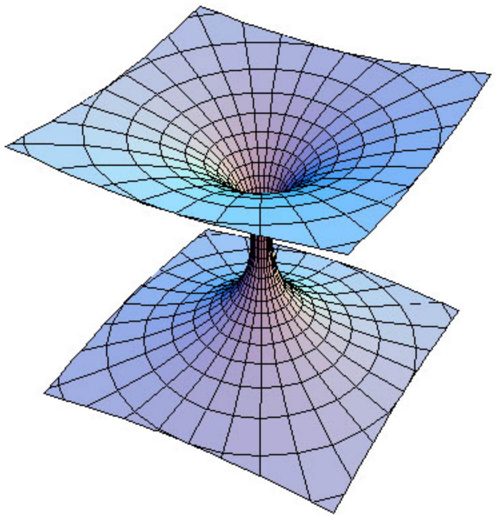

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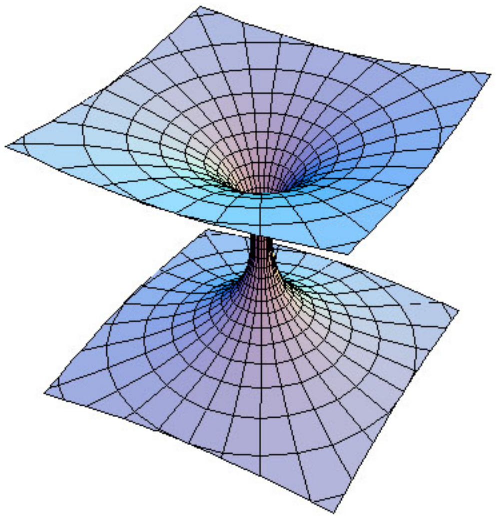

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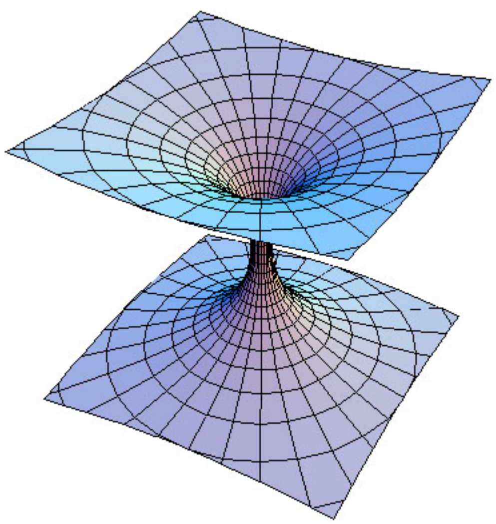

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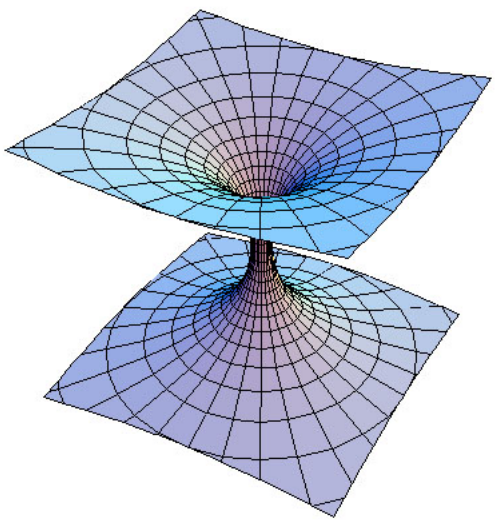

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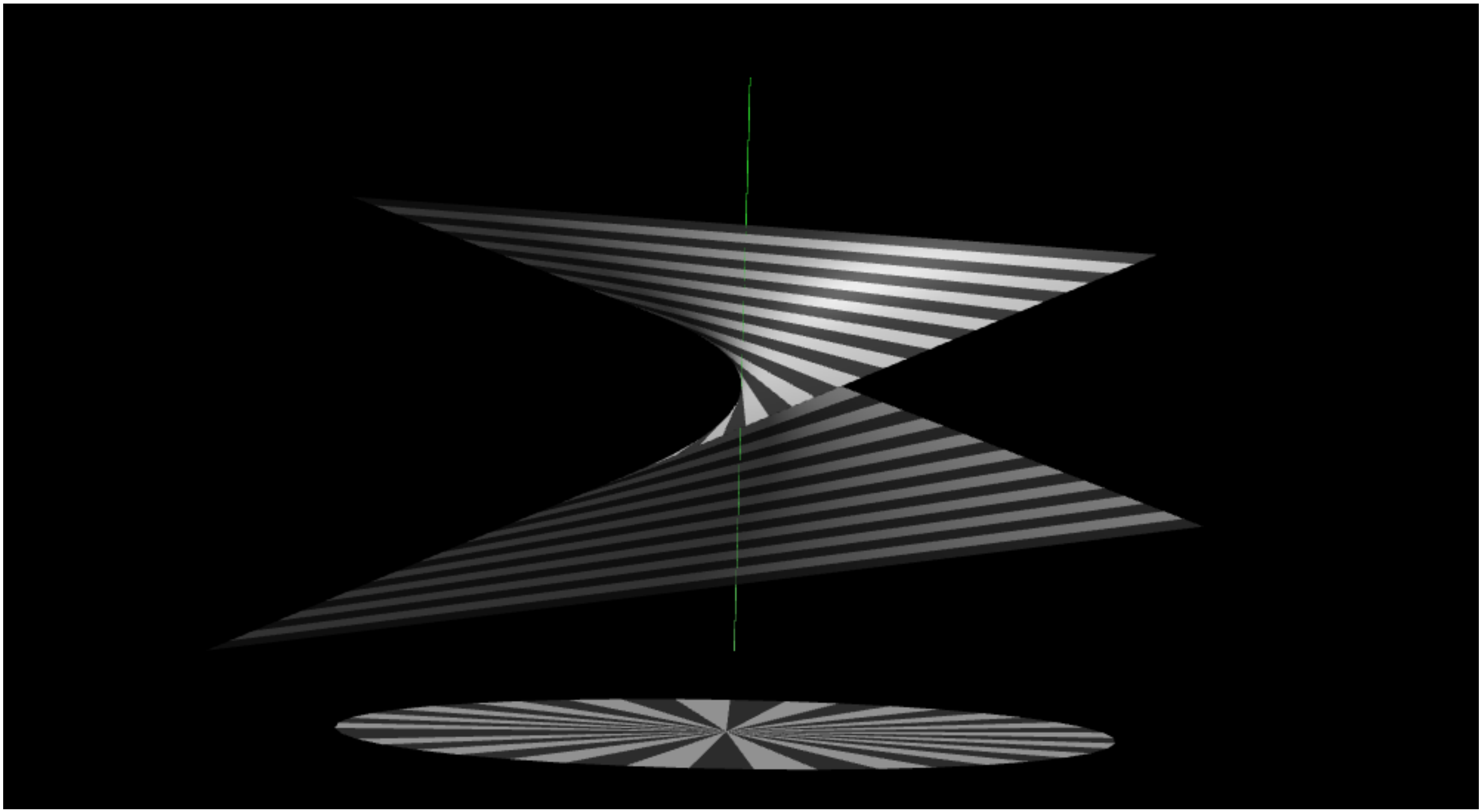

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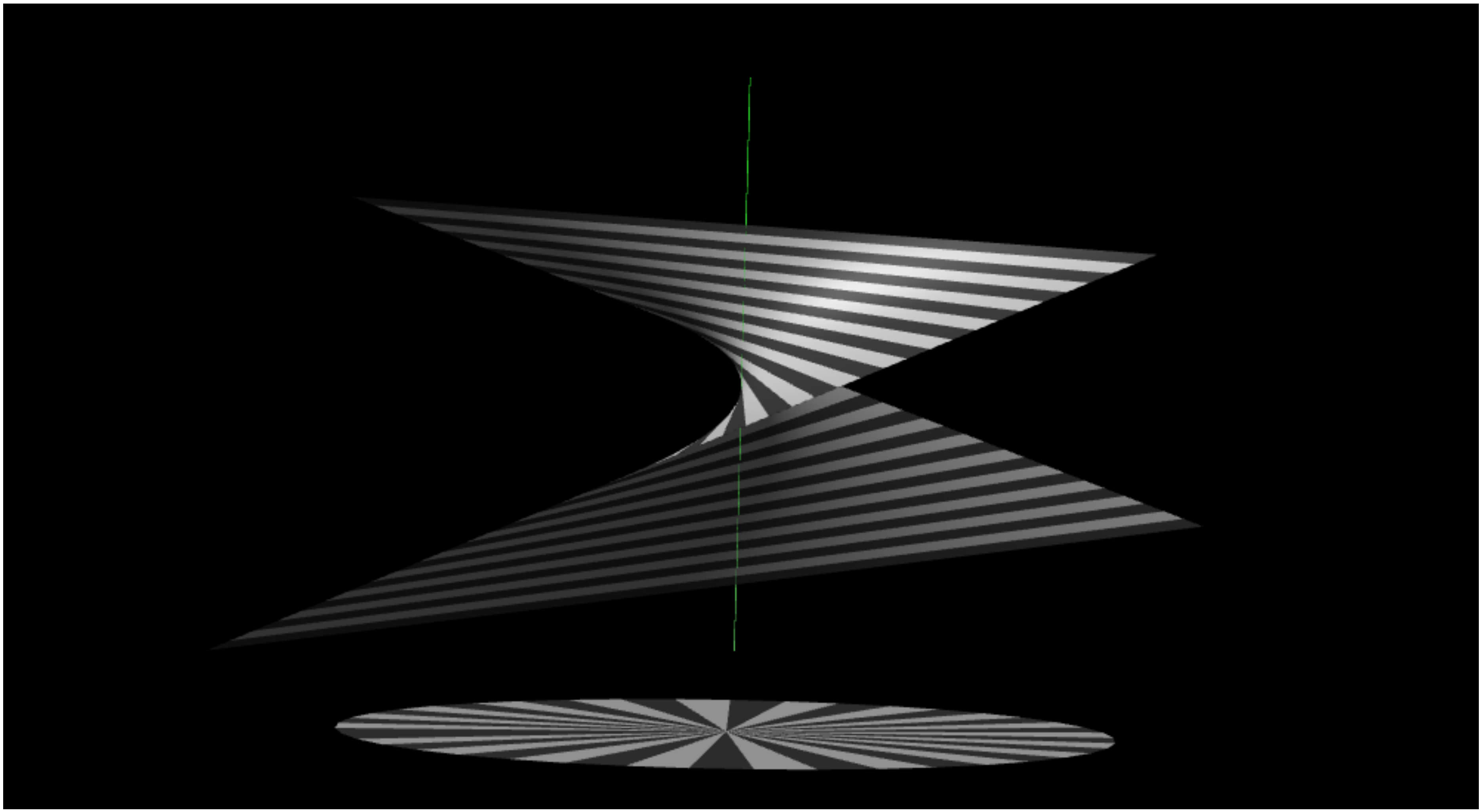

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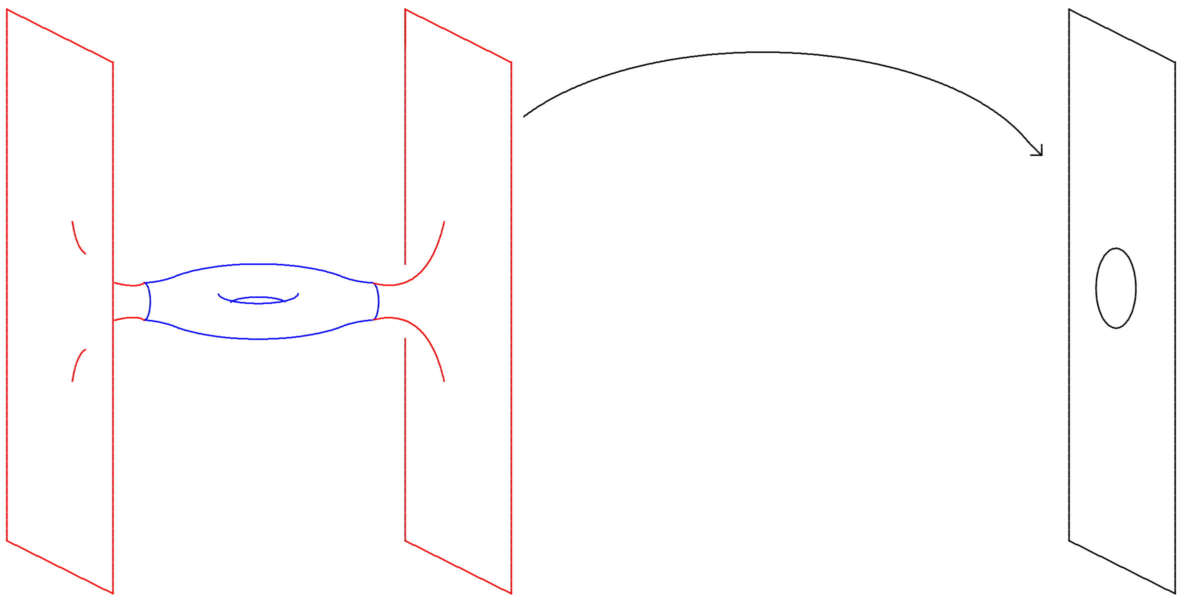

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Interesting generalization. . .

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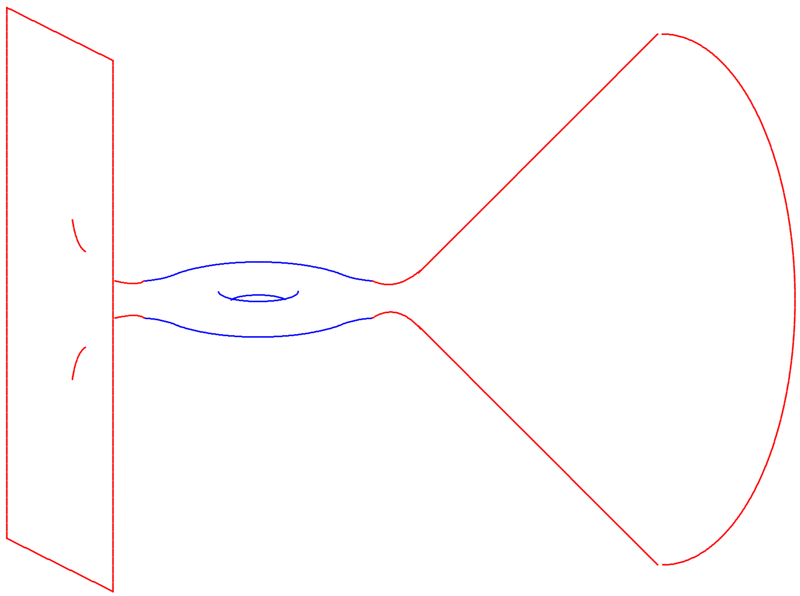

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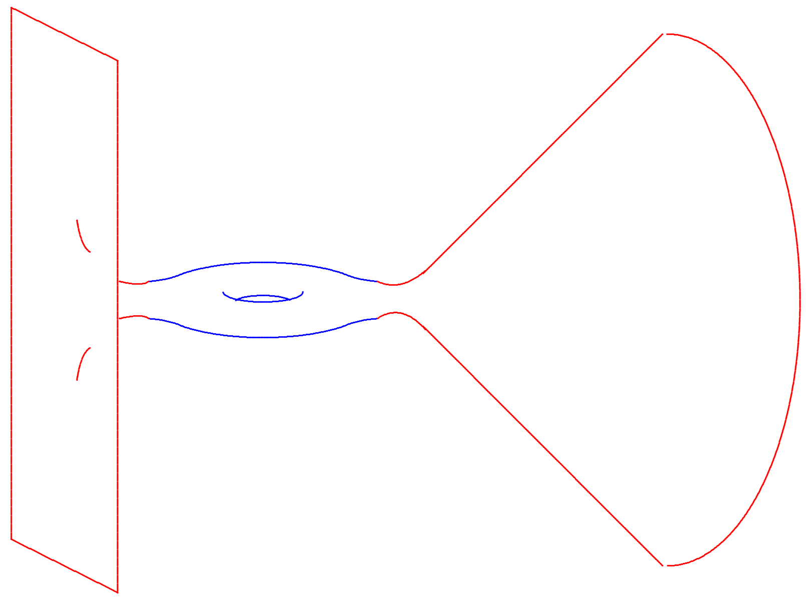

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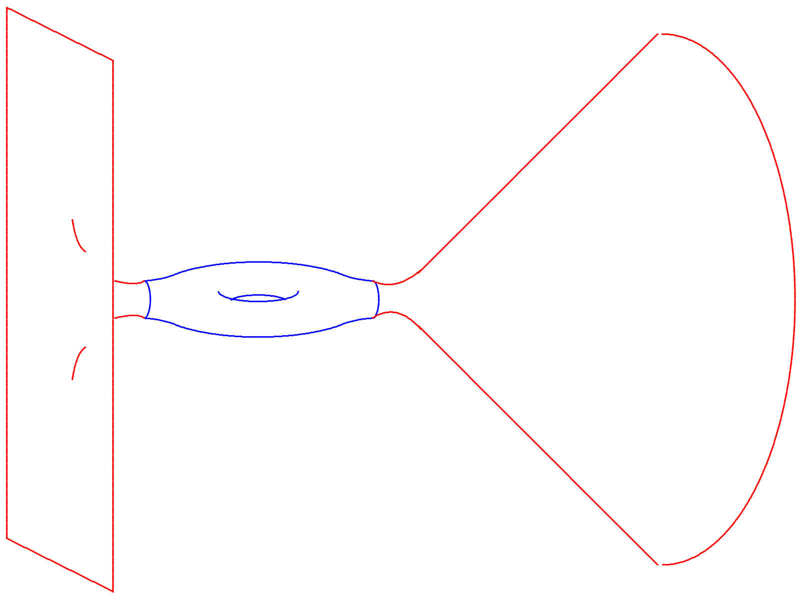

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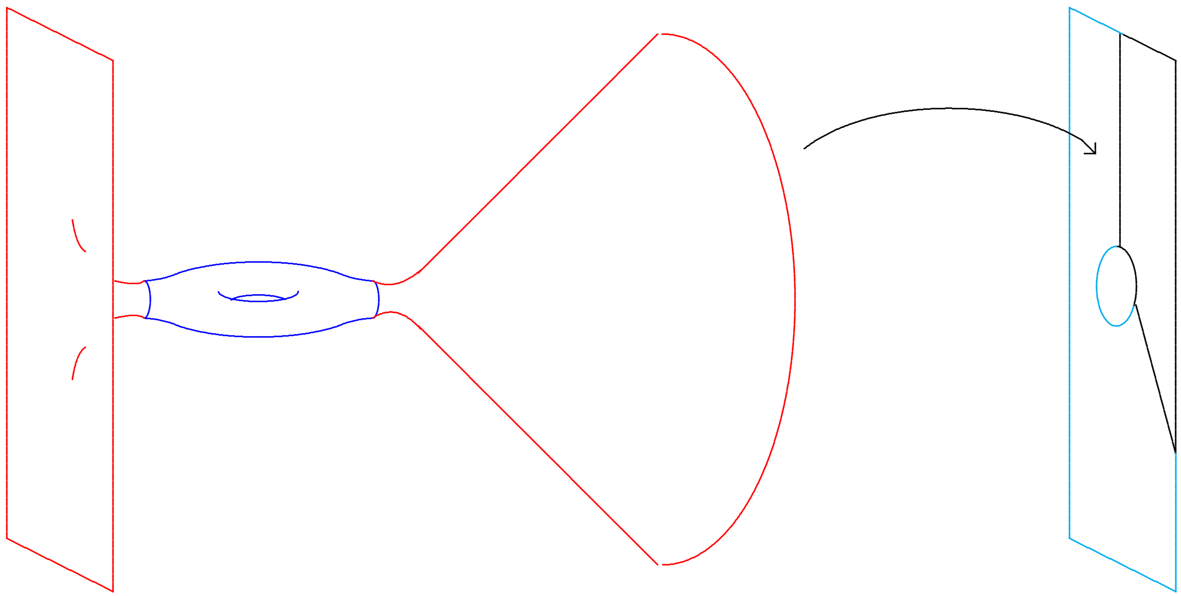

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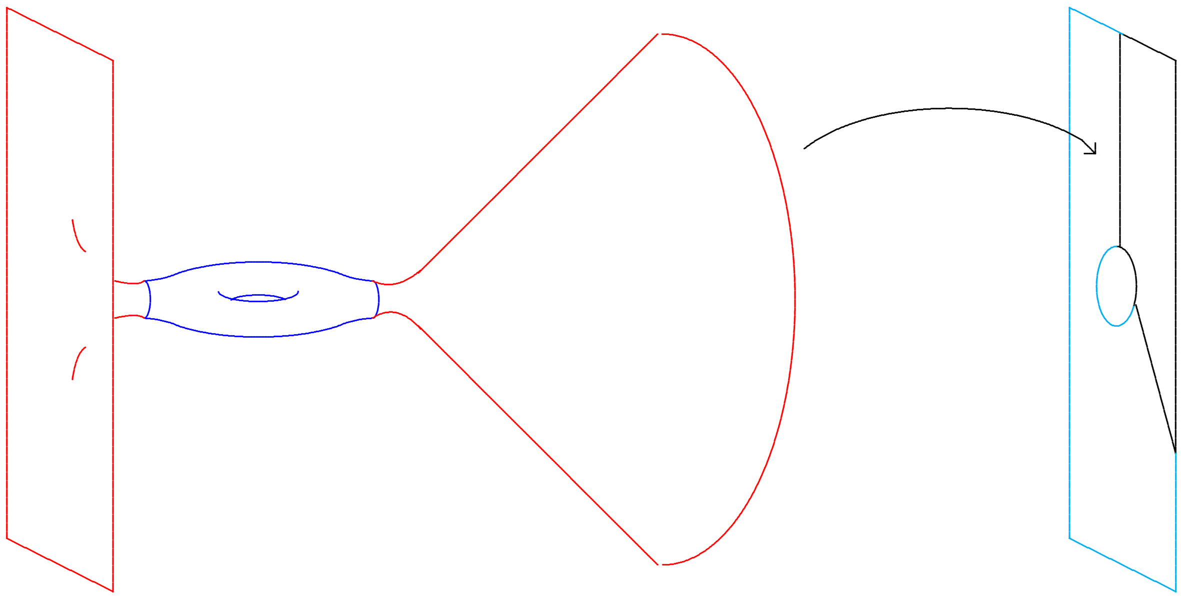

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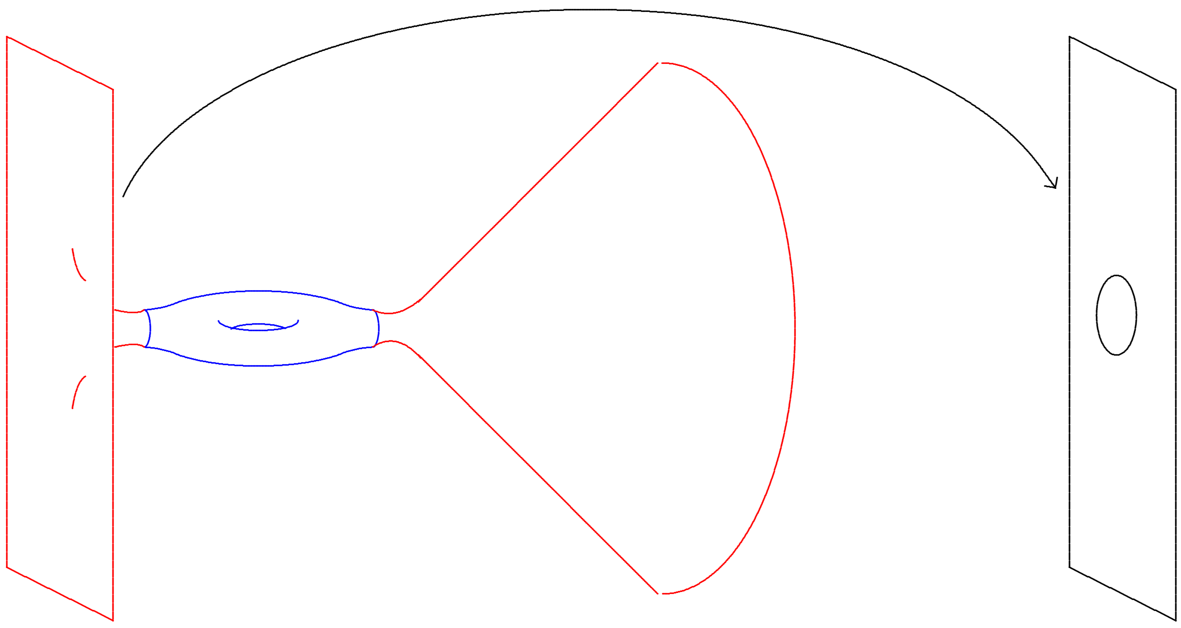

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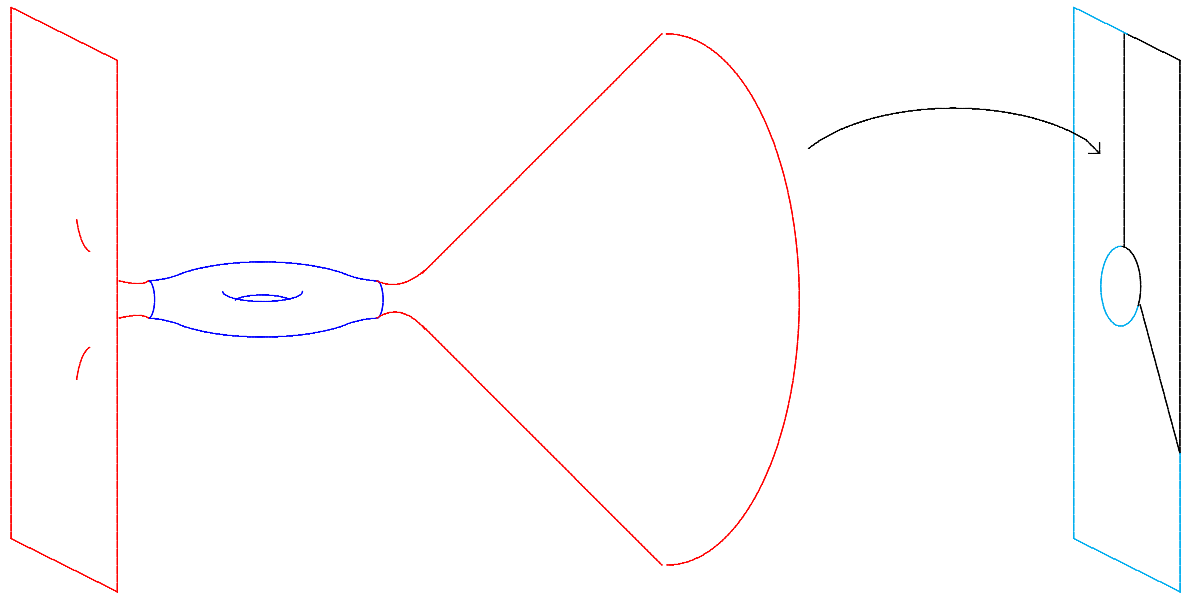

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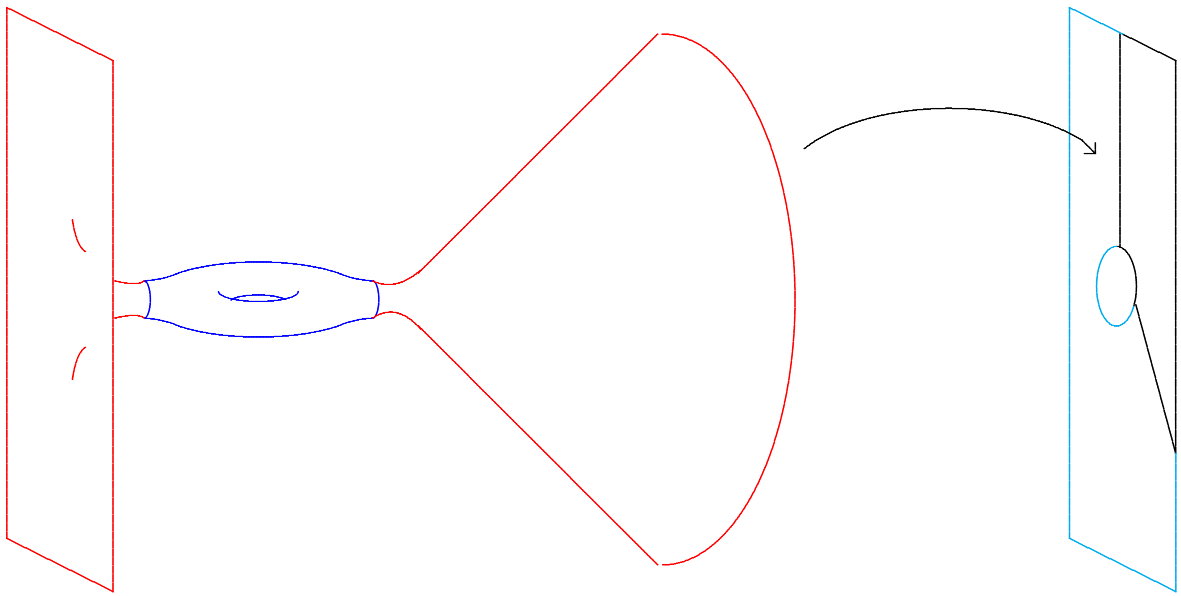

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Why consider ALE spaces?

Key examples:

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Gravitational Instantons:

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Bubbling modes for sequences of Einstein metrics.

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Theory used to construct compact Einstein 4-manifolds.

Will discuss some examples in next lecture.

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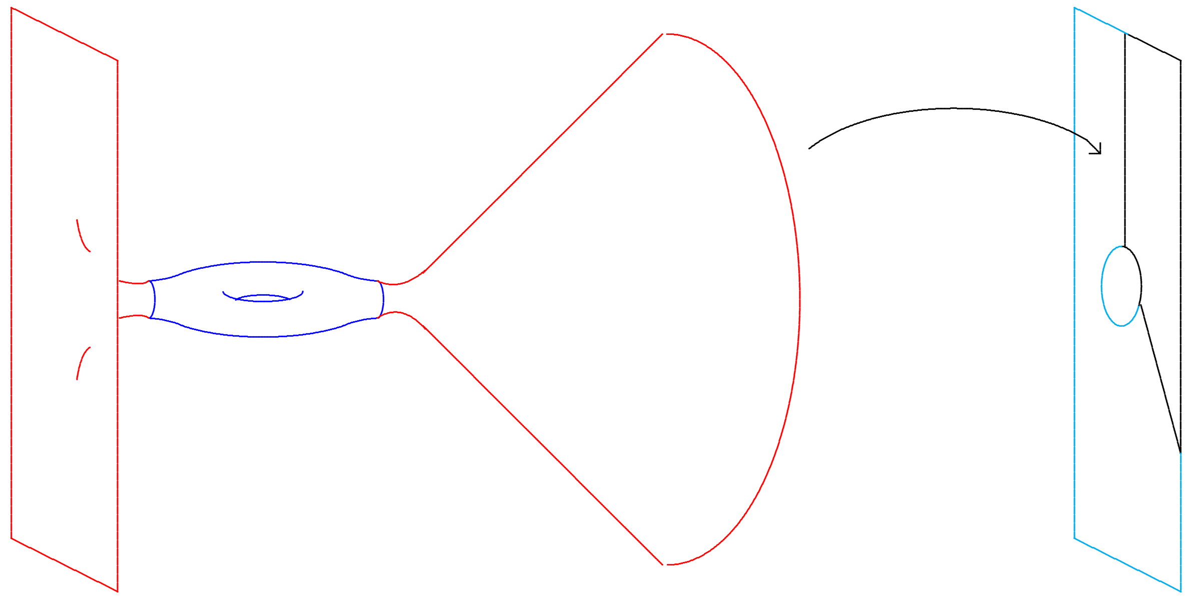

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Mass still meaningful in this context. . .

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$$m(M, g) := \qquad\qquad \left[g_{ij,i} - g_{ii,j}\right]$$

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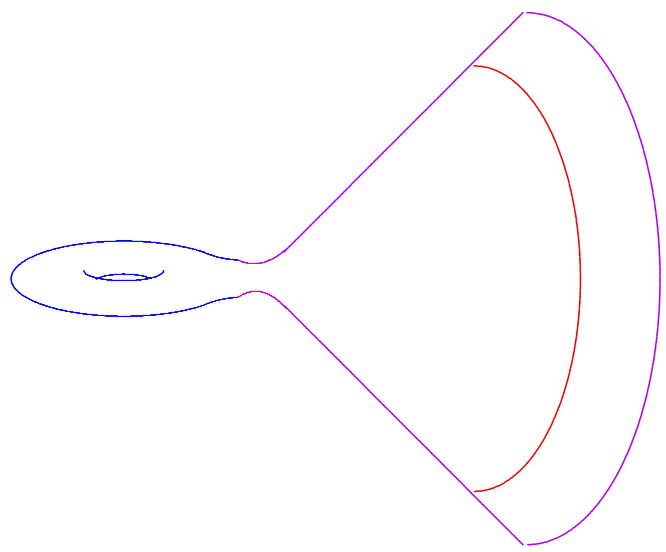

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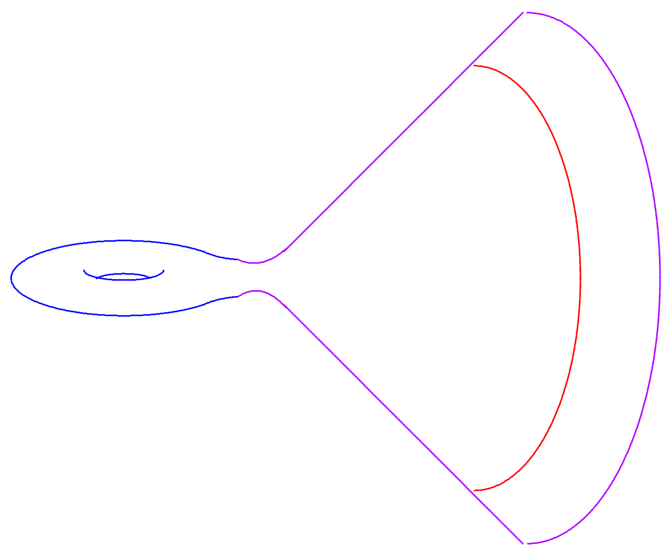

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General case in arbitrary dimension n.

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on line bundles $L \to \mathbb{CP}_1$ of Chern-class ≤ -3.

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Lemma.

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Lemma. *Any ALE Kähler manifold*

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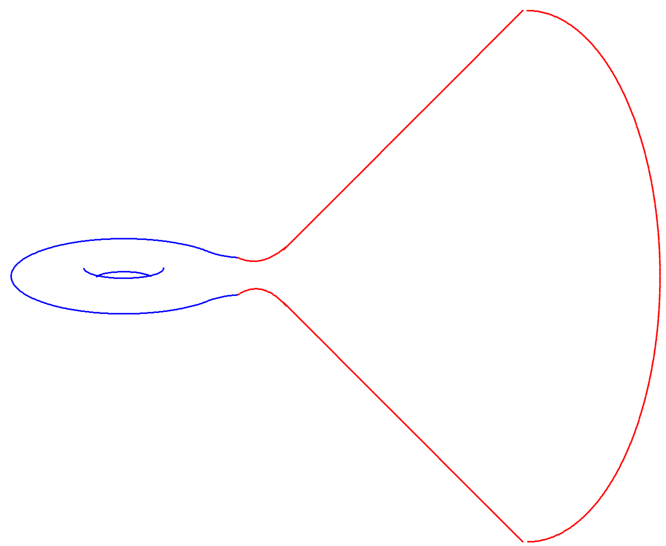

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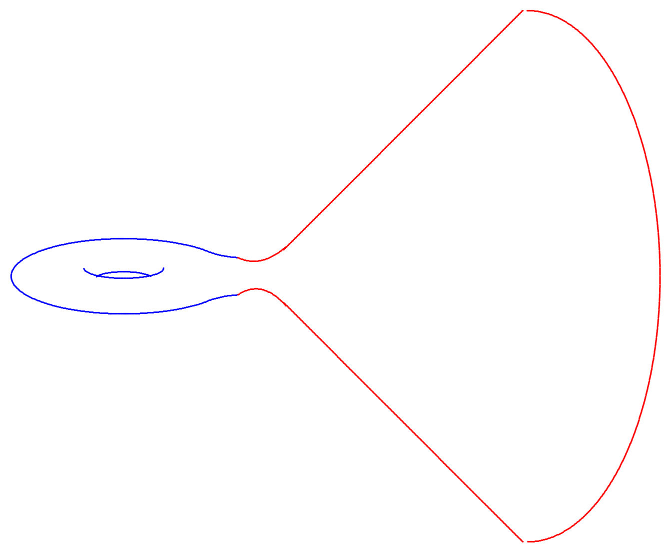

$$n = 2m \geq 4$$

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Upshot:

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Mass of an ALE Kähler manifold is unambiguous.

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Lemma. *Any ALE Kähler manifold has only one end.*

Upshot:

Mass of an ALE Kähler manifold is unambiguous.

Does not depend on the choice of an end!

We begin with the scalar-flat Kähler case.

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Theorem A.

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Theorem A. *The mass*

We begin with the scalar-flat Kähler case.

Theorem A. *The mass of an ALE*

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

The explicit formula reproduces the mass in cases where it previously had been laboriously computed from the definition.

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Non-minimal resolutions typically admit families of such metrics for which the mass can be continuously deformed from negative to positive.

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- $\langle\ ,\ \rangle$ *is pairing between* $H^2_c(M)$ *and* $H^{2m-2}(M)$.

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$$\frac{4\pi^m(2m-1)}{(m-1)!}\mathfrak{m}(M,g) = -\frac{4\pi}{(m-1)!}\langle \clubsuit(c_1), [\omega]^{m-1}\rangle + \int_M s_g d\mu_g$$

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So the mass is a "boundary correction" to the topological formula for the total scalar curvature.

Theorem C. *Any ALE Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

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Corollary. *Any ALE scalar-flat Kähler manifold (M, g, J) of complex dimension m has mass given by*

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So **Theorem A** is an immediate consequence!

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Proof actually shows something stronger!

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canonical $\leftrightsquigarrow$ holomorphic section of $K = \Lambda^{m,0}$

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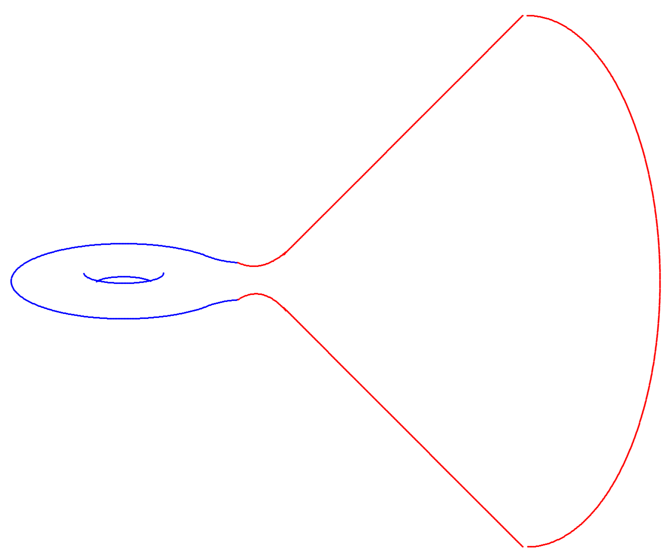

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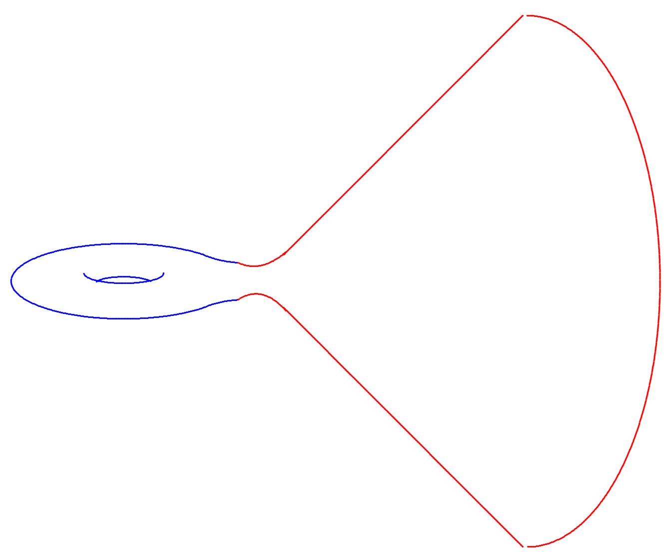

End, Part I