Bach-Flat

Kähler Surfaces

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$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2} \mathring{r}^{[a}{}_{[c} \delta^{b]}{}_{d]} + \frac{2}{n(n-1)} \, s \, \delta^{a}_{[c} \delta^{b}_{d]}$$

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$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

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Locally-conformally-flat: $g = u^2 \sum_{j=1}^{n}(dx^j)^{\otimes 2}$ in suitable local coordinates near any point.

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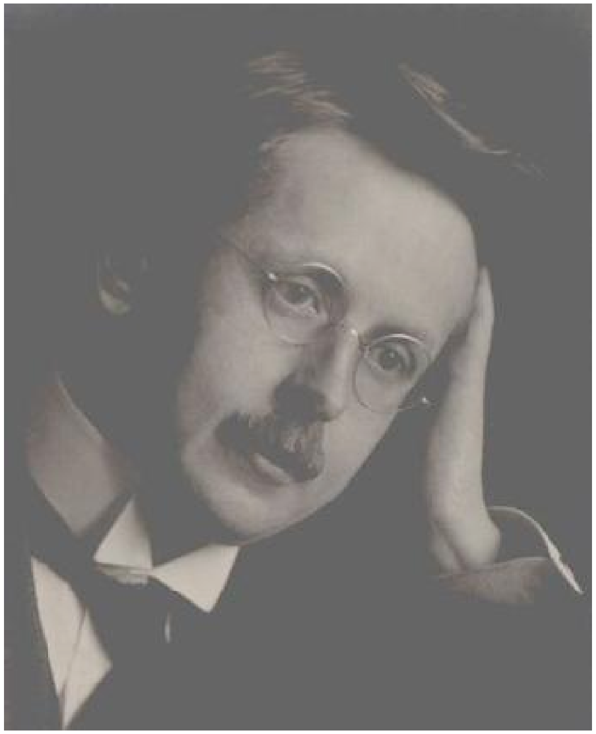

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- Are there any critical points?
- Can we classify them?

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when $\ell > 0$, because $\mathscr{W} \propto \mathrm{Vol}(T^\ell)$!

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Why is Dimension Four Exceptional?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

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$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right)$$

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	Λ^{+*}	Λ^{-*}
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$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W_+ &= \text{self-dual Weyl curvature} \\ W_- &= \text{anti-self-dual Weyl curvature} \end{aligned}$$

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More generally,

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More generally, anti-self-dual 4-mnfds are Bach-flat.

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Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

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$W_+ := \frac{1}{2}(W + \star W)$ called self-dual Weyl tensor.

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Anti-self-dual 4-manifolds: $\Leftrightarrow W = -\star W$.

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Anti-self-dual 4-manifolds: $\Leftrightarrow W = W_-$.

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$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

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for signature $\tau(M) = b_+(M) - b_-(M)$, where $b_\pm(M) = \max \dim$ subspaces $\subset H^2(M, \mathbb{R})$ on which intersection pairing

$$\begin{array}{ccc} H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) & \longrightarrow & \mathbb{R} \\ ([\varphi] , [\psi]) & \mapsto & \int_M \varphi \wedge \psi \end{array}$$

is positive (resp. negative) definite.

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Anti-self-dual 4-manifolds: $W_+ \equiv 0$.

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$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

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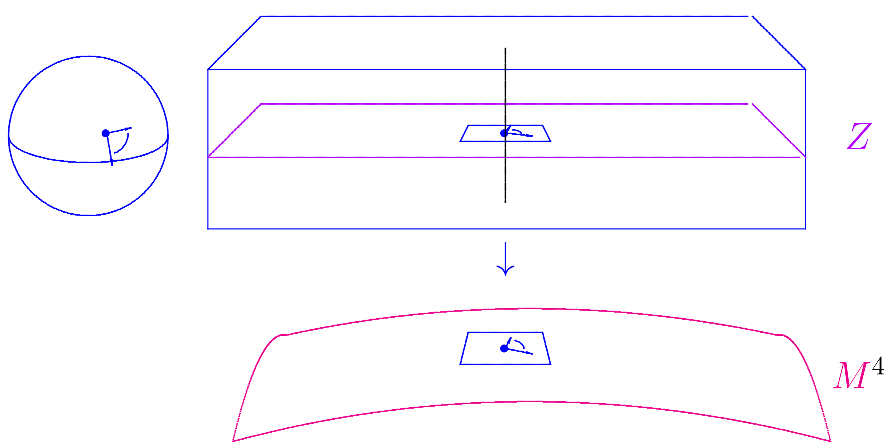

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Violate Hitchin-Thorpe, so $\nexists$ Einstein on such M.

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L-Singer '93, Kim-L-Pontecorvo '97 Any rational/ruled (M, J) has blow-ups admitting SFK.

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$f : M \to \mathbb{R}$ with $df \neq 0$ along $f^{-1}(0) \neq \varnothing$.

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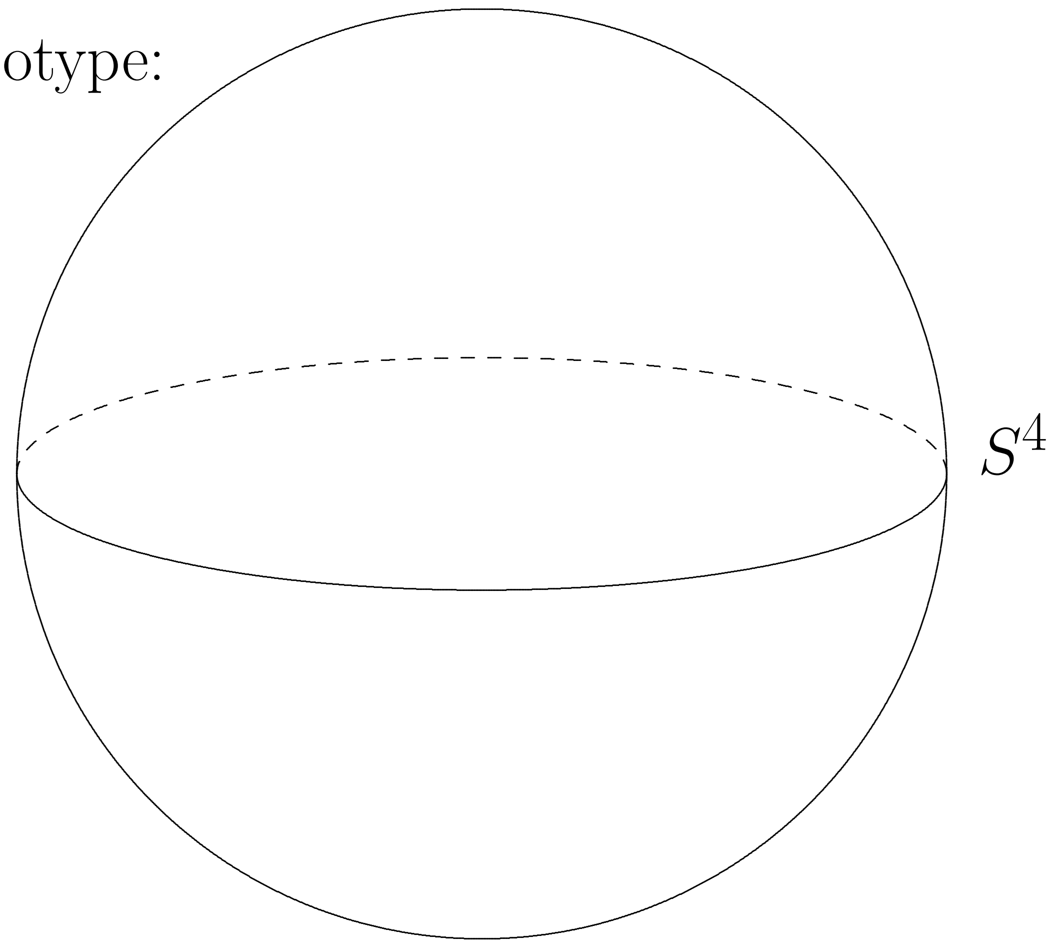

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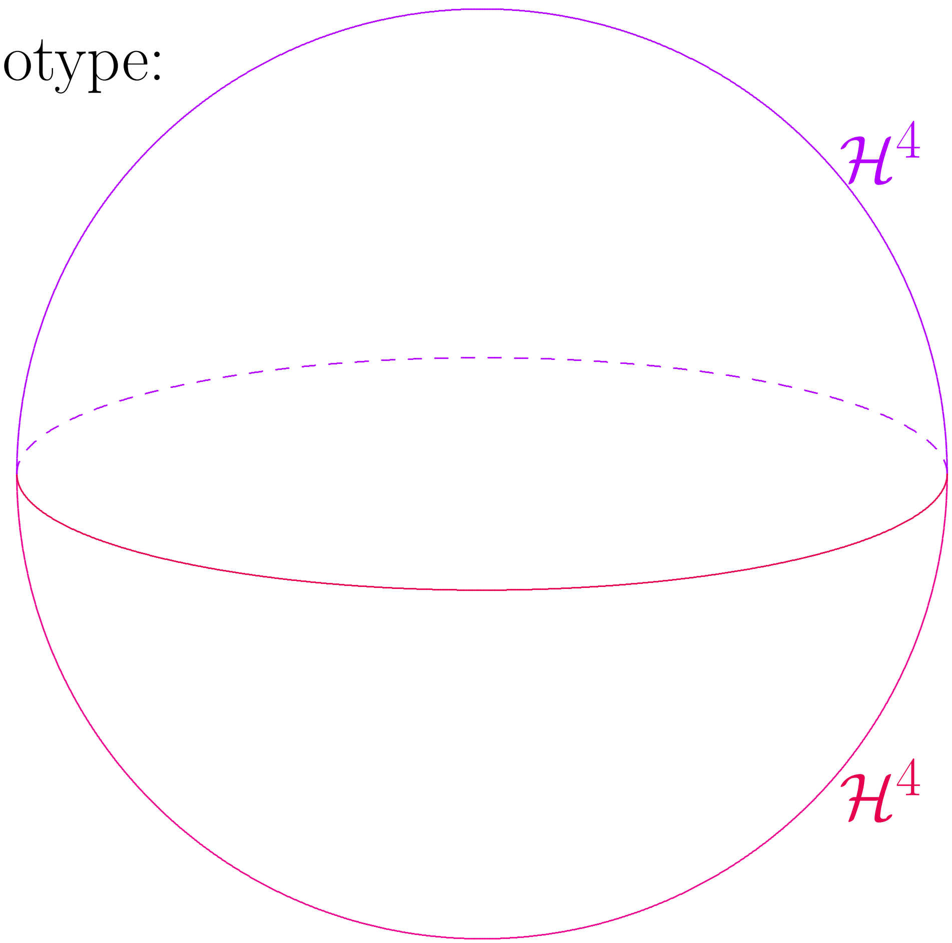

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S^4 is also Einstein, ASD.

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No! anti-self-dual 4-manifolds: $W_+ \equiv 0$.

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But $\exists$ genuine examples that aren't.

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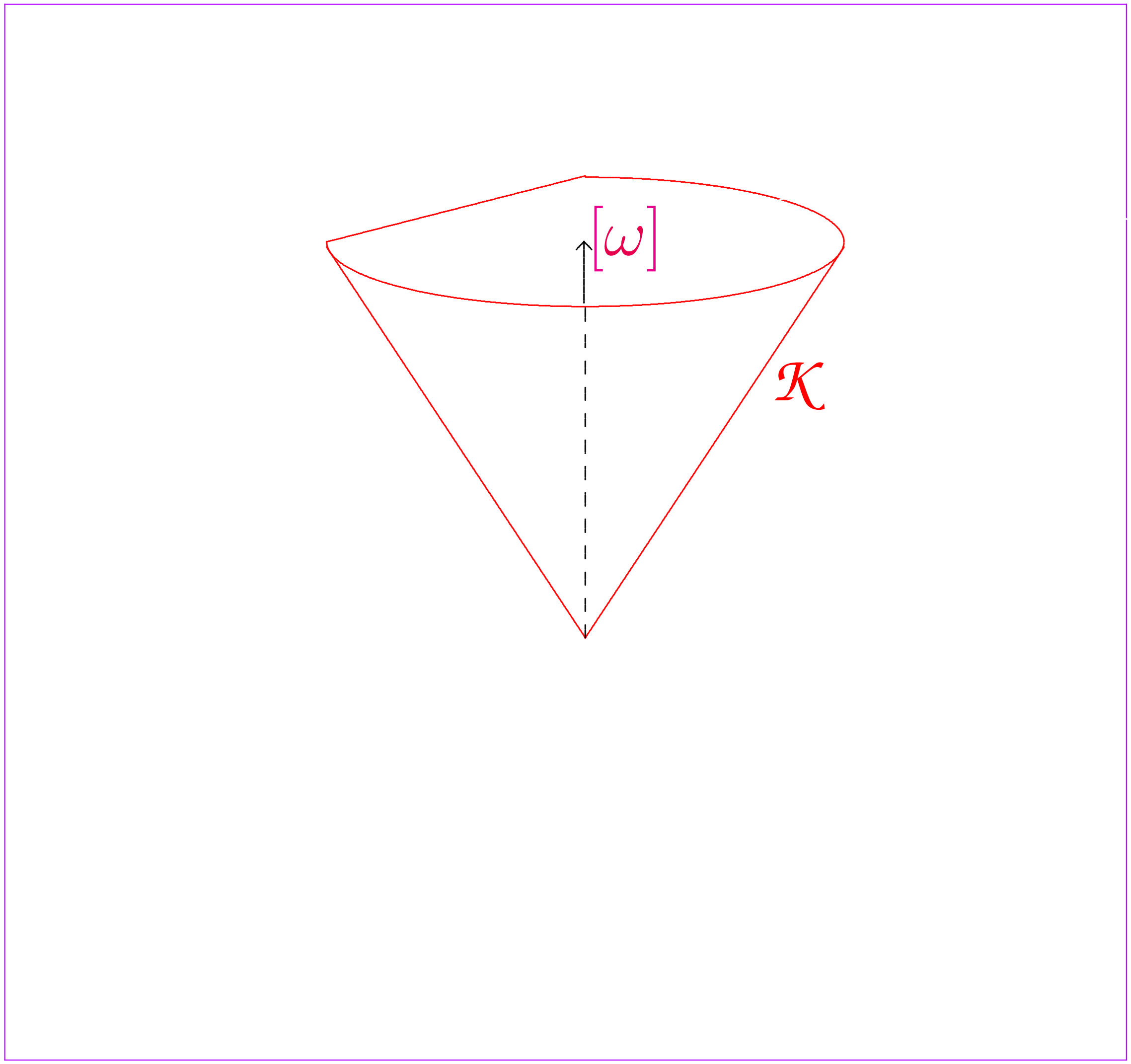

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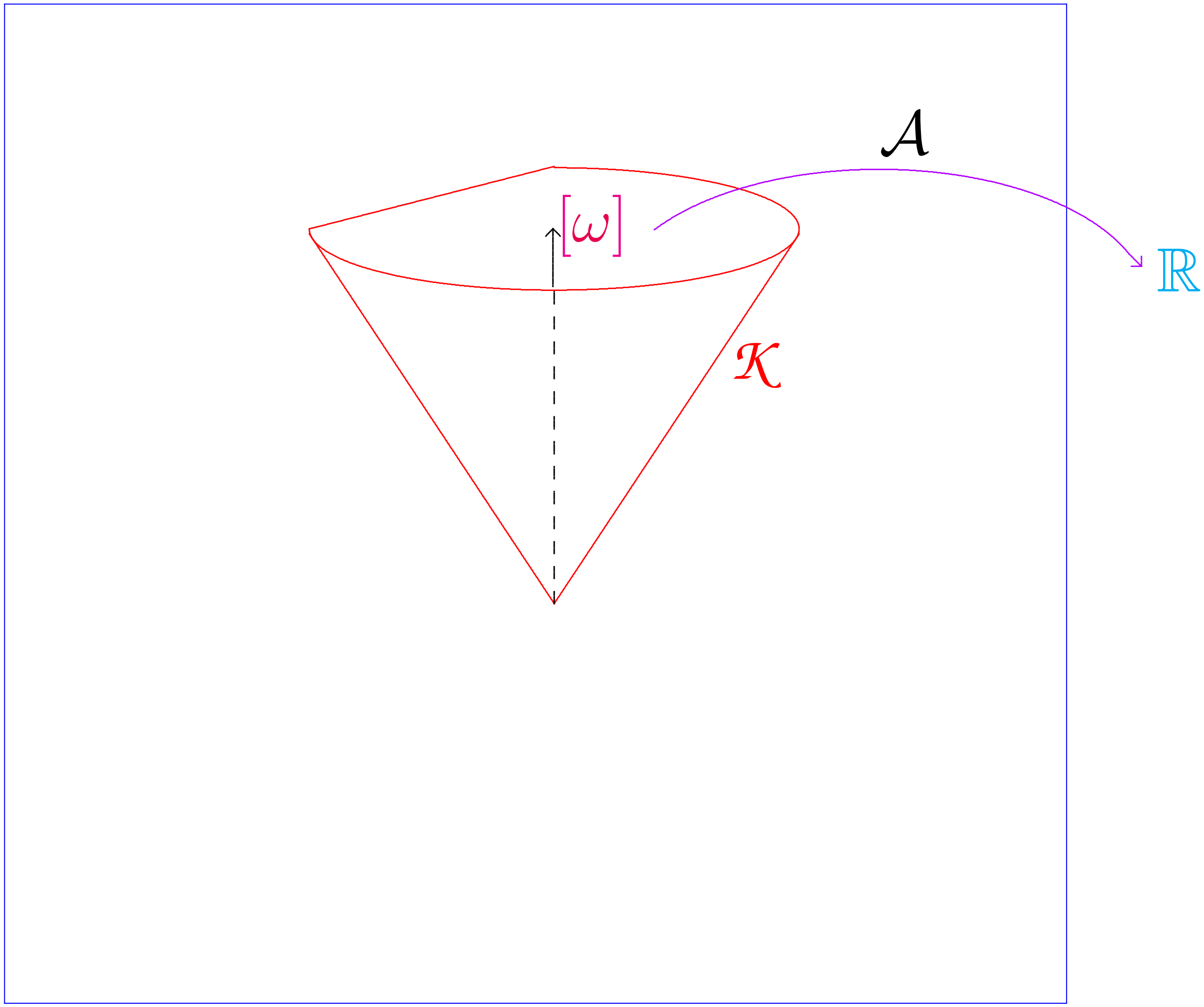

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For any extremal Kähler (M^4, g, J),

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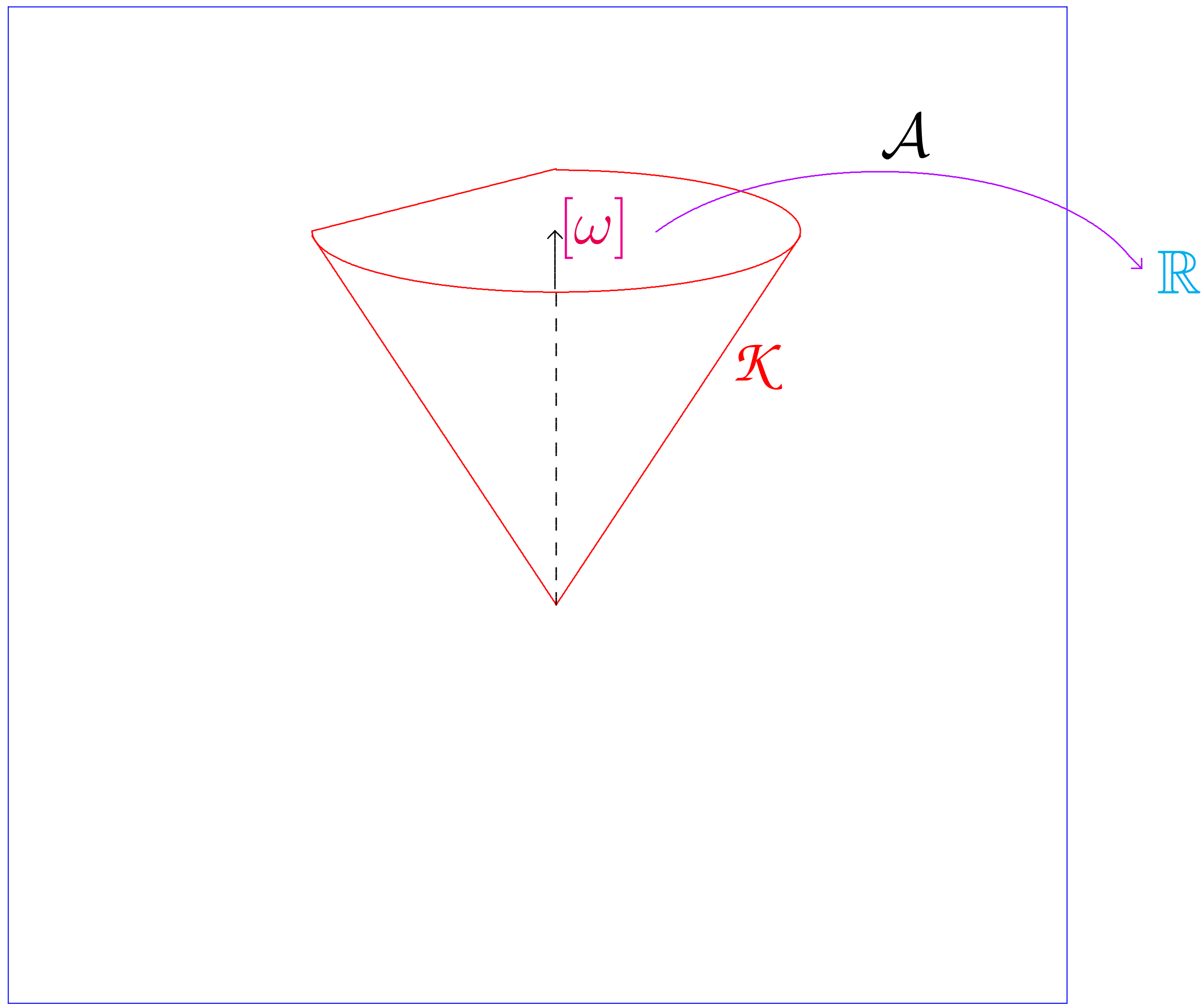

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(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$, *and* $M - \mathcal{Z}$ *has exactly two components.*

I. $s > 0$ *everywhere. Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein,* $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda = 0$*; or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $s < 0$ *somewhere. Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda < 0$*; or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$*, and* $M - \mathcal{Z}$ *has exactly two components.*

If not Kähler-Einstein:

I. s *is positive. Then*

$(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. s *is zero. Then*

(M, g, J) *SFK, but not Ricci-flat.*

III. s *changes sign. Then*

$(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathscr{Z}^3$, *and* $M - \mathscr{Z}$ *has exactly two components.*

I. $\min s > 0$. *Then*

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This happens $\Longleftrightarrow$ $c_1 > 0$.

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This happens $\iff$ $c_1 > 0$.

$\iff$ (M^4, J) is a Del Pezzo surface.

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Existence: Chen-L-Weber '08, et. al:

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Page, Siu, Yau, Tian, Odaka-Spotti-Sun, ...

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L '12: Bach-flat Kähler g uniquely determined by J
up to complex automorphisms and homothety.

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Inspired by numerical experiments of Gideon Maschler.

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(a) when $\mathrm{Aut}_0(M, J)$ reductive.

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This happens $\iff$ $c_1 > 0$.

$\iff$ (M^4, J) is a Del Pezzo surface.

(a) when $\mathrm{Aut}_0(M, J)$ reductive.

(b) when $\mathrm{Aut}_0(M, J)$ non-reductive.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

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This happens $\iff$ $c_1 > 0$.

$\iff$ (M^4, J) is a Del Pezzo surface.

(a) when $\mathrm{Aut}_0(M, J)$ reductive.

(b) when $M = \mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ or $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, $Hol = \mathbf{SO}(4)$.

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II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

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Previously discussed this case: $W_+ = 0$.

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Main point: if $\min s = 0$,

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(a) (M, g, J) *Kähler-Einstein,* $\lambda > 0$*; or else*

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Previously discussed this case: $W_+ = 0$.

Main point: if $\min s = 0$, then $s \equiv 0$.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

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(a) $\Longrightarrow$ $\text{Kod}\,(M, J) = 0$.

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Previously discussed this case: $W_+ = 0$.

(a) $\Longrightarrow$ Kod $(M, J) = 0$.

(b) $\Longrightarrow$ Kod $(M, J) = -\infty$.

I. $\min s > 0$. *Then*

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III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$; *or else*

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If $\min s < 0$, then s either constant, or changes sign.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda > 0$*; or else*

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(a) $\Longrightarrow$ $\mathrm{Kod}\,(M, J) = 2$.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

(b) (M, g, J) *anti-self-dual, but not Einstein.*

III. $\min s < 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$; *or else*

(b) $(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$, *and* $M - \mathcal{Z}$ *has exactly two components.*

(a) $\Longrightarrow$ $\mathrm{Kod}\,(M, J) = 2$. (b) $\Longrightarrow$ $\mathrm{Kod}\,(M, J) = -\infty$.

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

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Examples of (b): Hwang-Simanca, Tønnesen-Friedman

A few words about the proof. . .

For Kähler metrics g,

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$$B = \frac{1}{12}\left[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\right]$$

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Lemma. *Suppose* (M^4, g, J) *Bach-flat Kähler, with* s *non-constant.*

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Lemma. *Suppose* (M^4, g, J) *Bach-flat Kähler, with* s *non-constant. Then* $s : M \to \mathbb{R}$ *is a Morse-Bott function, with critical submanifolds either complex curves, or isolated points.*

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$$(\nabla s)\Big|_p = 0 \quad \Longrightarrow \quad (\nabla\nabla s)\Big|_p \neq 0,$$

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$$\Delta s \neq 0 \quad \text{at } \min s \text{ and } \max s.$$

For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\right]$$

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Lemma. *Suppose (M^4, g, J) Bach-flat Kähler, with s non-constant. Then $s : M \to \mathbb{R}$ is a Morse-Bott function, with critical submanifolds either complex curves, or isolated points.*

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so Bach-flat $\Longrightarrow$ g extremal and

$$0 = s\mathring{r} + 2\mathrm{Hess}_0(s).$$

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On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

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On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3,$$

For Kähler metrics g,

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Define

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where $\Delta = -\nabla^a\nabla_a$.

For Kähler metrics g,

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Define

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Lemma. *The function κ is constant,*

For Kähler metrics g,

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On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.

Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3.$$

Lemma. *The function κ is constant, and has the same sign $(+, -, 0)$ as* $\min s$.

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Lemma. *The function κ is constant, and has the same sign $(+, -, 0)$ as* $\min s$.

Obvious if s constant.

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Lemma. *The function κ is constant, and has the same sign $(+, -, 0)$ as* $\min s$.

Otherwise $\kappa = (s^2 - 6\Delta s)s$ at $\min s$.

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$$B = \frac{1}{12}\left[2s\mathring{r} + \text{Hess}_0(s) + 3J^*\text{Hess}_0(s)\right]$$

so Bach-flat $\Longrightarrow$ g extremal and

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Lemma. *The function κ is constant, and has the same sign $(+, -, 0)$ as $\min s$. On set where $s \neq 0$, the constant $\kappa =$ scalar curvature of $s^{-2}g$.*

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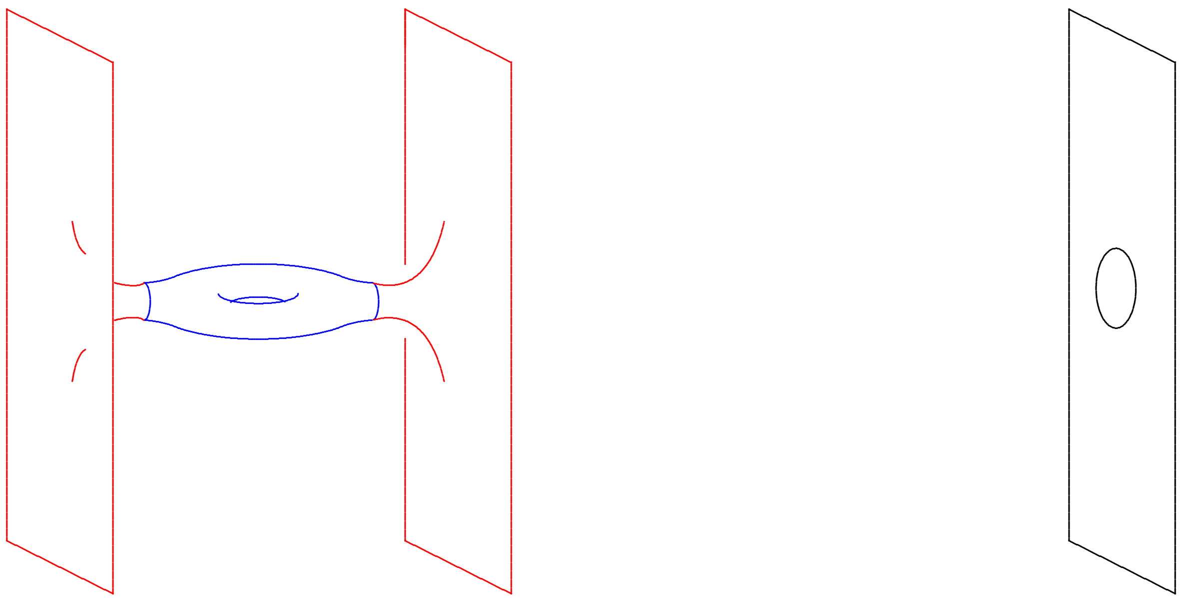
Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g) *is called* asymptotically Euclidean *(AE) if there is a compact set* $K \subset M$ *such that each component of* $M - K$ *is diffeomorphic to* $\mathbb{R}^n - D^n$ *in such a manner that*



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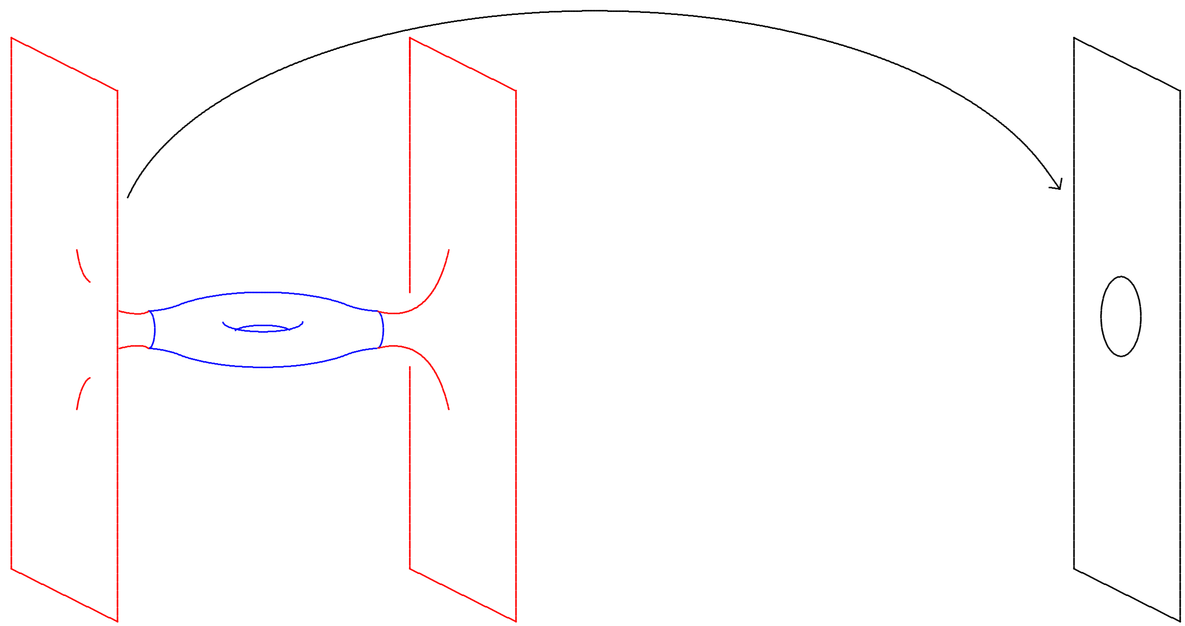

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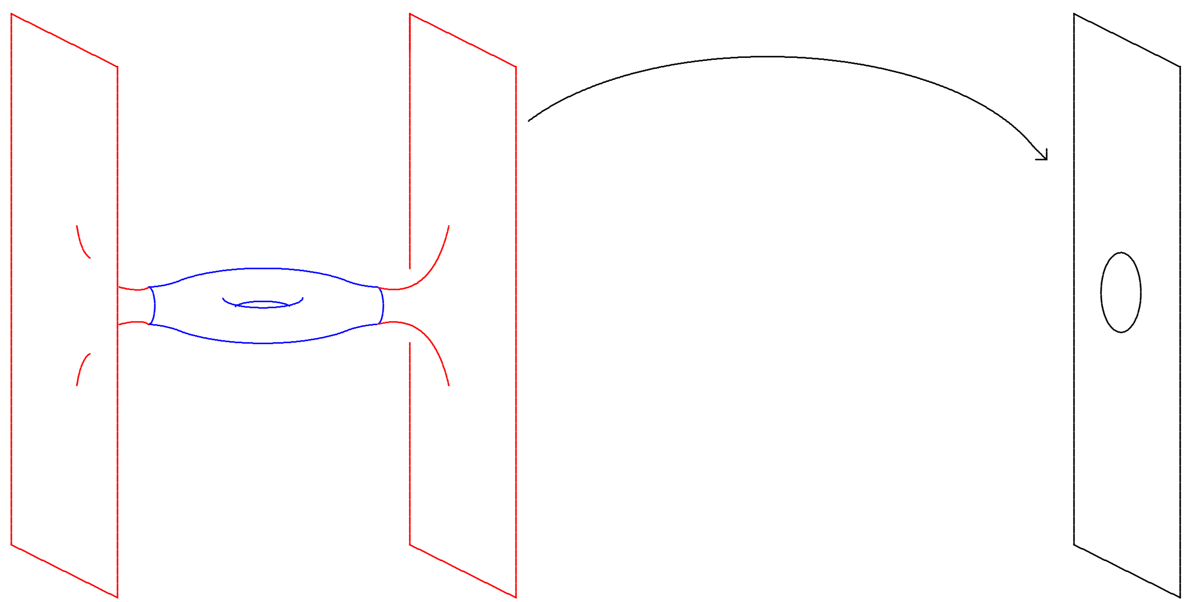

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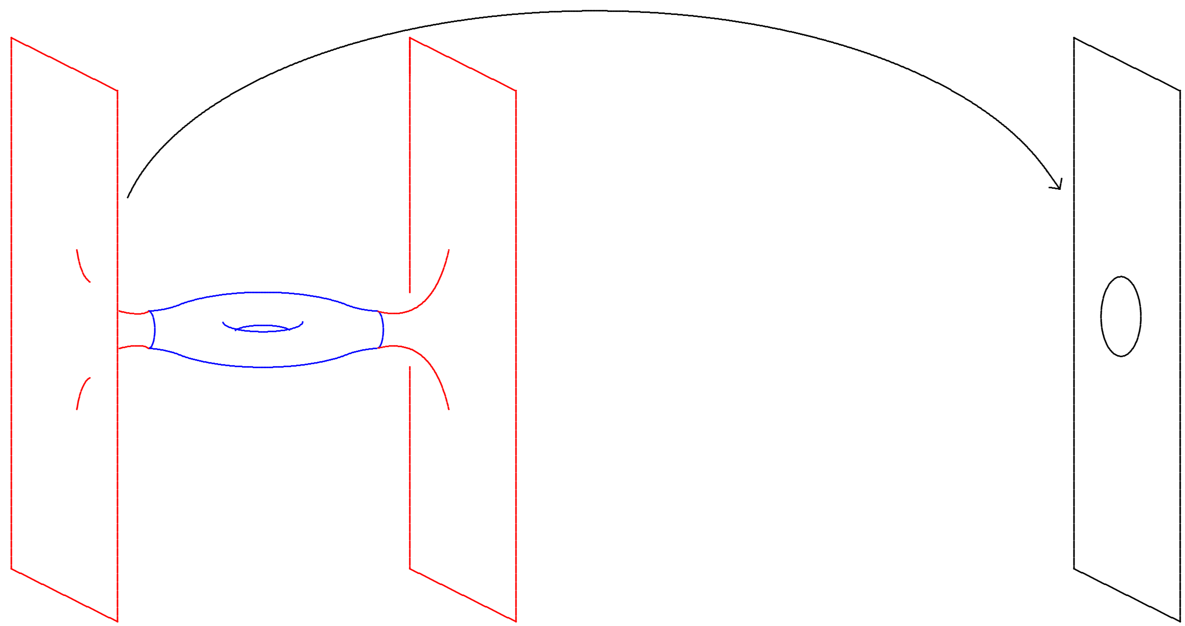

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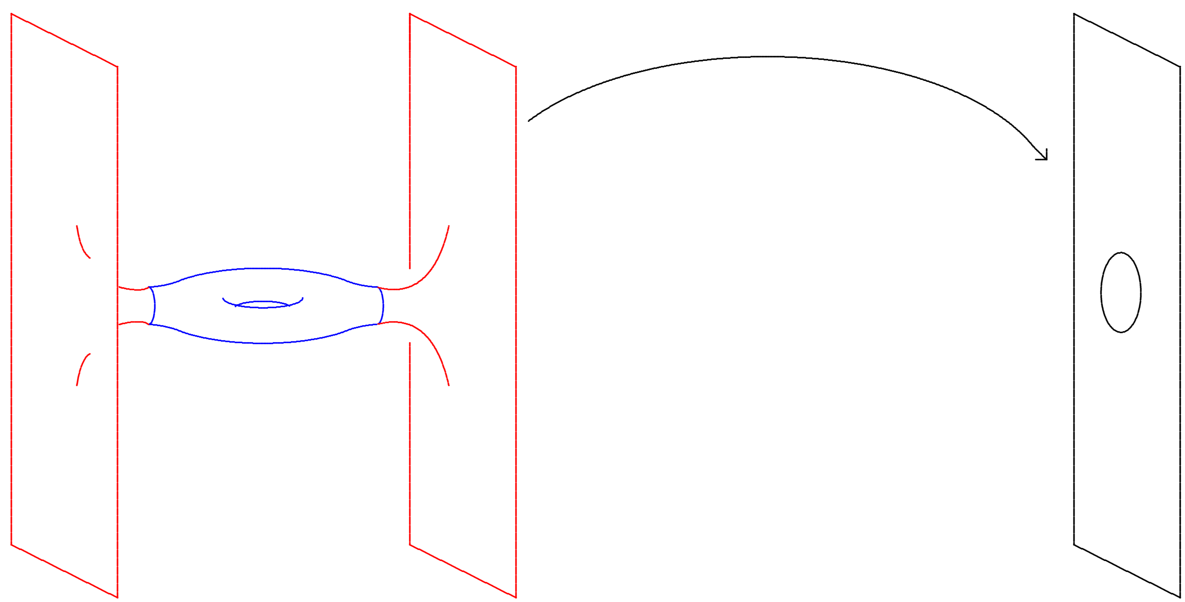

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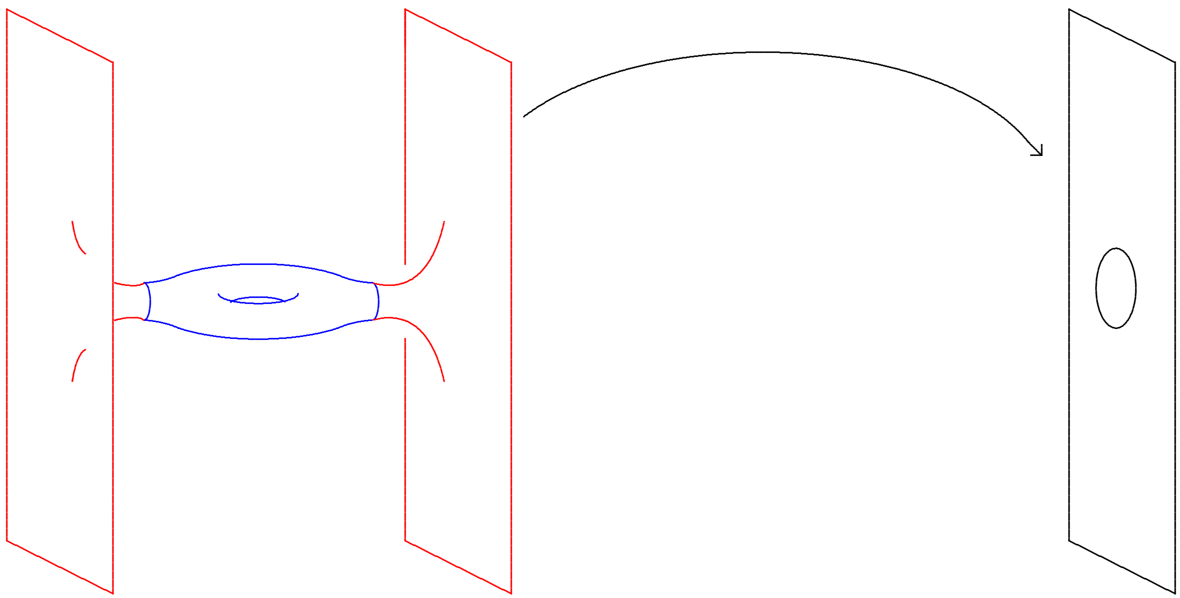

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Contradiction! So $s \equiv 0$.

Theorem. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein,* $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein,* $\lambda = 0$*; or else*

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III. $\min s < 0$. *Then*

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$(M, s^{-2}g)$ double Poincaré-Einstein. Here, $s = 0$ defines smooth connected $\mathcal{Z}^3$, and $M - \mathcal{Z}$ has exactly two components.

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¡Muchas Gracias por la Invitación!