Bach-Flat 4-Manifolds,

Quasi-Fuchsian Groups, &

Almost-Kähler Geometry

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Stony Brook University

Special Metrics and Gauge Theory
ICMAT, December 10, 2018

Some of results discussed are from joint work.

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Collaborators:

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Collaborators:

Xiuxiong Chen, Brian Weber, Chris Bishop.

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$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

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Proposition. *Assume* $n \geq 4$. *Then* (M^n, g) *locally conformally flat* $\Longleftrightarrow W \equiv 0$.

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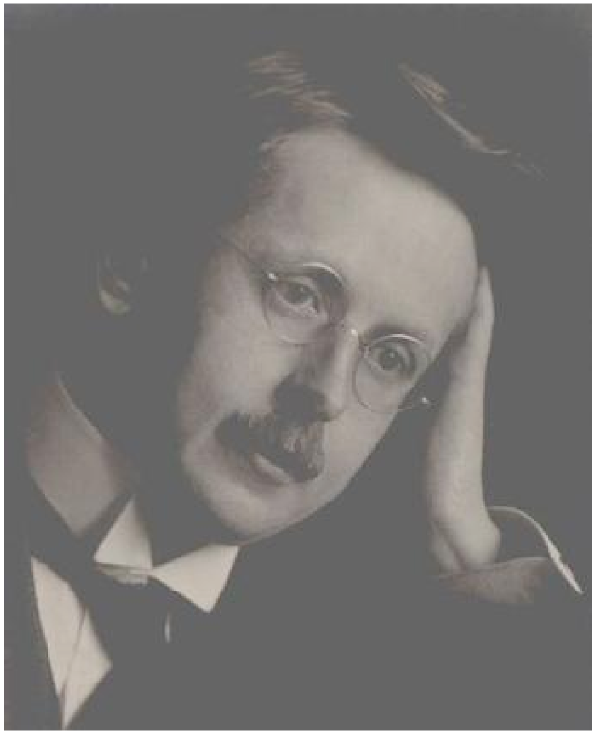

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- What is the moduli space of solutions?

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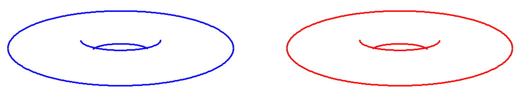

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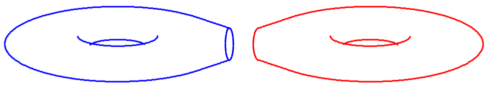

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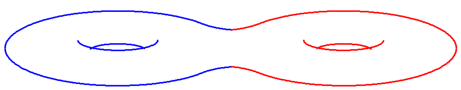

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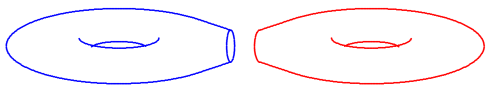

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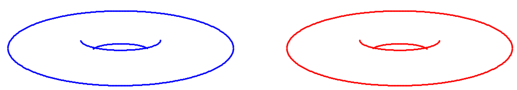

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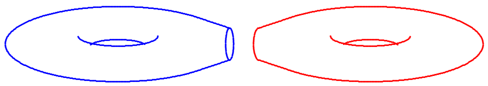

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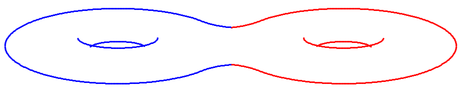

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Seiberg-Witten & Hitchin-Thorpe: Only candidates.

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Existence: conformally Kähler Einstein metrics.

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Λ^+ self-dual 2-forms.
Λ^- anti-self-dual 2-forms.

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4-dimensional signature formula

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

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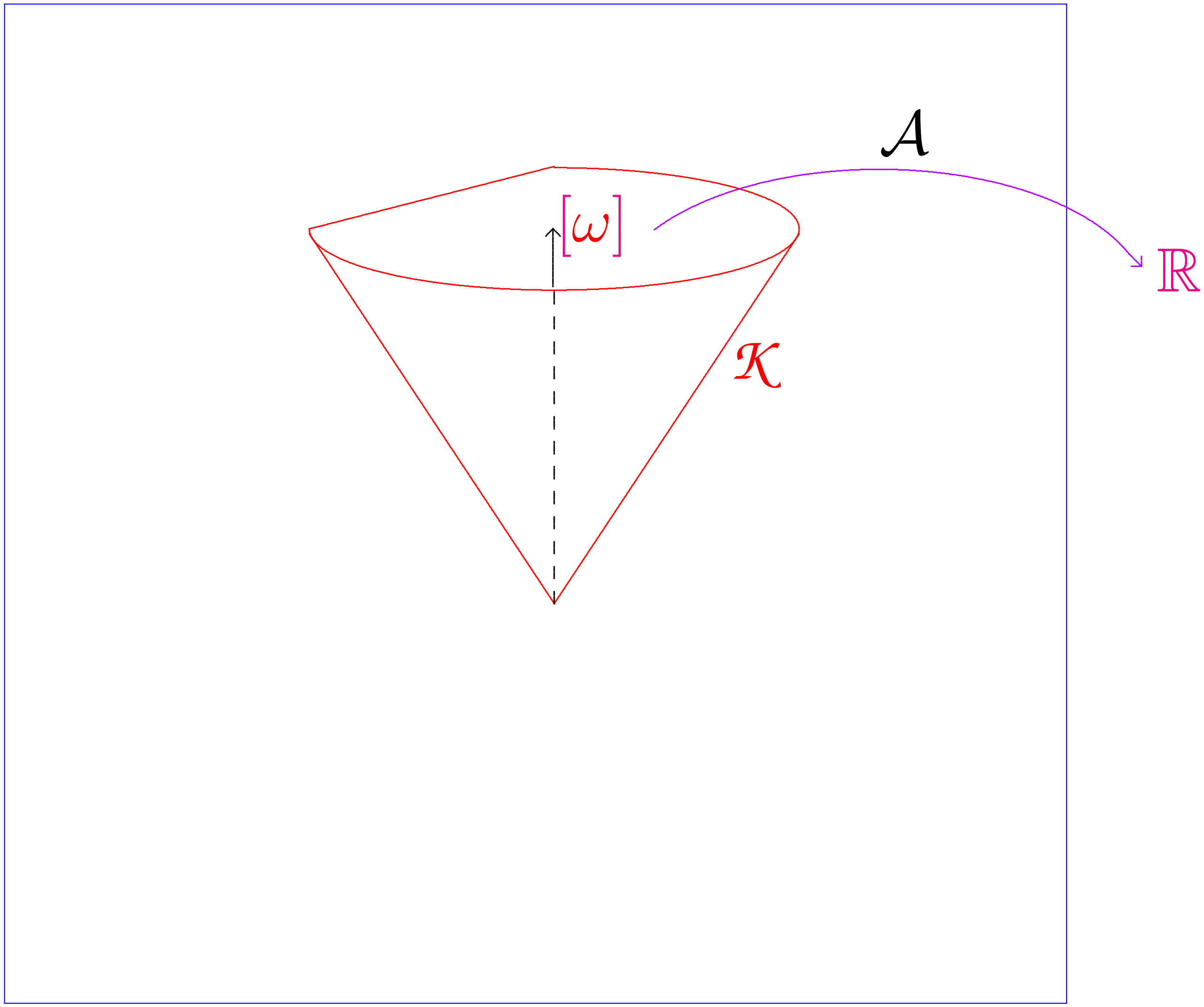

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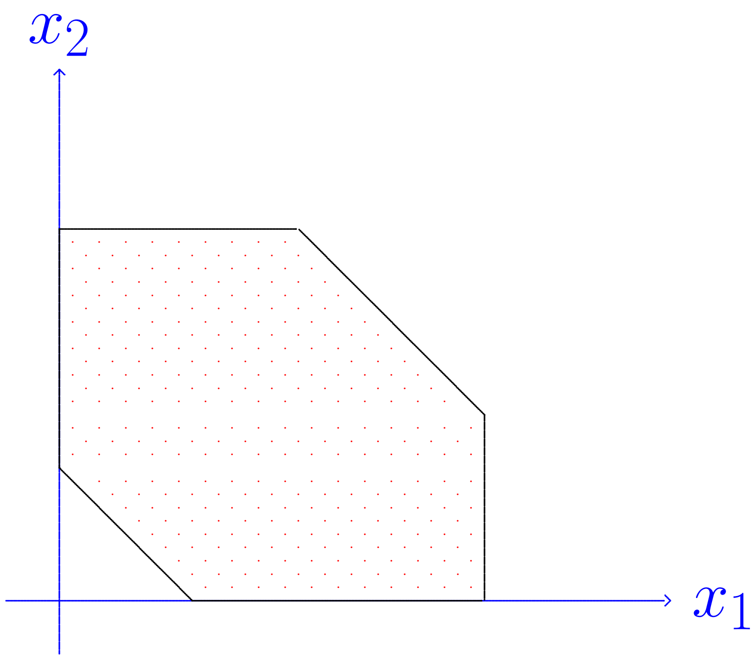

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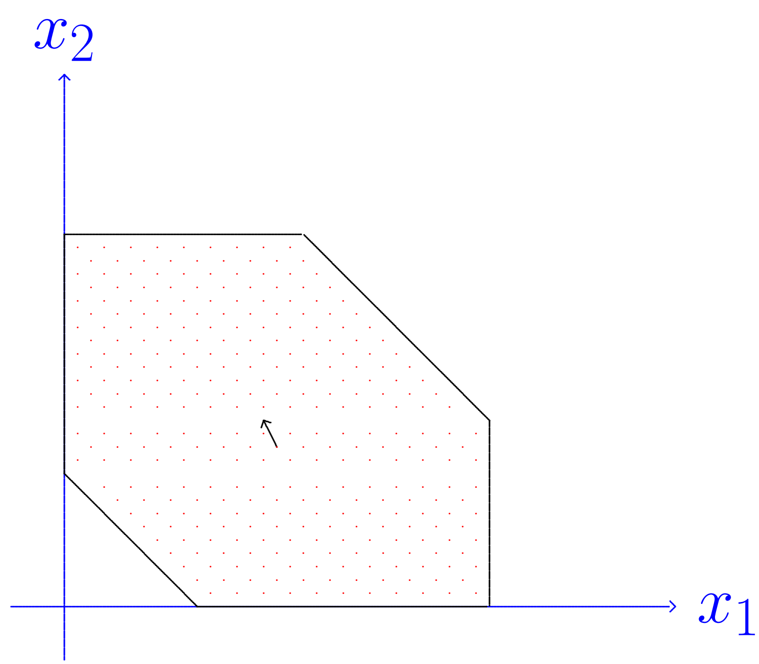
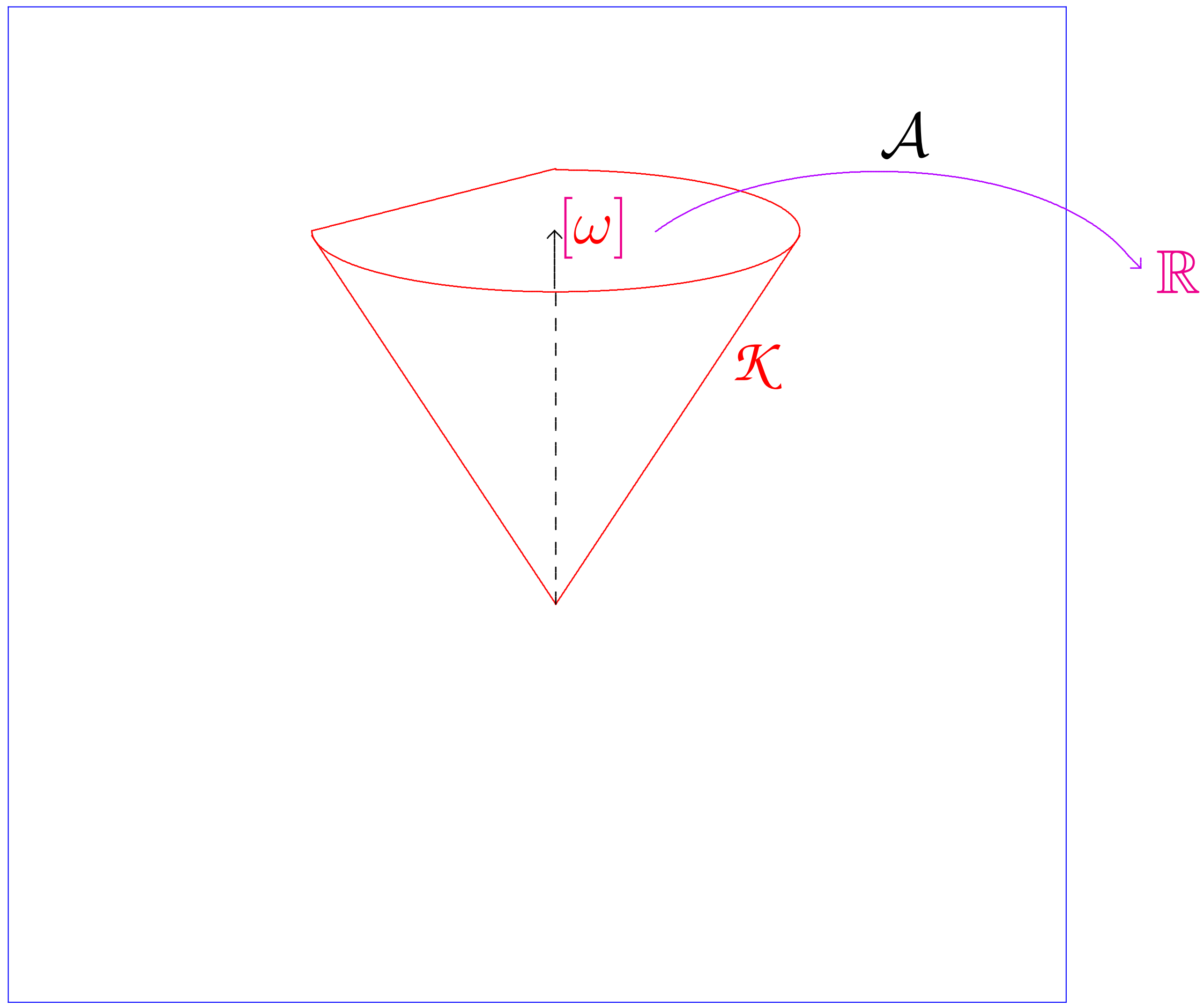


$$\mathcal{A}([\omega]) = \frac{|\partial P|^2}{2}\left(\frac{1}{|P|} + \vec{\mathfrak{D}} \cdot \Pi^{-1}\vec{\mathfrak{D}}\right)$$



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The fact that 4-dimensional Einstein metrics are Bach-flat can sometimes be used to construct them:

Theorem (Chen-L-Weber '08). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic form ω. Then M admits an Einstein metric g with $\lambda > 0$*

$$\Longleftrightarrow \quad M \approx \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \quad or & \\ S^2 \times S^2 & \end{cases}$$

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These are the diffeotypes of the Del Pezzo surfaces.

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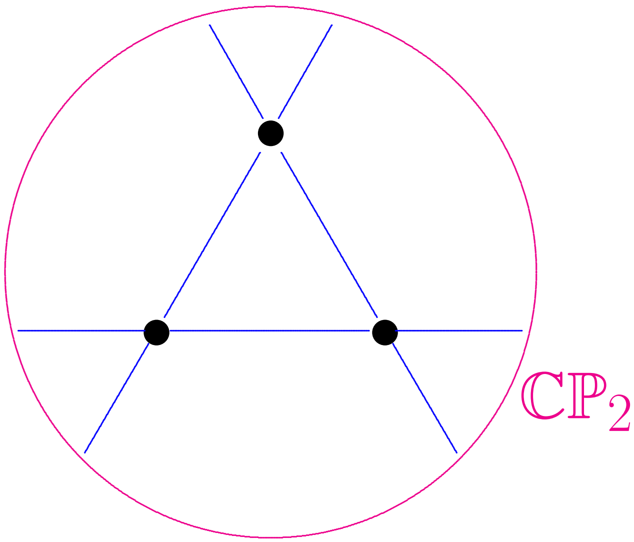

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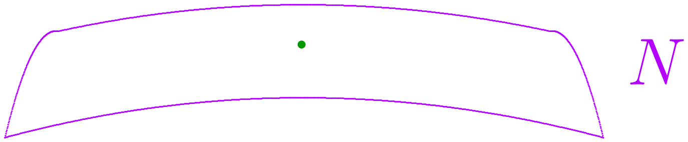

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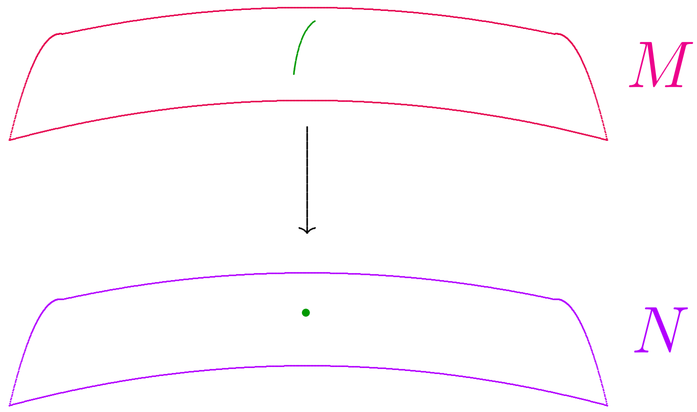

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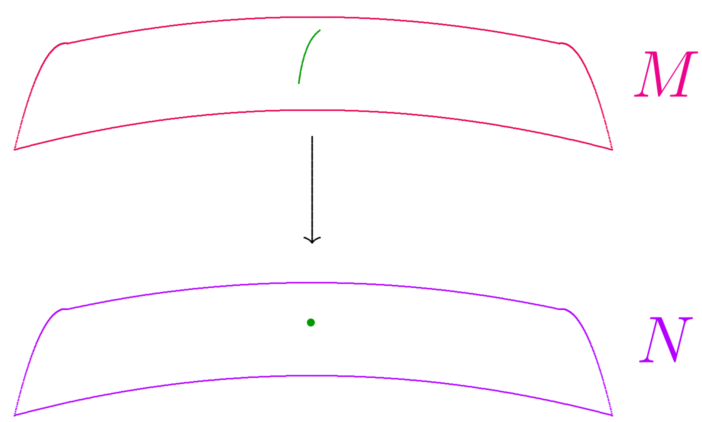

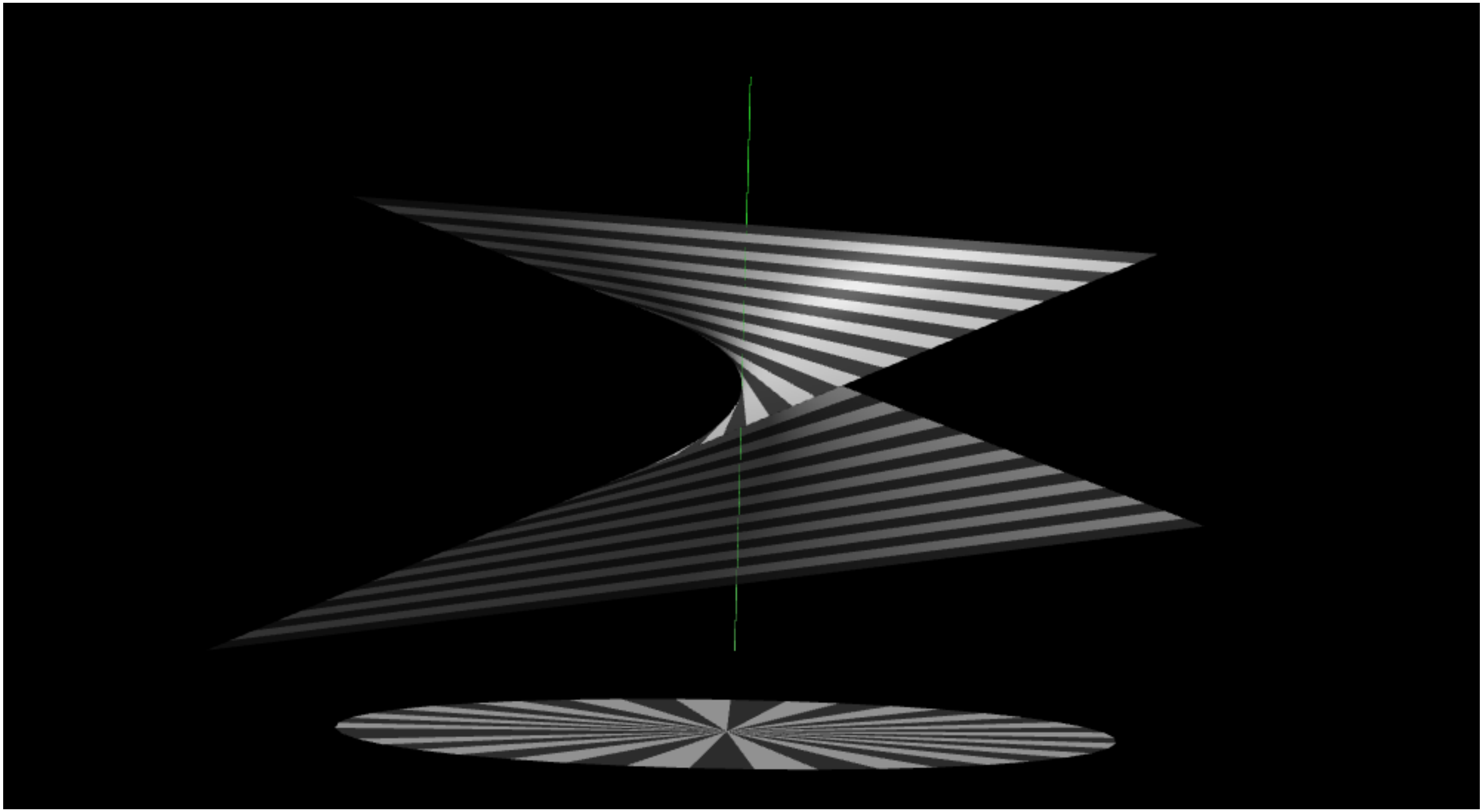

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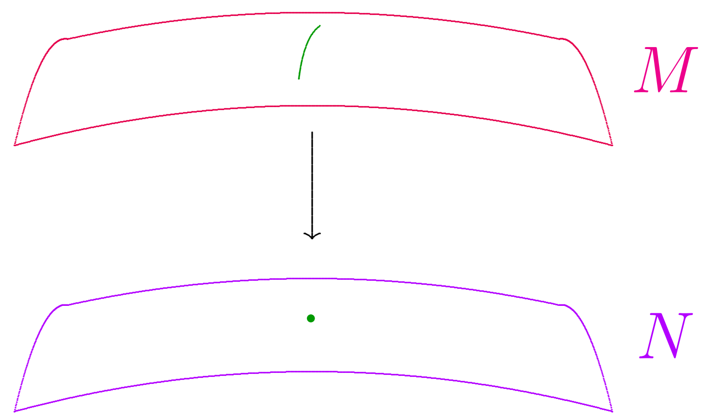

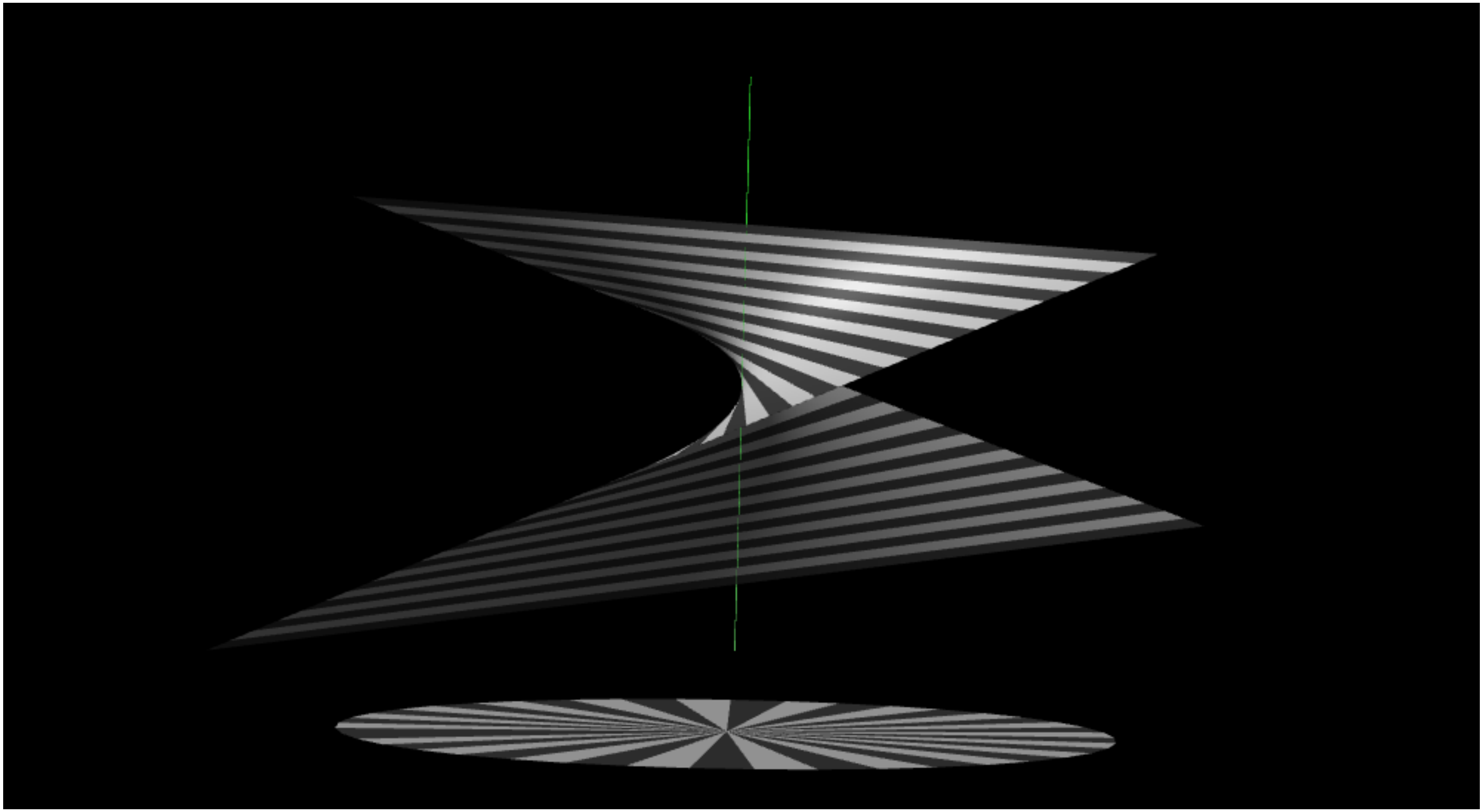

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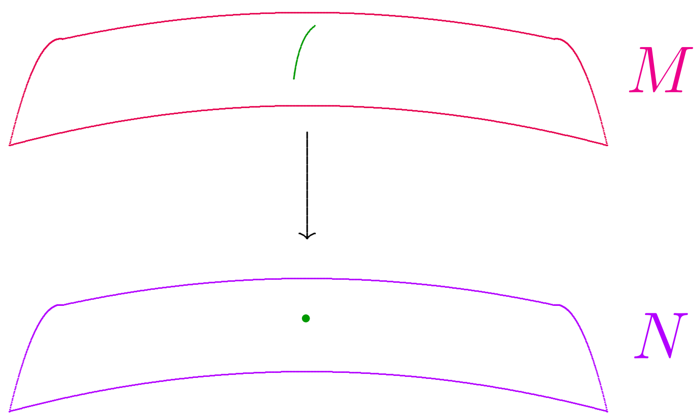

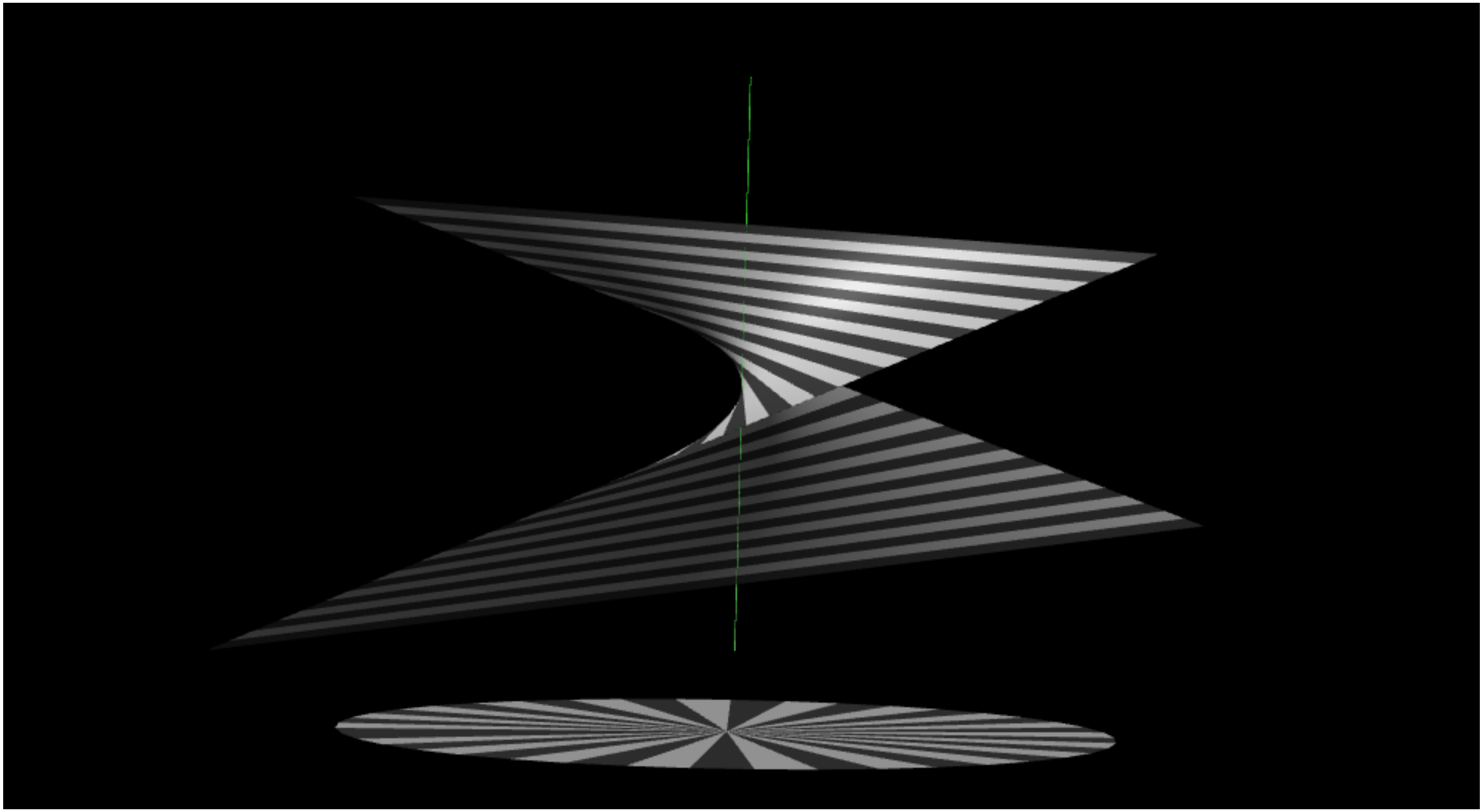

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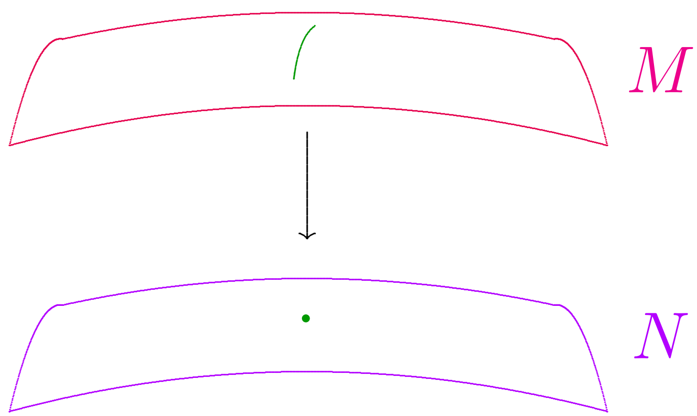

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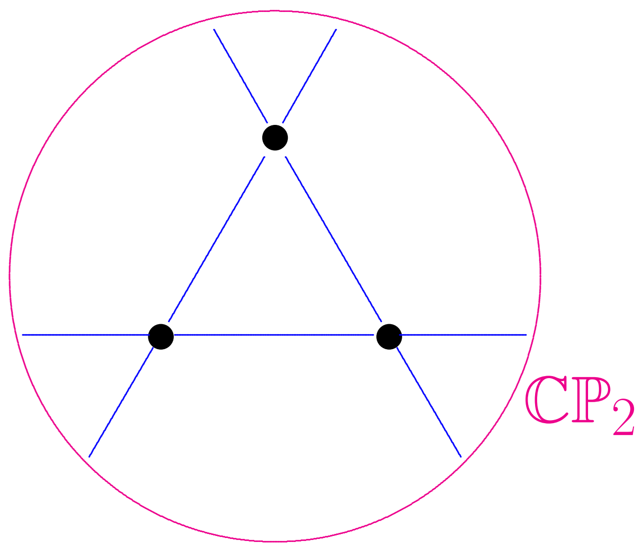

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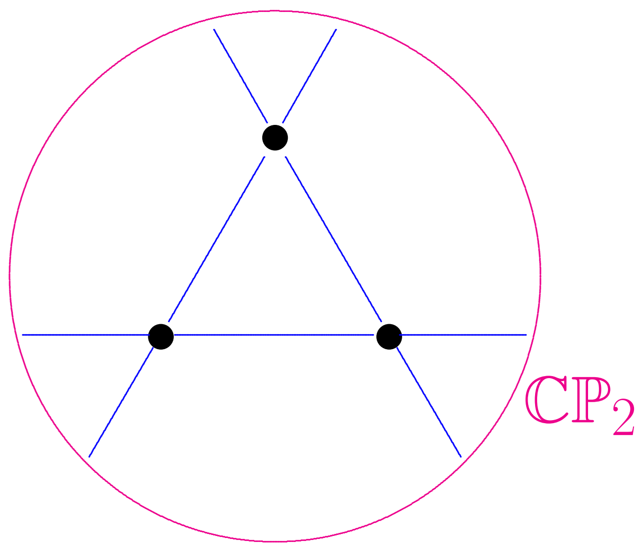

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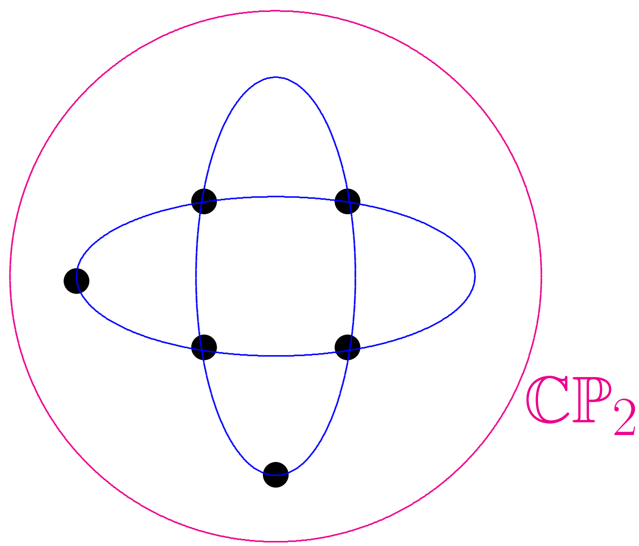

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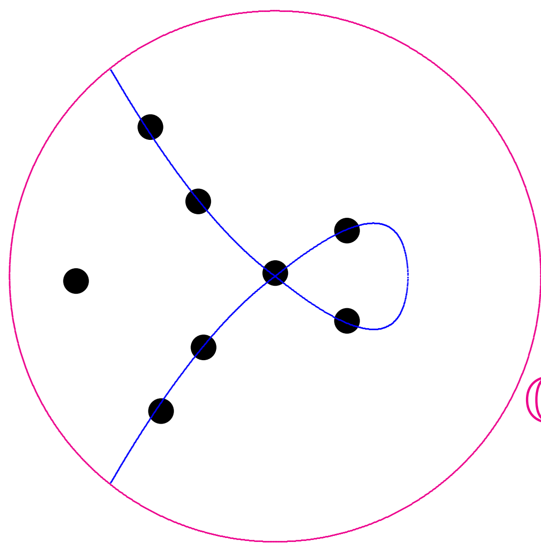

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Existence: Tian, Odaka-Spotti-Sun, Chen-L-Weber…

Uniqueness: Bando-Mabuchi, L '12…

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Einstein metrics satisfying this: connected.

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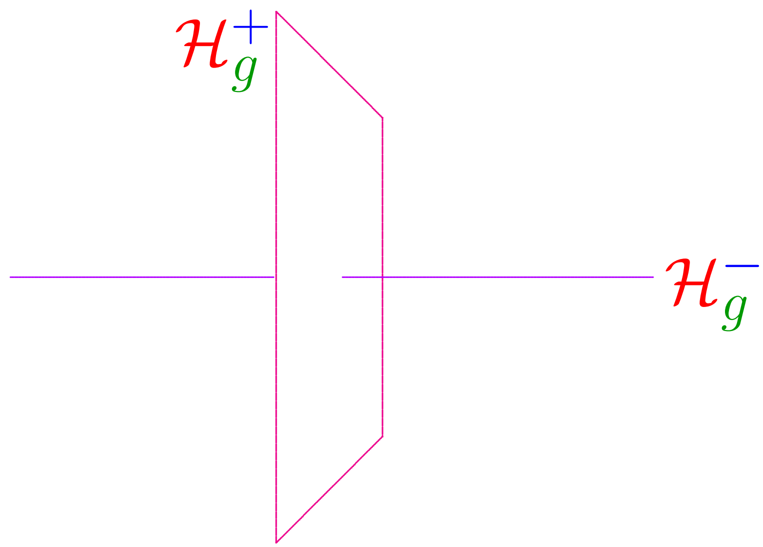
$$b_\pm(M) = \dim \mathcal{H}_g^\pm.$$



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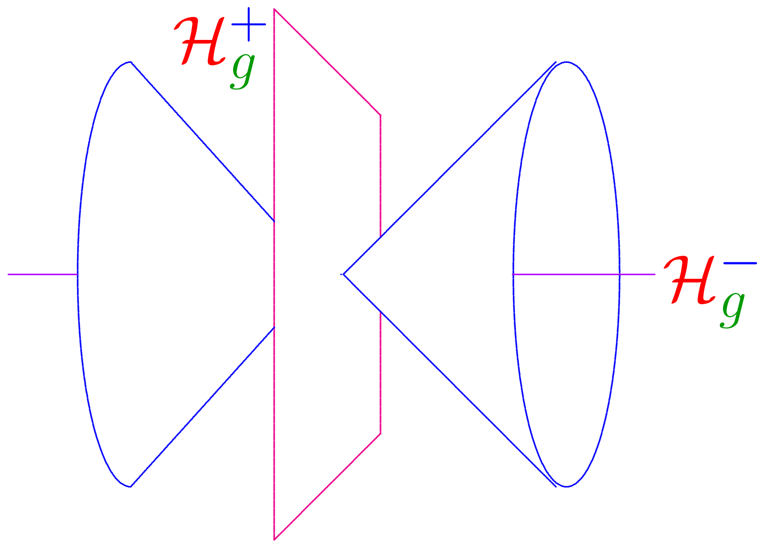

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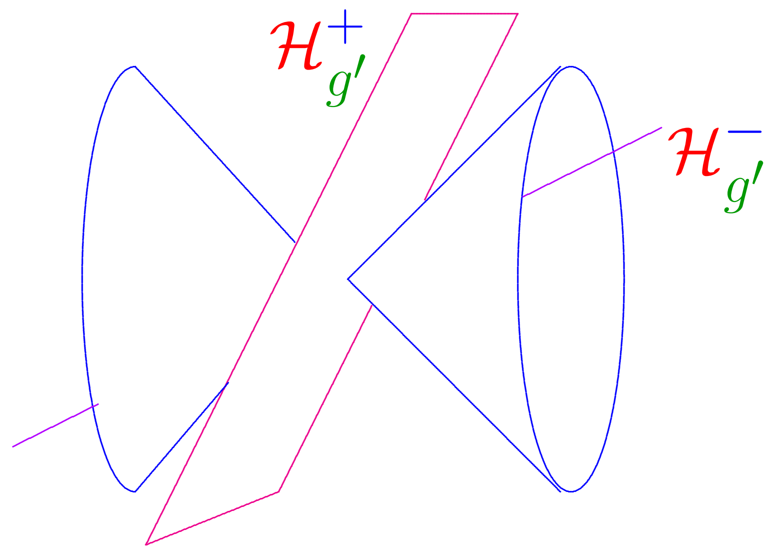

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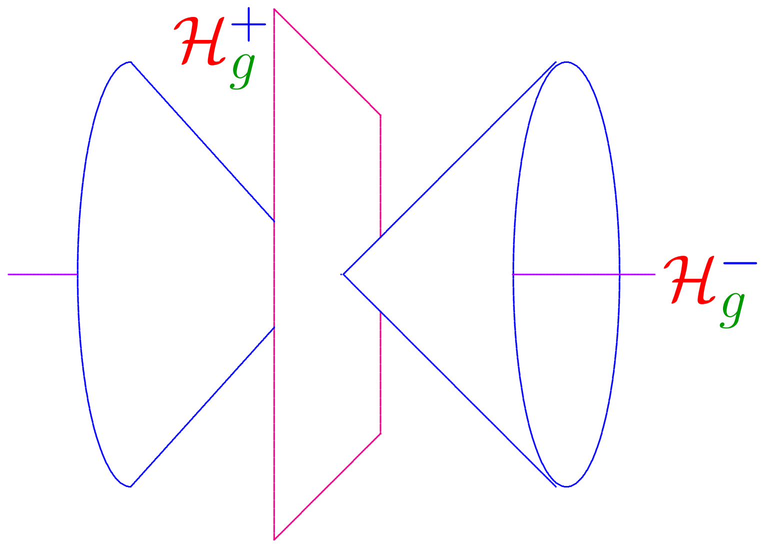

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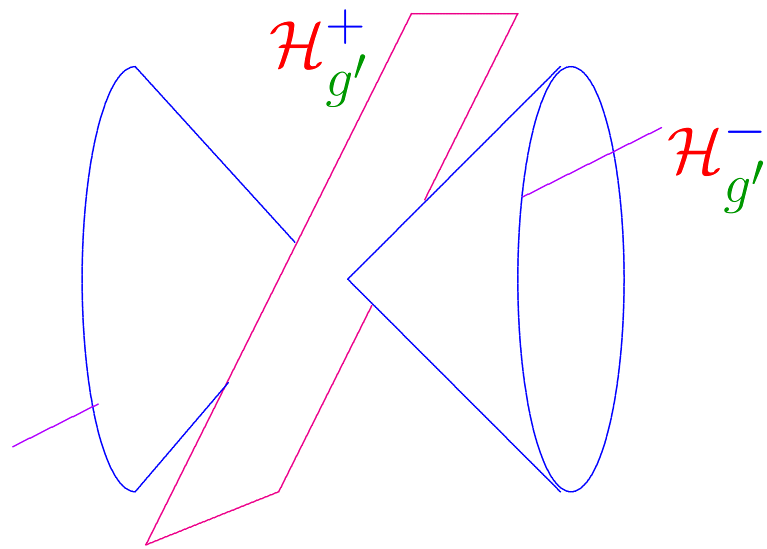

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Criterion $\Longrightarrow$ $\omega \neq 0$ everywhere.

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"Conformal classes of symplectic type"

Notice that when $b_+ = 1$, ω is unique up to scale.

Minimizers?

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Represented by metric with $s > 0$.

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Most of these have negative Yamabe constant!

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Thus, symplectic methods allow one to prove non-trivial results in 4-dimensional conformal geometry.

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What are the limitations of this method?

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Another class of Bach-flat metrics is illuminating. . .

For M^4 compact, the Weyl functional

$$\mathscr{W}([g]) = \int_M \left(|W_+|^2 + |W_-|^2 \right) d\mu_g$$

measures the deviation from conformal flatness, because (M^4, g) is locally conformally flat $\Longleftrightarrow$ its Weyl curvature $W = W_+ + W_-$ vanishes.

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But we've already noted that

$$12\pi^2\tau(M) = \int_M \left(|W_+|^2 - |W_-|^2\right) d\mu_g$$

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In particular, metrics with $W_+ \equiv 0$ minimize $\mathscr{W}$.

If g has $W_+ \equiv 0$, it is said to be anti-self-dual.
(ASD)

Twistor picture of anti-self-duality condition:

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Oriented $(M^4, g) \leftrightsquigarrow (Z, J)$.

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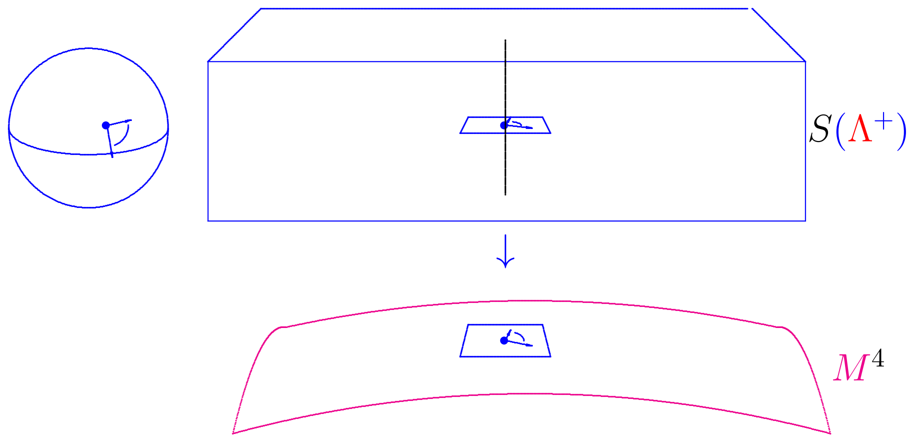

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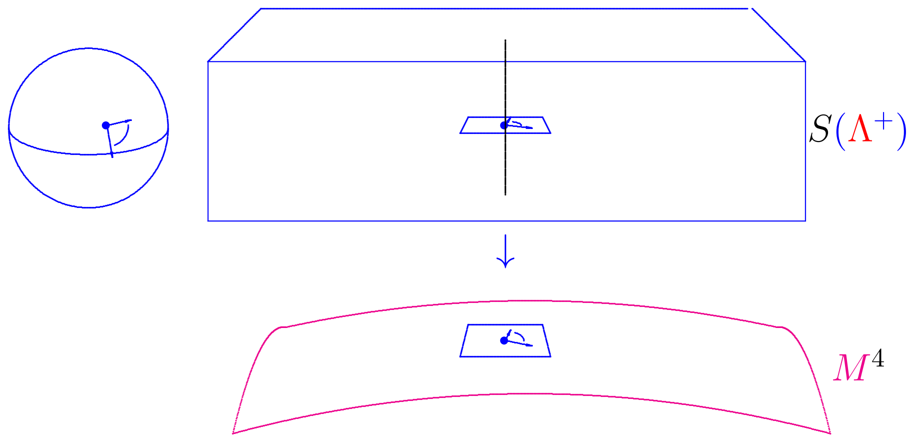

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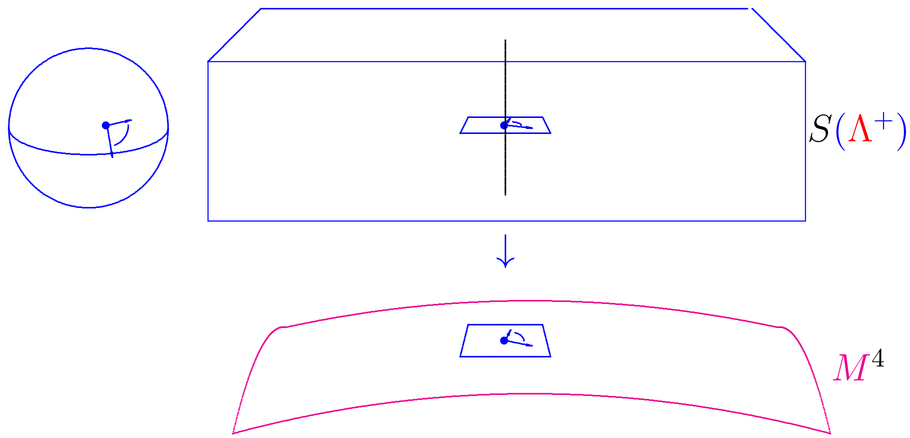

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Reconceptualizes earlier work by Penrose.

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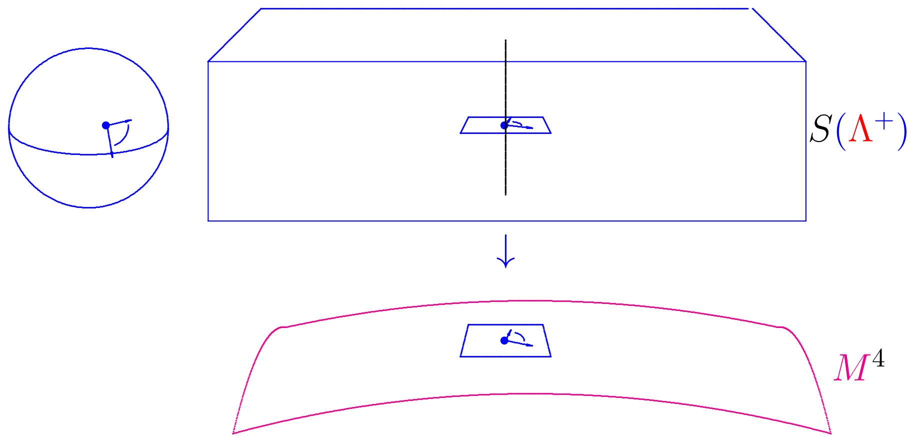

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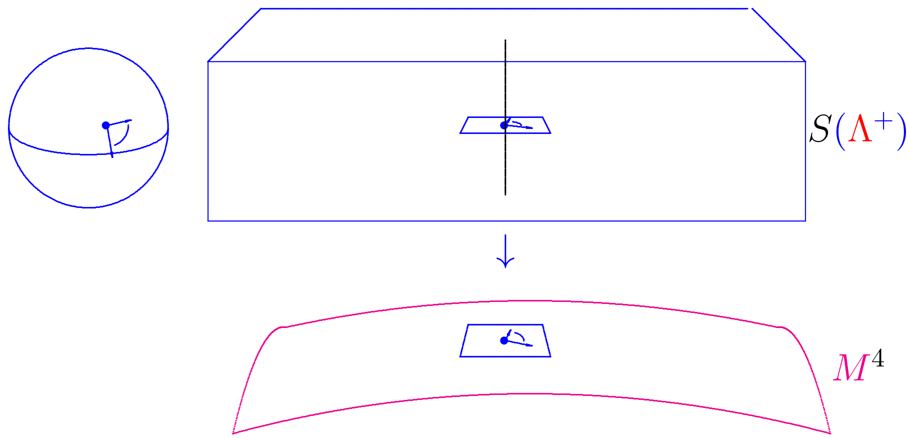

Theorem (Atiyah-Hitchin-Singer). (Z, J) *is a complex 3-manifold iff* $W_+ = 0$.

Motivates study of ASD metrics, and yields methods for constructing them.

So ASD metrics are linked to complex geometry. . .

A different link with complex geometry:

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Results proved about SFK in '90s foreshadowed many more recent results about general case.

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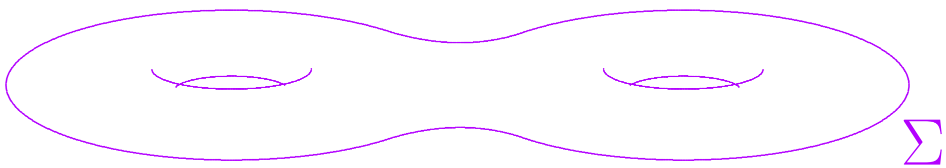
Does this say anything about general ASD metrics?

Almost-Kähler ASD metrics sweep out an open set in the ASD moduli space.

Example.



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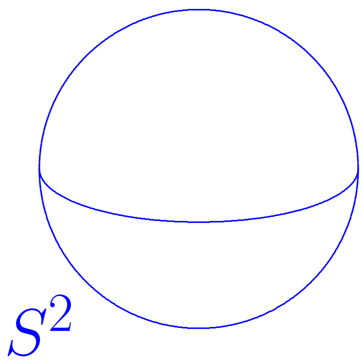
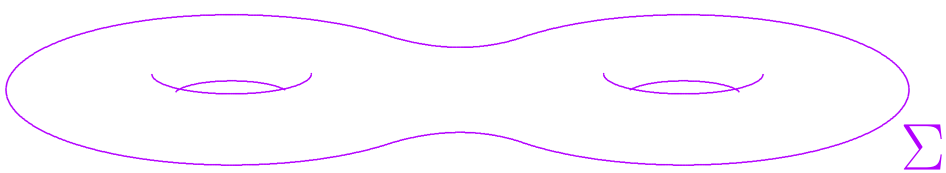
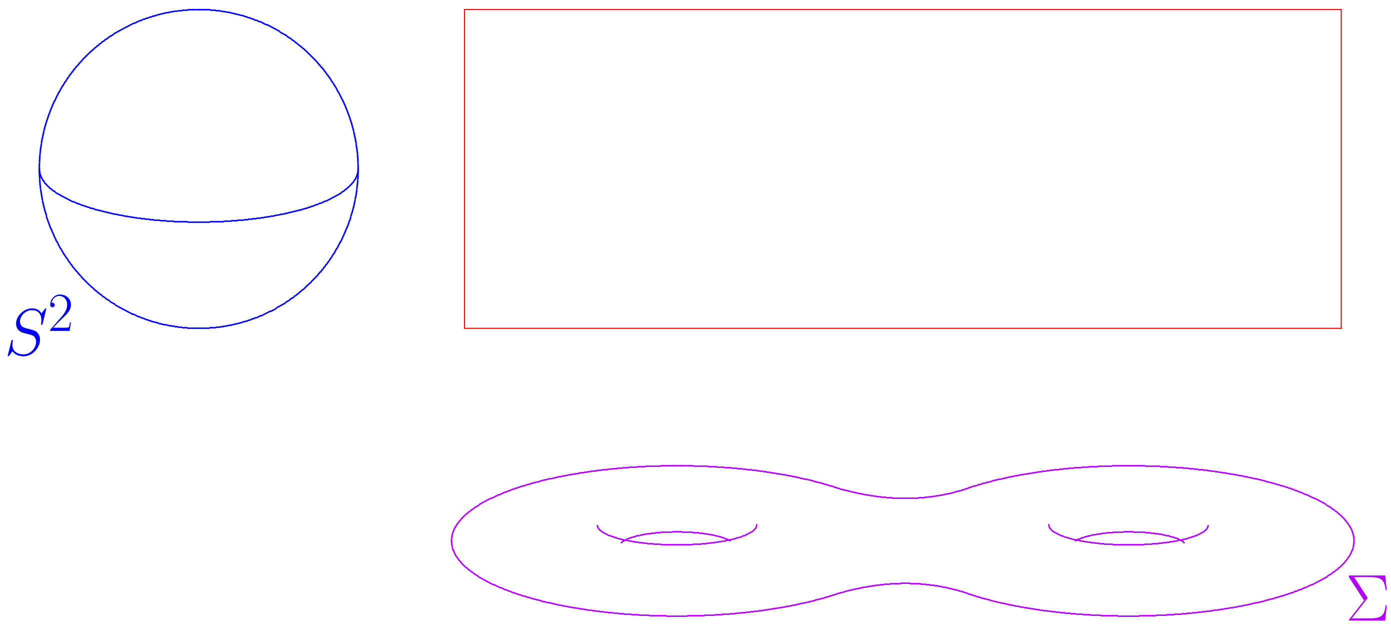




Example.

$$M = \Sigma \times S^2$$



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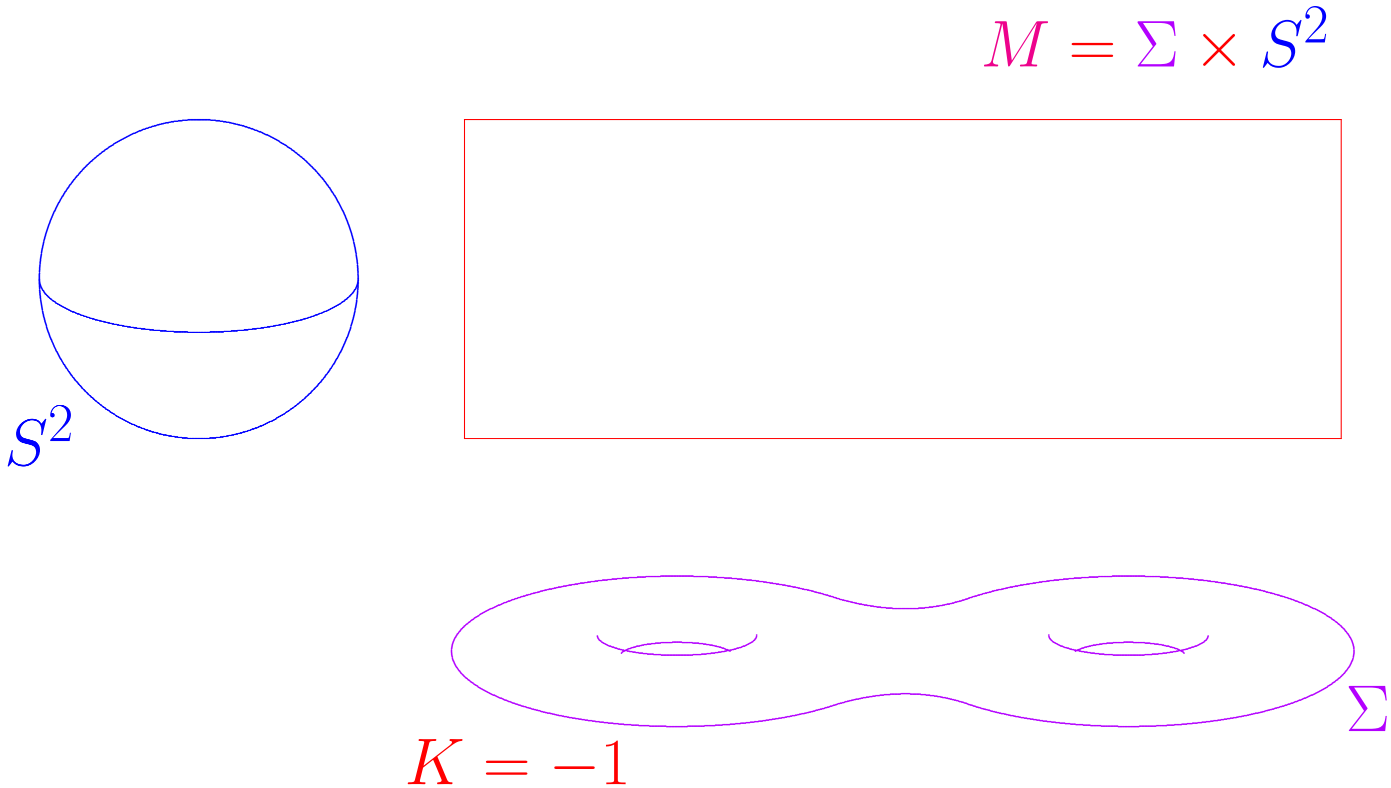

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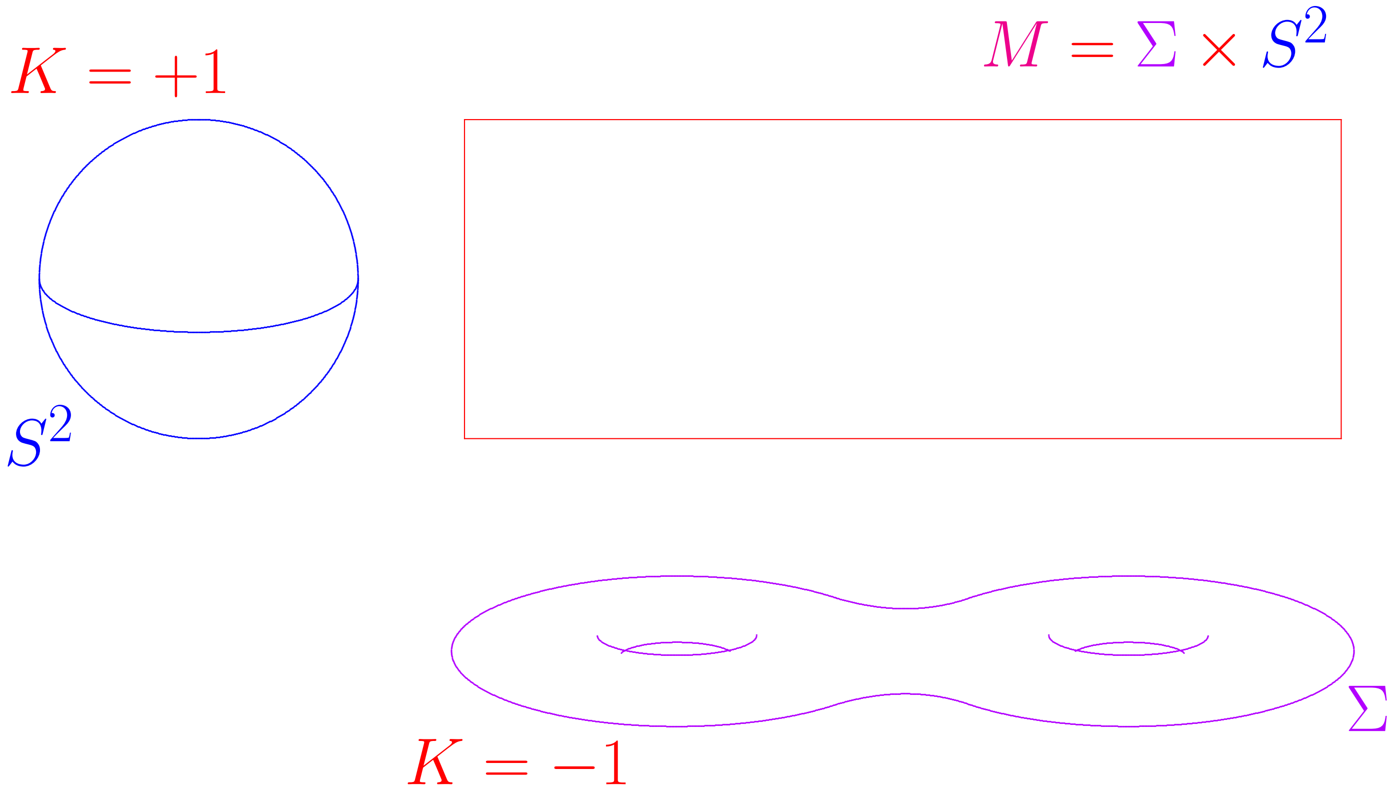

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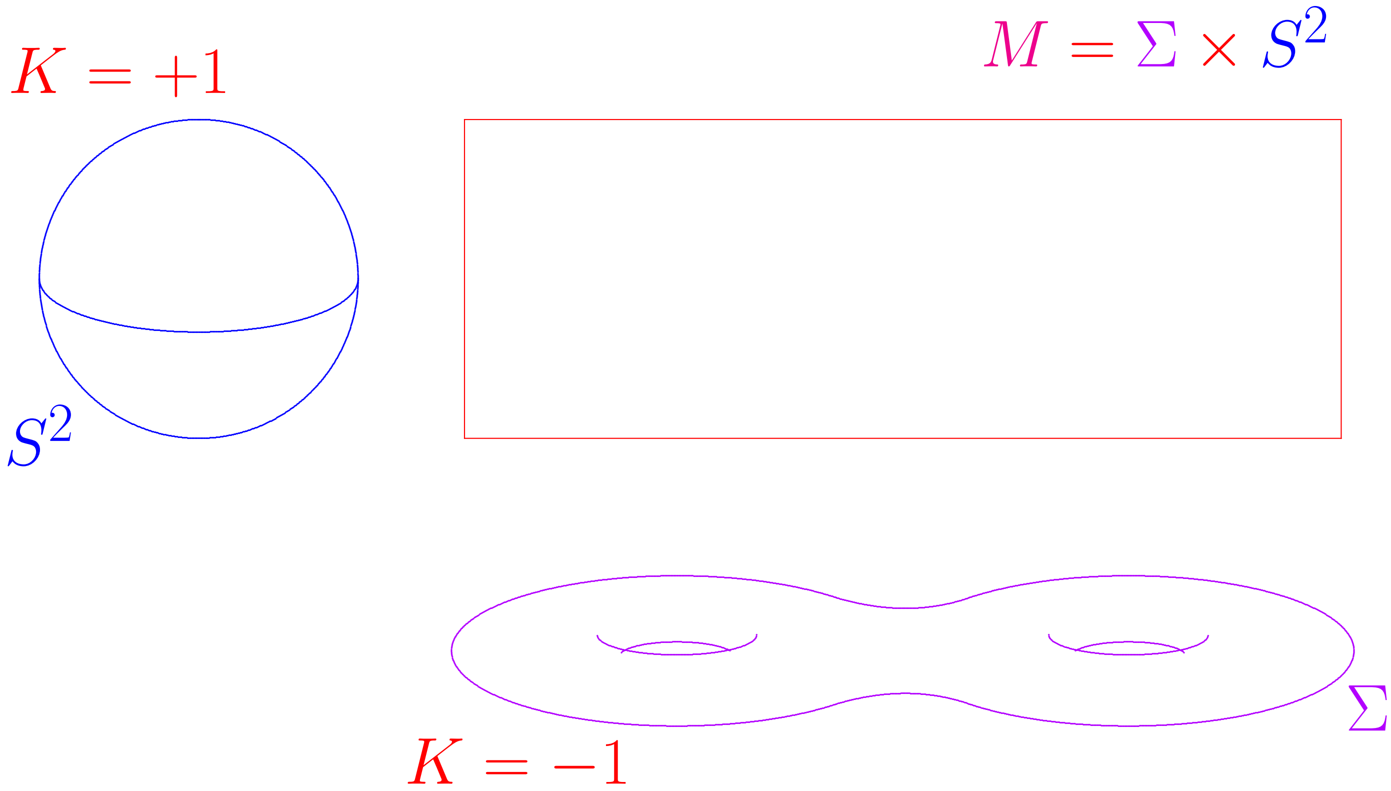

Product is scalar-flat

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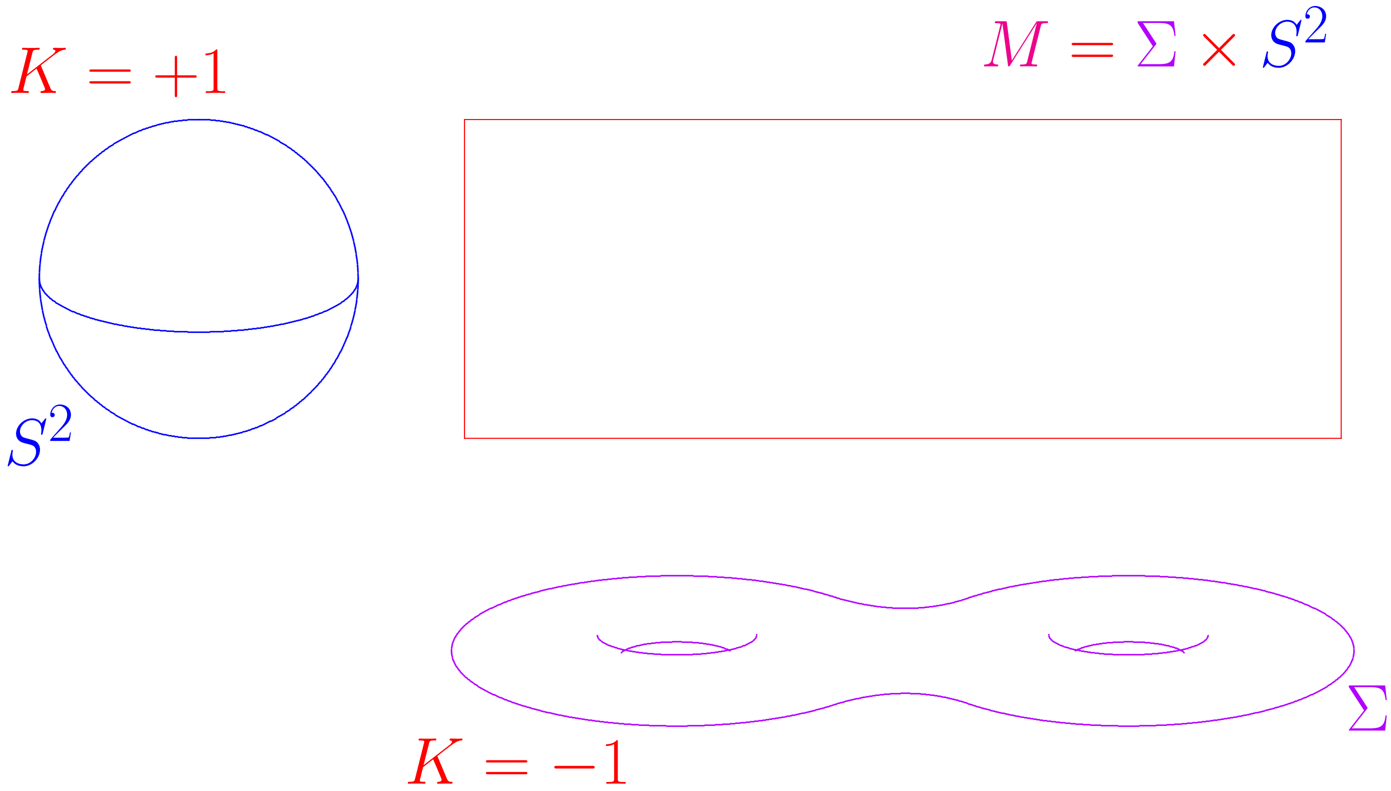

Product is scalar-flat Kähler.

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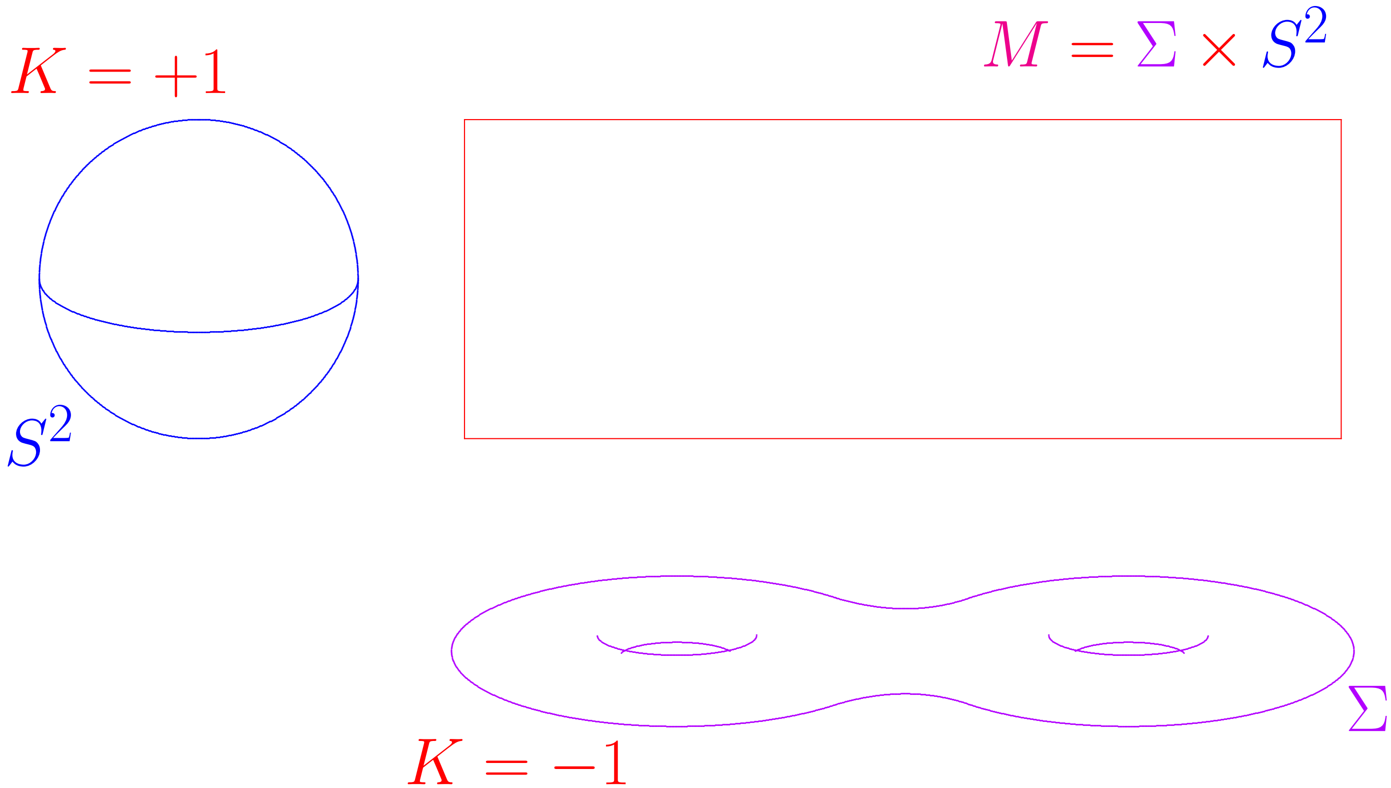

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For both orientations!

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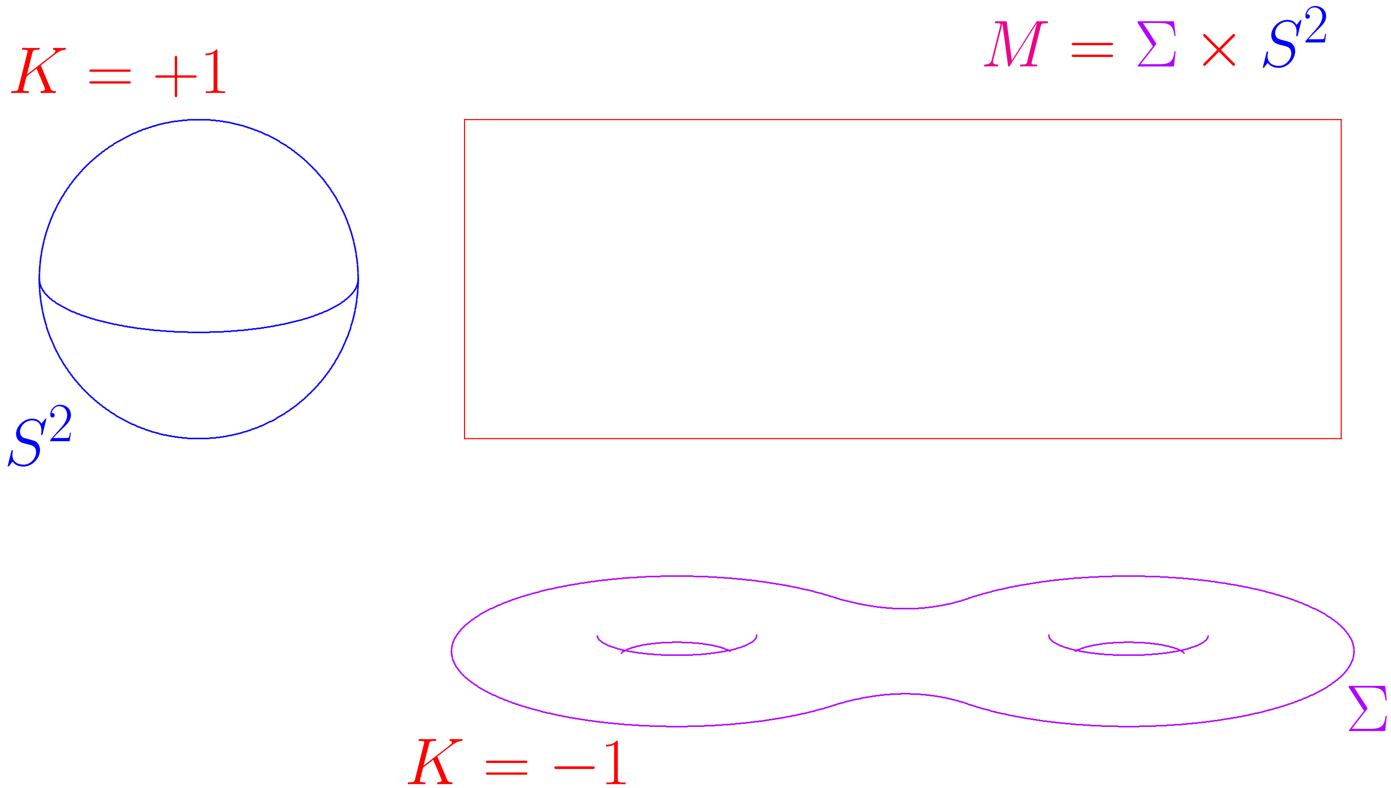

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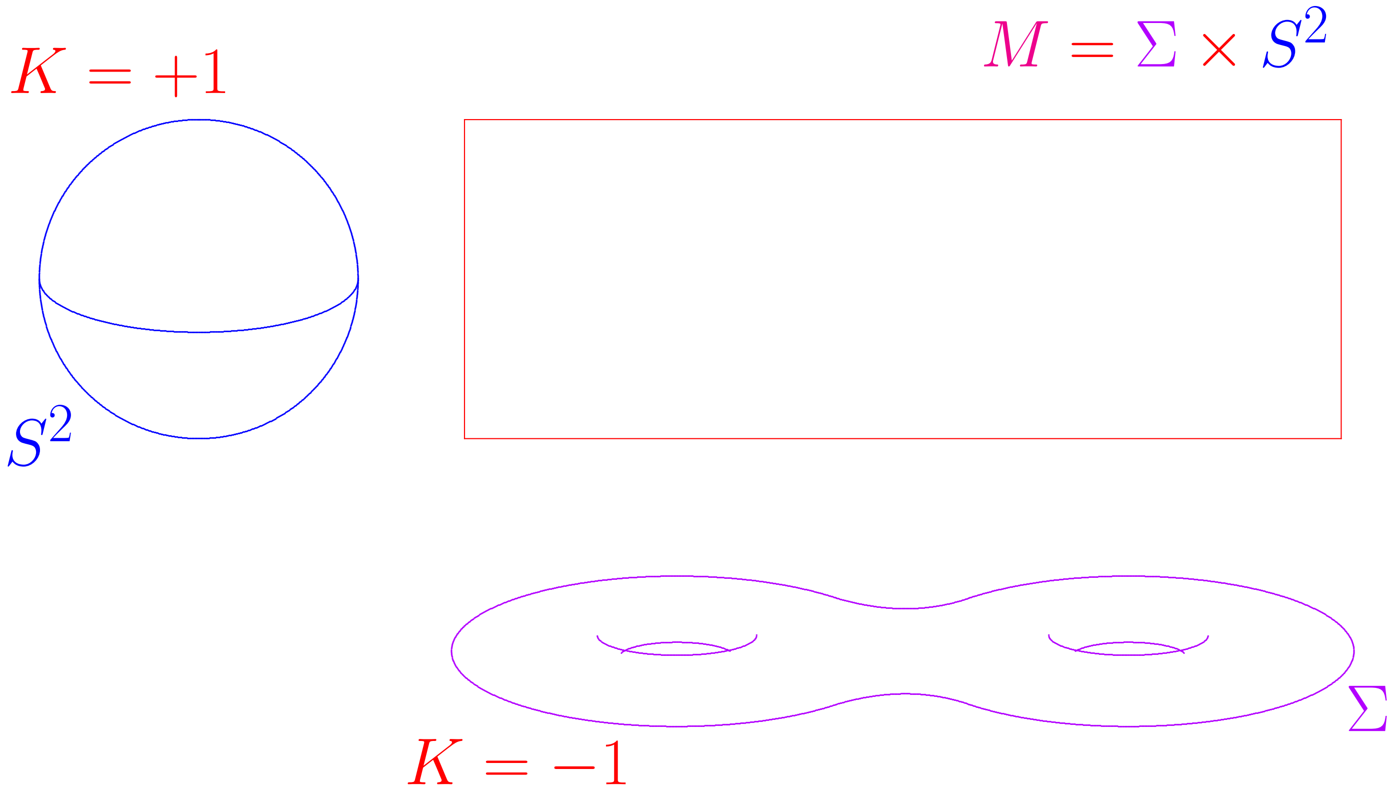

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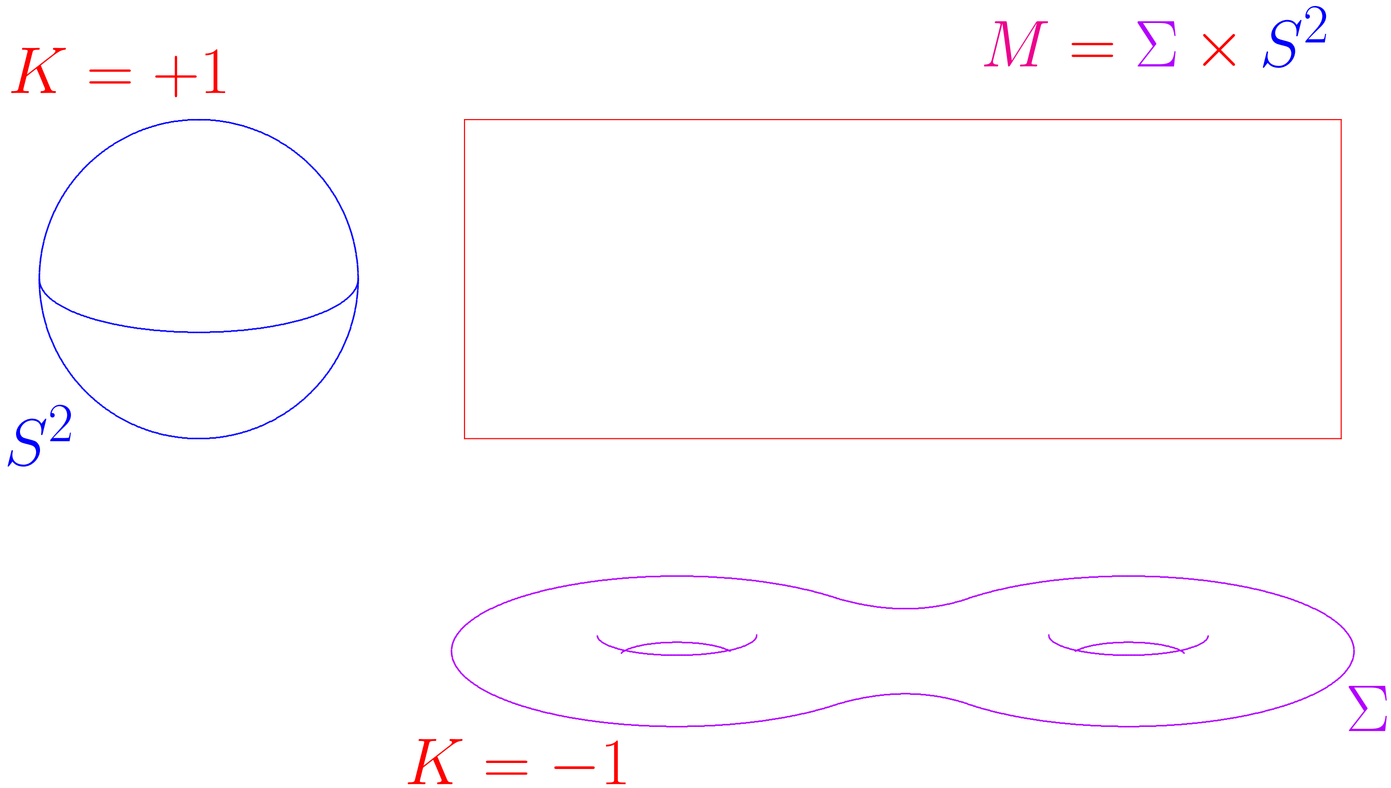

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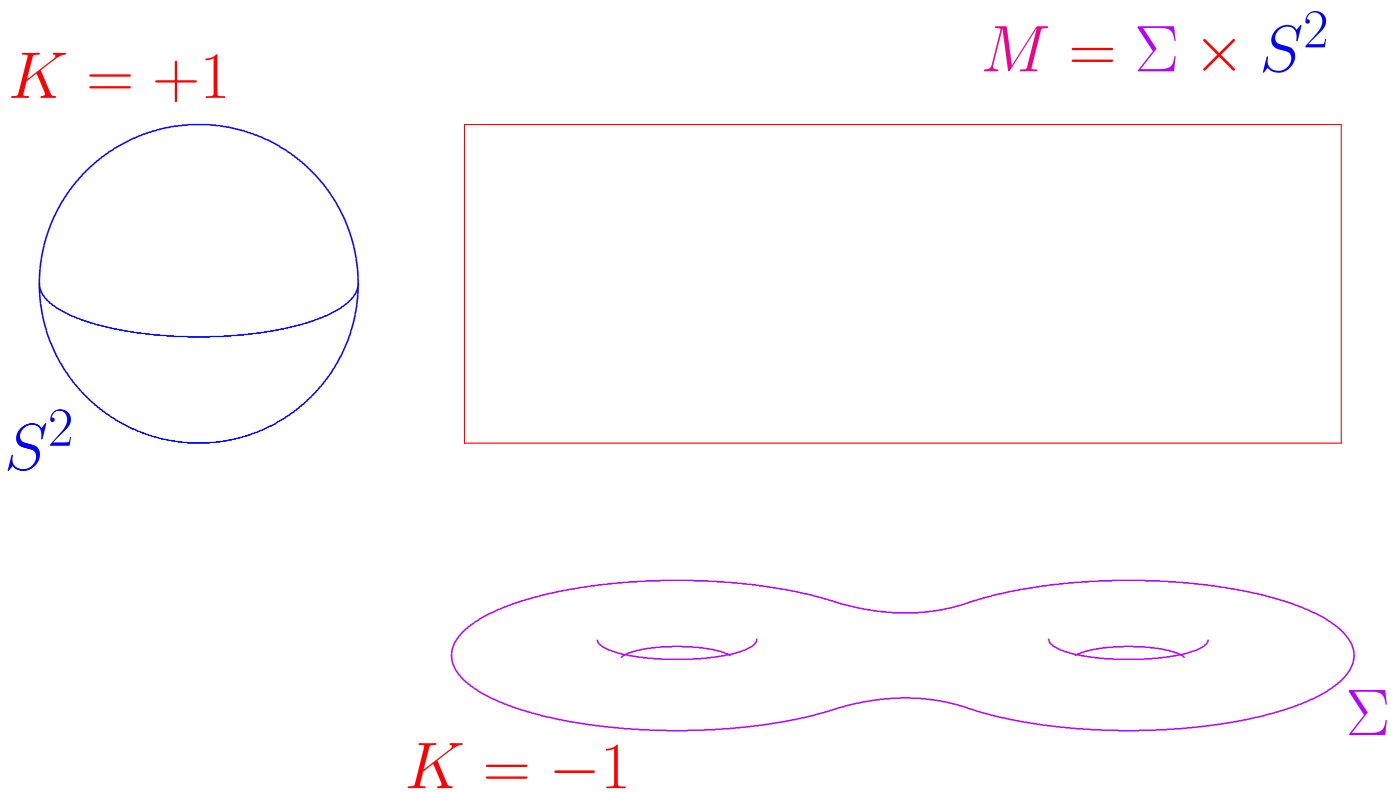
Locally conformally flat!

Example.



$$\widetilde{M} = \mathcal{H}^2 \times S^2$$

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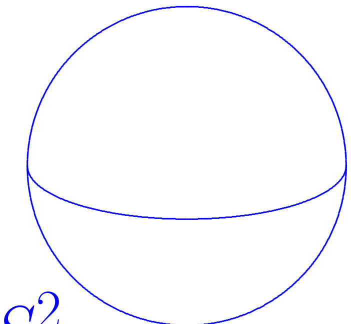

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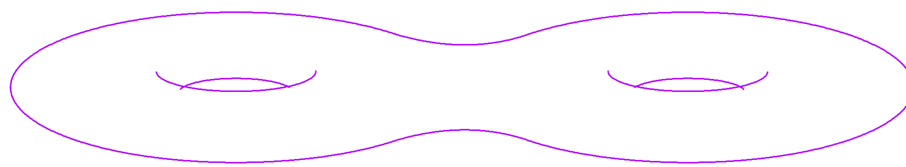
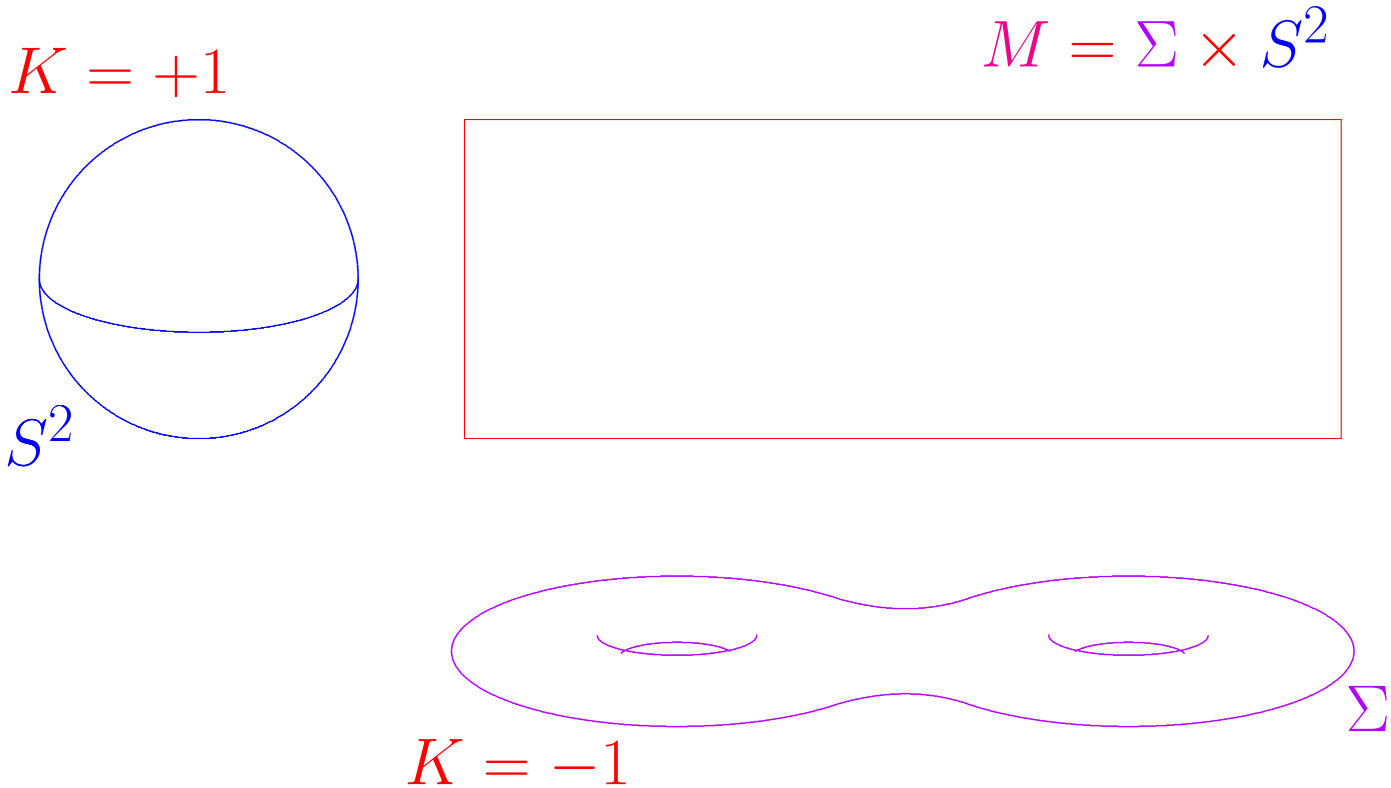


$$\widetilde{M} = \mathcal{H}^2 \times S^2 = S^4 - S^1$$

Example.



$$\widetilde{M} = \mathcal{H}^2 \times S^2 = S^4 - S^1$$

$$\pi_1(\Sigma) \hookrightarrow \mathbf{SO}_+(1,2)$$

Example.

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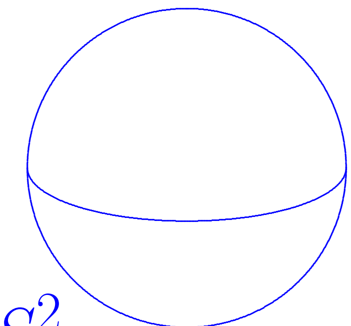
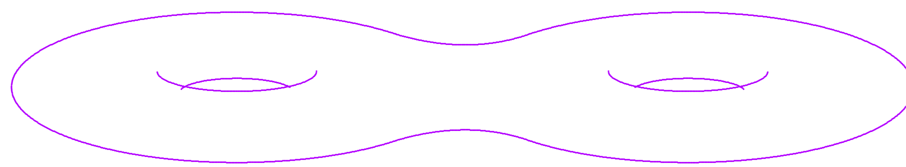


S^2

$M = \Sigma \times S^2$



Σ

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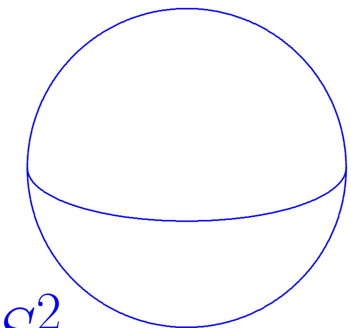
$$\pi_1(\Sigma) \hookrightarrow \mathbf{SO}_+(1,2) \times \mathbf{SO}(3)$$

Example.



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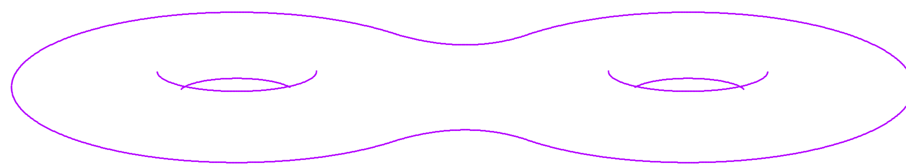

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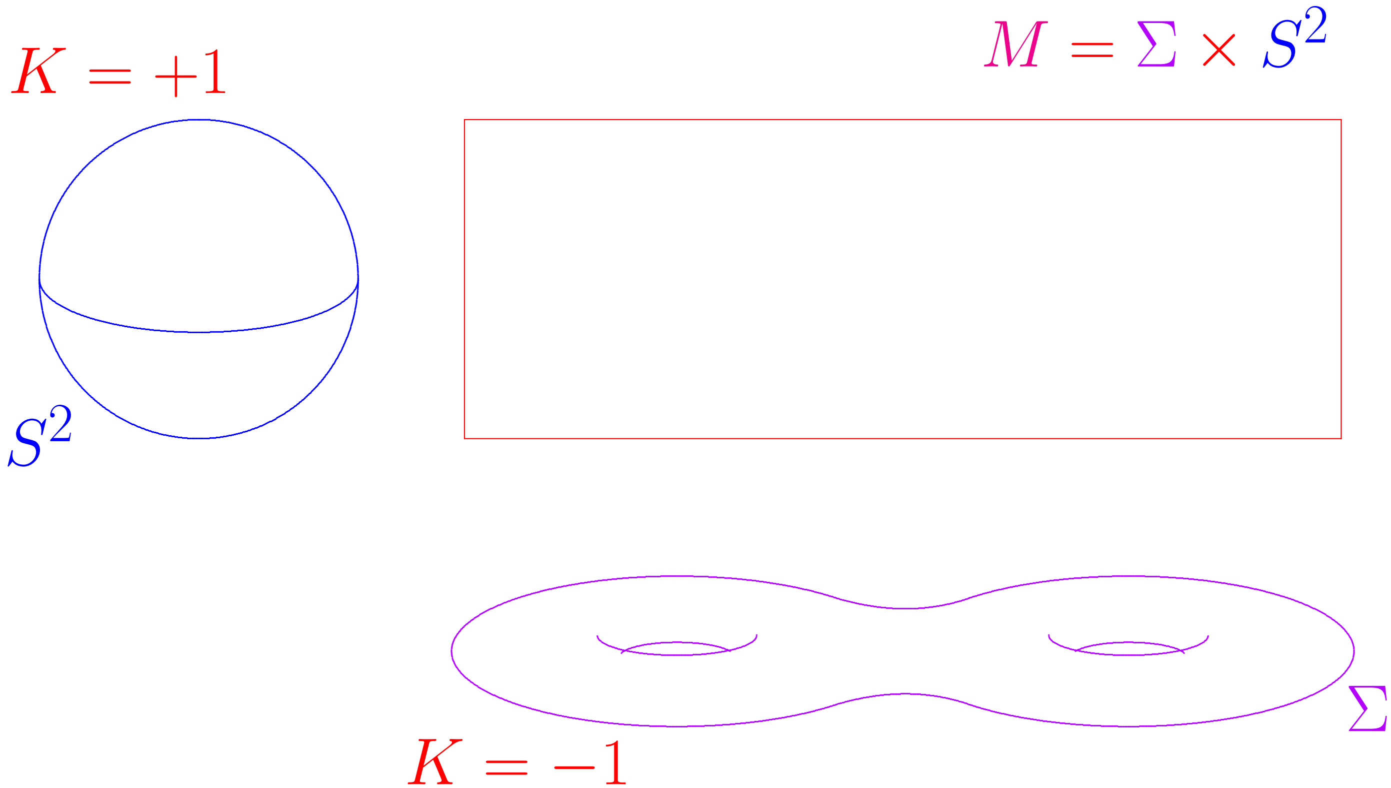

Scalar-flat Kähler deformations: $12(\mathbf{g}-1)$ moduli

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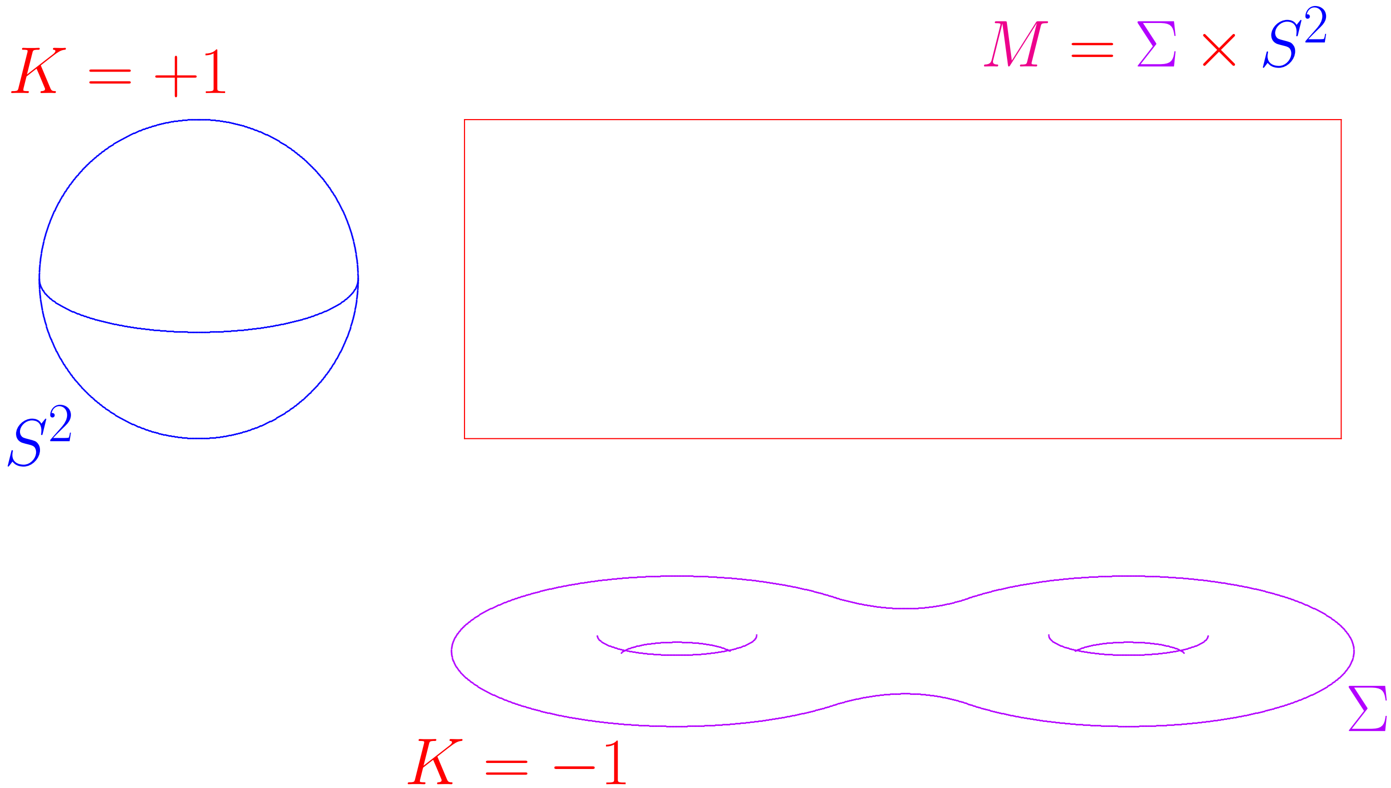
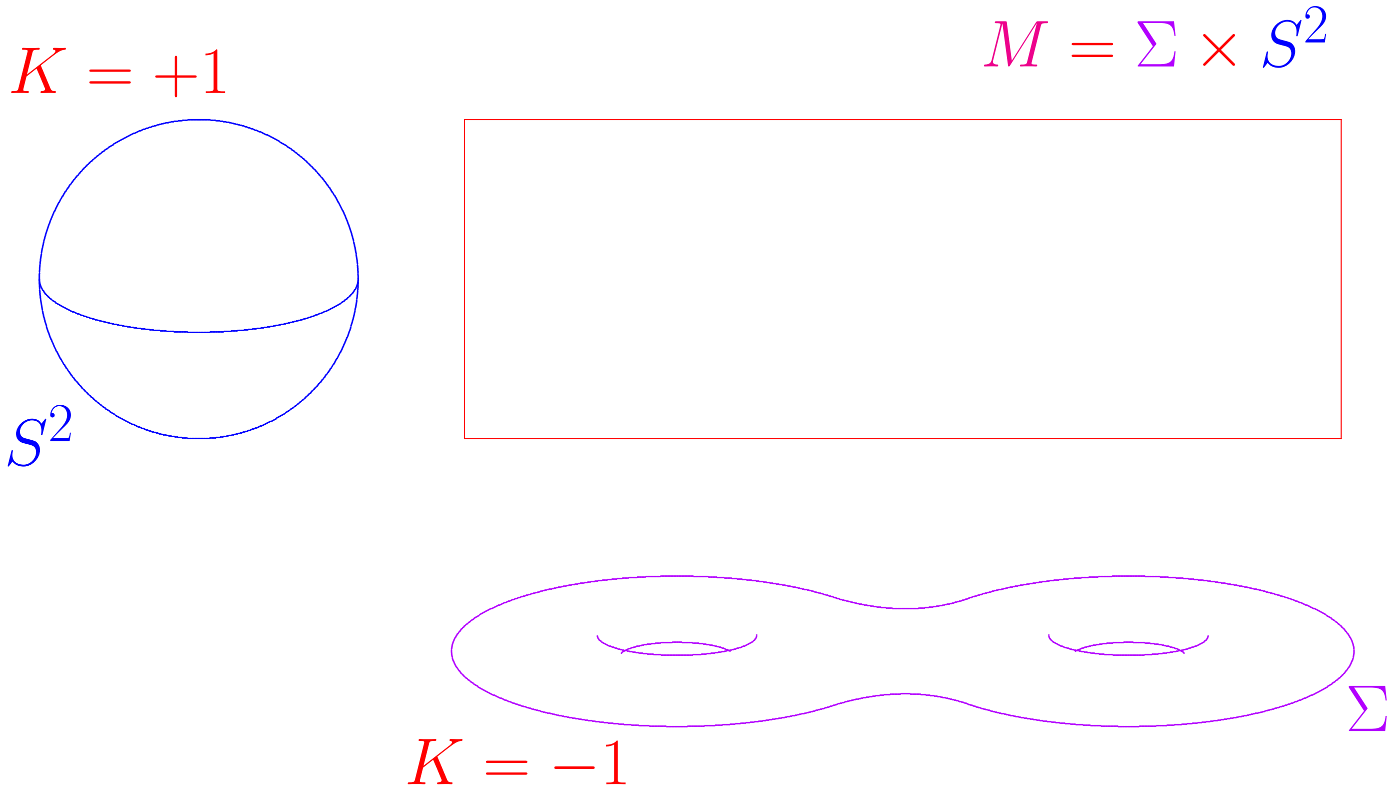


Scalar-flat Kähler deformations: $12(\mathfrak{g} - 1)$ moduli

Locally conformally flat def'ms: $30(\mathfrak{g} - 1)$ moduli

Example.



Scalar-flat Kähler deformations: $12(\mathfrak{g}-1)$ moduli
almost-Kähler ASD deformat'ns: $30(\mathfrak{g}-1)$ moduli

Almost-Kähler condition gives extra control on ASD conformal geometry.

Inyoung Kim '16: classification of almost-Kähler ASD roughly the same as in scalar-flat Kähler case.

Does this say anything about general ASD metrics?

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Alas, **No!**

Theorem.

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Same method simultaneously proves…

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Proof hinges on a construction of hyperbolic 3-manifolds.

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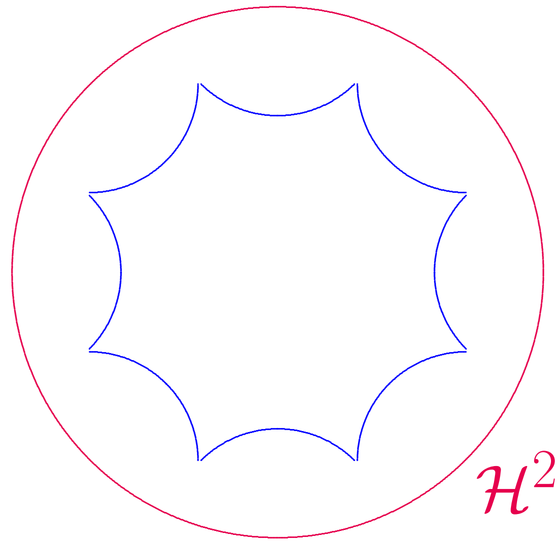
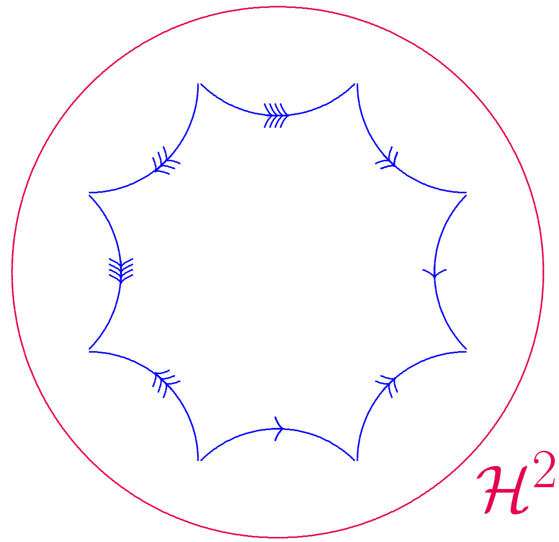
Proof hinges on a construction of hyperbolic 3-manifolds.

We begin by revisiting hyperbolic metrics on Σ.


$\mathcal{H}^2$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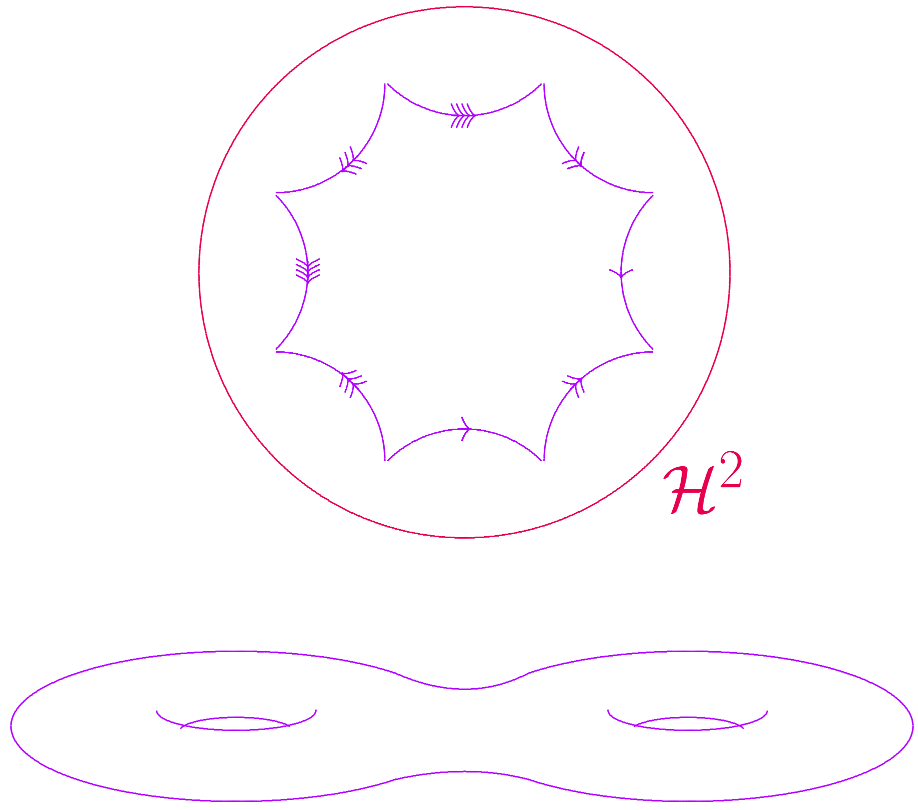
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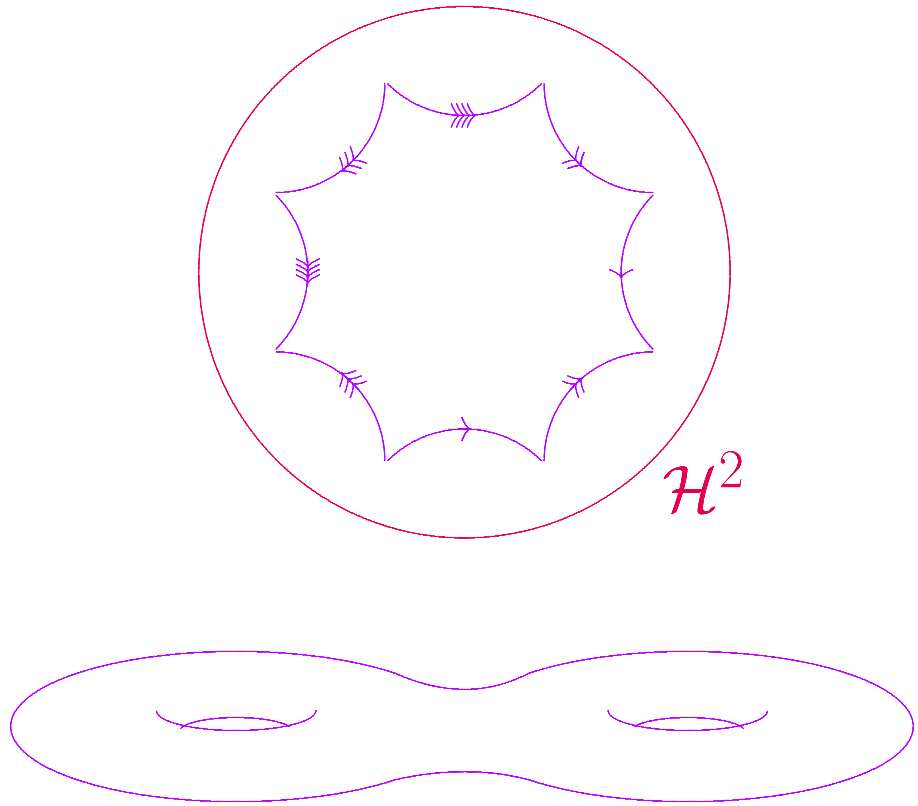
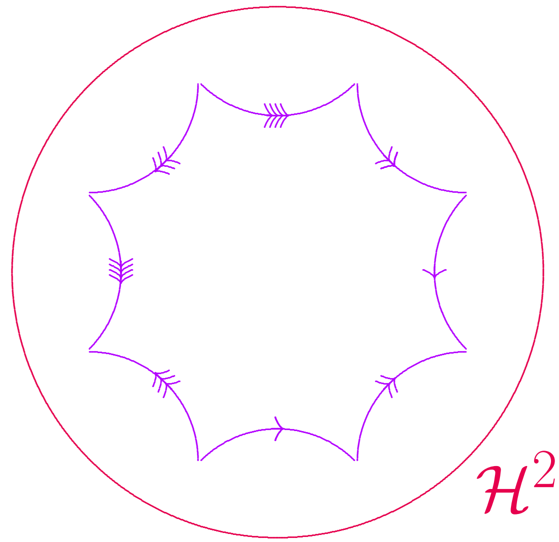
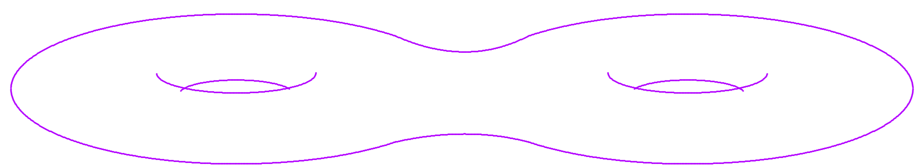
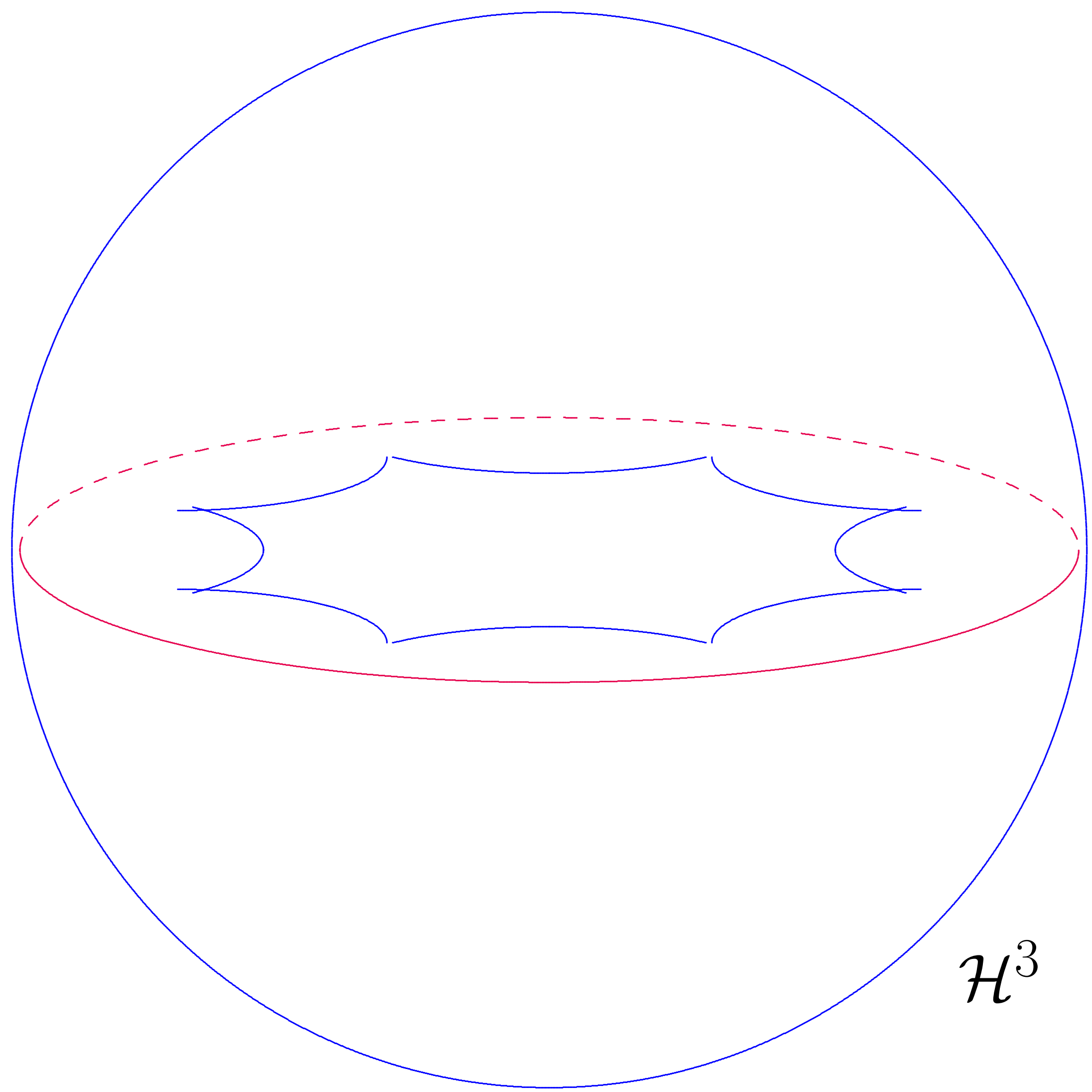


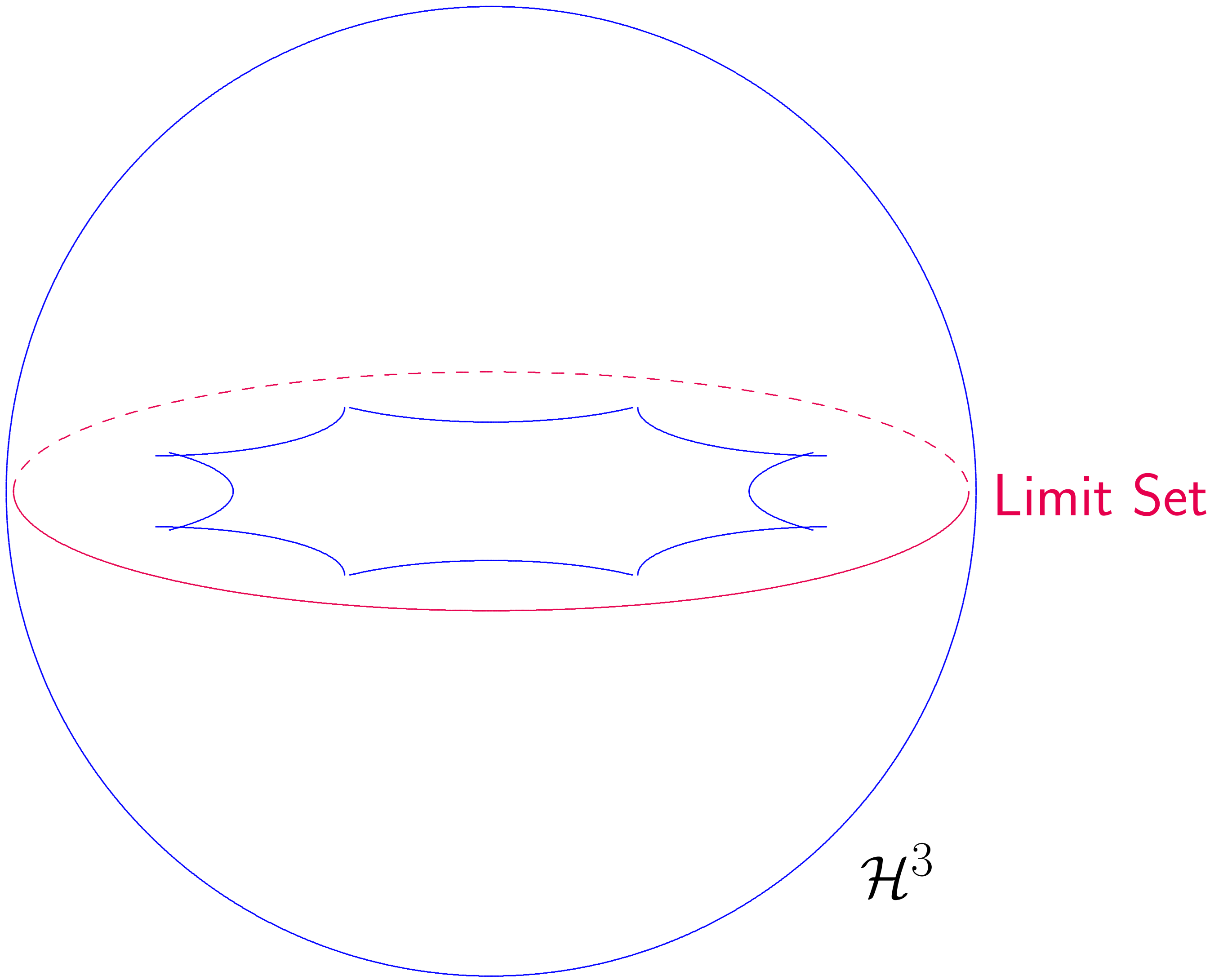
$$\pi_1(\Sigma) \hookrightarrow \mathbf{SO}_+(1,2) = \mathbf{PSL}(2,\mathbb{R})$$

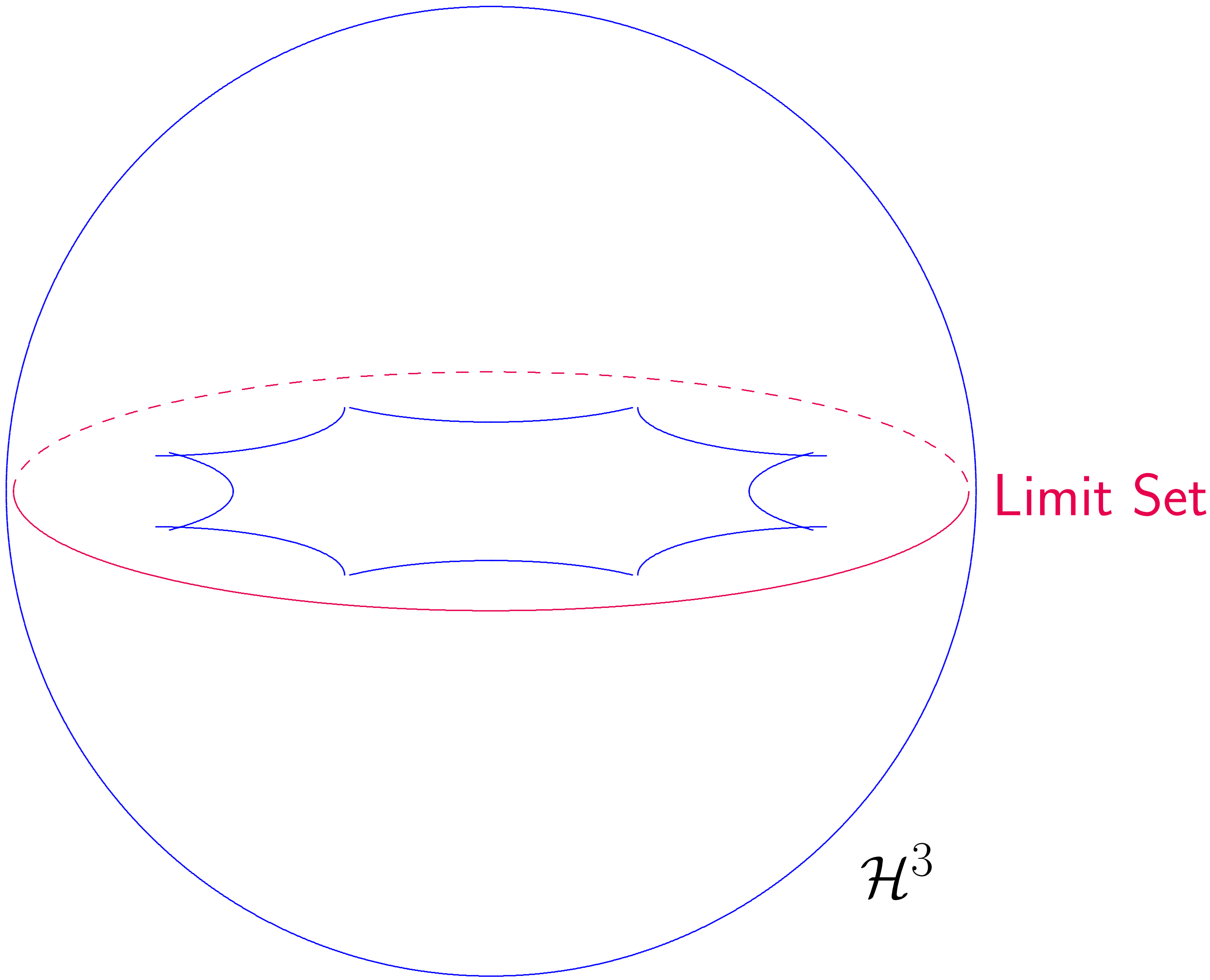




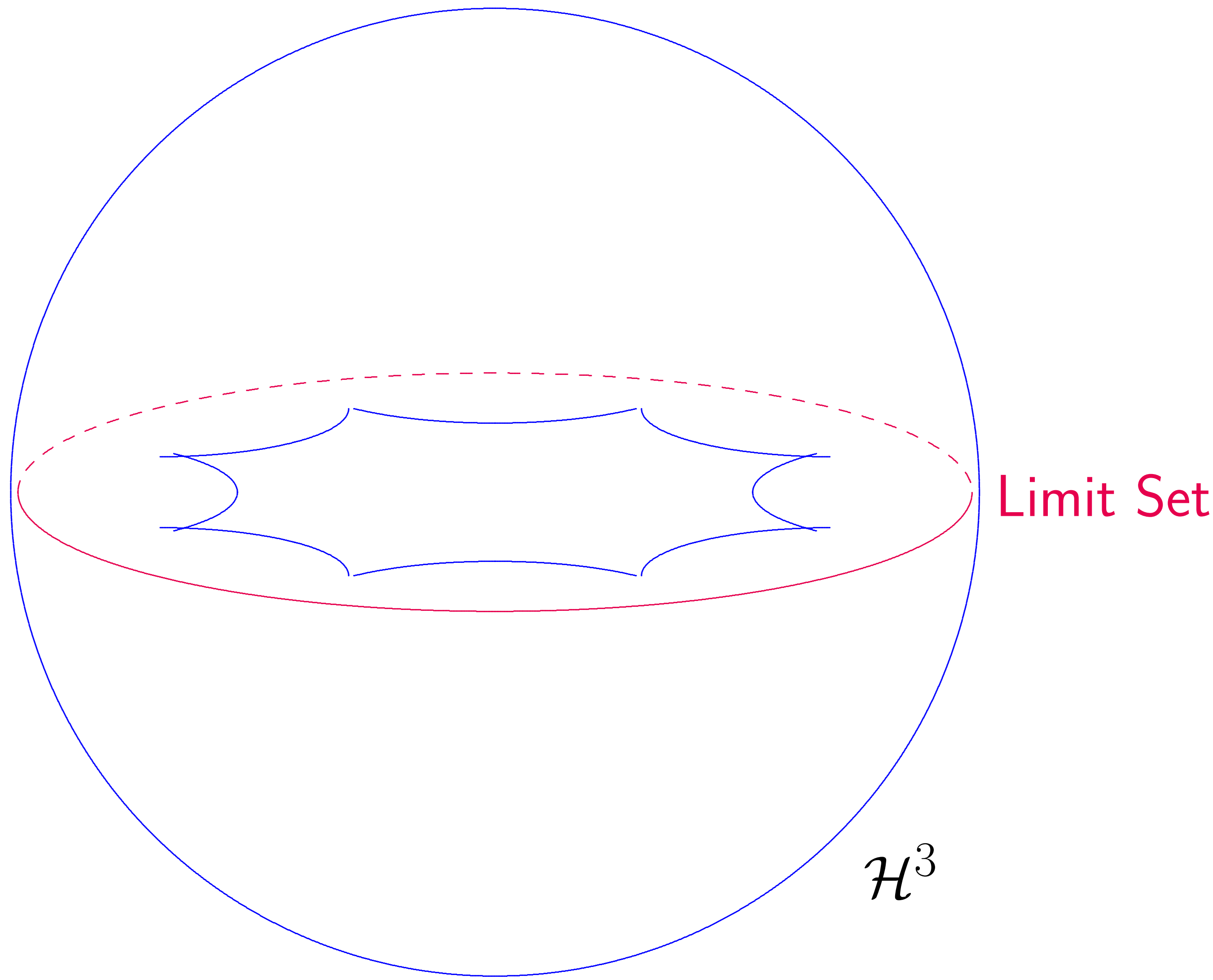
$$\begin{array}{ccccc} \pi_1(\Sigma) & \hookrightarrow & \mathbf{SO}_+(1,2) & = & \mathbf{PSL}(2,\mathbb{R}) \\ & & \cap & & \cap \\ & & \mathbf{SO}_+(1,3) & = & \mathbf{PSL}(2,\mathbb{C}) \end{array}$$


$\mathcal{H}^3$


Limit Set
$\mathcal{H}^3$



$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2,\mathbb{R})$$ Fuchsian group



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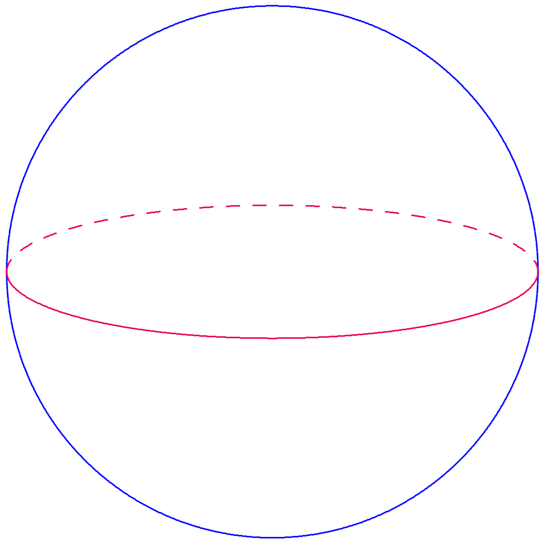
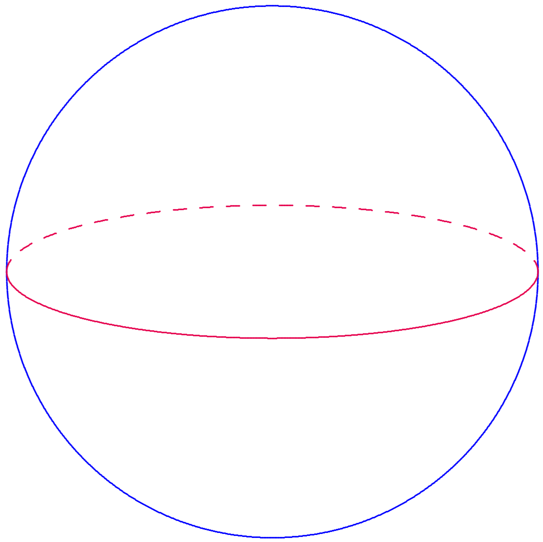
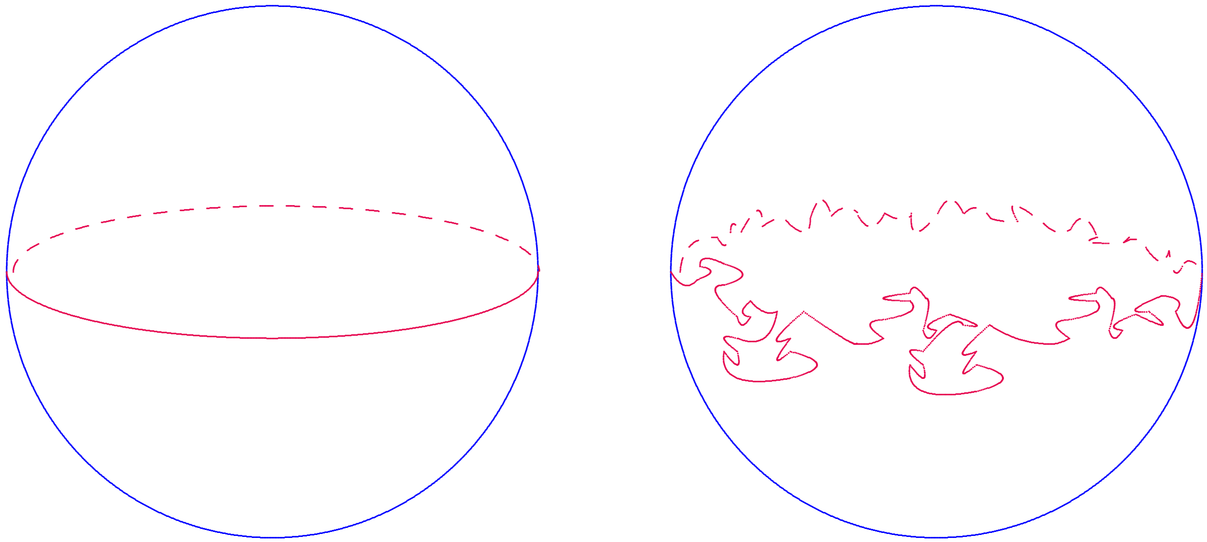




Fuchsian



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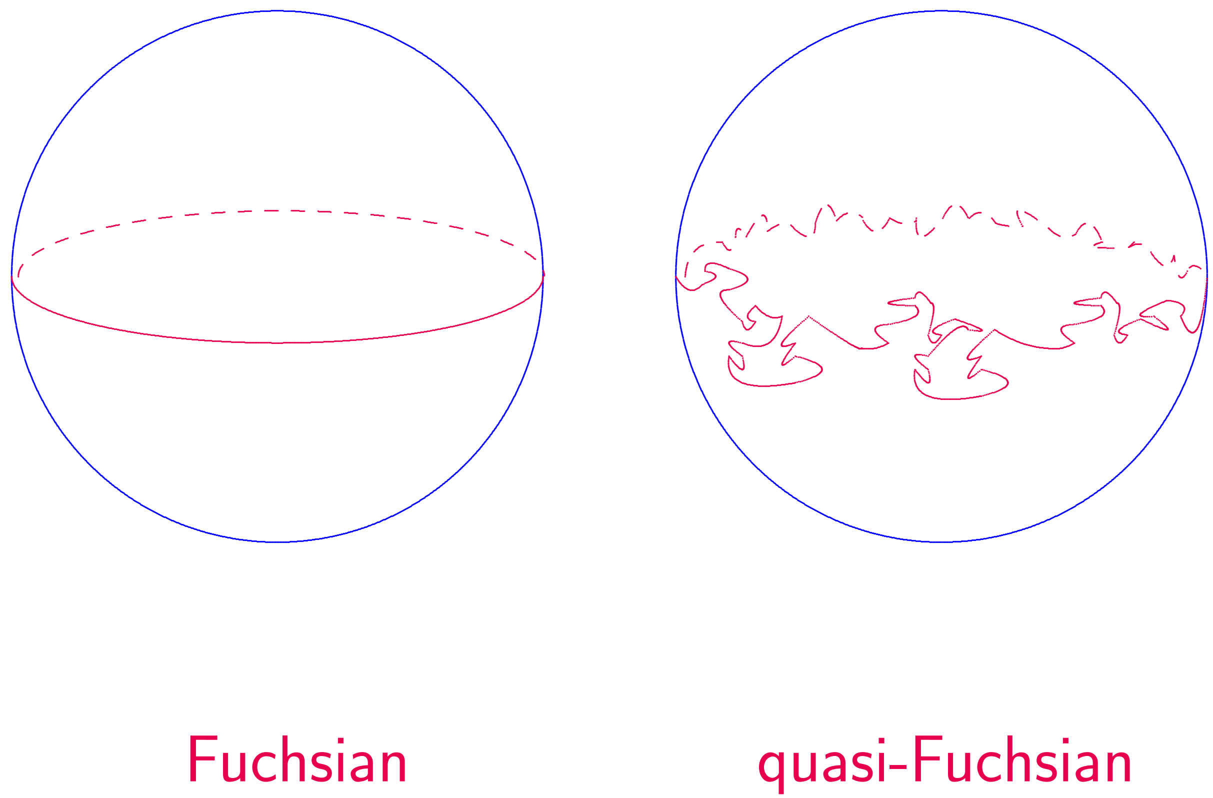
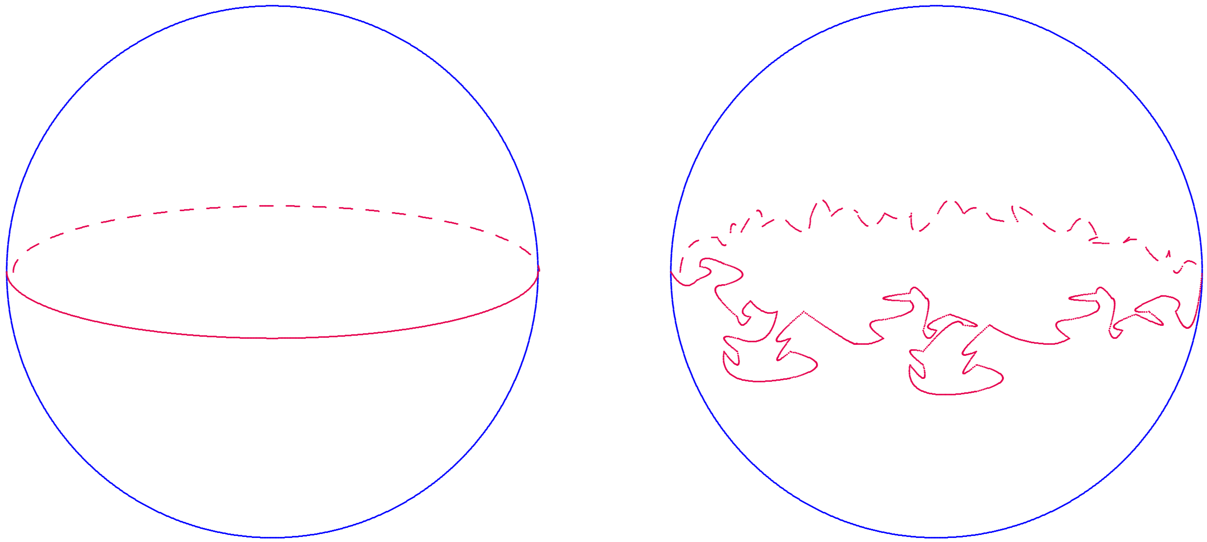

Fuchsian
quasi-Fuchsian



Fuchsian quasi-Fuchsian

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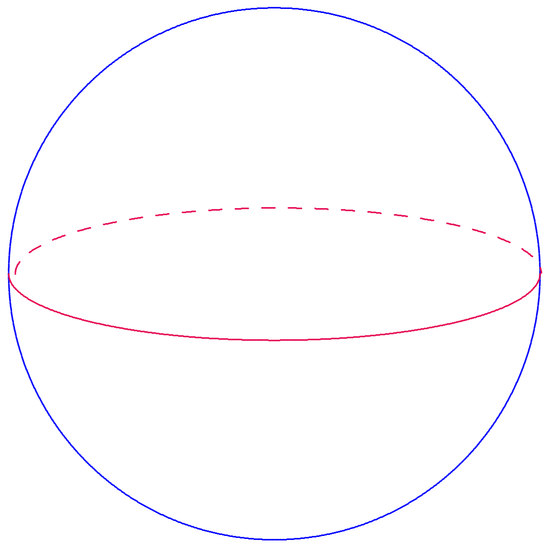

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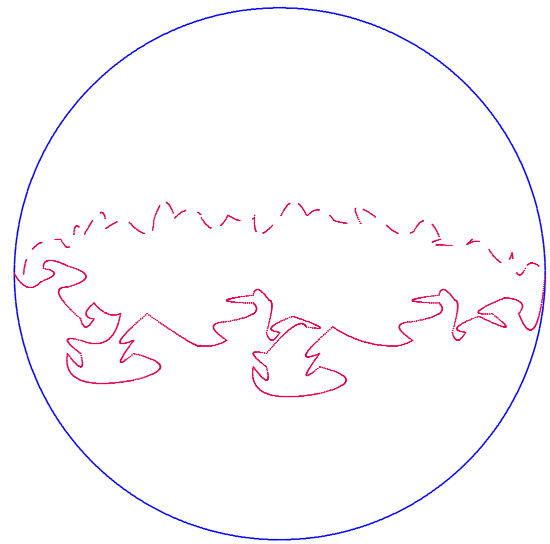

quasi-Fuchsian

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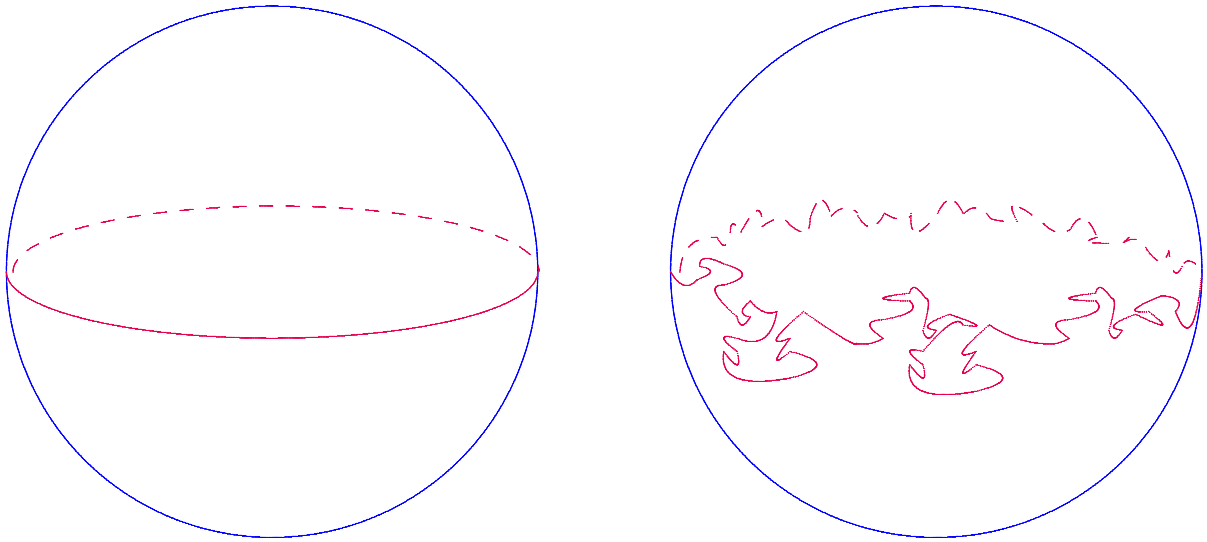

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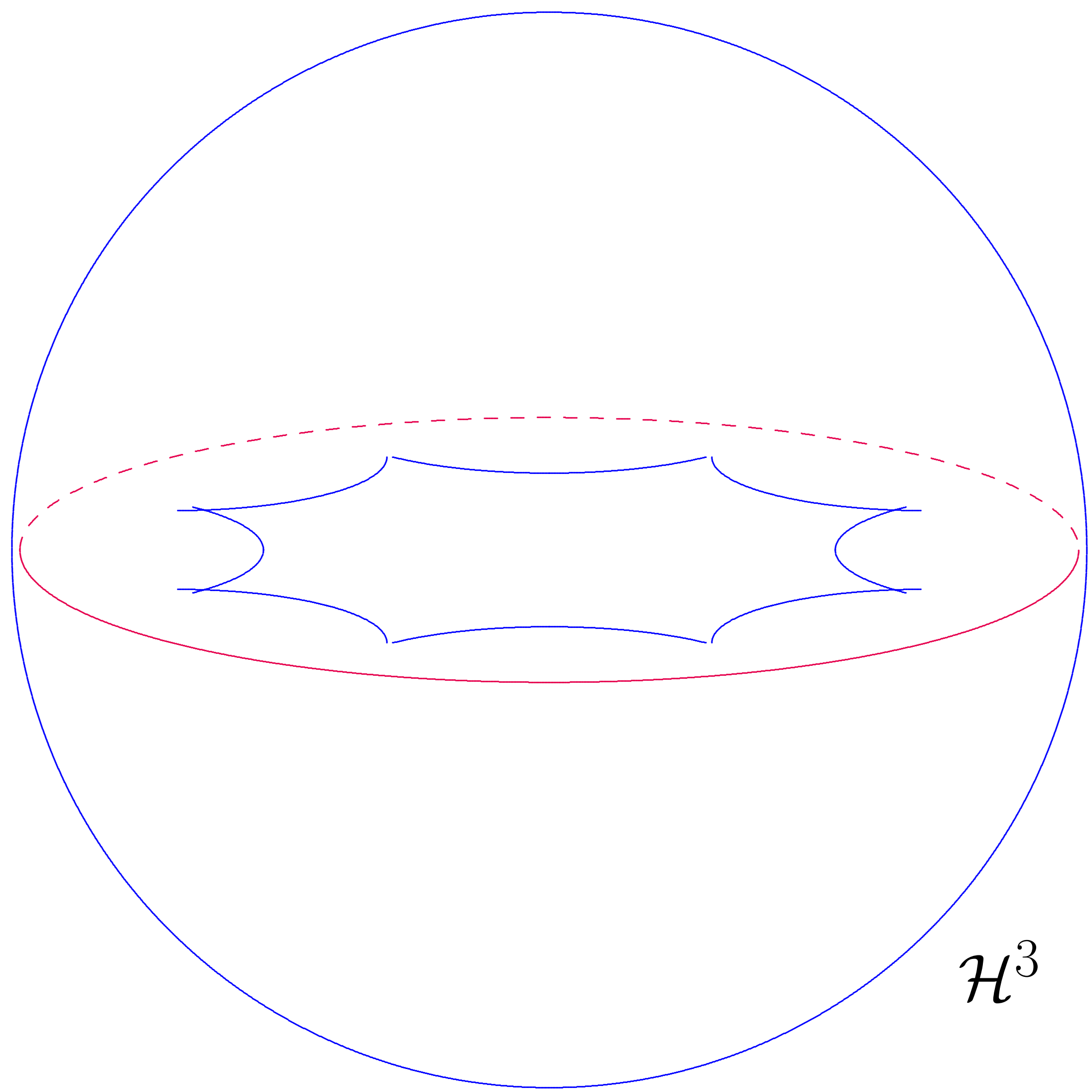
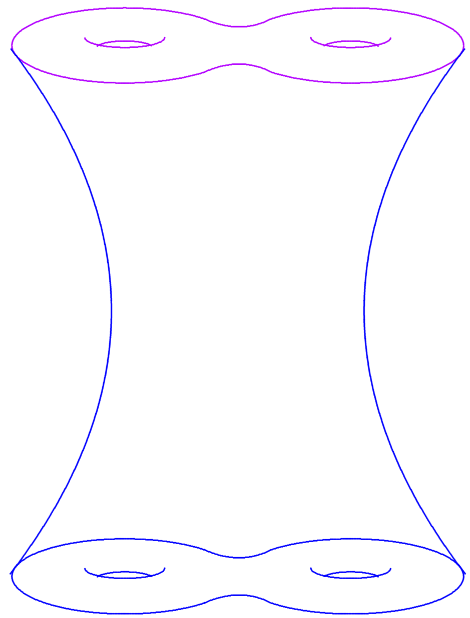
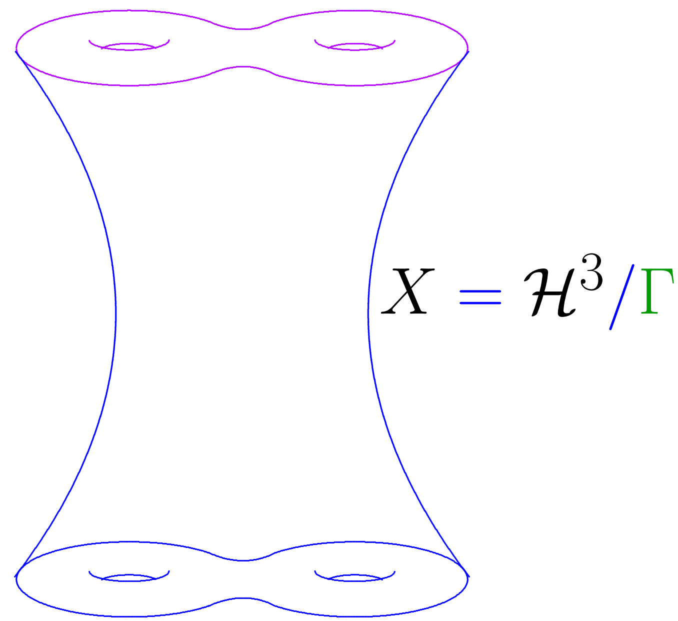
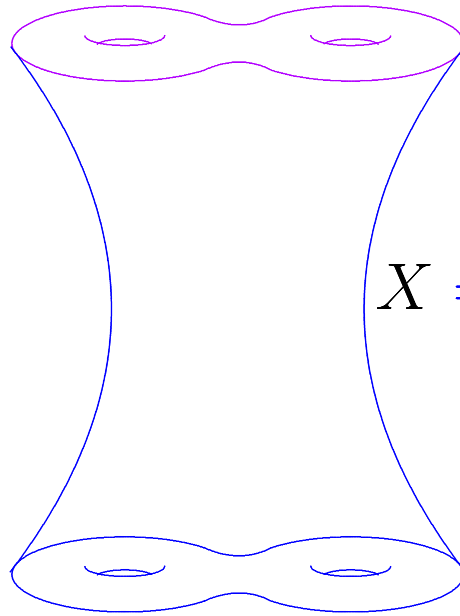
Quasi-conformally conjugate to Fuchsian.


$\mathcal{H}^3$




$X = \mathcal{H}^3/\Gamma$



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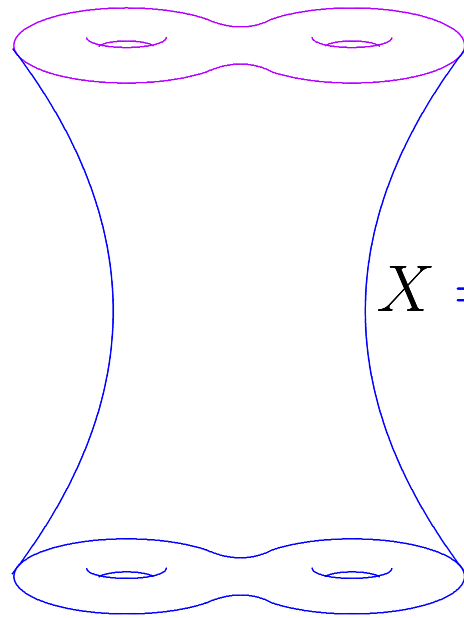
Γ Fuchsian



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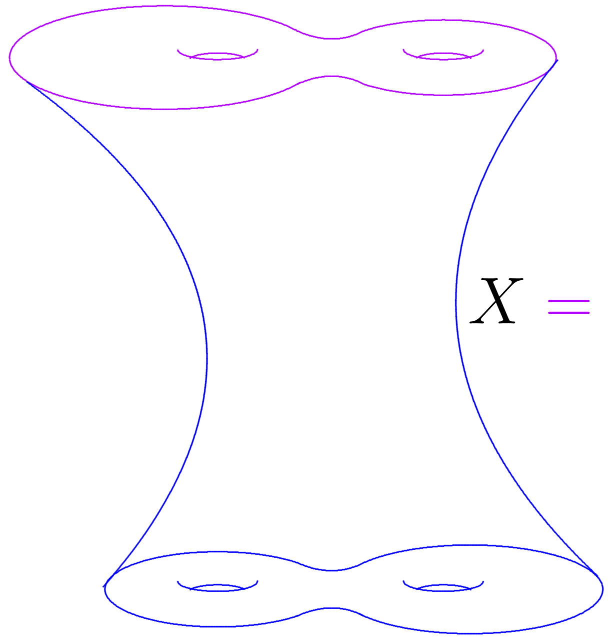
Γ Fuchsian

$$X \approx \Sigma \times \mathbb{R}$$



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Γ quasi-Fuchsian

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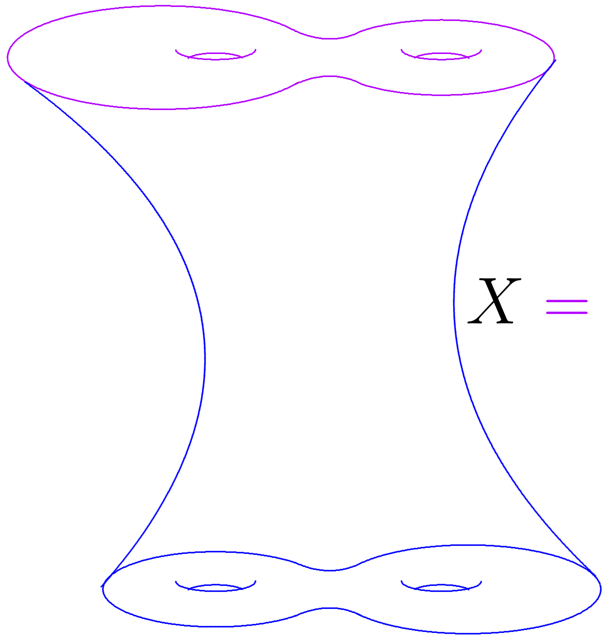
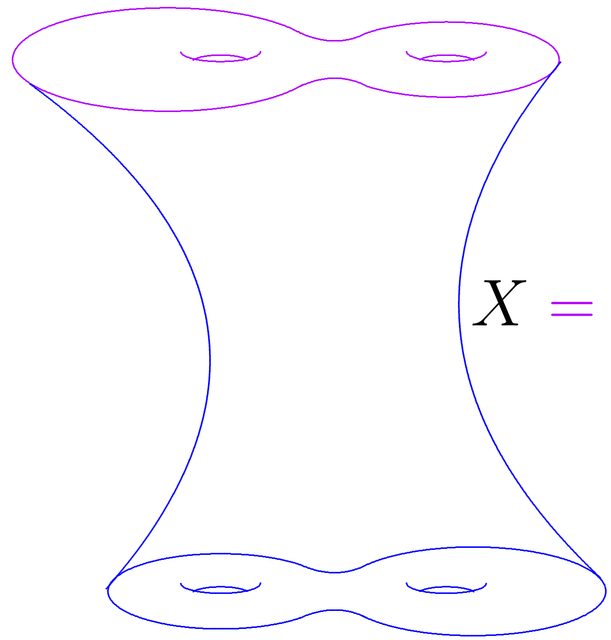


$X = \mathcal{H}^3/\Gamma$

Γ quasi-Fuchsian

$$X \approx \Sigma \times \mathbb{R}$$

Freedom: two points in Teichmüller space.



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Γ quasi-Fuchsian

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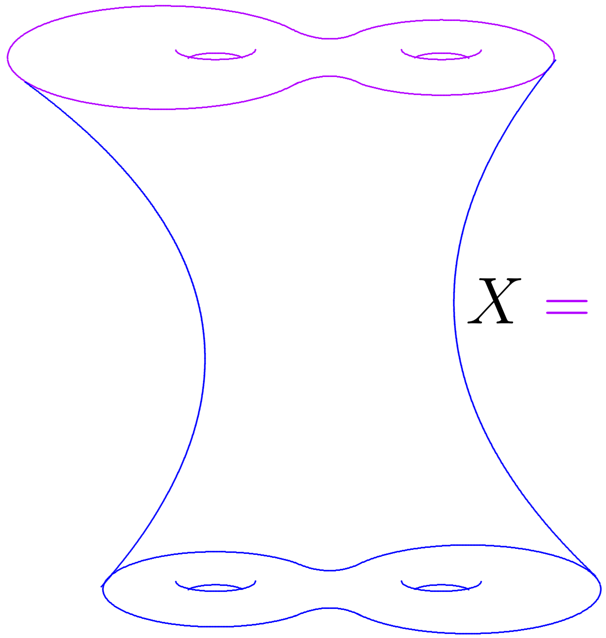
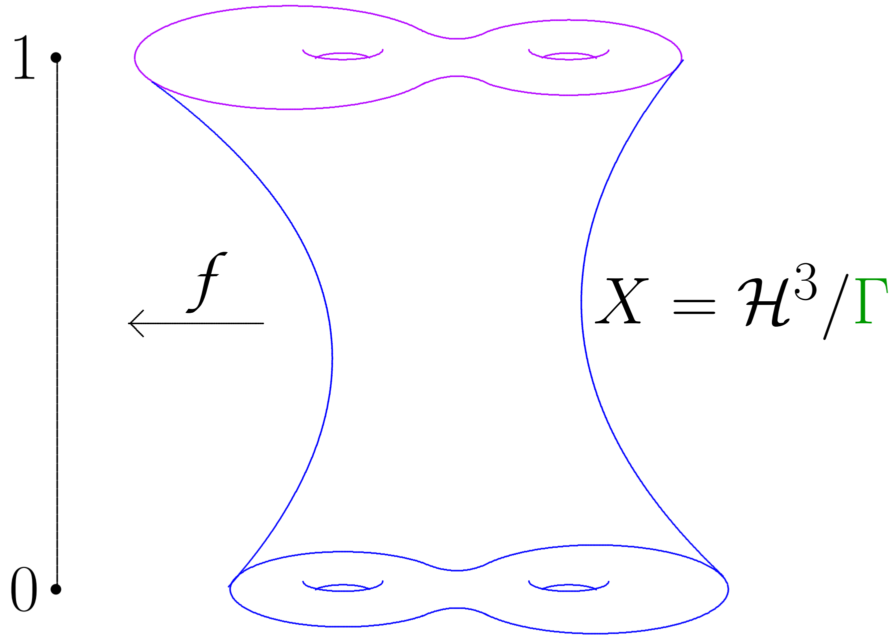


$X = \mathcal{H}^3/\Gamma$

Γ quasi-Fuchsian

$\overline{X} \approx \Sigma \times [0, 1]$

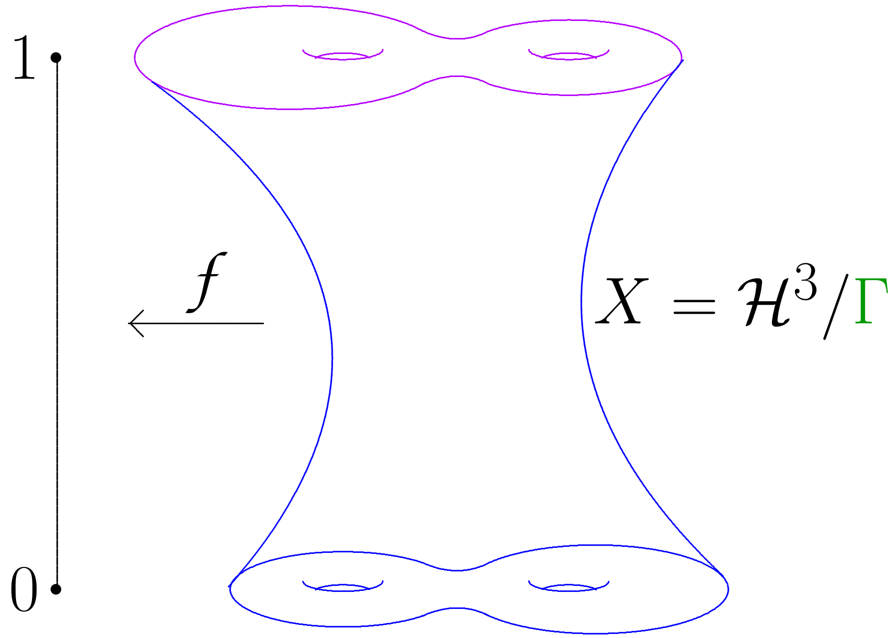


Γ quasi-Fuchsian

$$\overline{X} \approx \Sigma \times [0,1]$$

Tunnel-Vision function:

$$f : \overline{X} \to [0,1]$$



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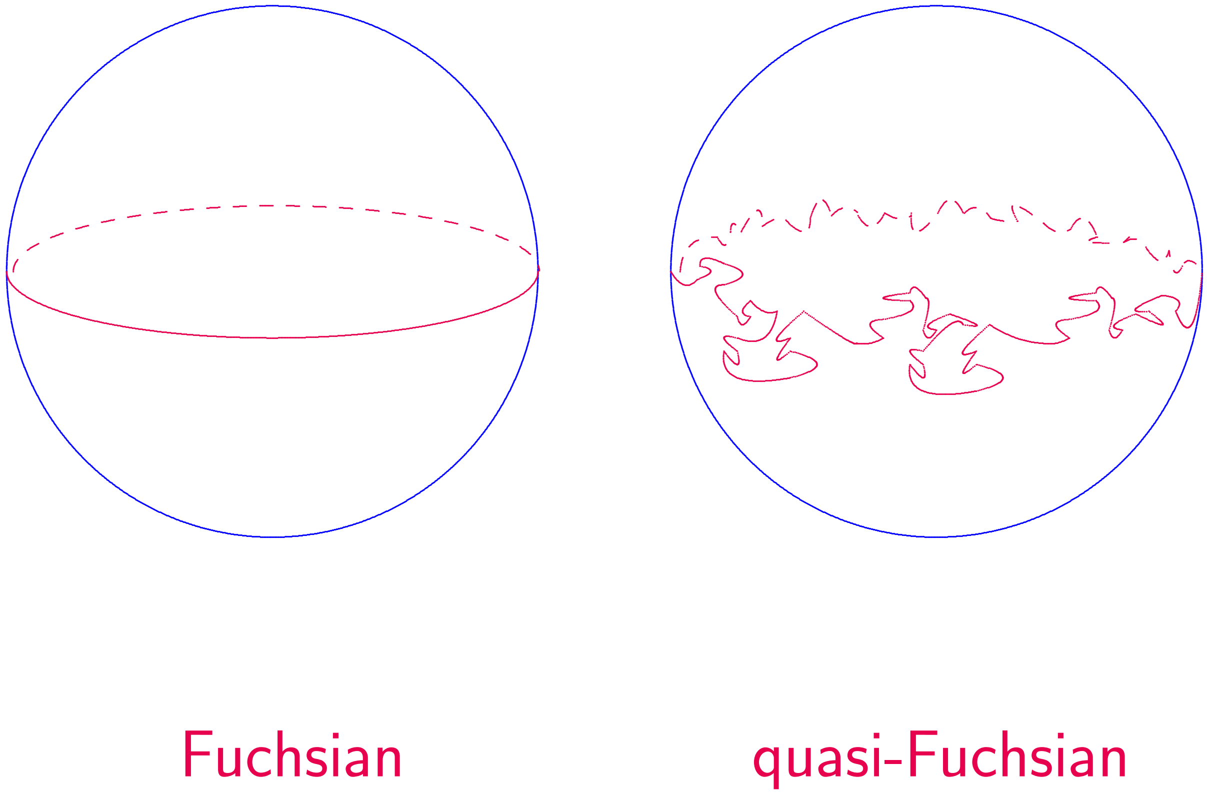
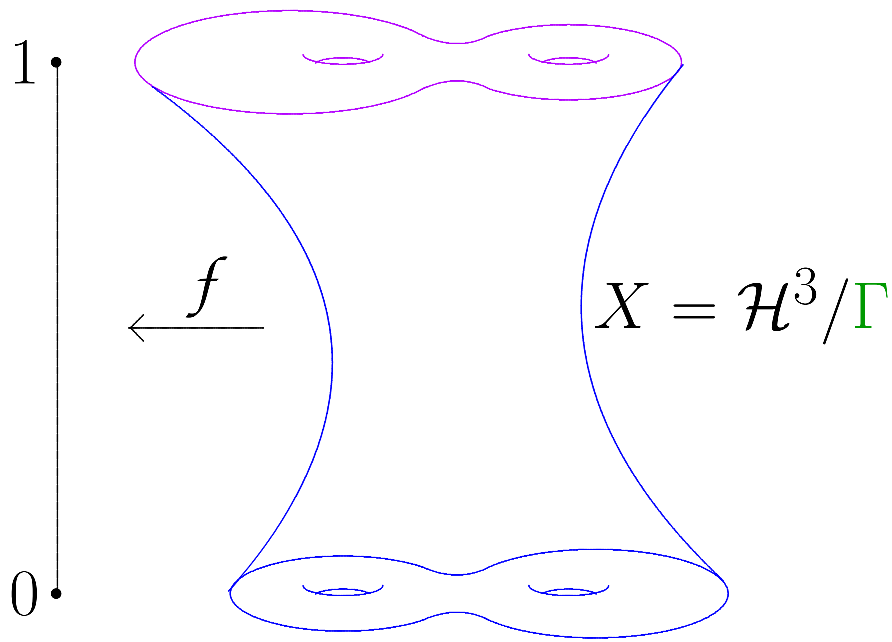

Fuchsian
quasi-Fuchsian



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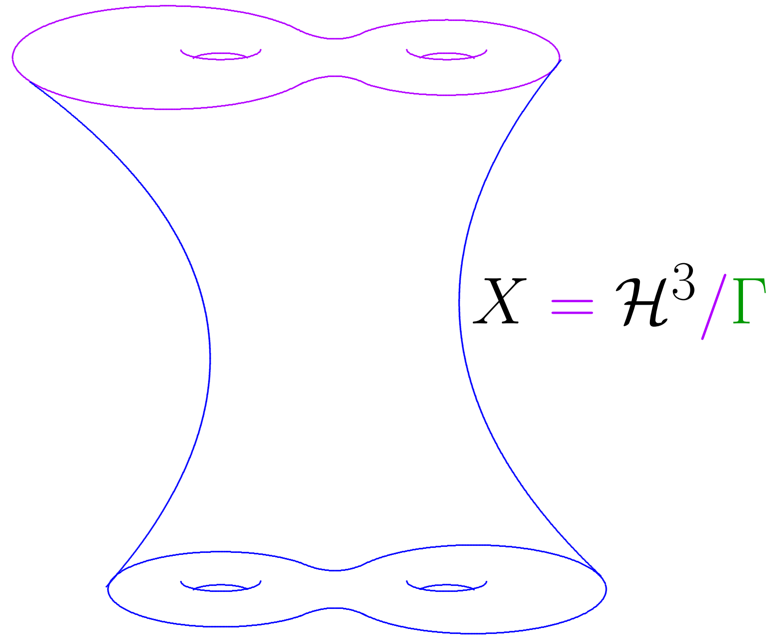
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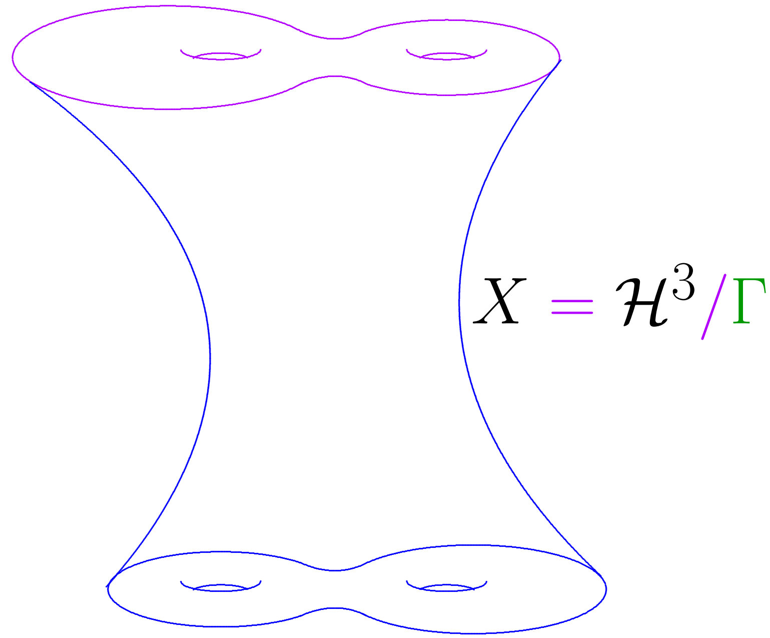
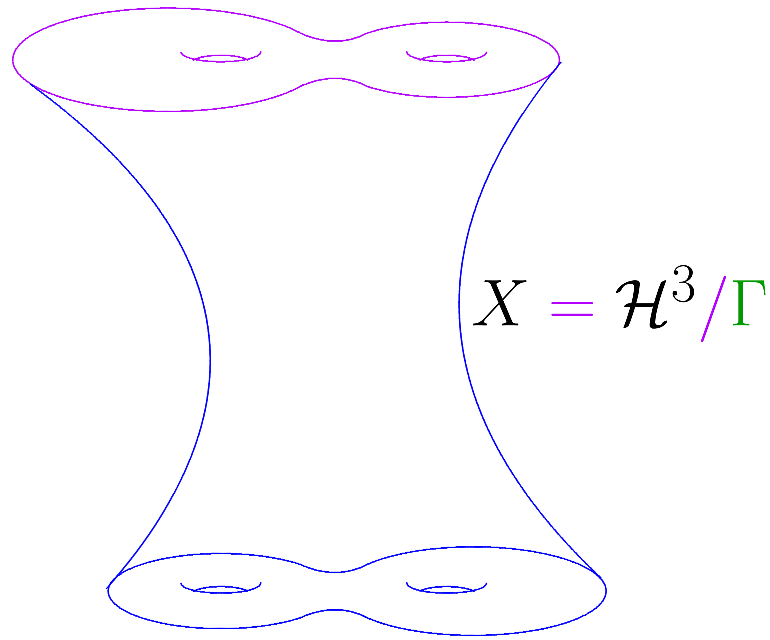
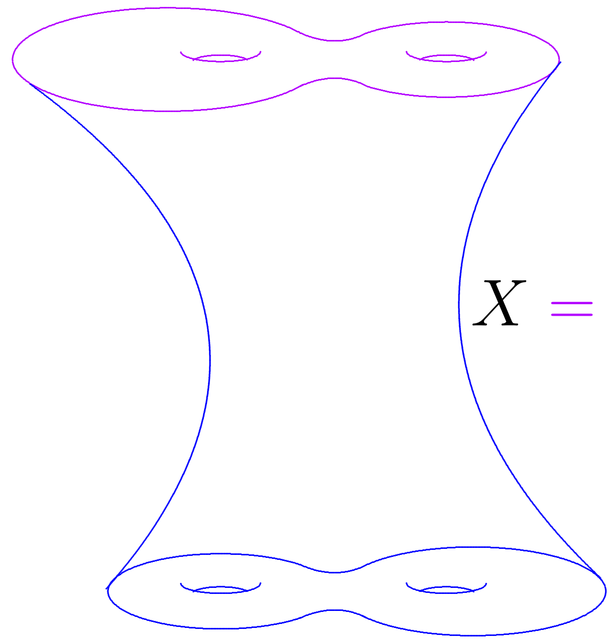


Construction of conformally flat 4-manifolds:



Construction of conformally flat 4-manifolds:

$$M = [\overline{X} \times S^1]/\sim$$

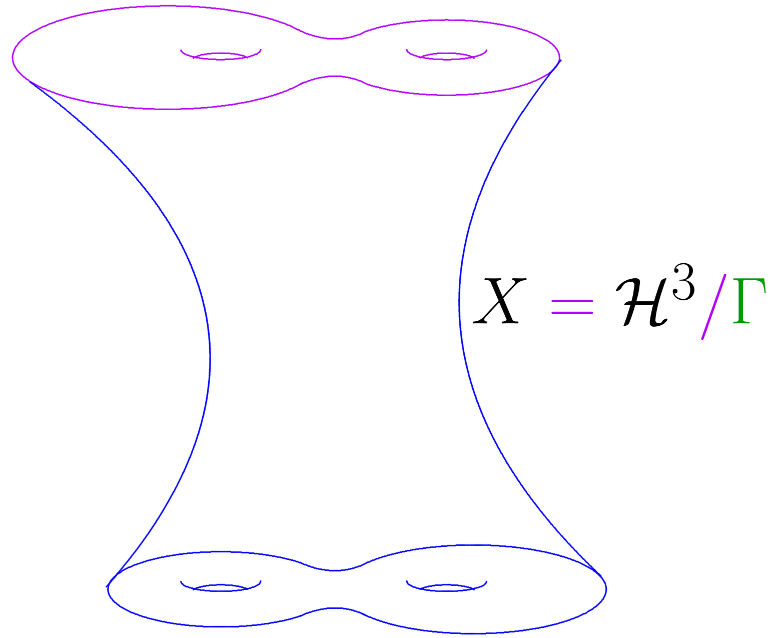


$X = \mathcal{H}^3/\Gamma$

Construction of conformally flat 4-manifolds:

$$M = [\overline{X} \times S^1]/\sim$$

$\sim$: crush $\partial\overline{X} \times S^1$ to $\partial\overline{X}$.



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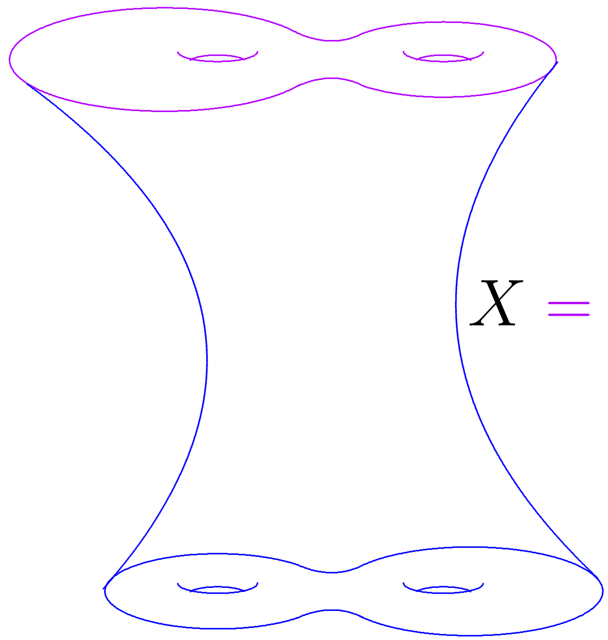

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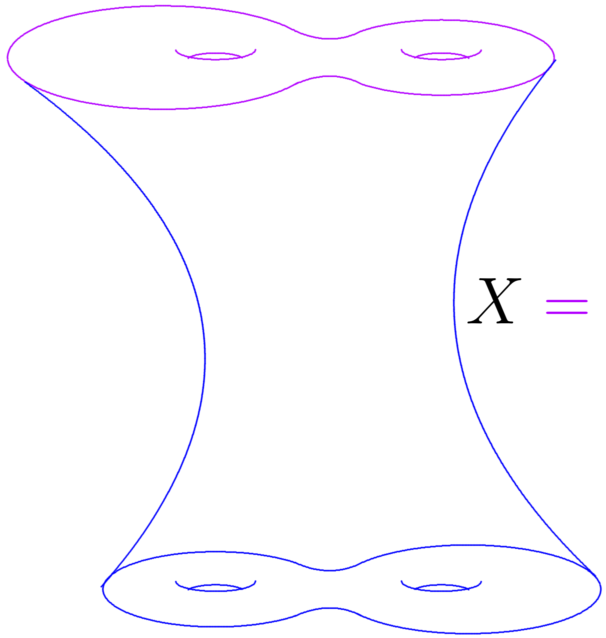

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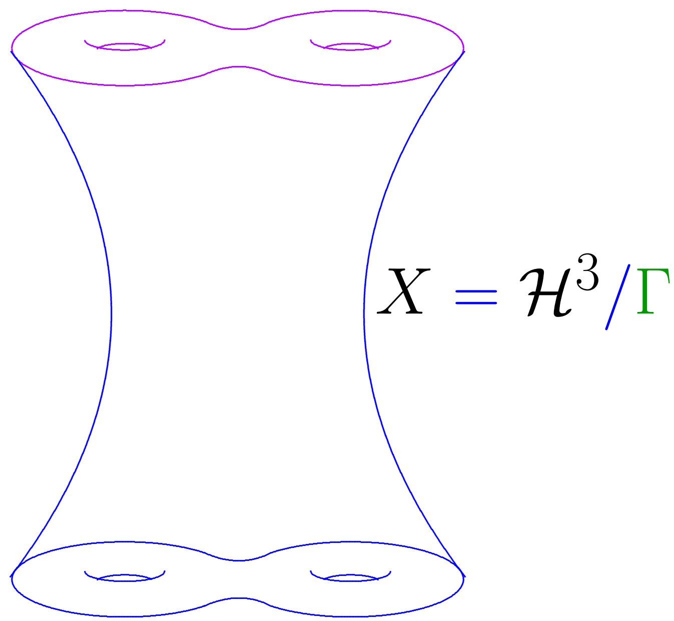

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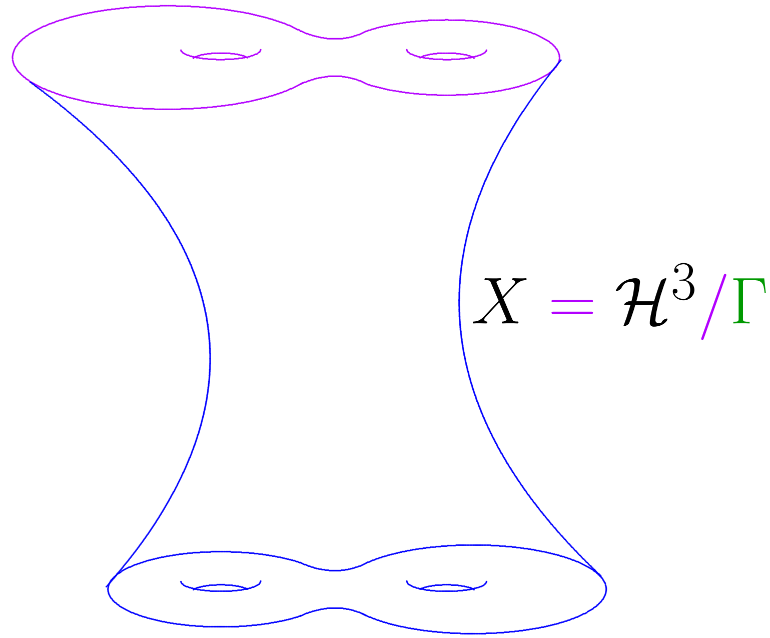
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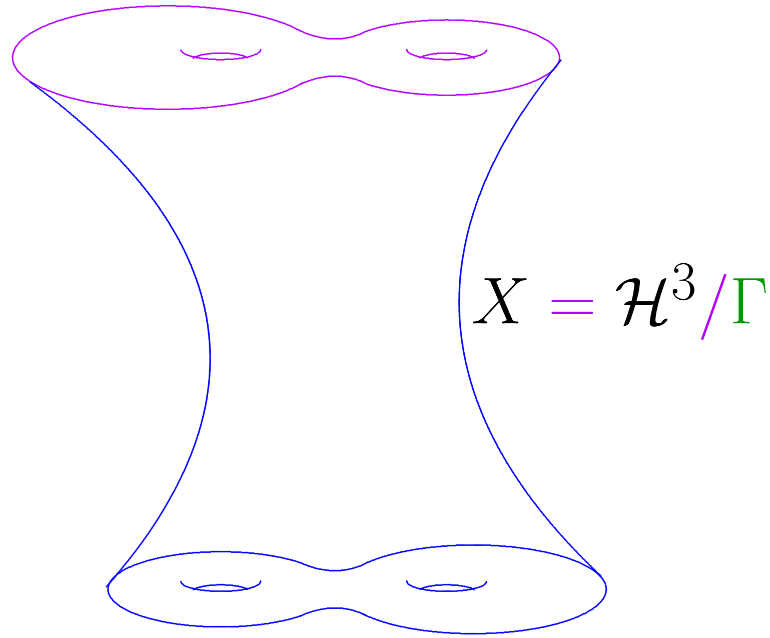

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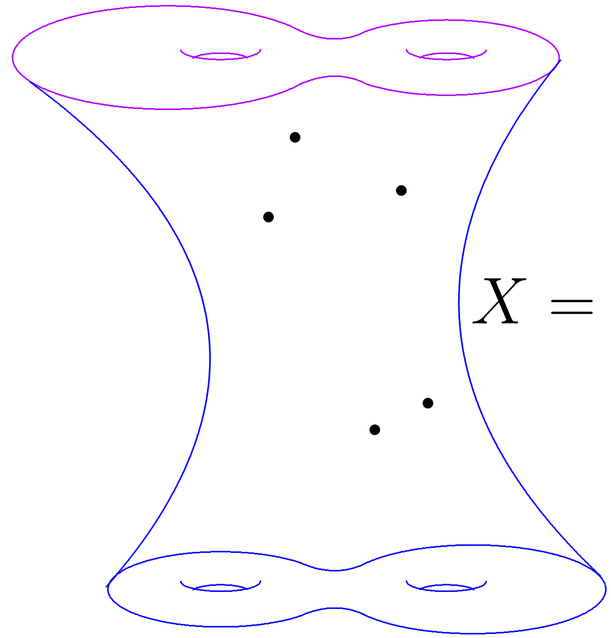

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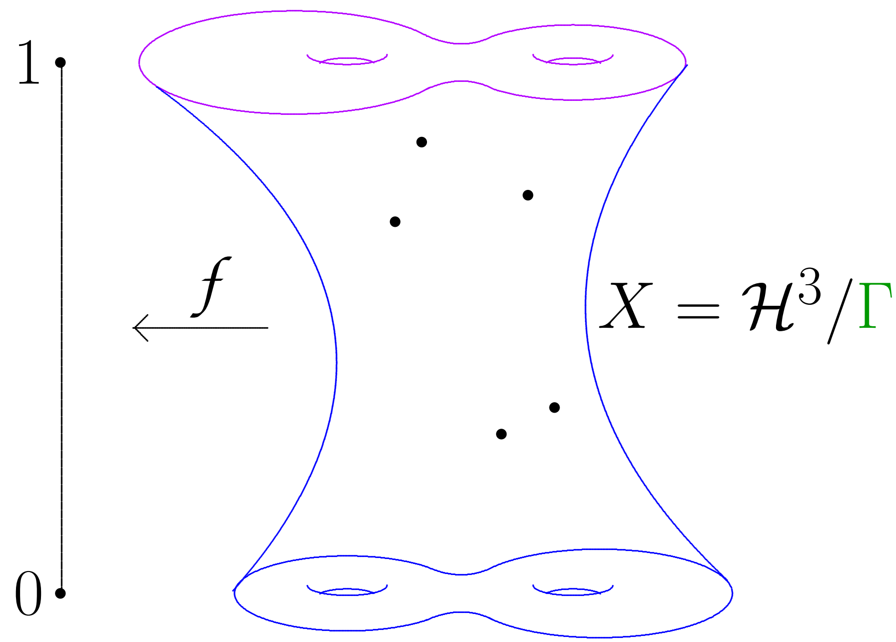
Choose k points $p_1, \ldots, p_k \in X$



Construction of ASD 4-manifolds:

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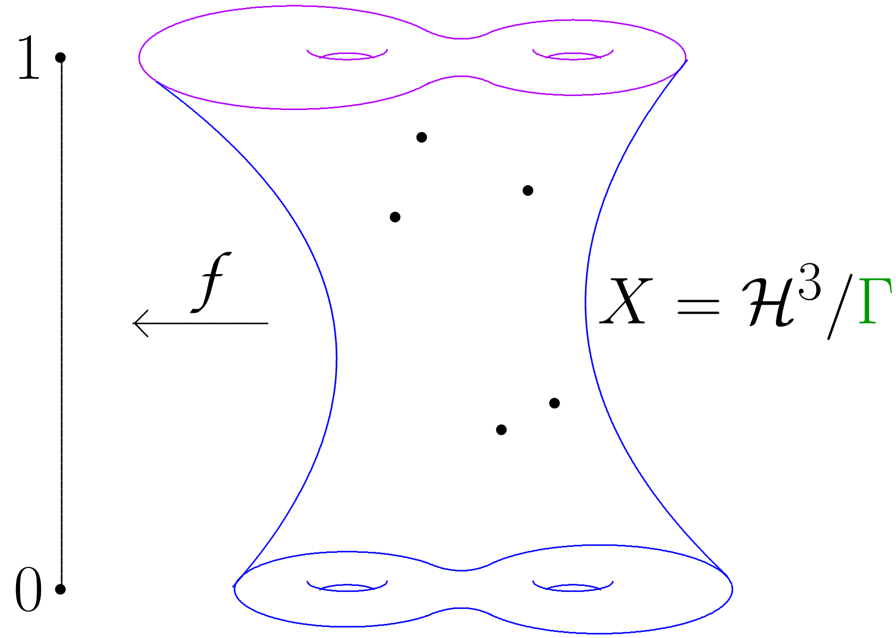
satisfying $\sum_{j=1}^{k} f(p_j) \in \mathbb{Z}$.



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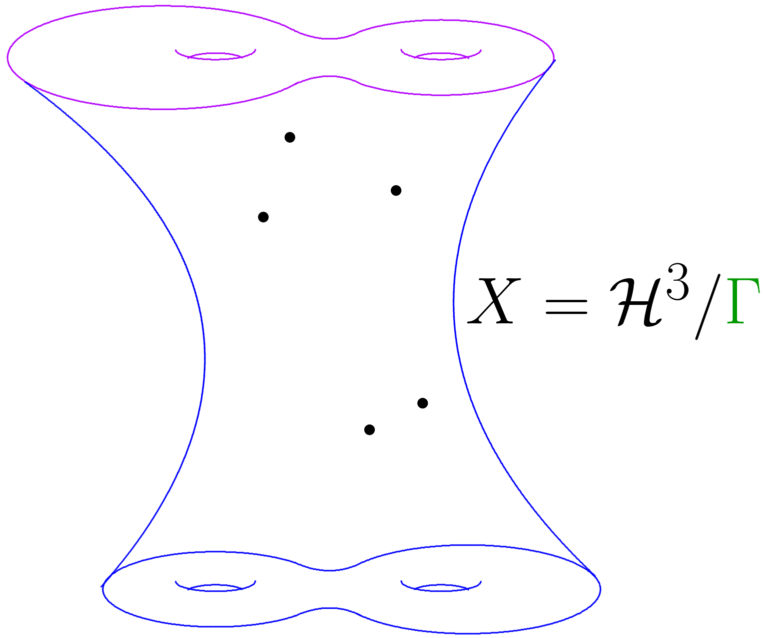
Can do if $k \neq 1$.



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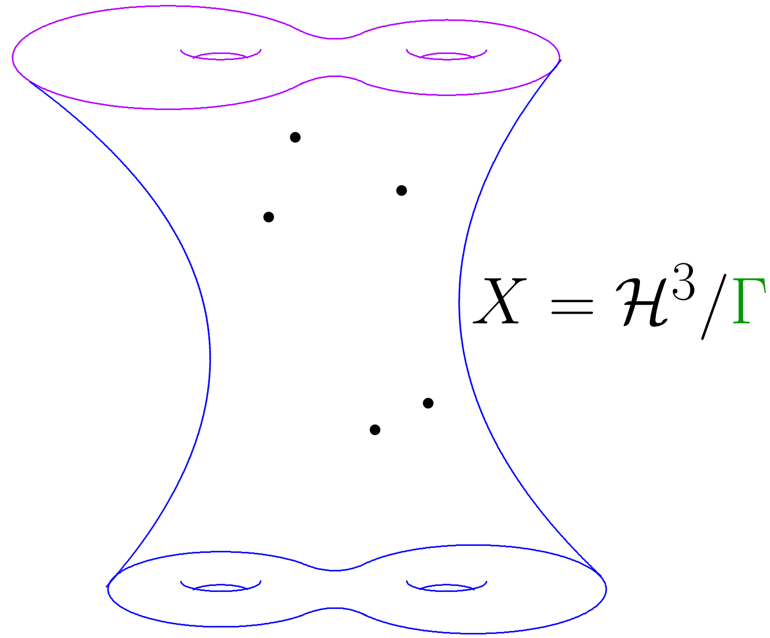
Let G_j be the Green's function of p_j:



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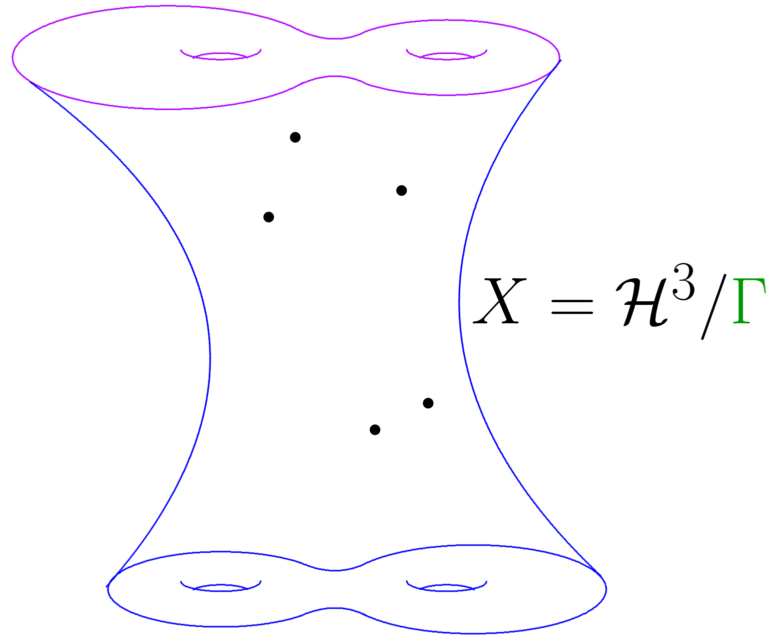
Let G_j be the Green's function of p_j:

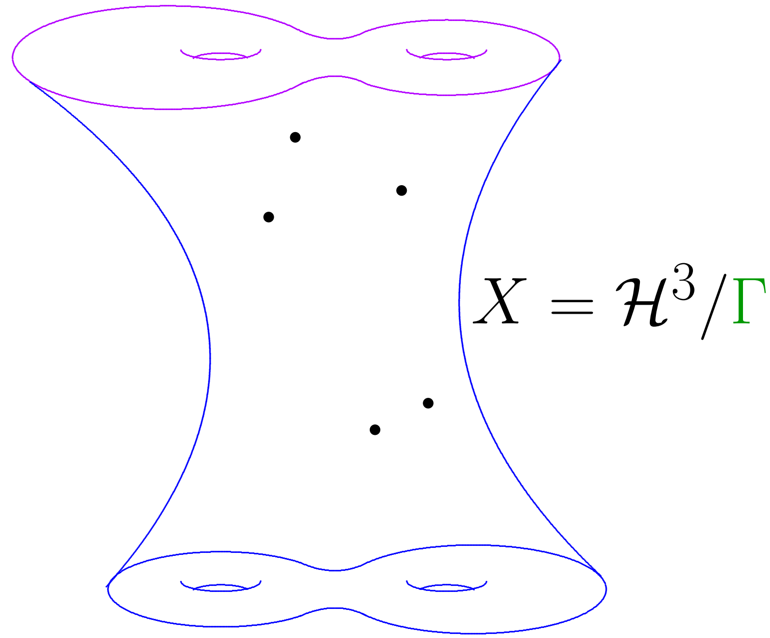
$$\Delta G_j = 2\pi\delta_{p_j}, \qquad G_j \to 0 \text{ at } \partial\overline{X}$$



Construction of ASD 4-manifolds:

Let G_j be the Green's function of p_j, and set

$$V = 1 + \sum_{j=1}^{k} G_j.$$



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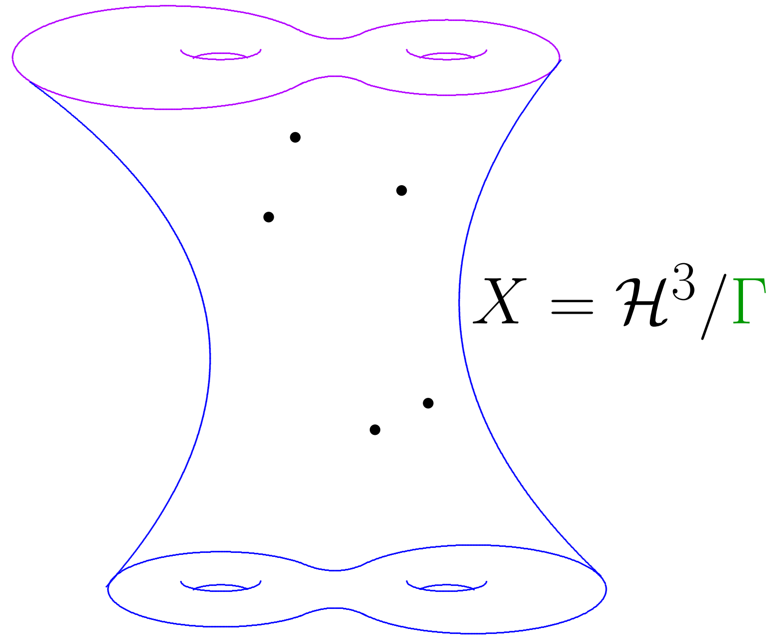

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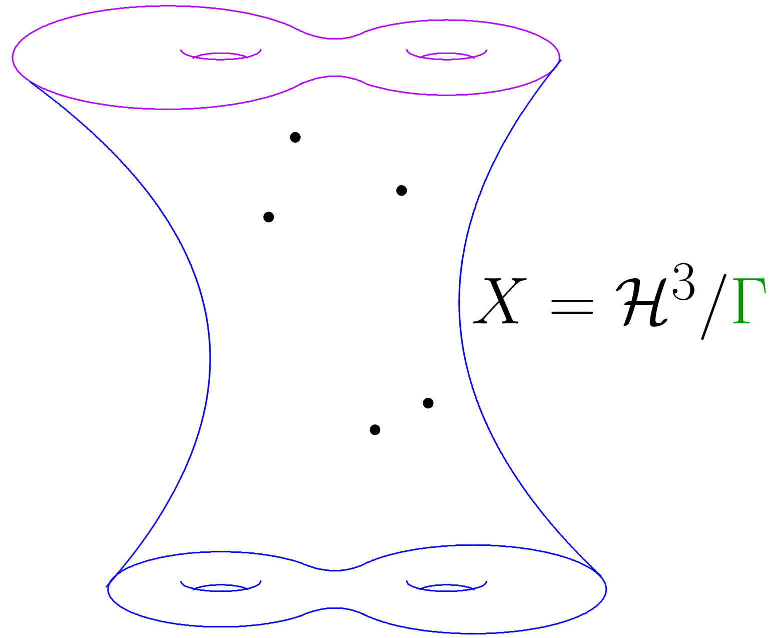
Choose $P \to (X - \{p_1, \ldots, p_k\})$ circle bundle with connection form θ such that

$$d\theta = \star dV.$$



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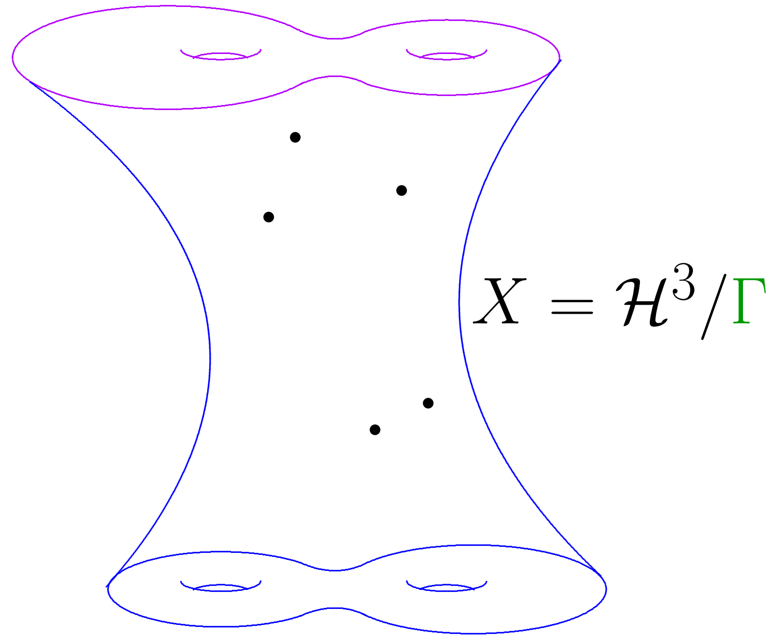

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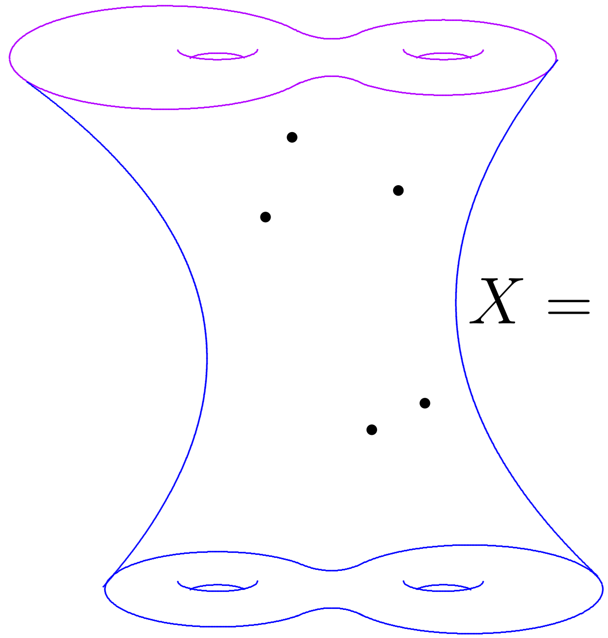

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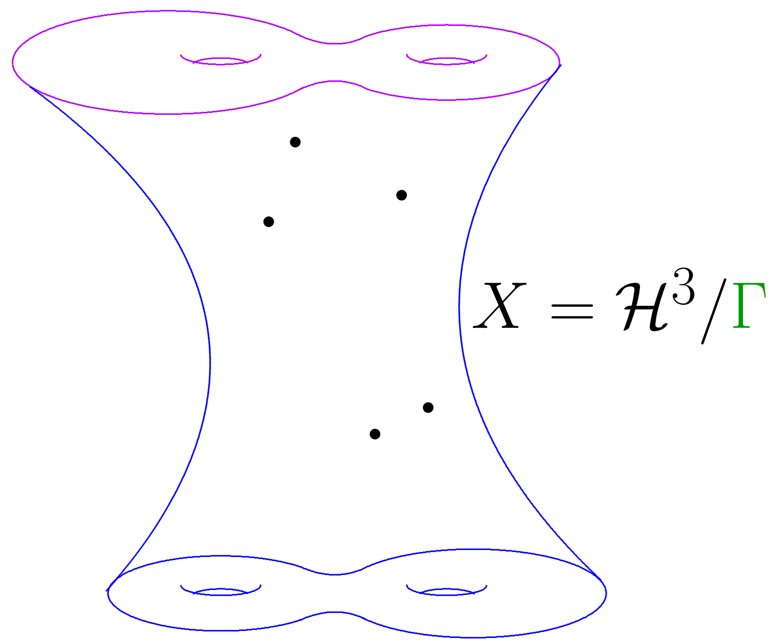

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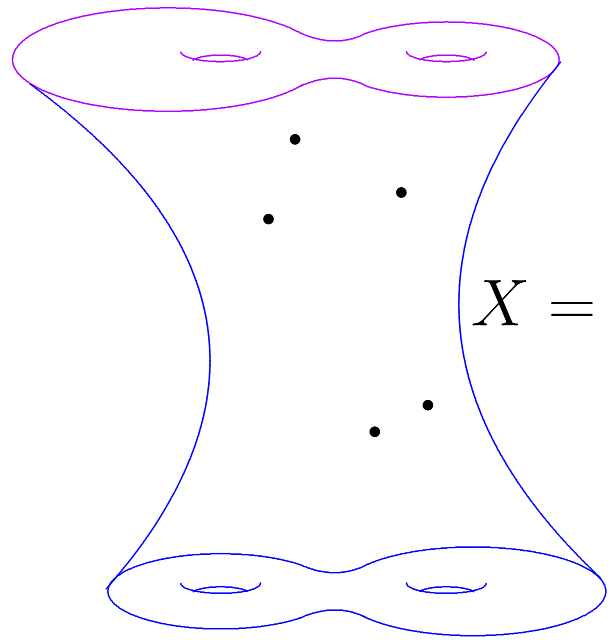
$$\begin{array}{ccccccc}
M & = & P & \cup & \{\hat{p}_1, \ldots, \hat{p}_k\} & \cup & \partial\overline{X} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{X} & = & X - \{p_1, \ldots, p_k\} & \cup & \{p_1, \ldots, p_k\} & \cup & \partial\overline{X}
\end{array}$$



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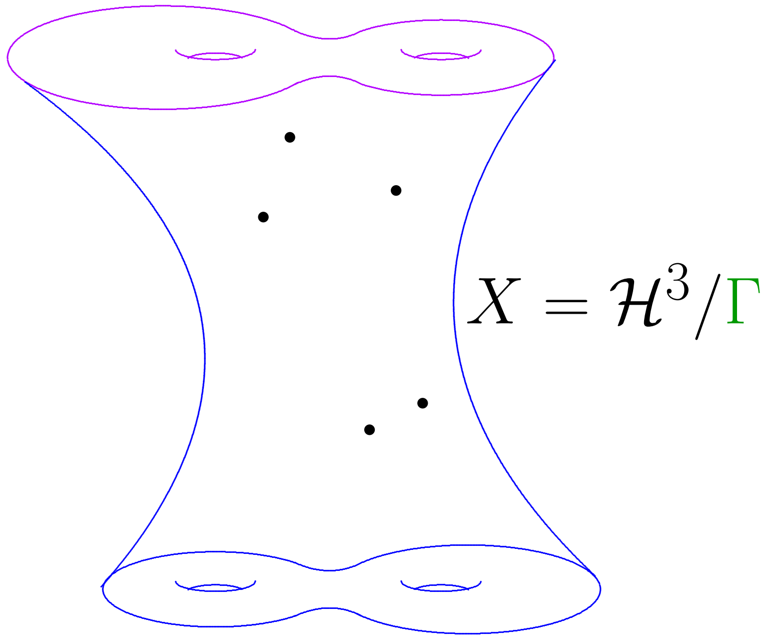

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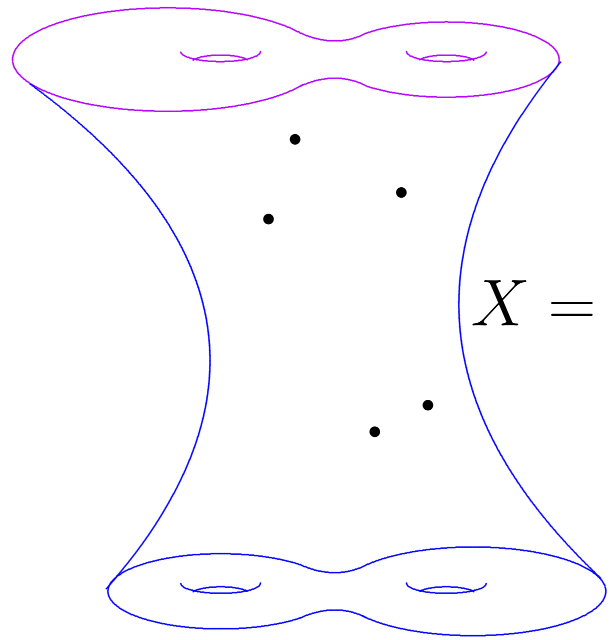

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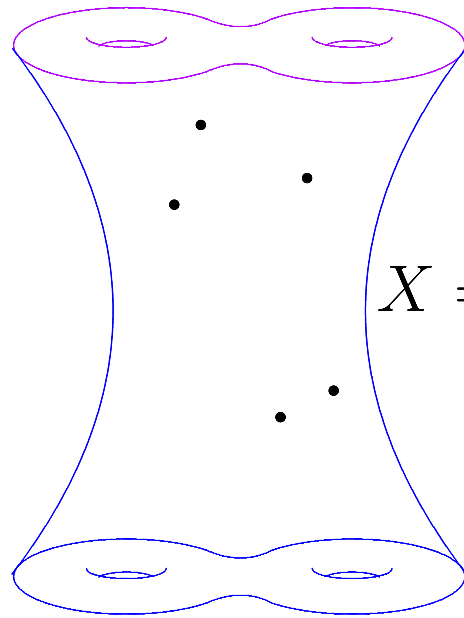

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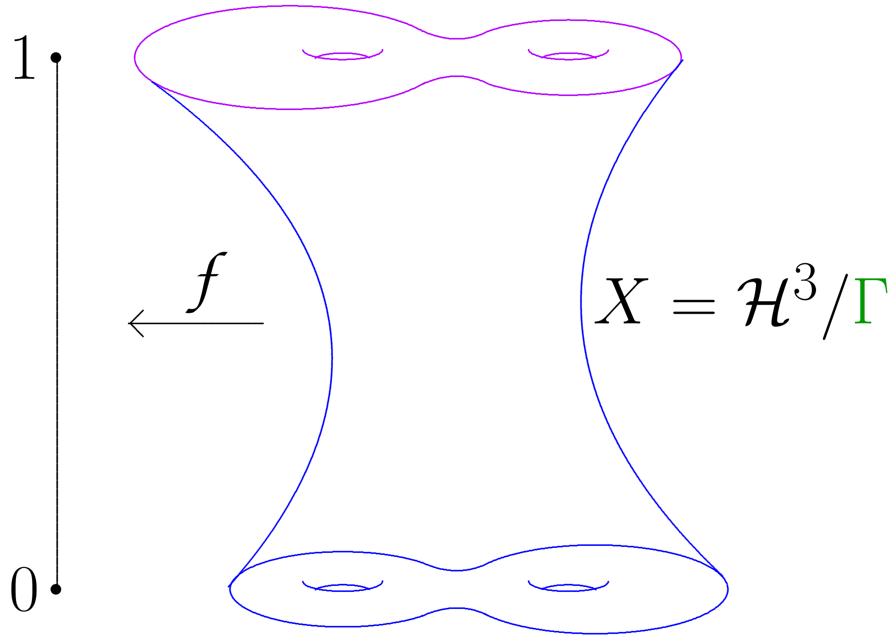

Γ quasi-Fuchsian

$$\overline{X} \approx \Sigma \times [0,1]$$

Tunnel-Vision function:

$$f : \overline{X} \to [0,1]$$

$$\Delta f = 0$$

Theorem.

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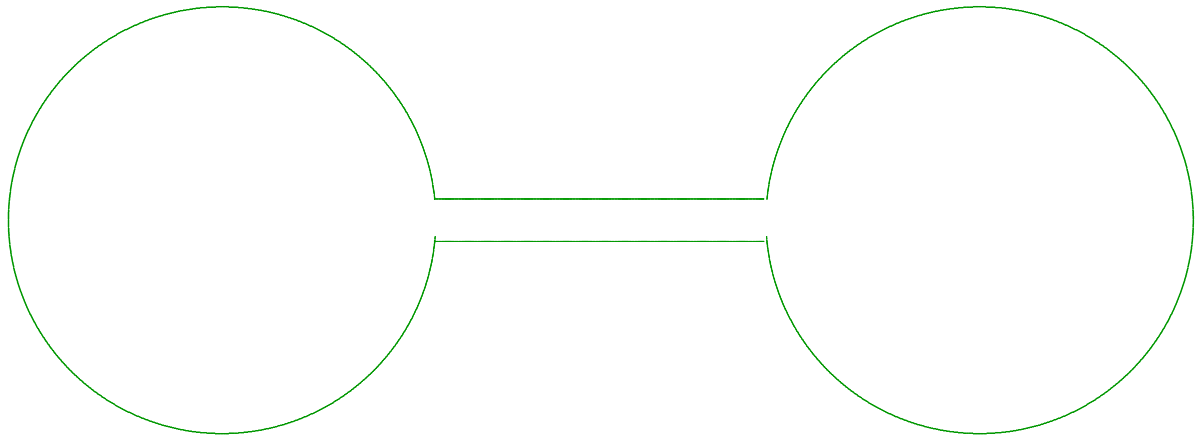
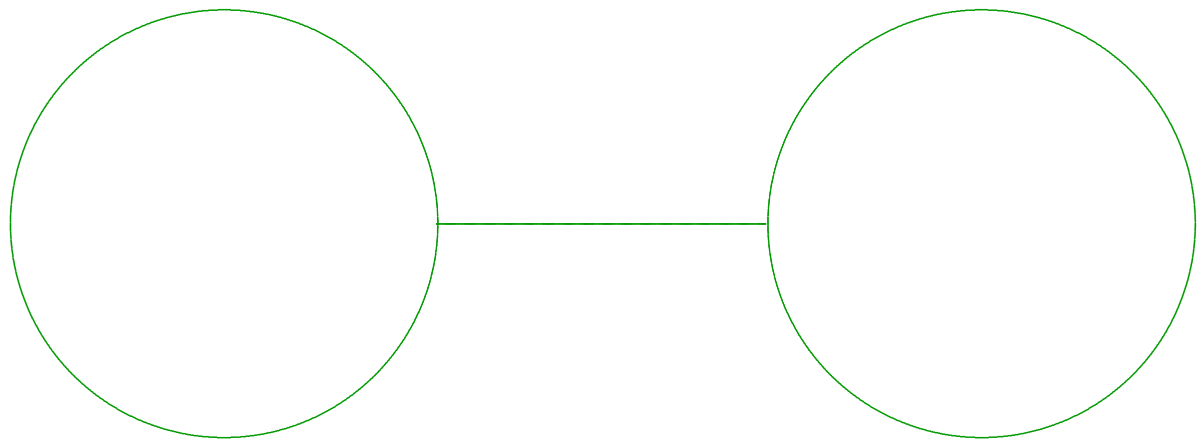
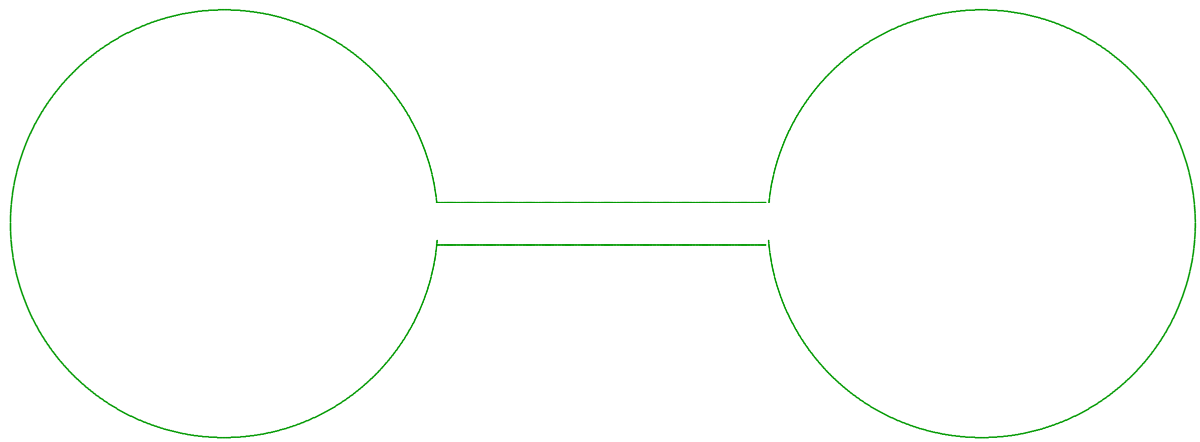
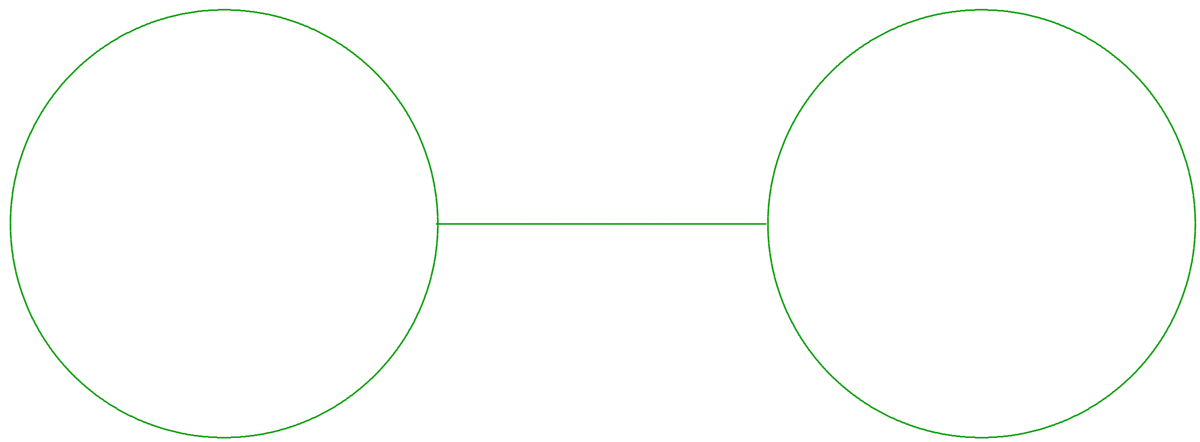
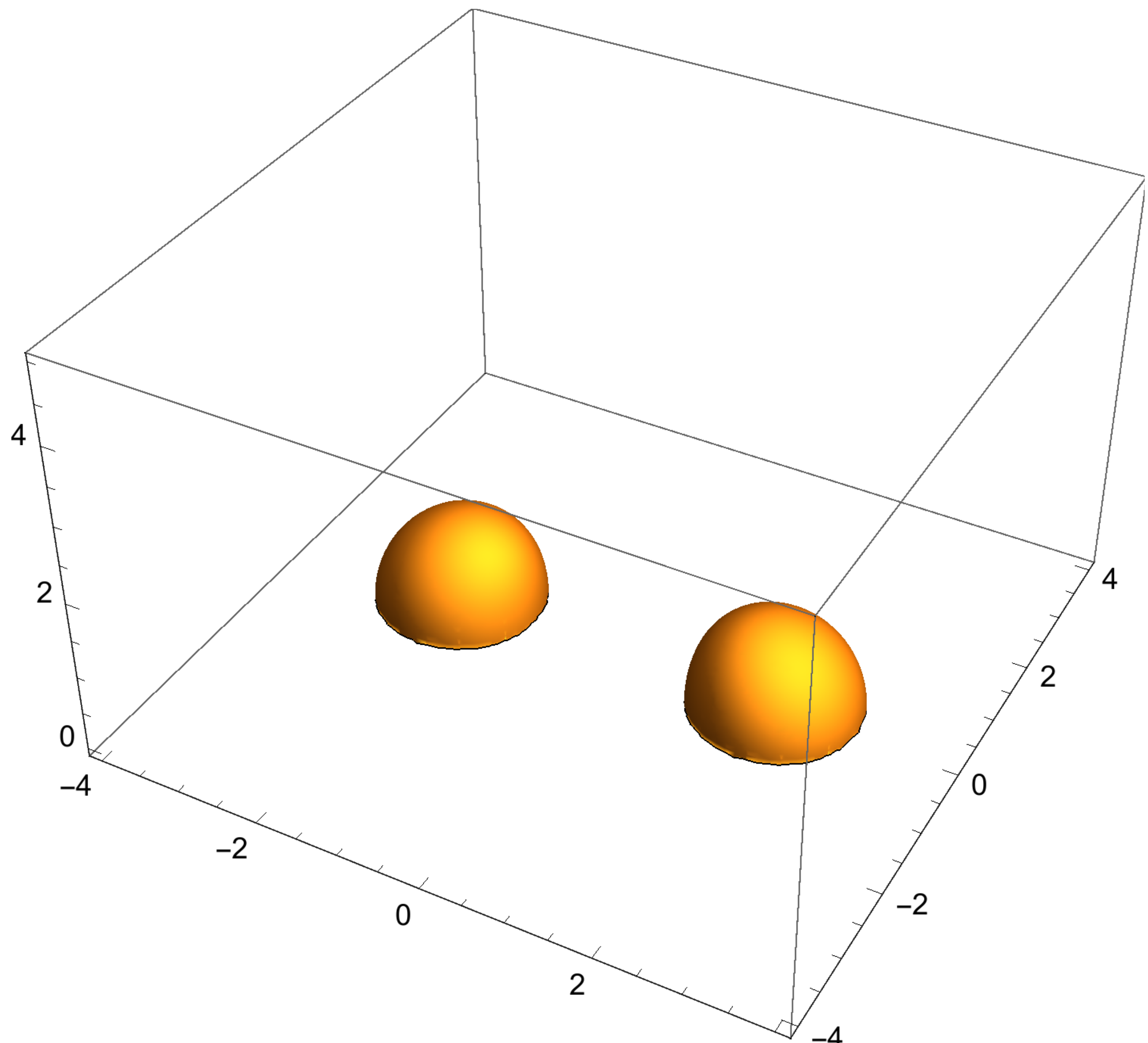
Ahlfors-Bers: Quasi-conformal mappings











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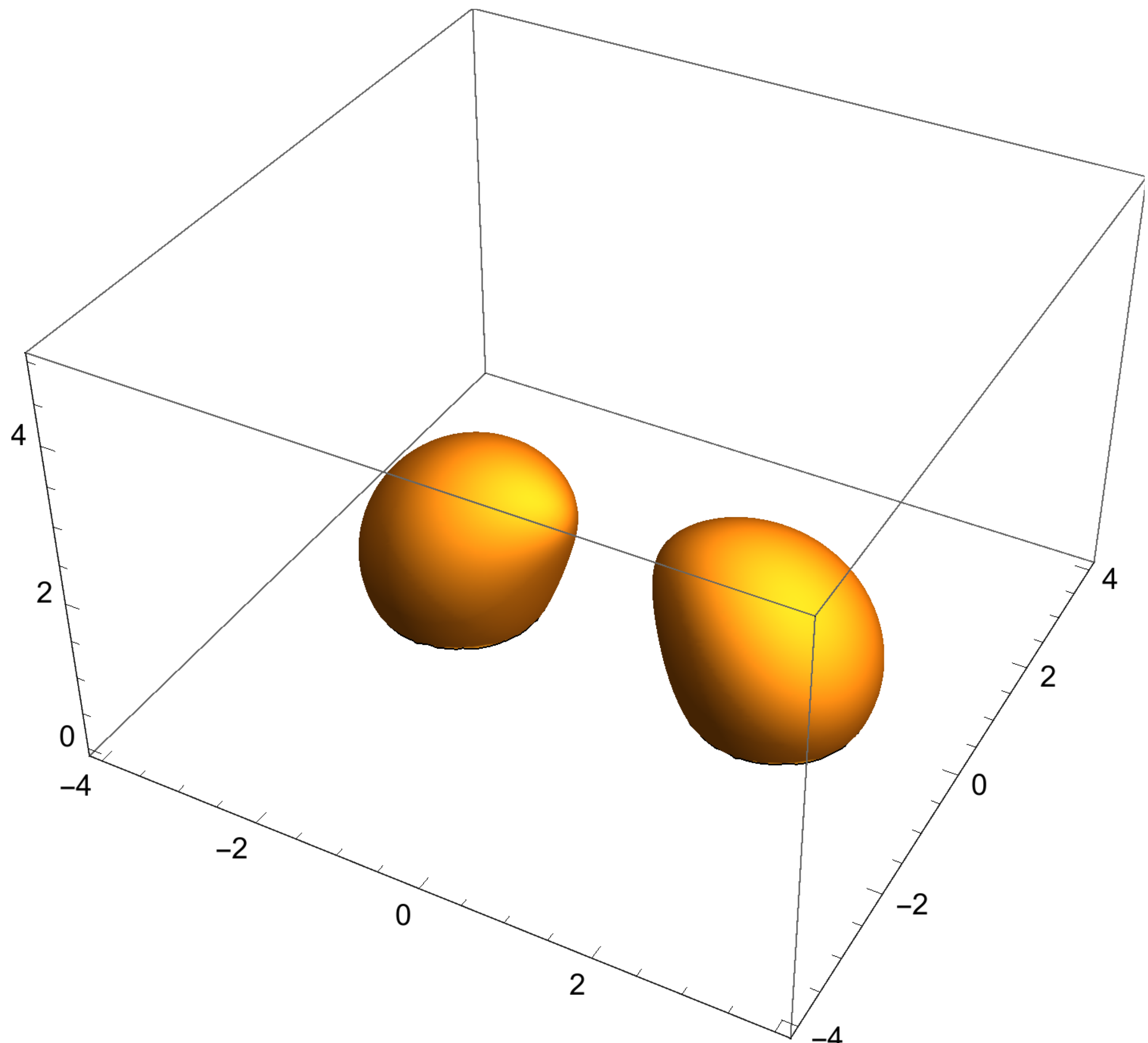

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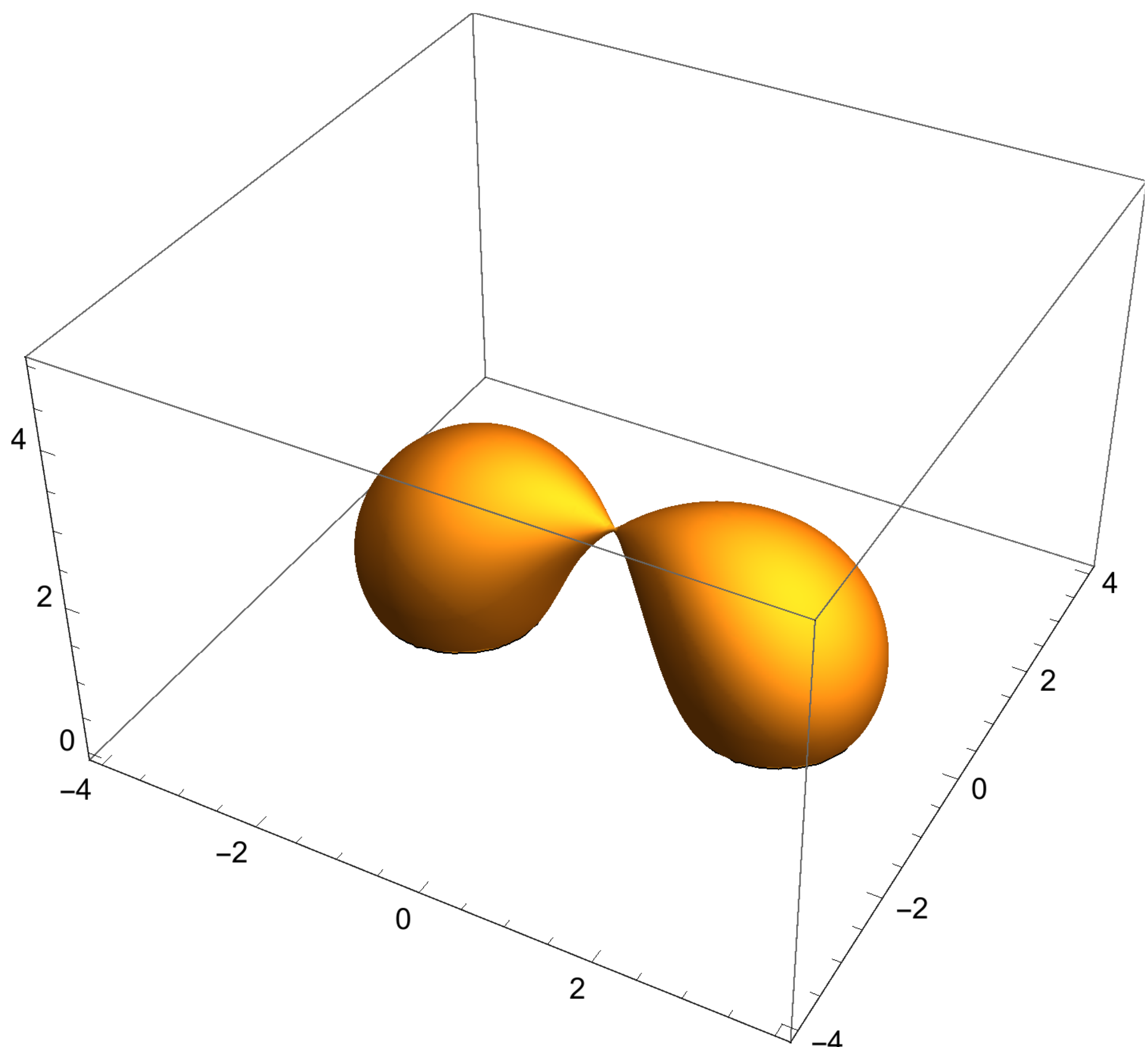
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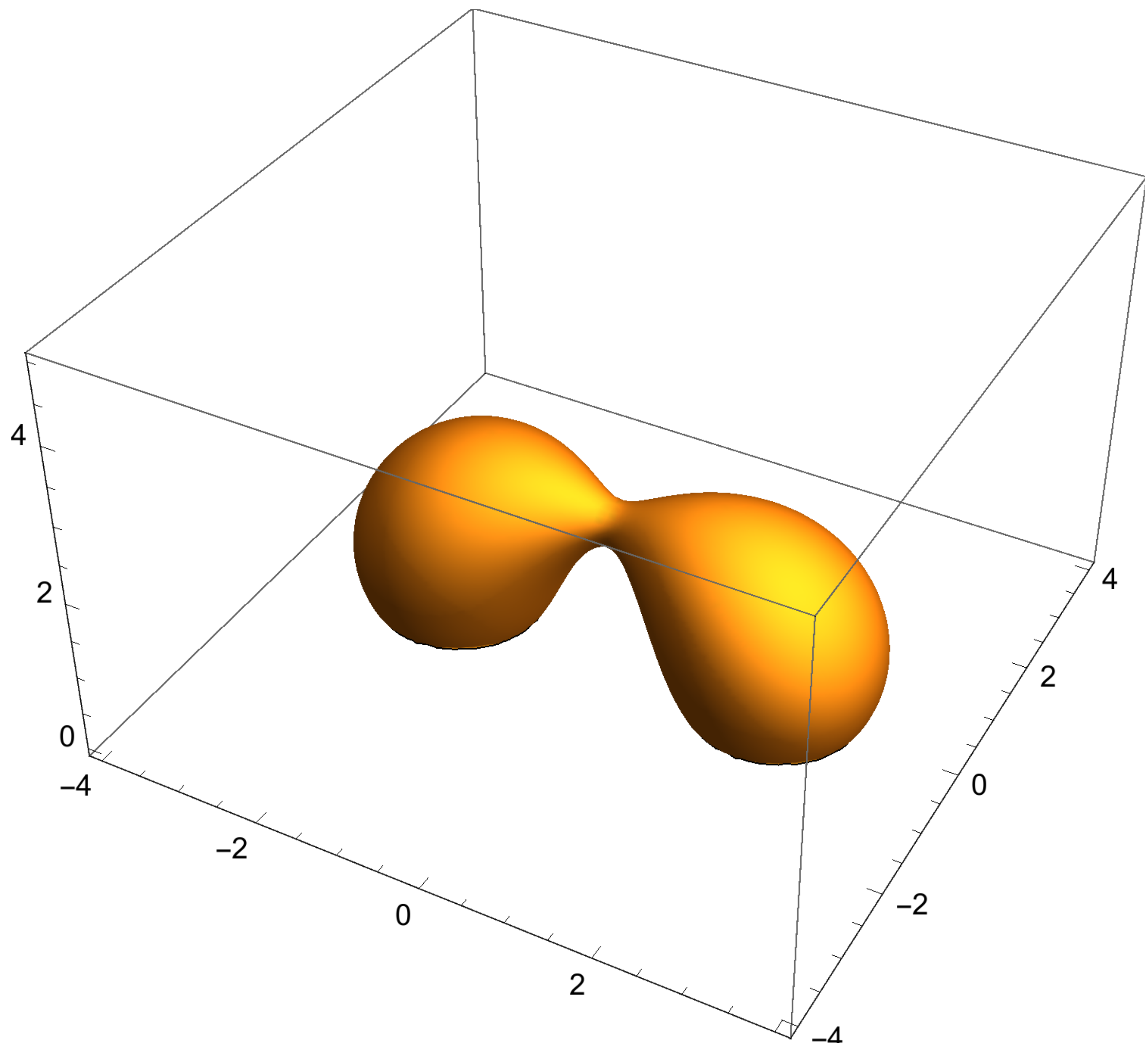
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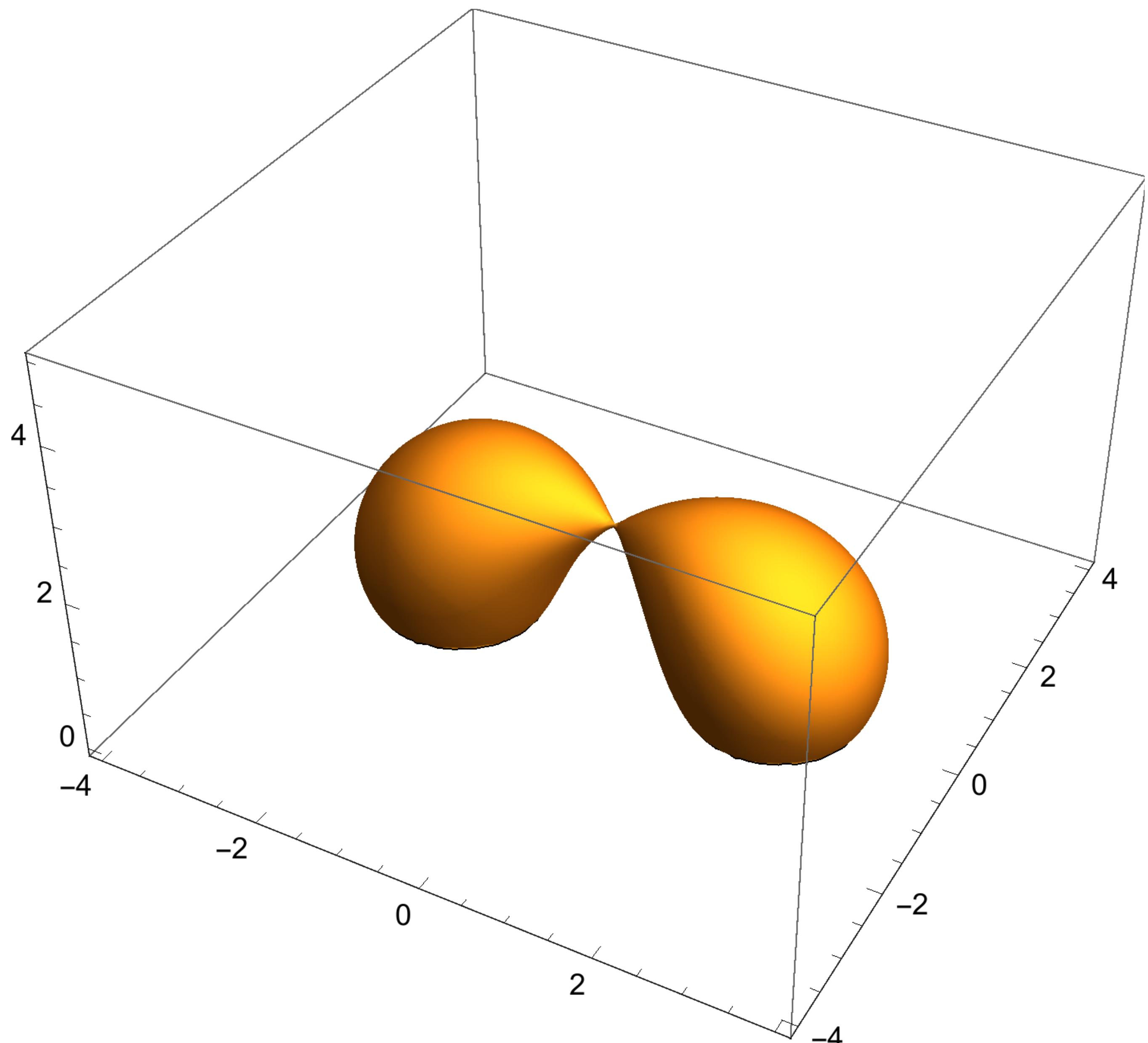

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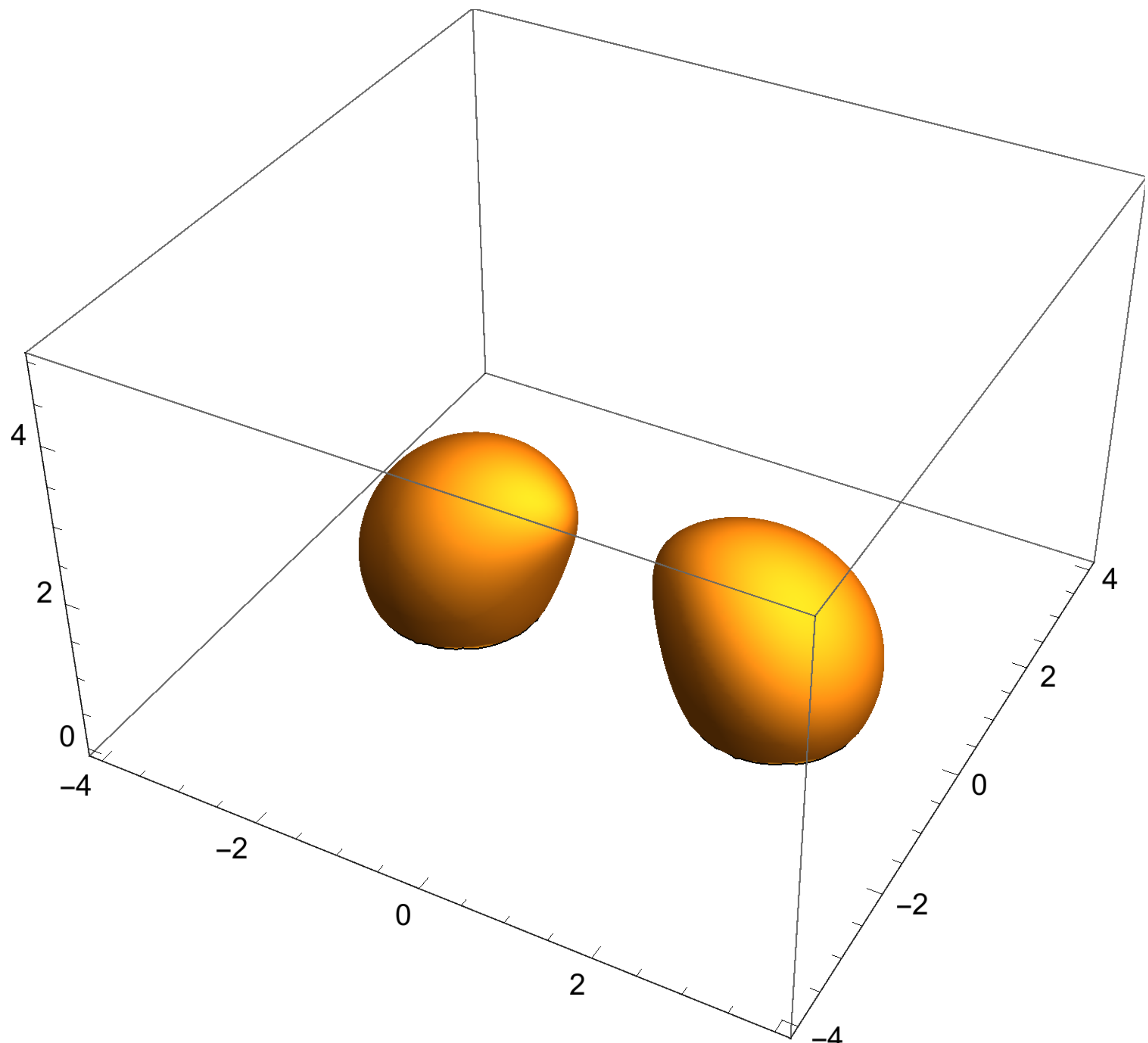

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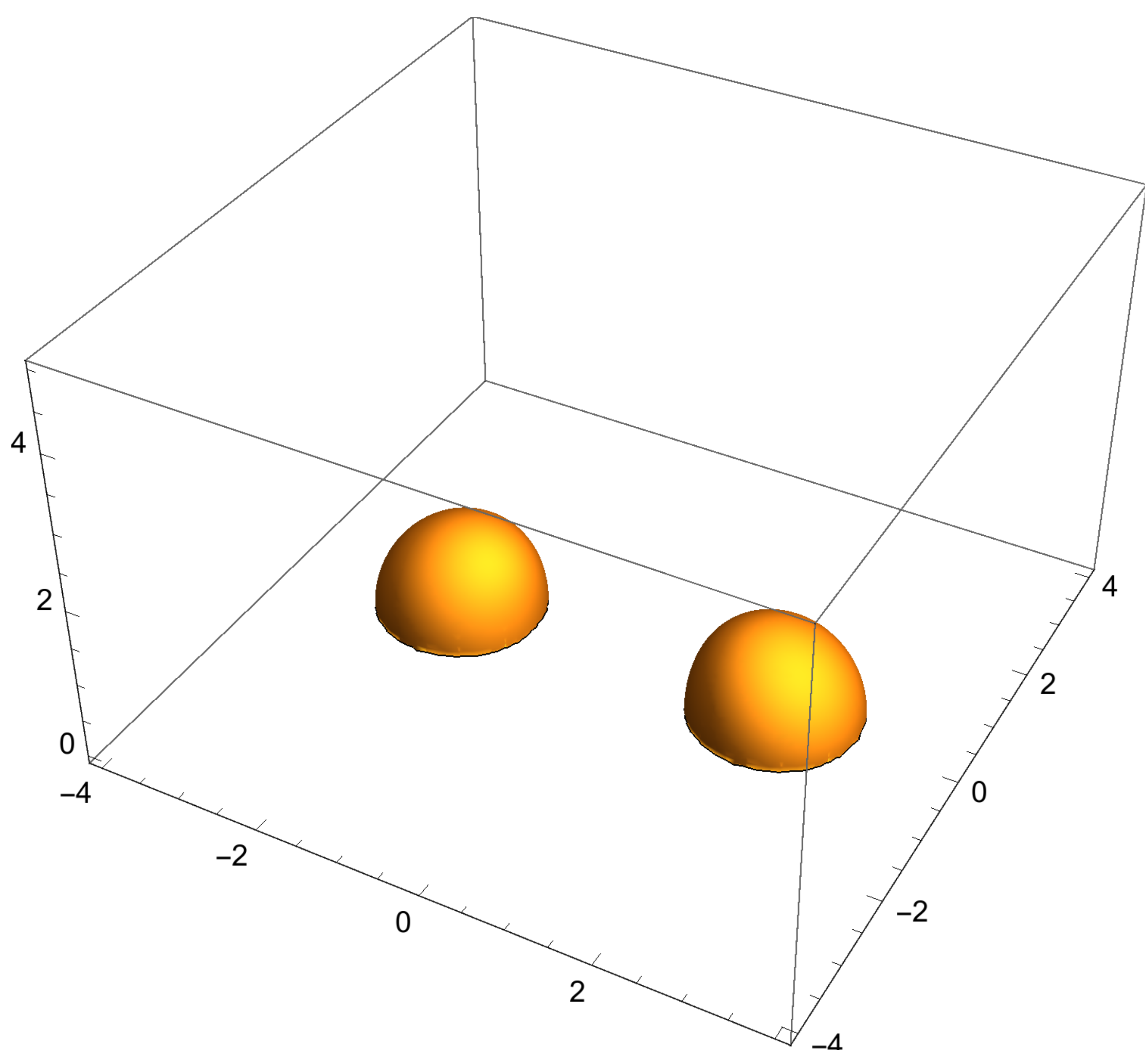

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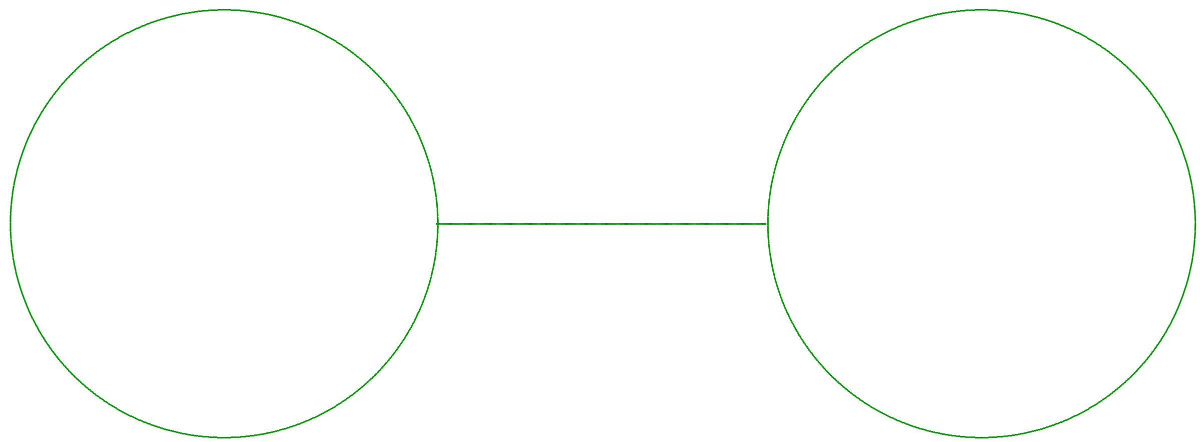
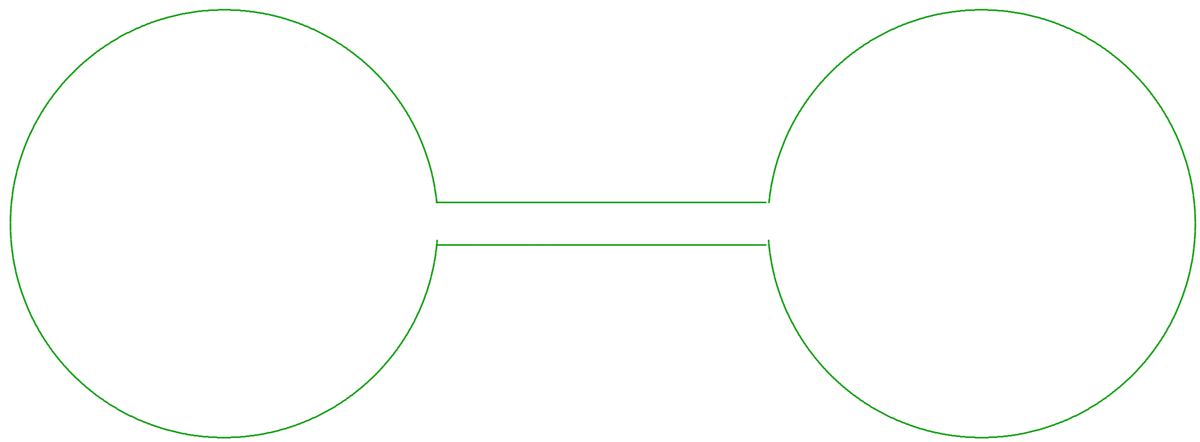
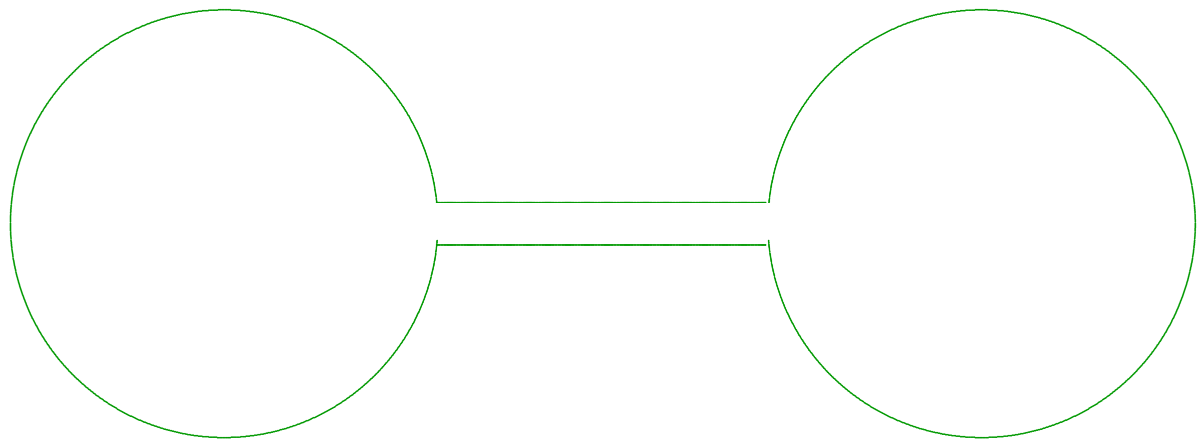
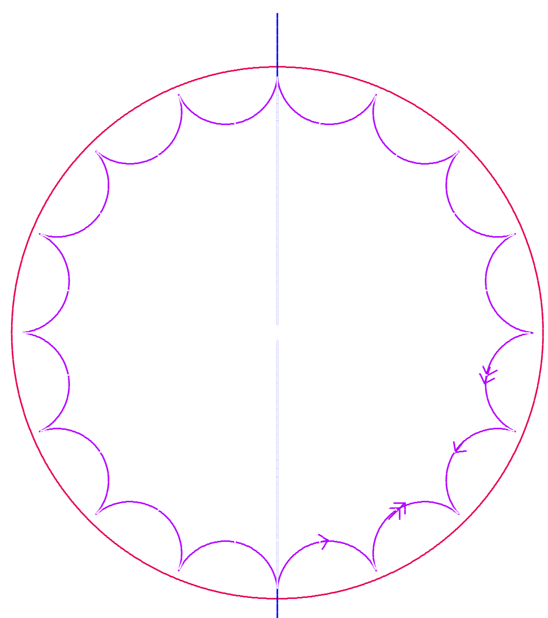
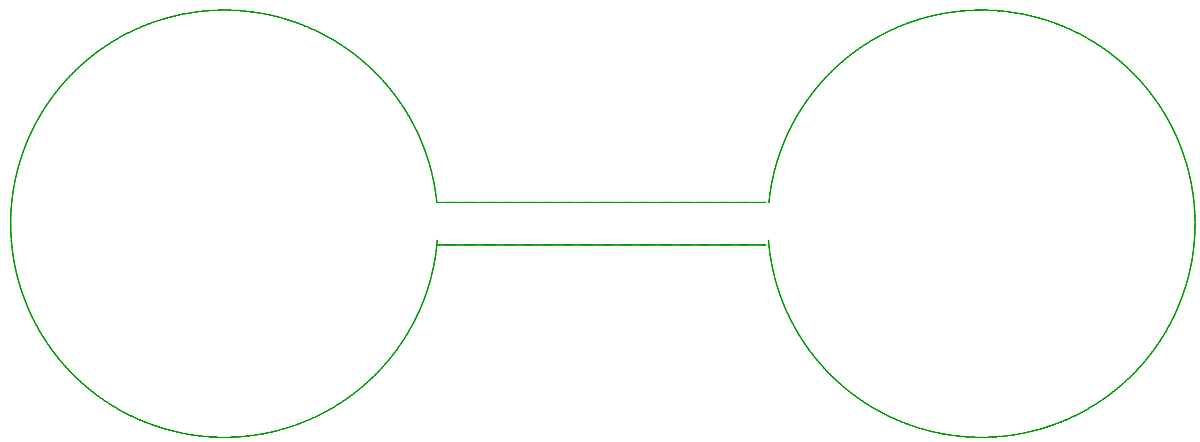
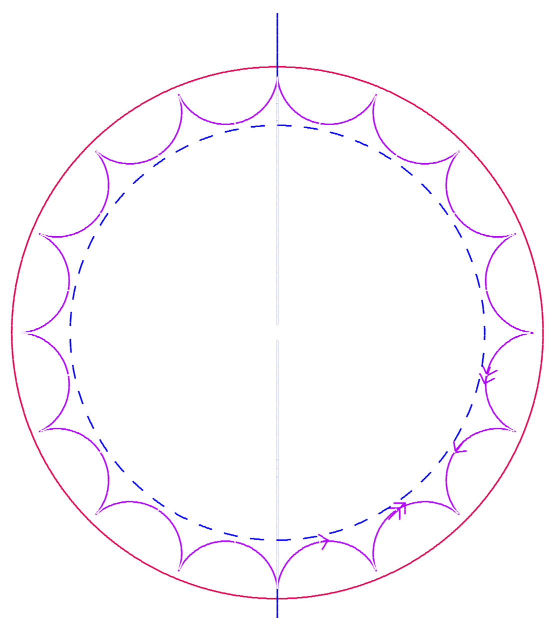


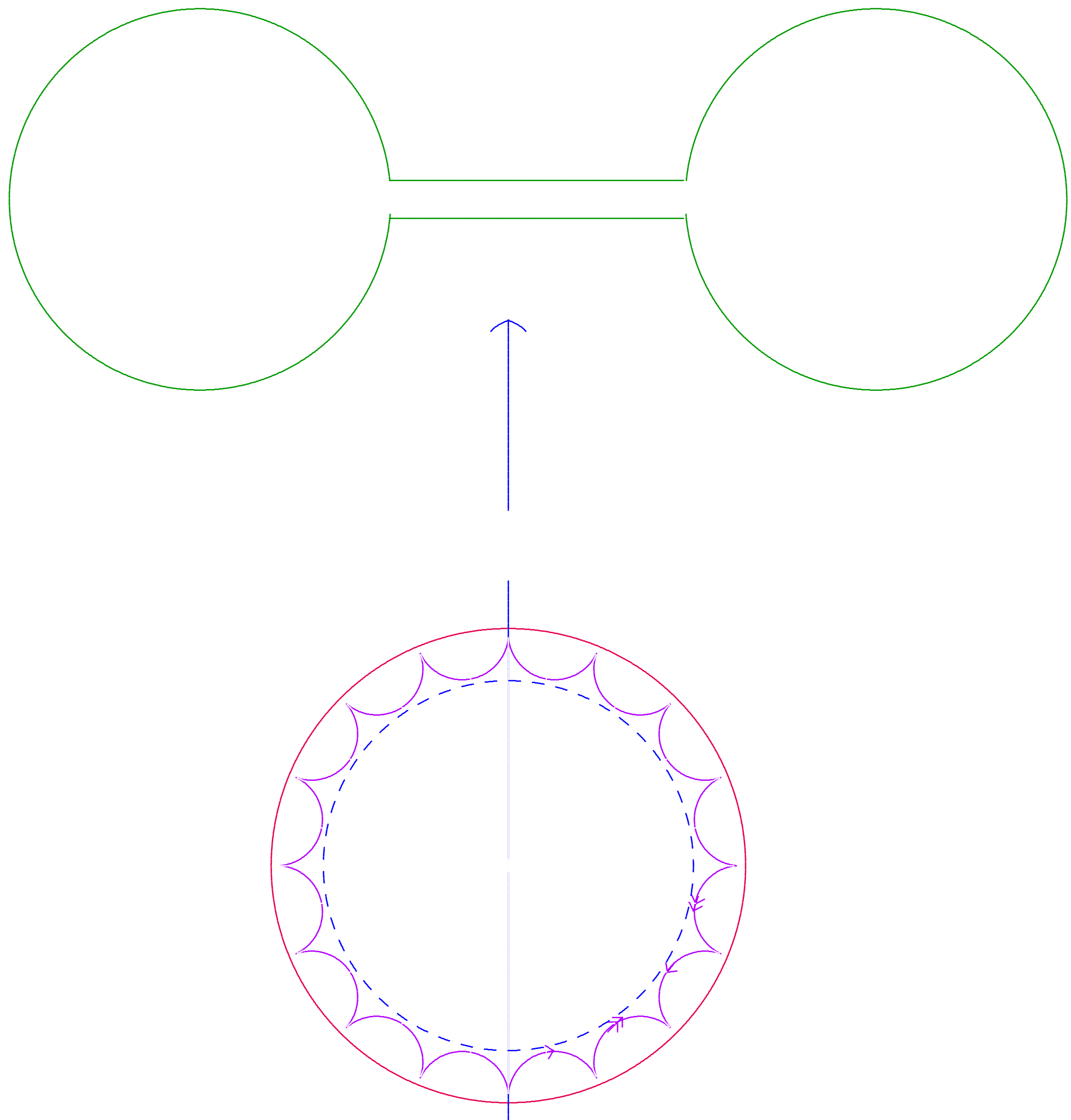


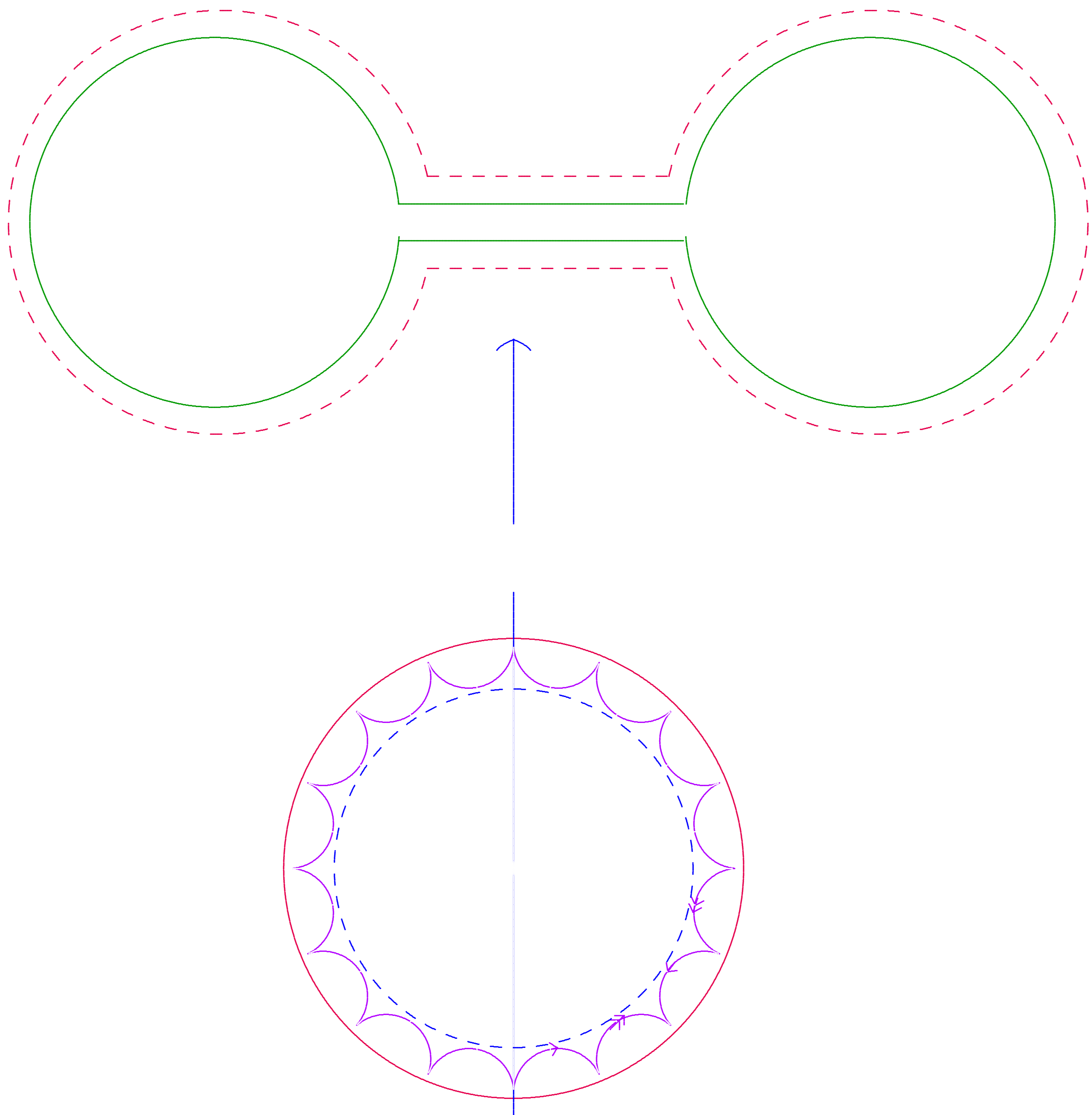


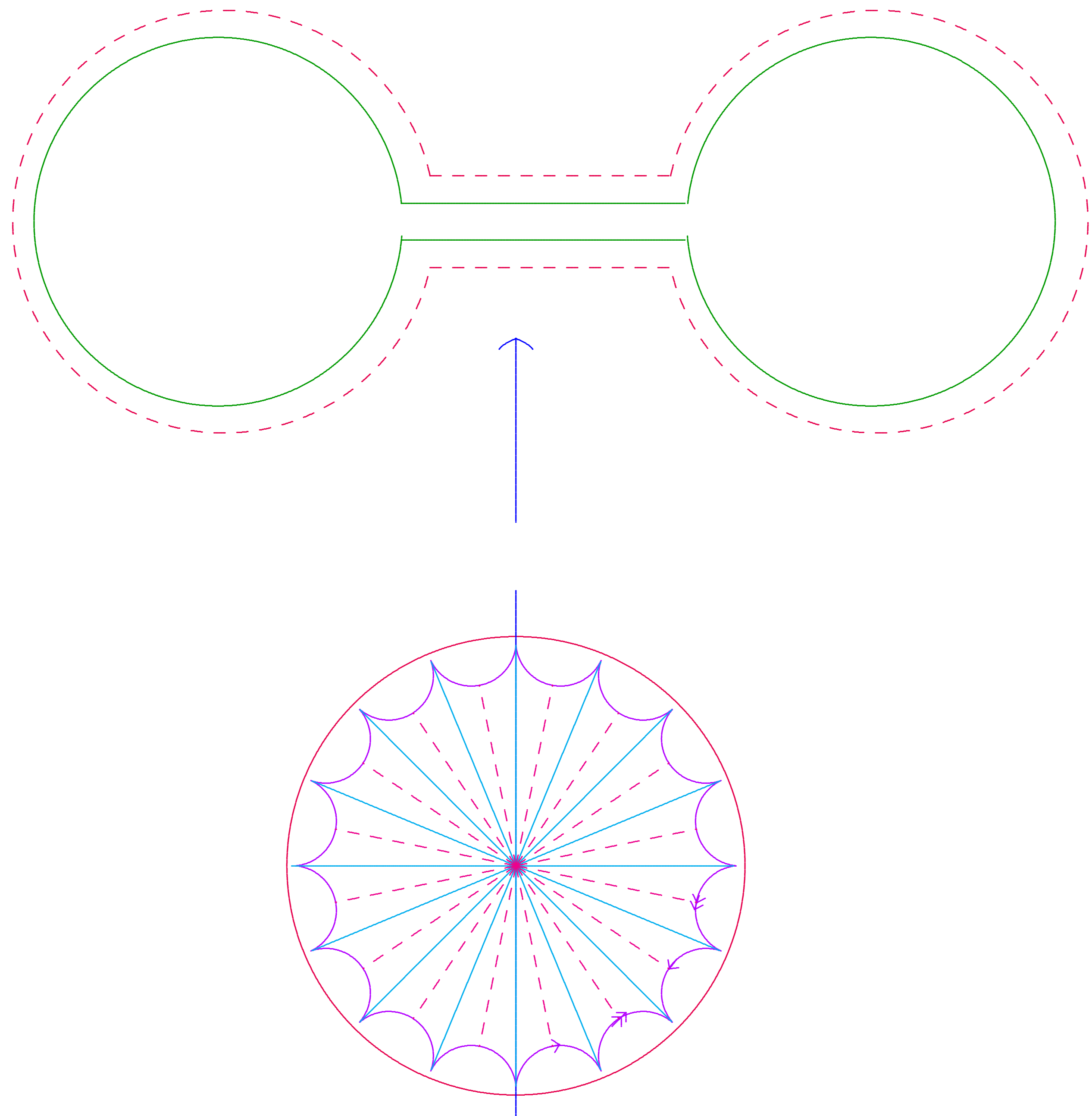


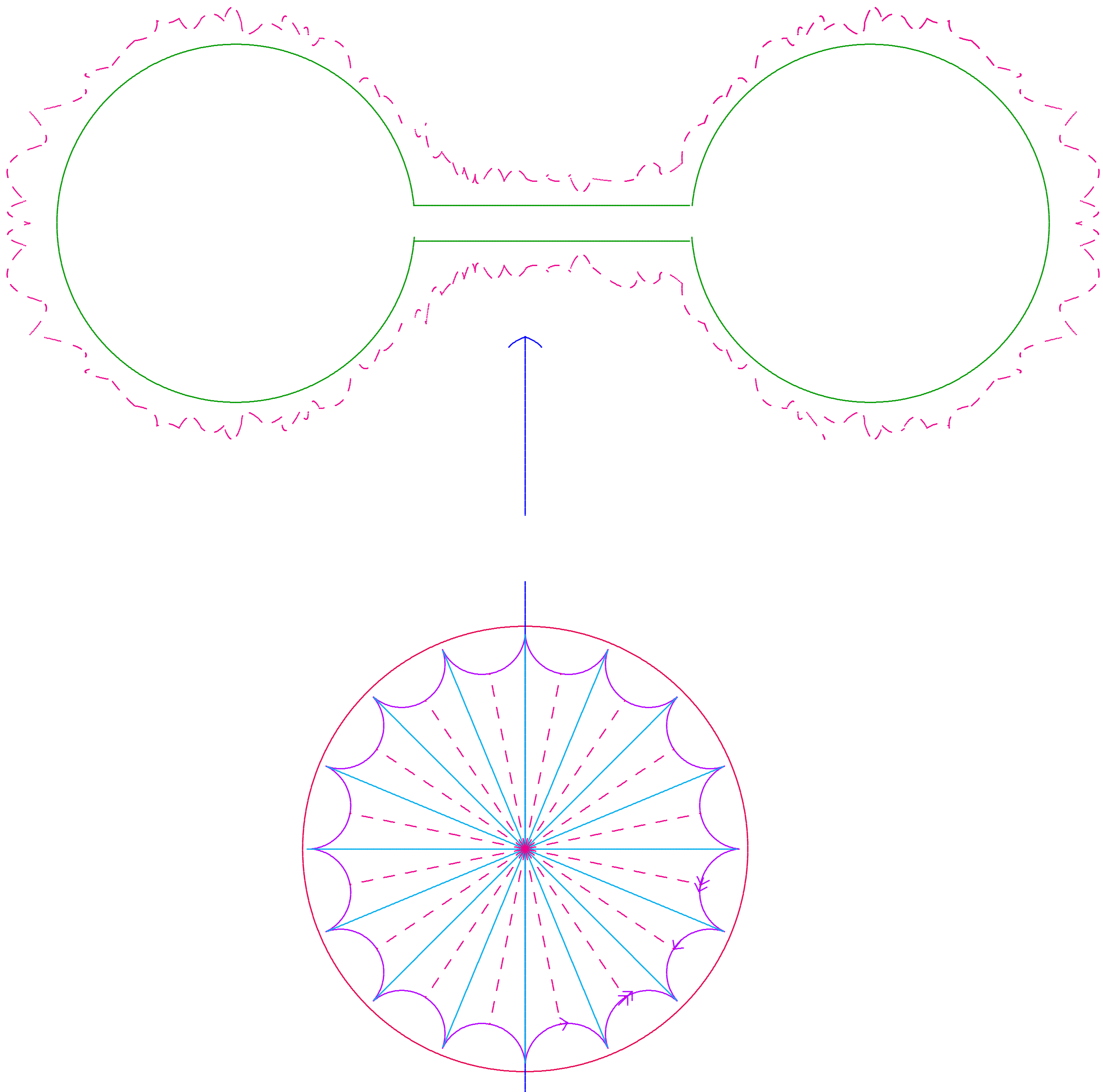








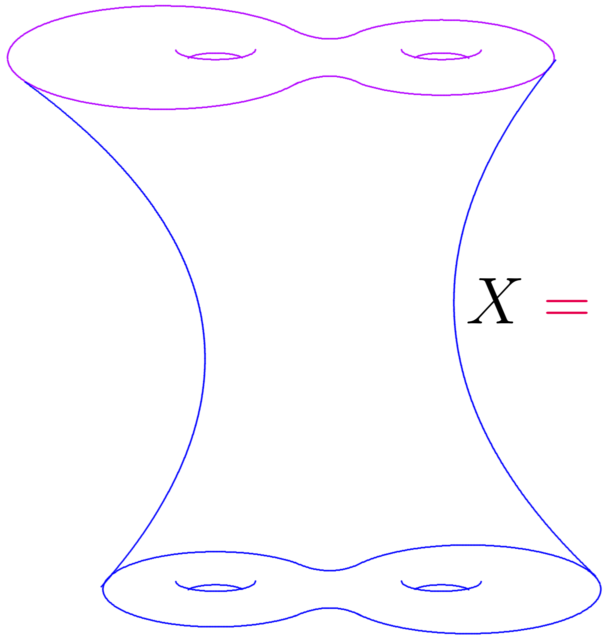




Theorem. *Consider 4-manifolds* $M = \Sigma \times S^2$, *where* Σ *compact Riemann surface of genus* $\mathbf{g}$.

Then $\forall$ *even* $\mathbf{g} \gg 0$, $\exists$ *family* $[g_t]$, $t \in [0, 1]$, *of locally-conformally-flat classes on* M, *such that*

- $\exists$ *scalar-flat Kähler metric* $g_0 \in [g_0]$; *but*
- $\nexists$ *almost-Kähler metric* $g \in [g_1]$.



$X = \mathcal{H}^3/\Gamma$

Construction of conformally flat 4-manifolds:

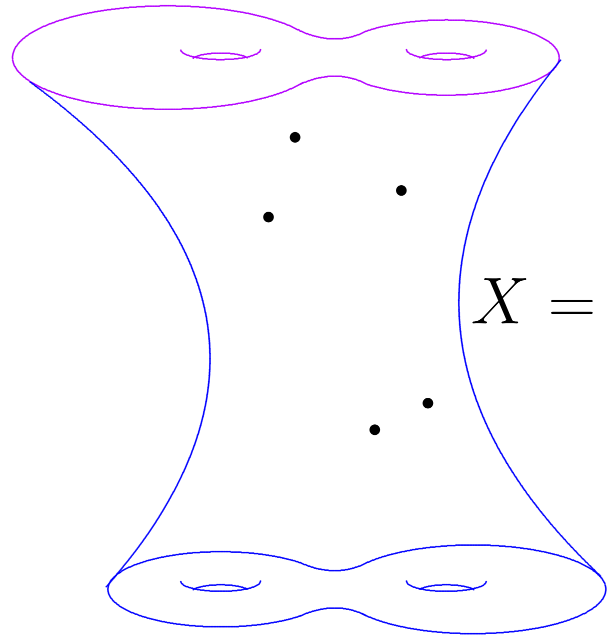
$$M = [\overline{X} \times S^1]/\sim$$

$$g = f(1-f)[h + dt^2]$$

Theorem. *Fix an integer* $k \geq 2$*, and then consider the 4-manifolds* $M = (\Sigma \times S^2)\#k\overline{\mathbb{CP}}_2$*, where* Σ *compact Riemann surface of genus* $\mathfrak{g}$*.*

Then $\forall$ *even* $\mathfrak{g} \gg 0$*,* $\exists$ *family* $[g_t]$*,* $t \in [0, 1]$*, of anti-self-dual conformal classes on* M*, such that*

- $\exists$ *scalar-flat Kähler metric* $g_0 \in [g_0]$*; but*
- $\nexists$ *almost-Kähler metric* $g \in [g_1]$*.*



Construction of ASD 4-manifolds:

$$g = f(1-f)[Vh + V^{-1}\theta^2]$$

$$V = 1 + \sum_{j=1}^{k} G_j$$

$$d\theta = \star dV$$

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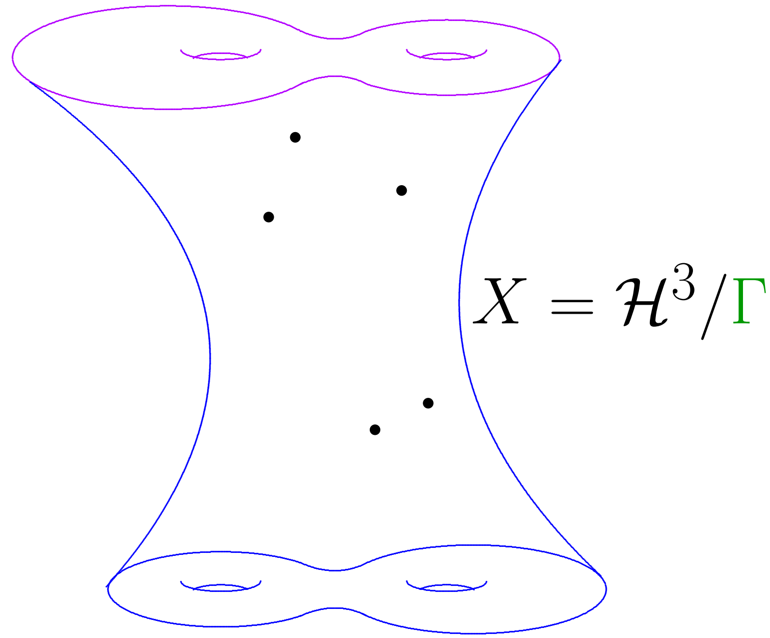

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¡Muchas Gracias por la Invitación!