Einstein Metrics,

Harmonic Forms, &

Conformally Kähler Geometry

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Stony Brook University

Seminario de Geometría
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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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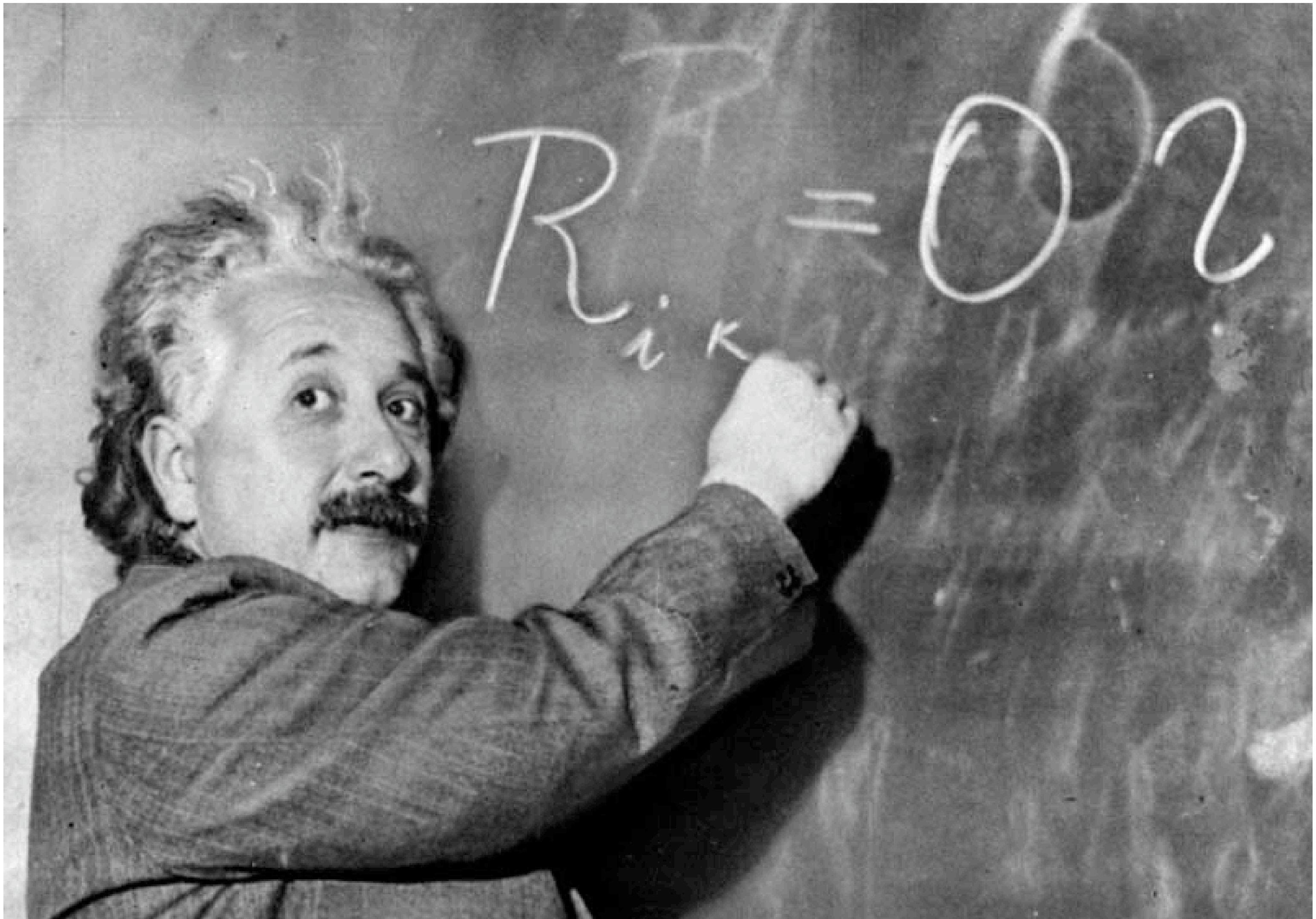

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Has same sign as the *scalar curvature*

$$s = r_j^{\,j} = \mathcal{R}^{ij}{}_{ij}.$$

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When $n = 4$, situation is more encouraging...

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One key question:

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One key question:

Does enough rigidity really hold in dimension four to make this a genuine geometrization?

Symplectic 4-manifolds:

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Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric h*

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Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric h (unrelated to ω)?*

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Some Suggestive Questions. *If* (M^4, ω) *is a symplectic* 4*-manifold, when does* M^4 *admit an Einstein metric* h *(unrelated to* ω*)? What if we also require* $\lambda \geq 0$*?*

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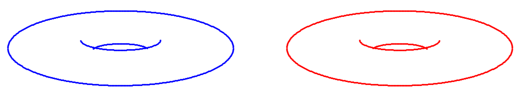

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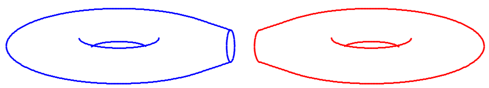

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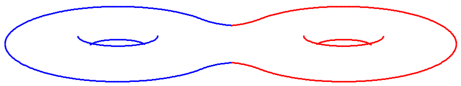

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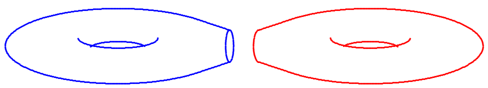

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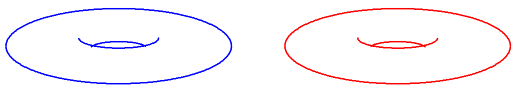

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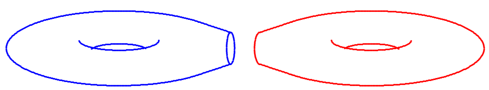

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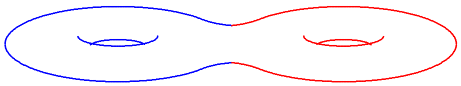

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Calabi/Yau: Admits Ricci-flat Kähler metrics.

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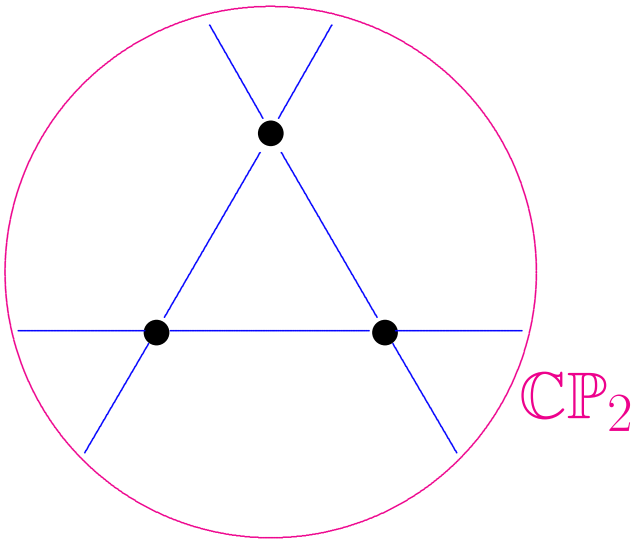

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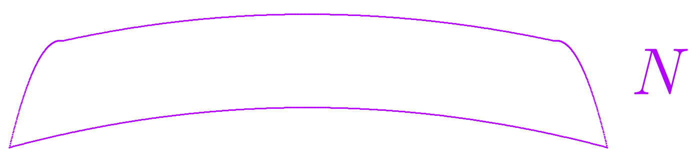

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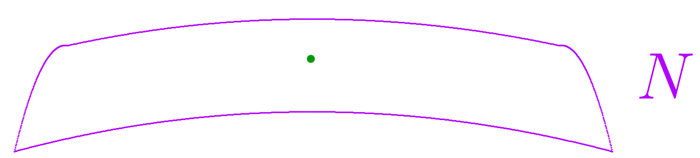

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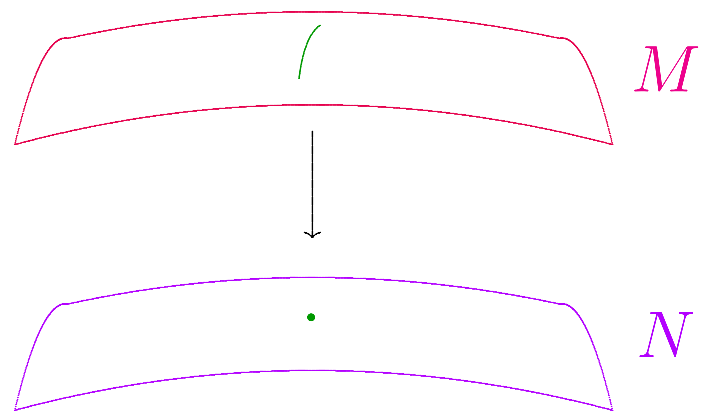

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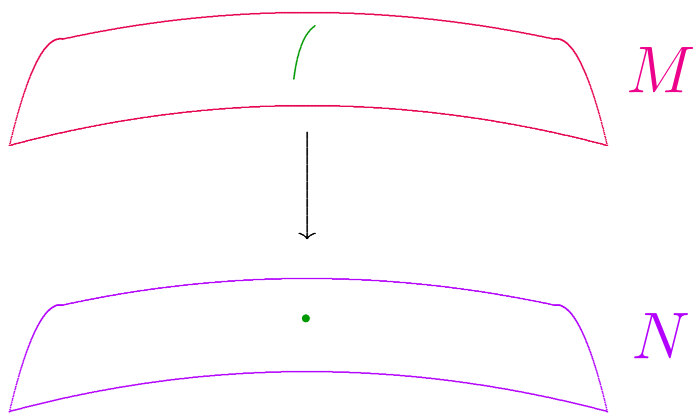
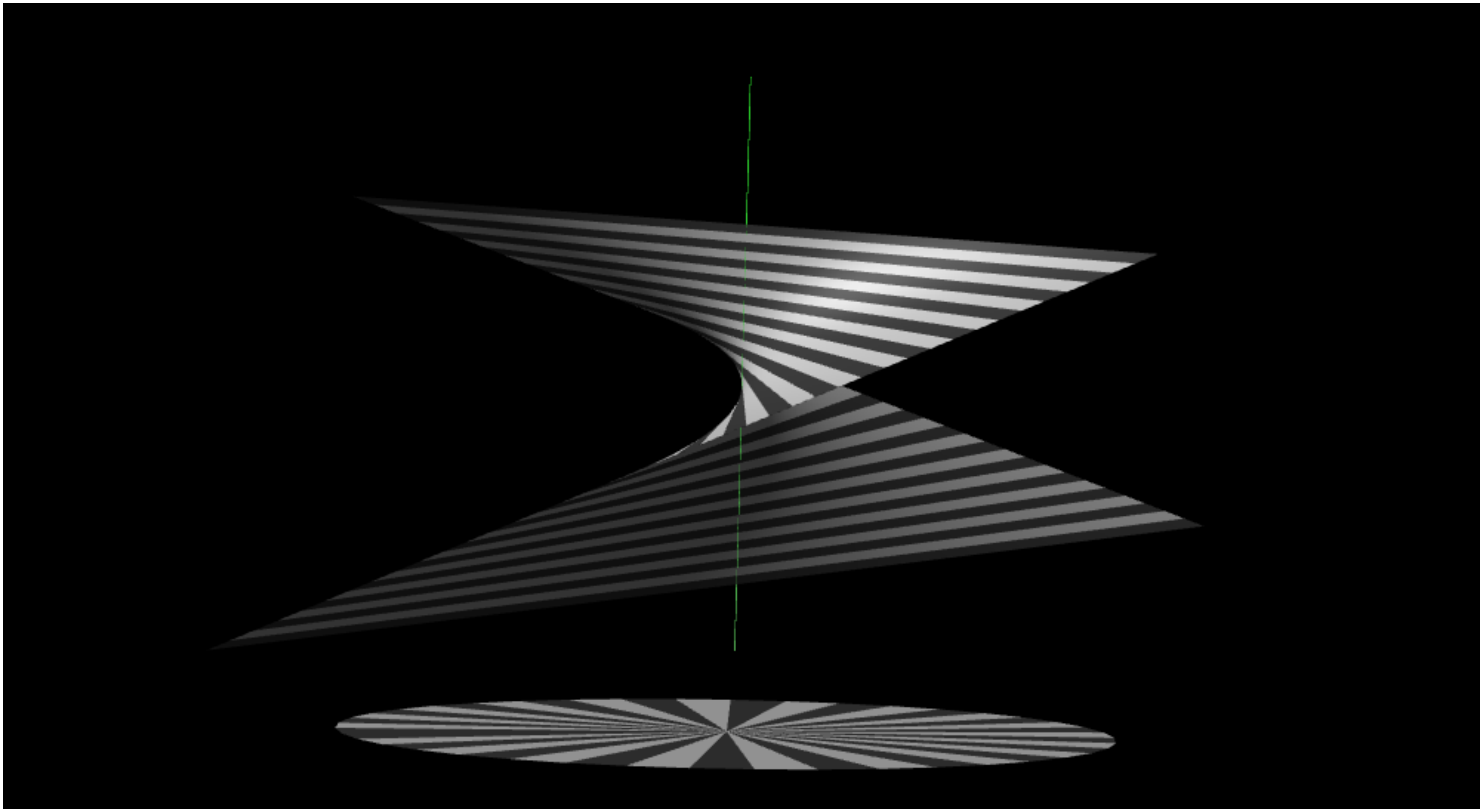
$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.





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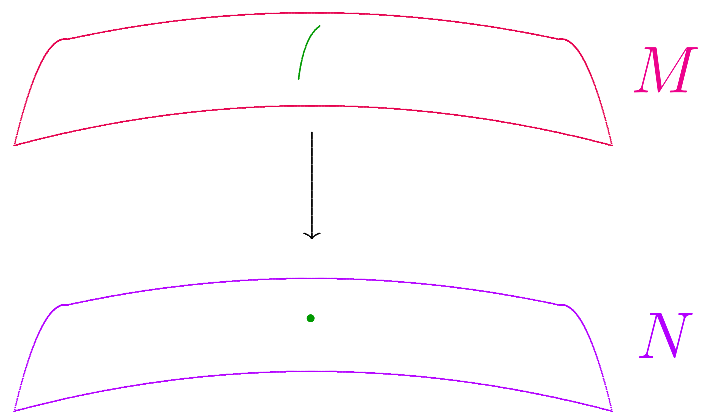

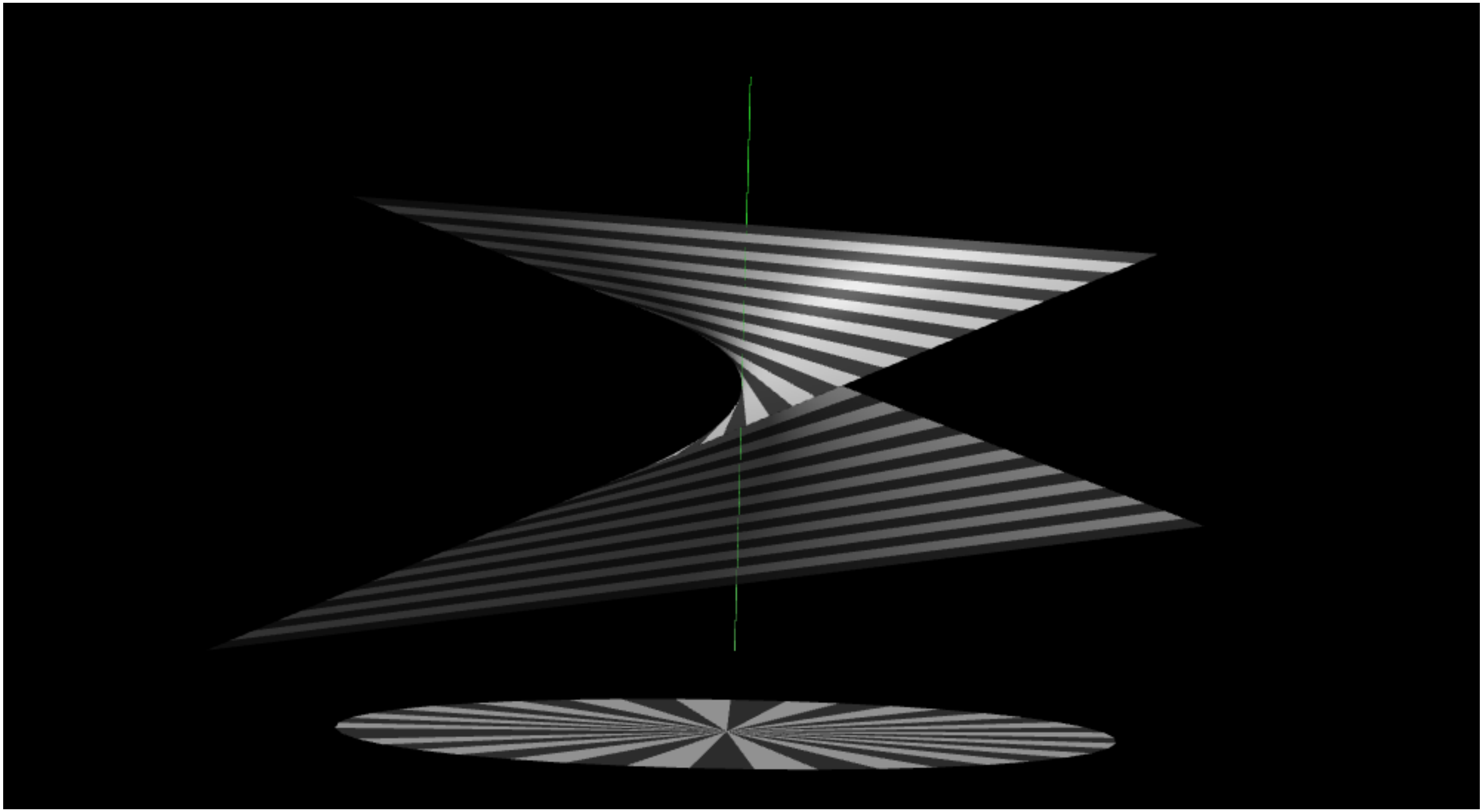

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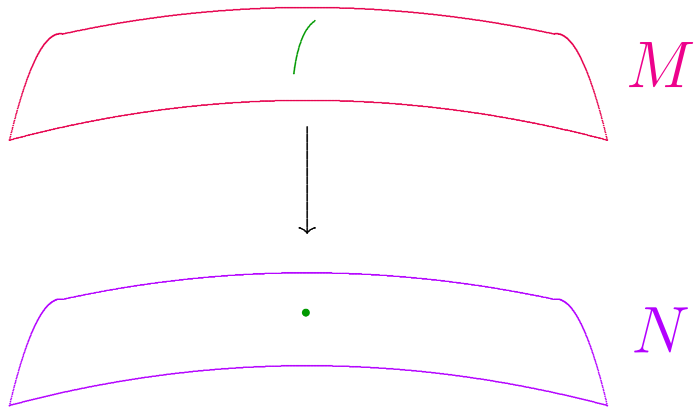

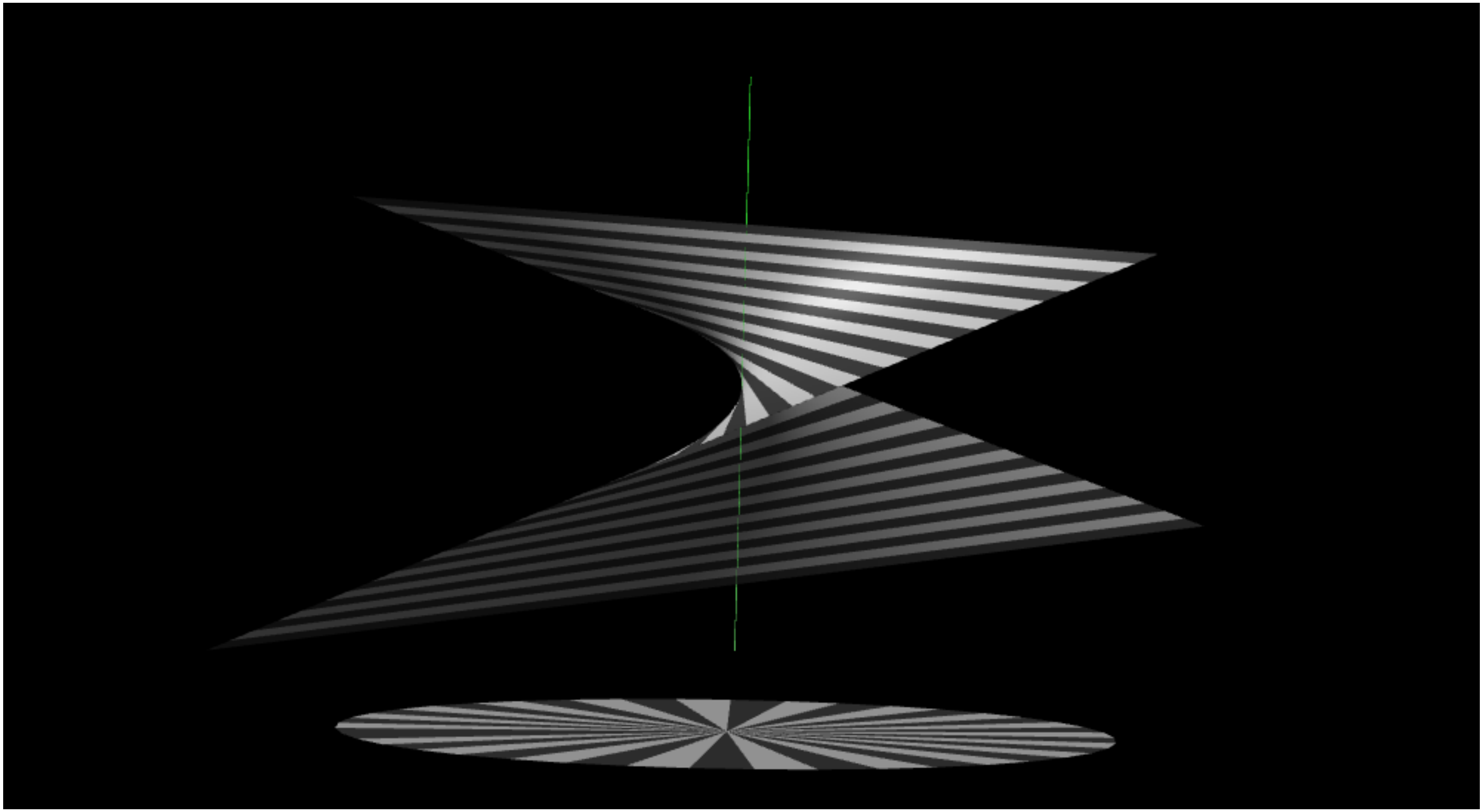

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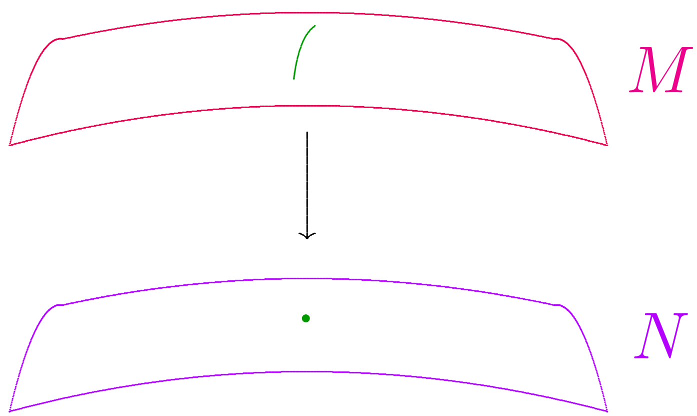

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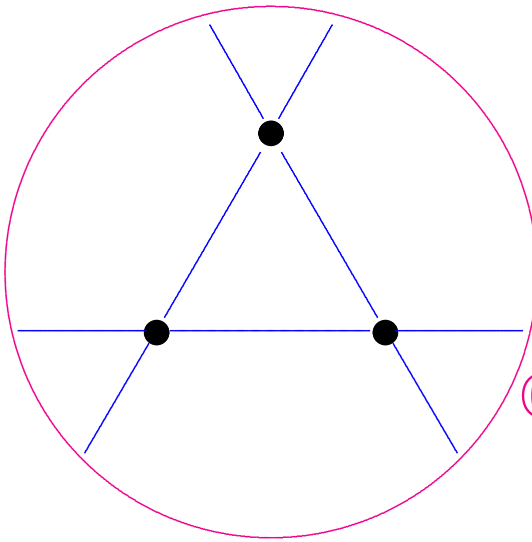

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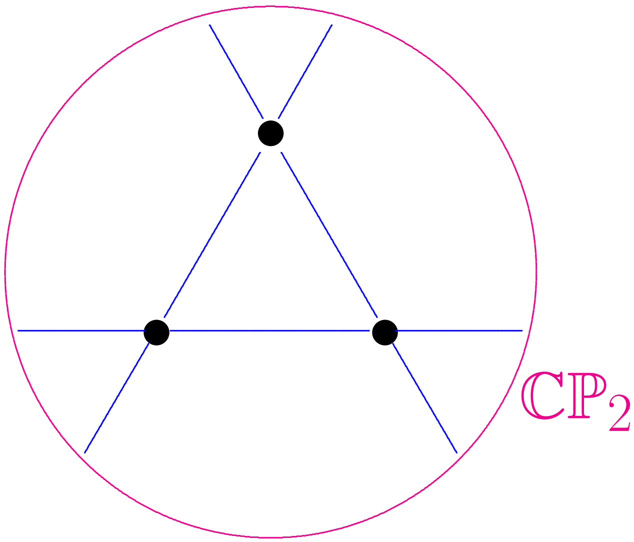

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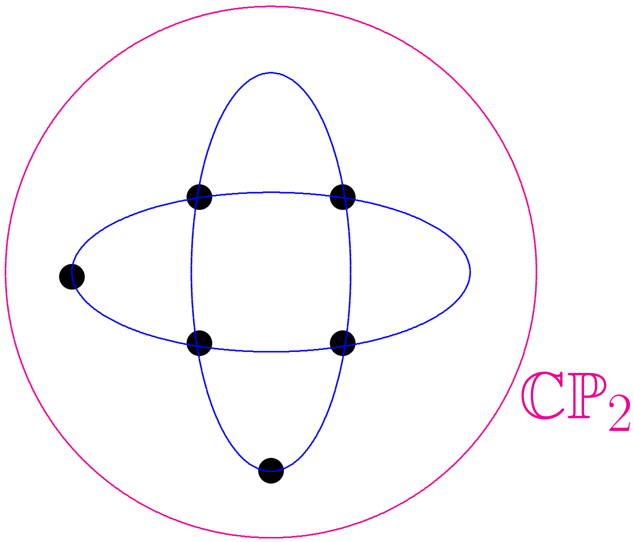

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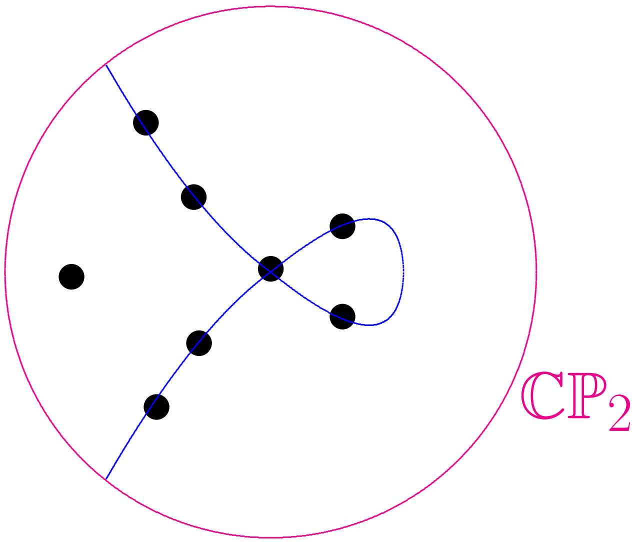

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Theorem. *Each Del Pezzo* (M^4, J) *admits a compatible conformally Kähler Einstein metric, and this metric is unique up to automorphisms.*

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Existence: Tian, Odaka-Spotti-Sun, Chen-L-Weber.

Uniqueness: Bando-Mabuchi '87, L '12.

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Moduli space $\mathscr{E}(M) \neq \varnothing$. But is it connected?

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Understand all Einstein metrics on Del Pezzos.

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Natural characterization of known Einstein metrics.

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Natural characterization of known Einstein metrics.

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Formulation depends on . . .

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Notice these spaces are conformally invariant.

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$$H^2(M,\mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0,\ d\star\varphi = 0\}.$$

Since $\star$ is involution of RHS, $\Longrightarrow$

$$H^2(M,\mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g,$$

where

$$\mathcal{H}^\pm_g = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

self-dual & anti-self-dual harmonic forms.

Notice these spaces are conformally invariant.

More generally, their dimensions

$$b_\pm(M) = \dim \mathcal{H}^\pm_g$$

are completely metric-independent, and
are oriented homotopy invariants of M.

Key background result:

Theorem (L '15)**.**

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Yes — with a reasonable extra hypothesis on ω...

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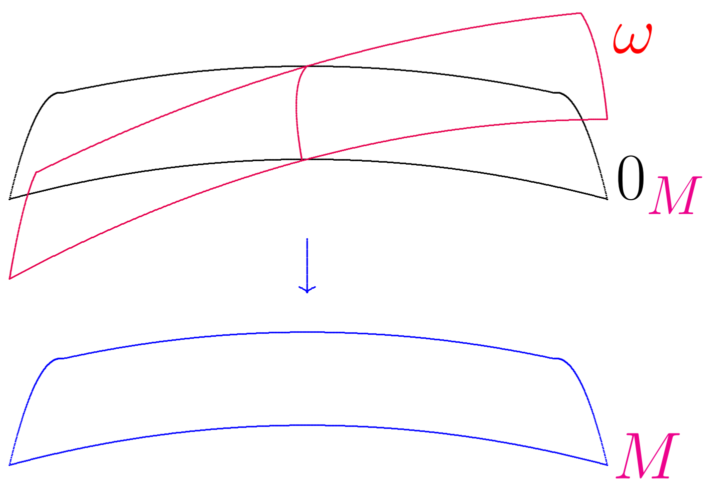

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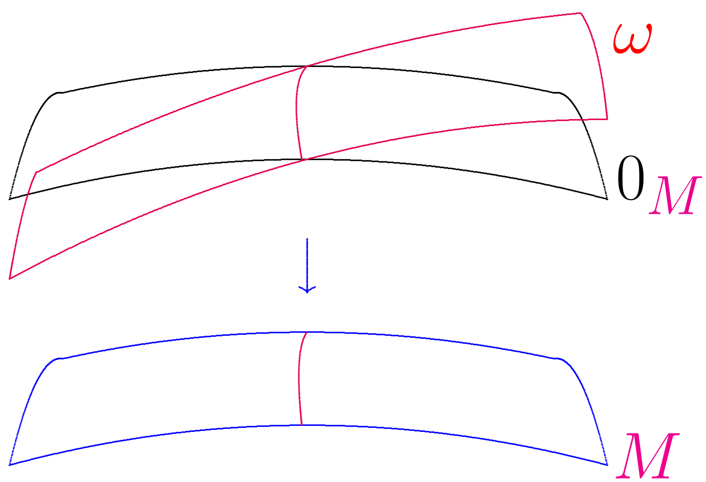

$\Longrightarrow$ Zero set Z of ω has codimension 3:

$$Z \approx \sqcup_{j=1}^{n} S^1.$$

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Theorem (Taubes, et al). *If* $b_+(M) \neq 0$, *such forms exist for an open dense set of metrics* h *on* M.

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Indeed, all known Einstein metrics on Del Pezzo surfaces arise this way!

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Conversely, these complex surfaces all admit $\lambda \geq 0$ *Einstein metrics* h *of the above type.*

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Moral: Taubes' genericity result does not guarantee genericity among metrics solving an equation!

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Kähler $\Longrightarrow$

$$W^+ = \begin{bmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{bmatrix}$$

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Let $\varphi \not\equiv 0$ be a holomorphic 2-form. Set $\omega = \Re e\varphi$.

Then ω harmonic, self-dual, but not near-symplectic.

$$\text{Kähler} \Longrightarrow \Lambda^+ = \mathbb{R}\Omega \oplus \Re e\Lambda^{2,0}.$$

$$W^+ = \begin{bmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{bmatrix}$$

Counter-examples promised by **Theorem C**:

Let (M, h) be a Kähler-Einstein manifold with

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$$W^+(\omega, \omega) \geq 0, \qquad W^+(\omega, \omega) \not\equiv 0$$

Before discussing **Theorems A & B**,

consider simpler case when $W^{+}(\omega, \omega) > 0$.

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Lemma. *There is a constant C, independent of $\epsilon \in (0,1)$, but depending on (M, h, ω), such that*

$$\left|\int_{X_\epsilon} \left[\langle \nabla^*\nabla(fW^+), \omega \otimes \omega\rangle - \langle fW^+, \nabla^*\nabla(\omega \otimes \omega)\rangle\right] d\mu_g\right| \leq C\epsilon^{-3/2} Vol^{(3)}(\partial X_\epsilon, h),$$

where $X_\epsilon =$ region where $|\omega|_h \geq \epsilon$.

¡Muchas Gracias por la Invitación!