Mass in

Kähler Geometry

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Differential Geometry Seminar
Harvard University
September 15, 2015

Joint work with

Joint work with

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University of Maryland

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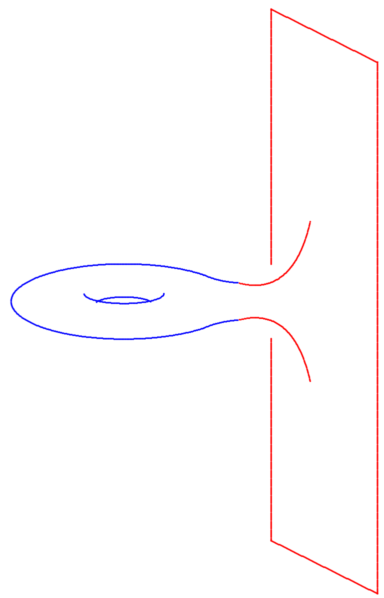
e-print: arXiv:1507.08885 [math.DG]

Definition. *A complete, non-compact Riemannian n-manifold* (M^n, g)



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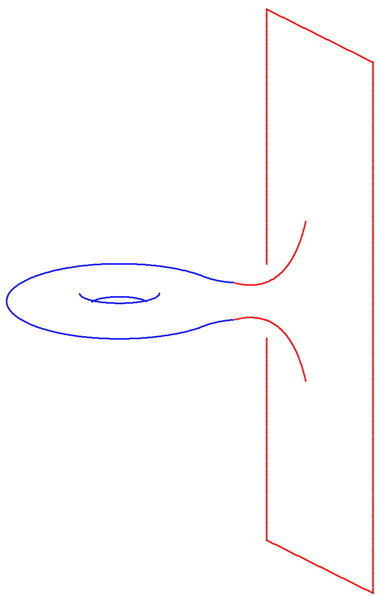

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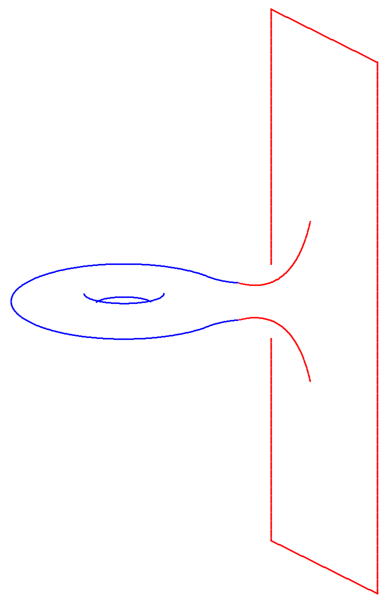

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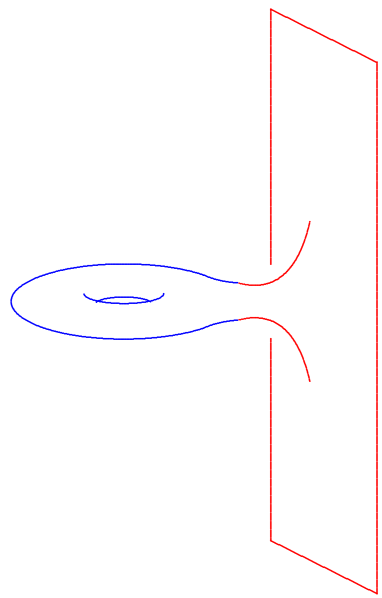

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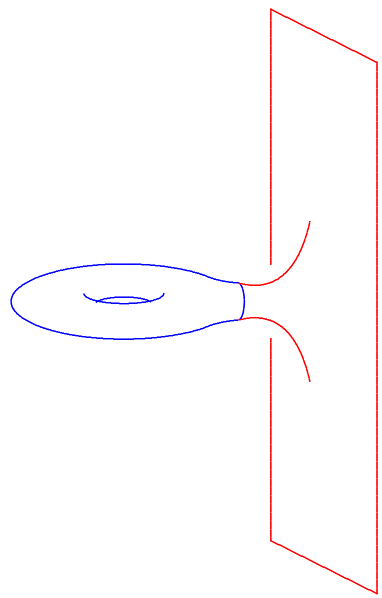

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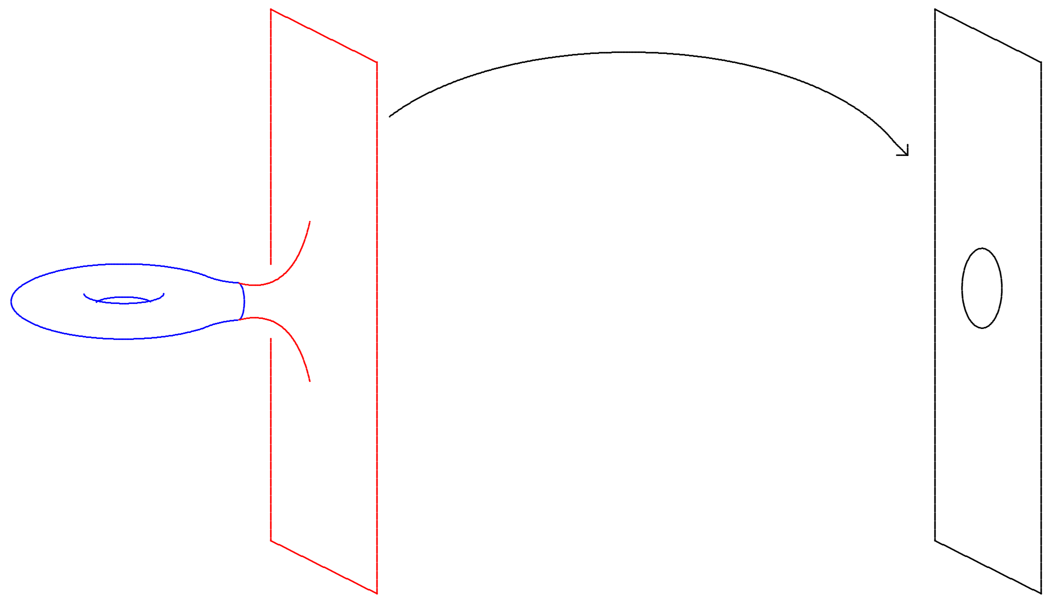

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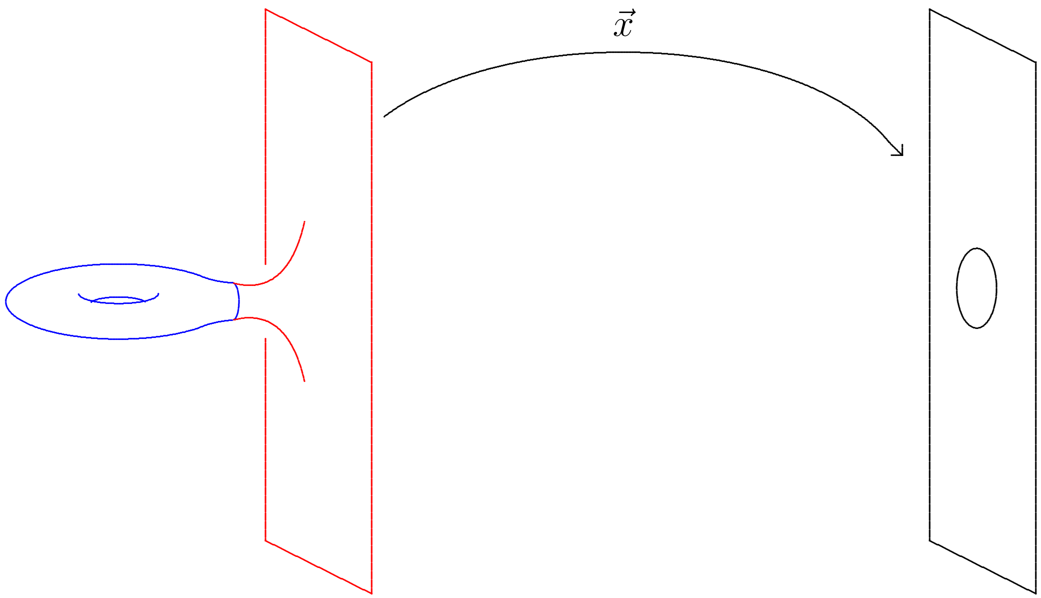

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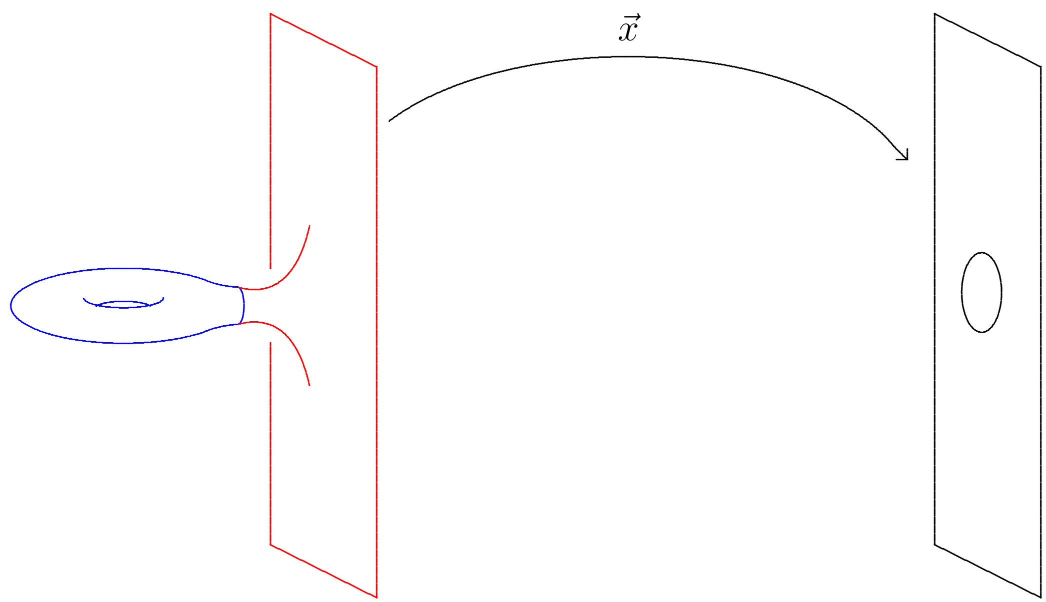

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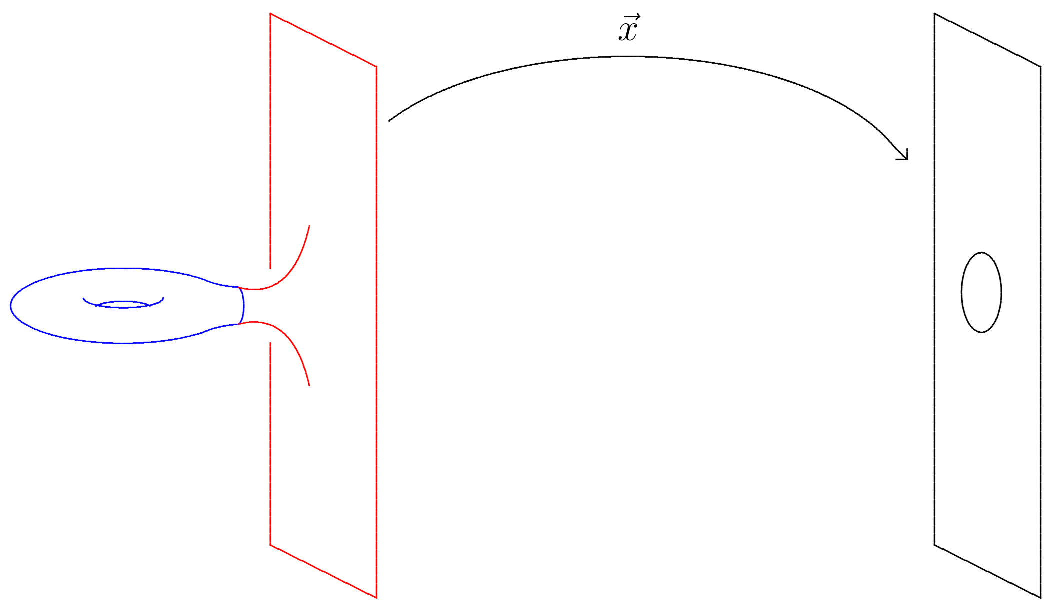

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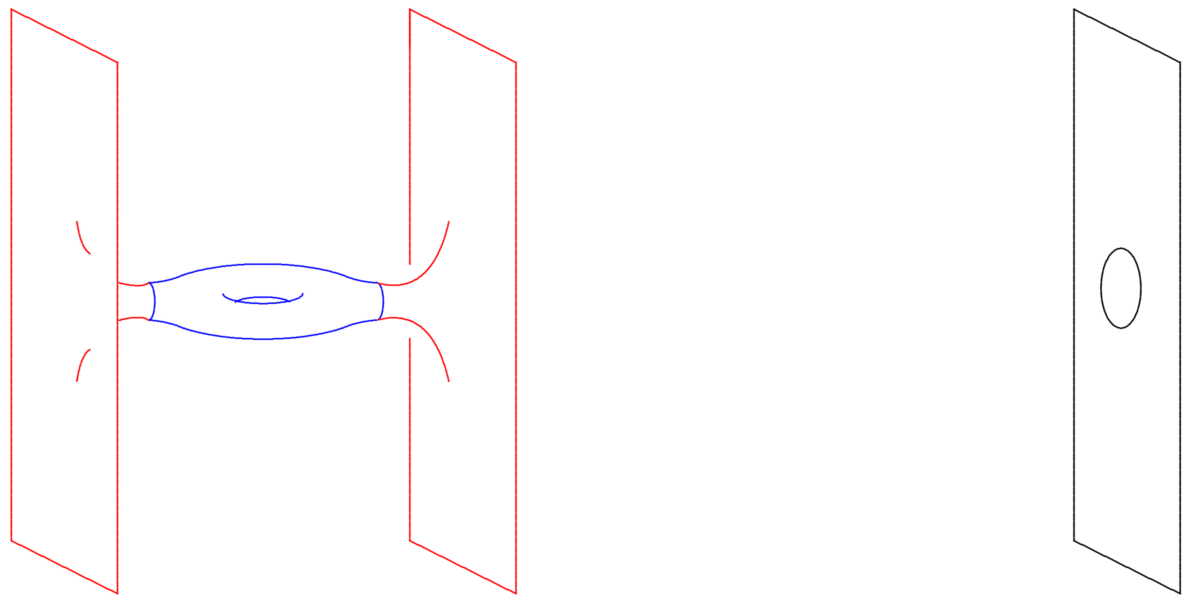

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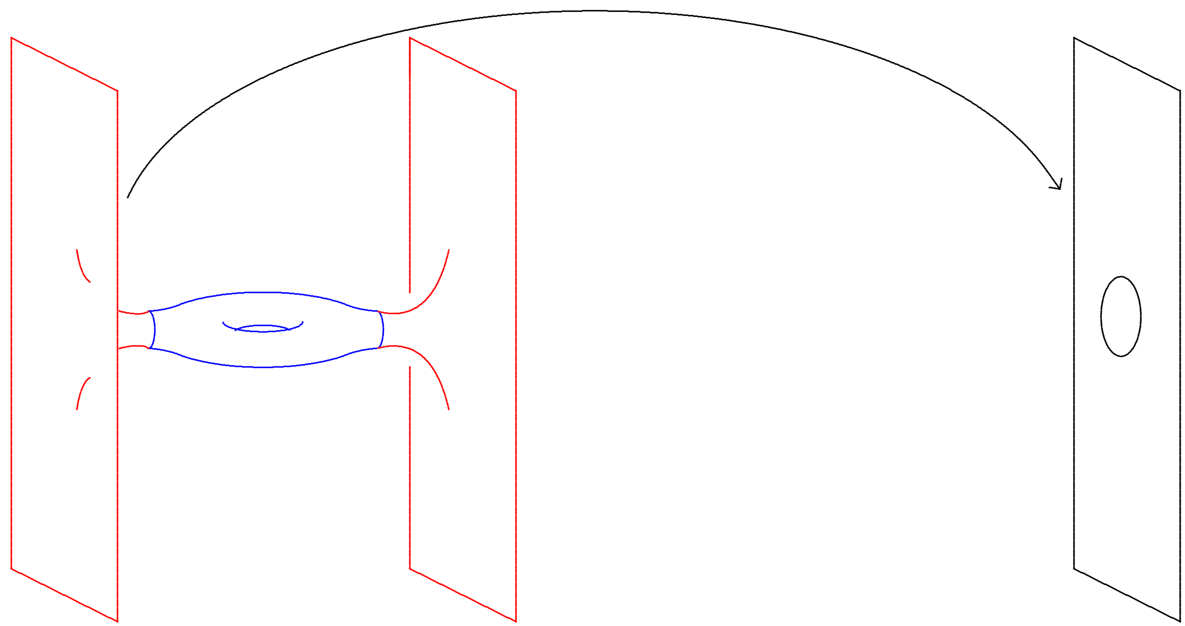

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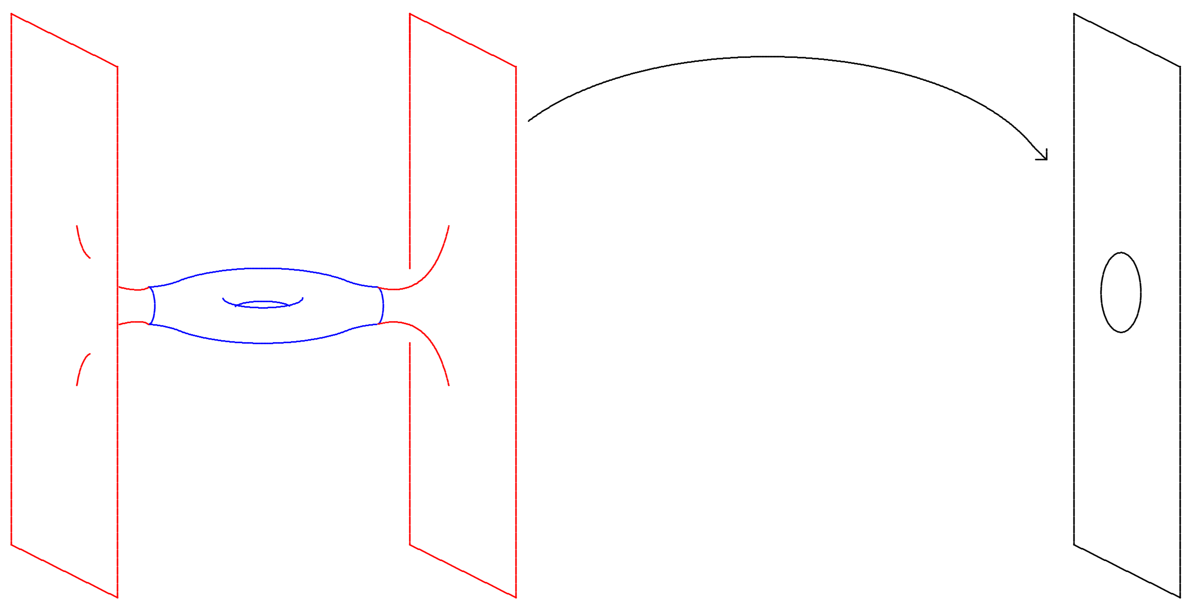

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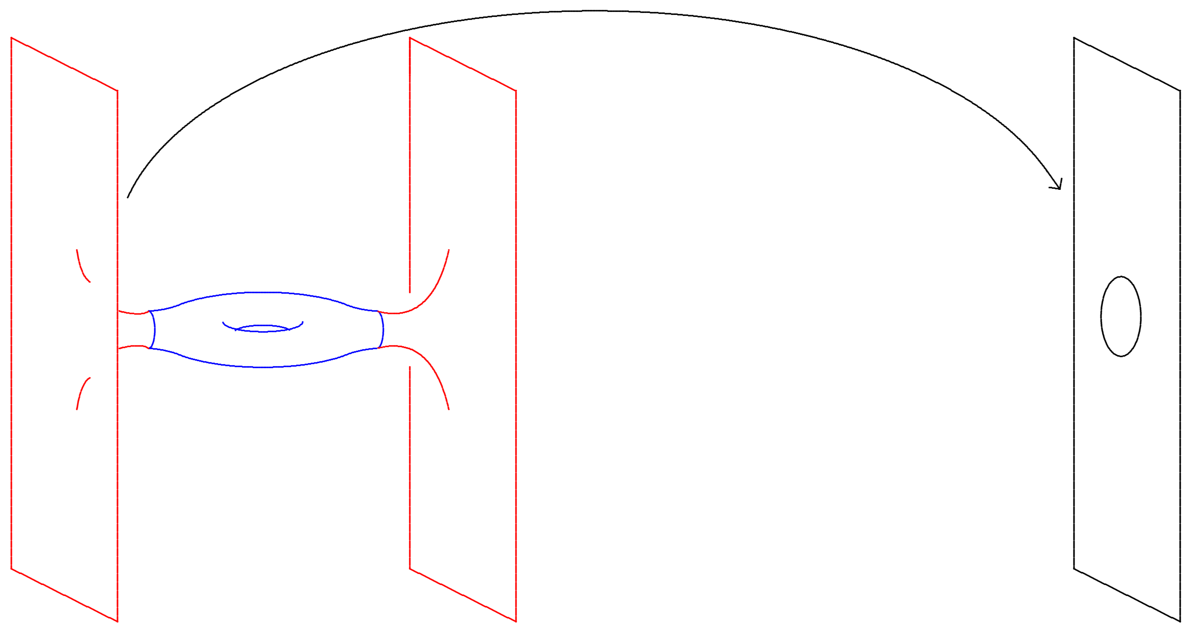

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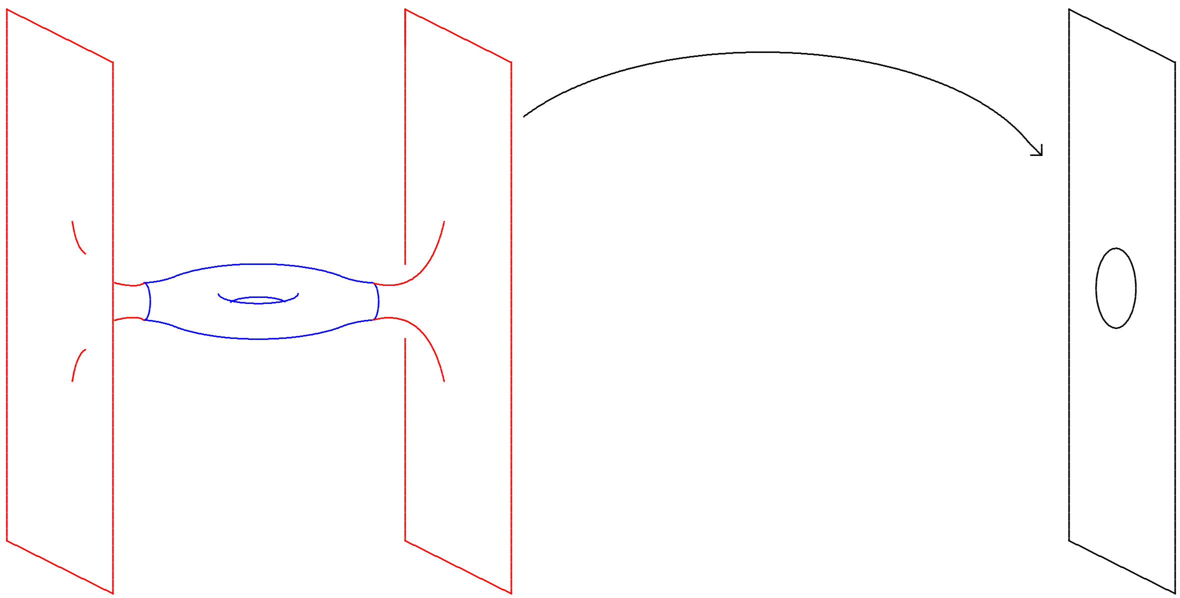

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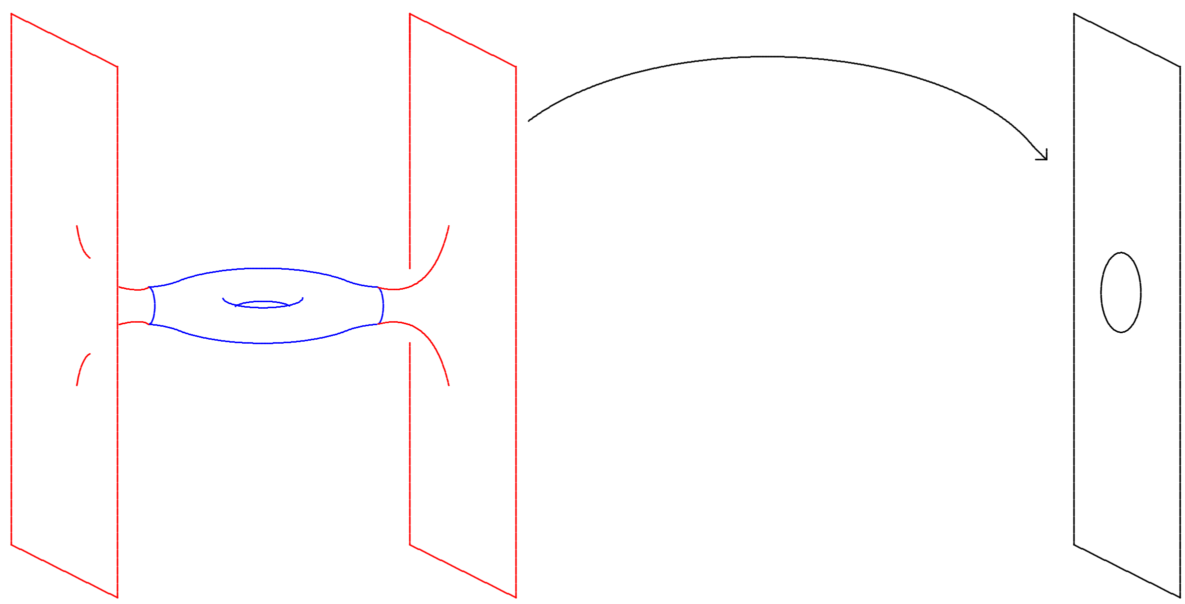

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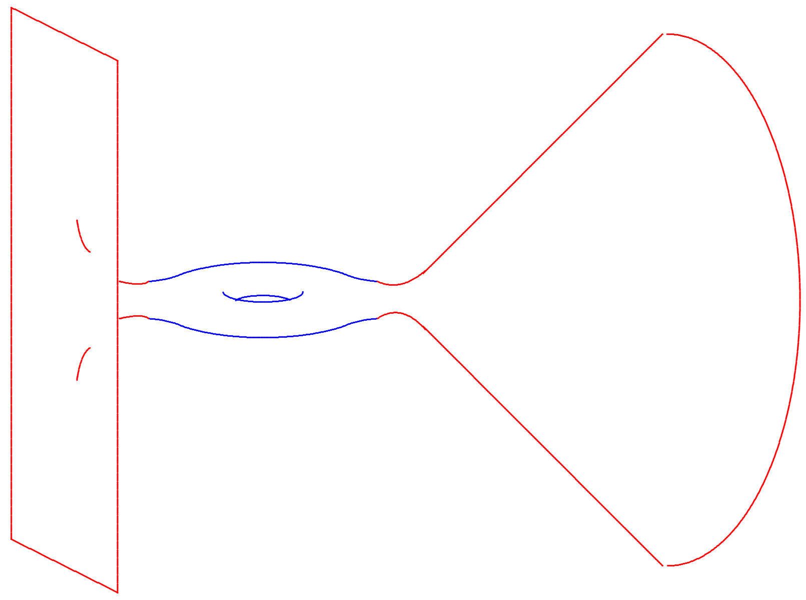

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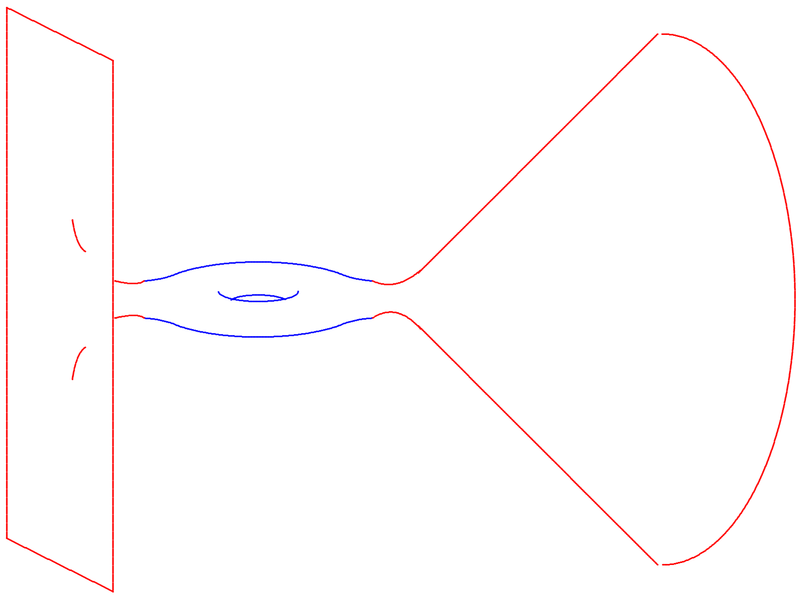

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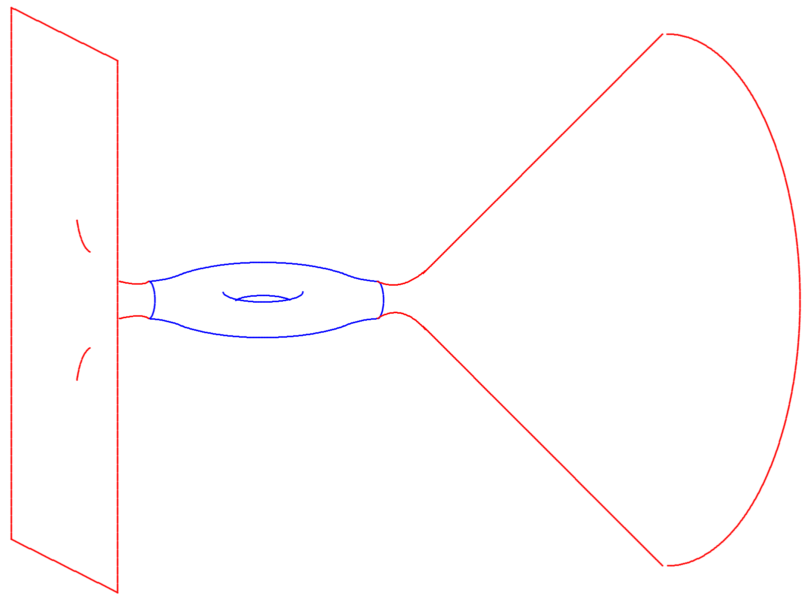

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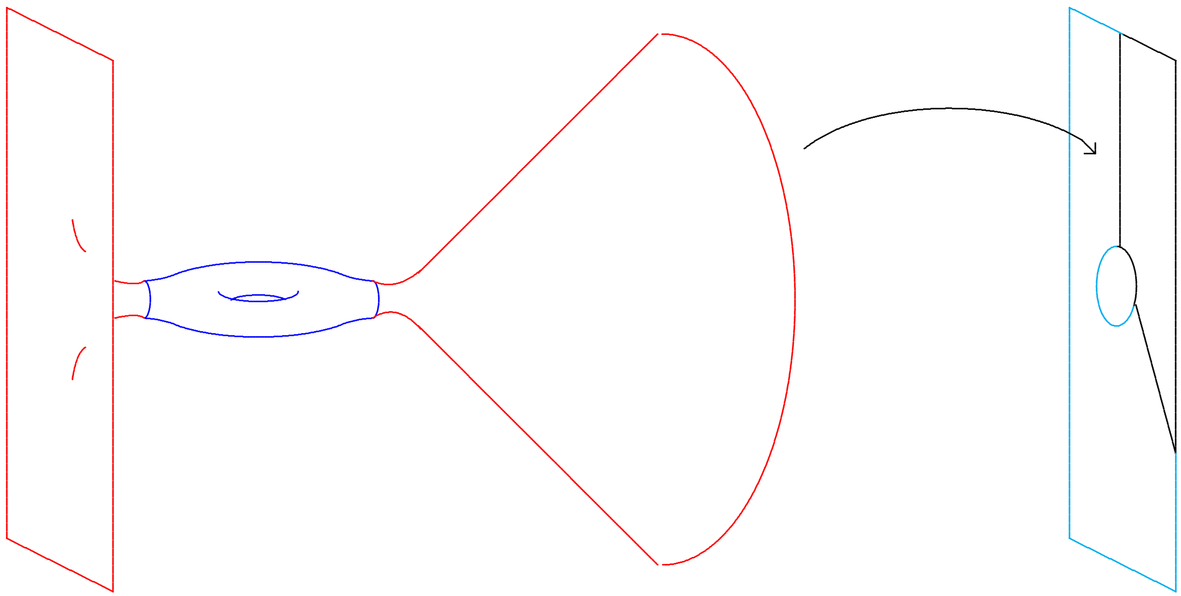

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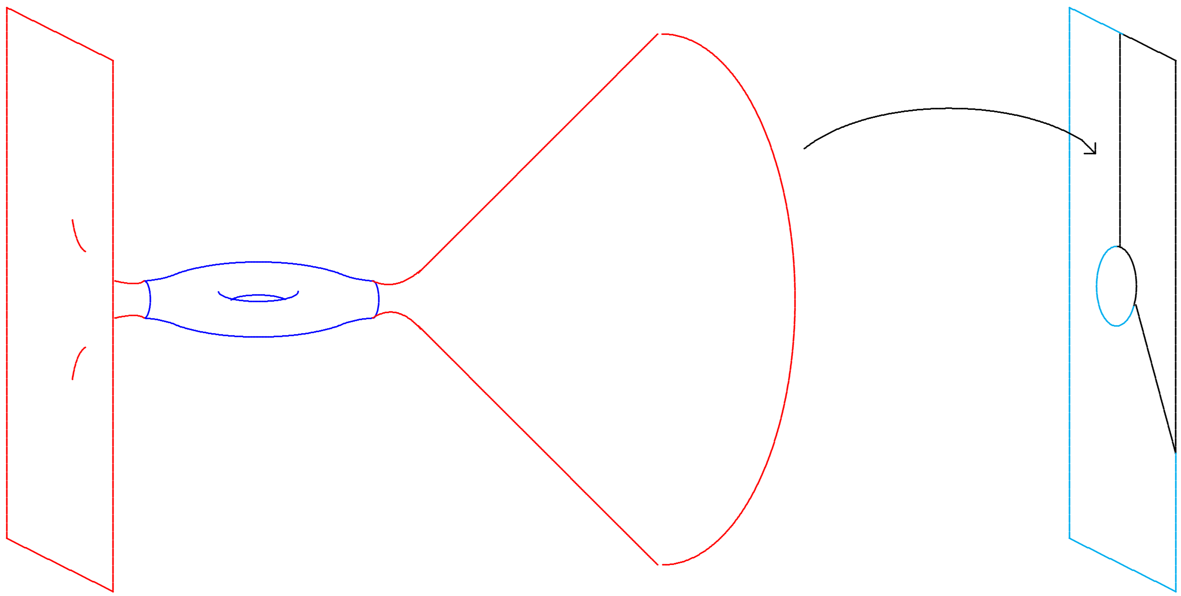

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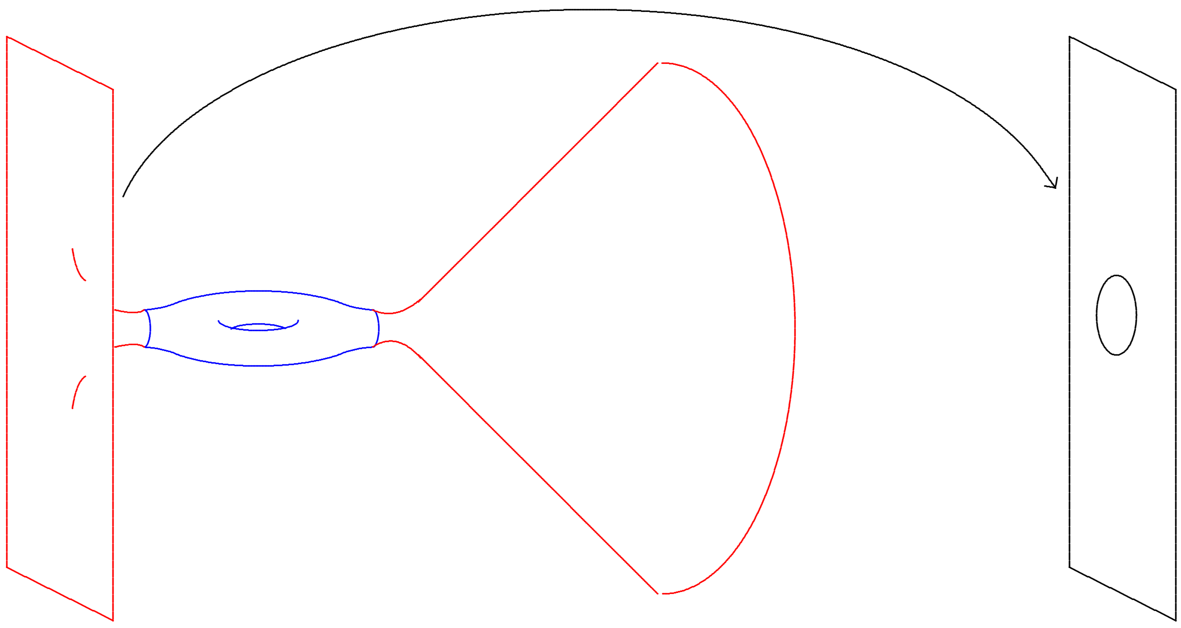

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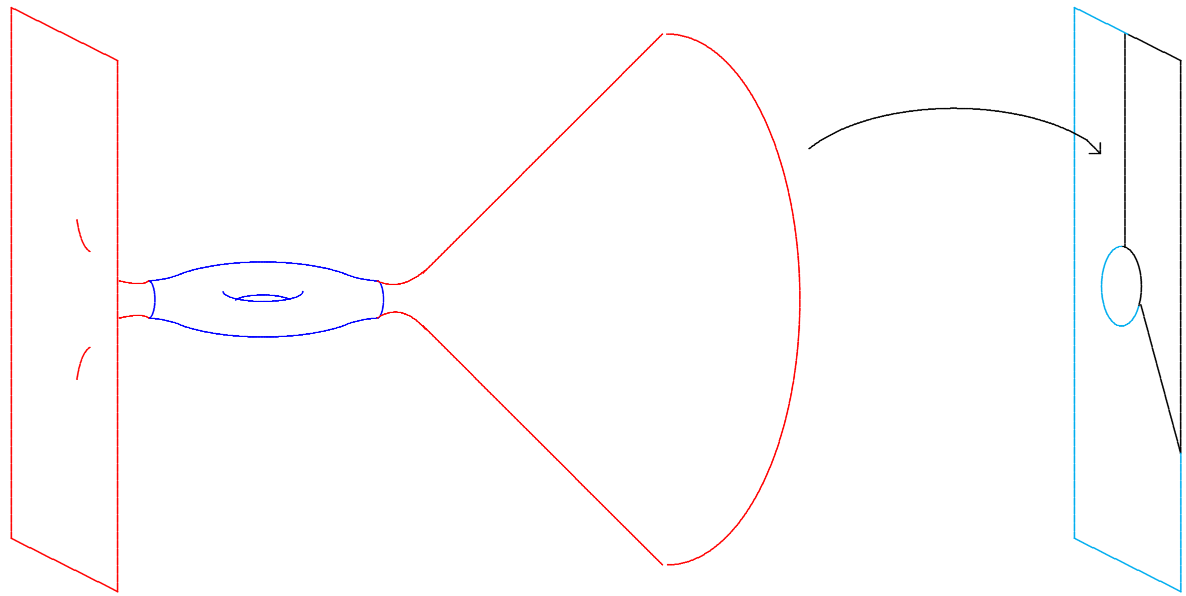

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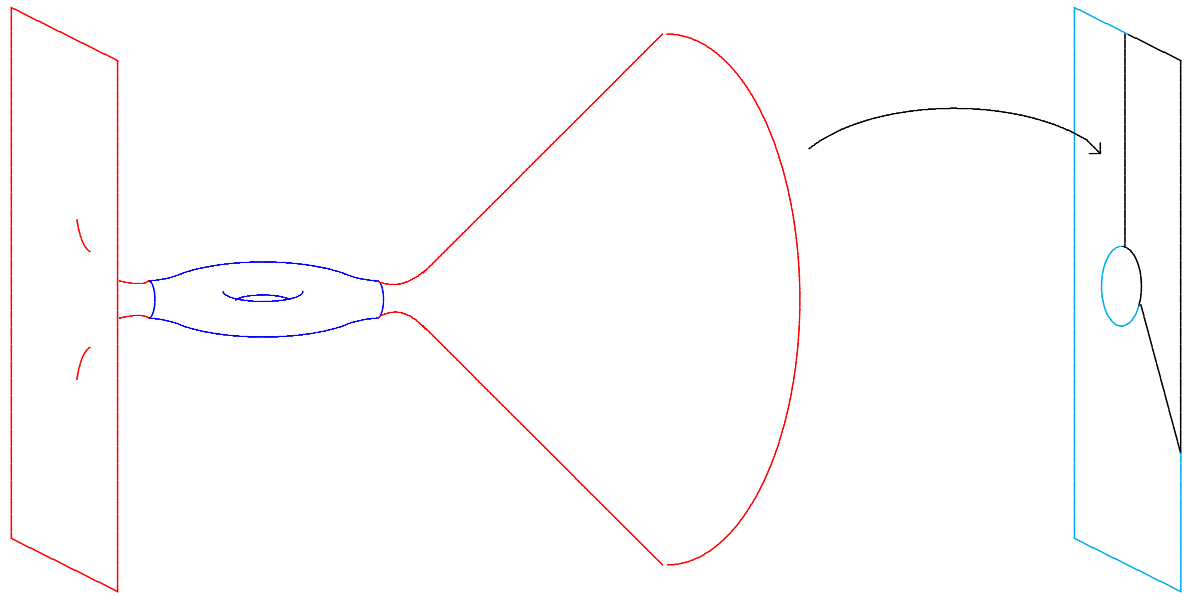

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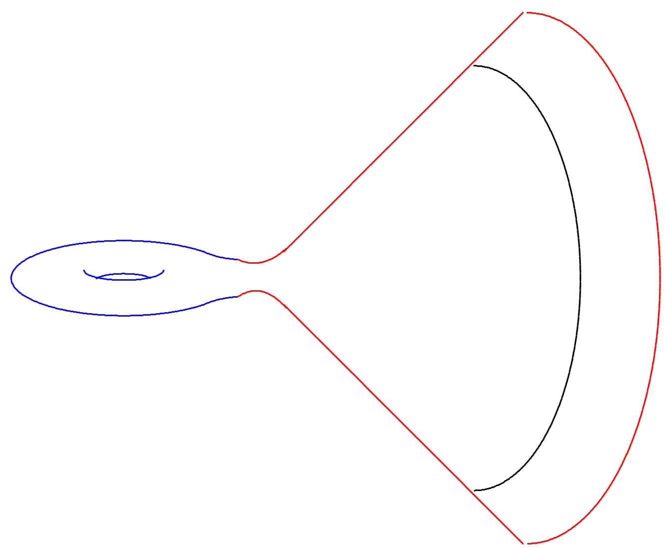

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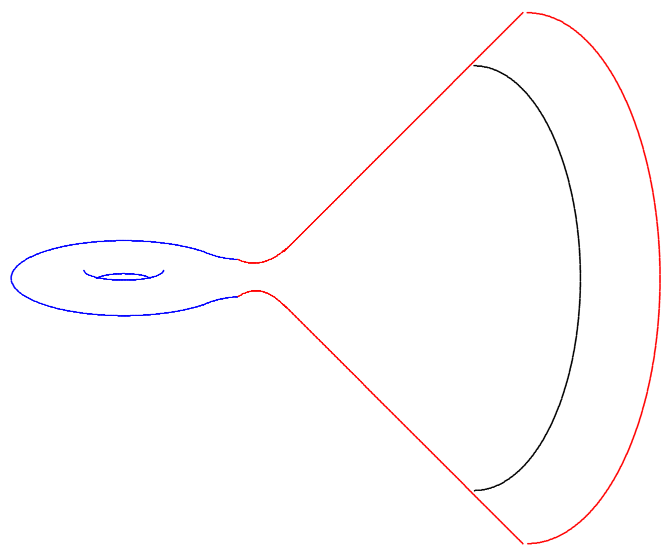

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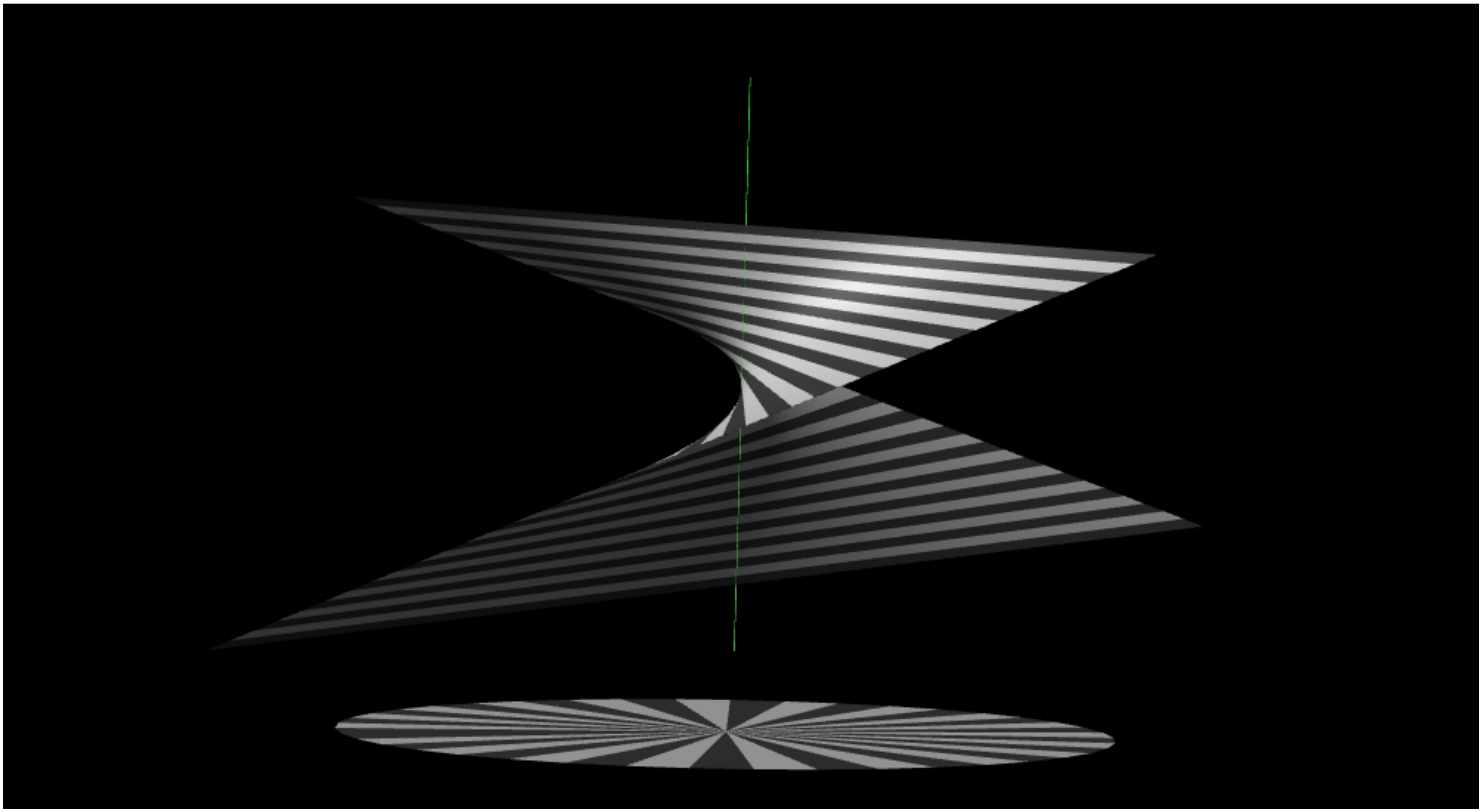

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Reads off "apparent mass" from strength of the gravitational field far from an isolated source.

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on line bundles $L \to \mathbb{CP}_1$ of Chern-class ≤ -3.

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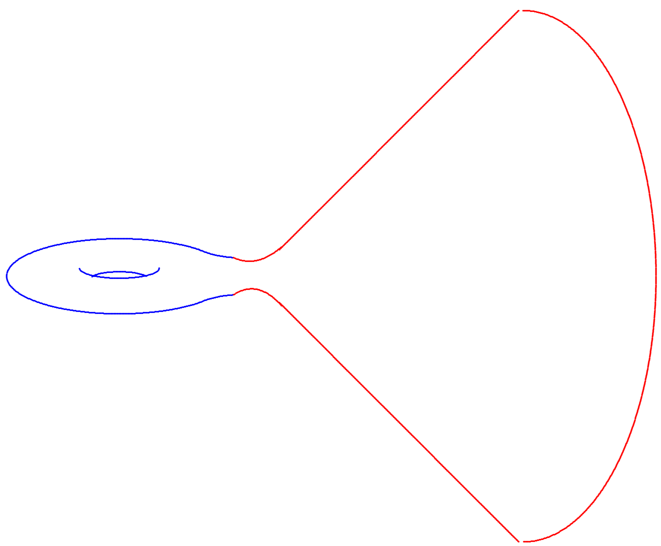

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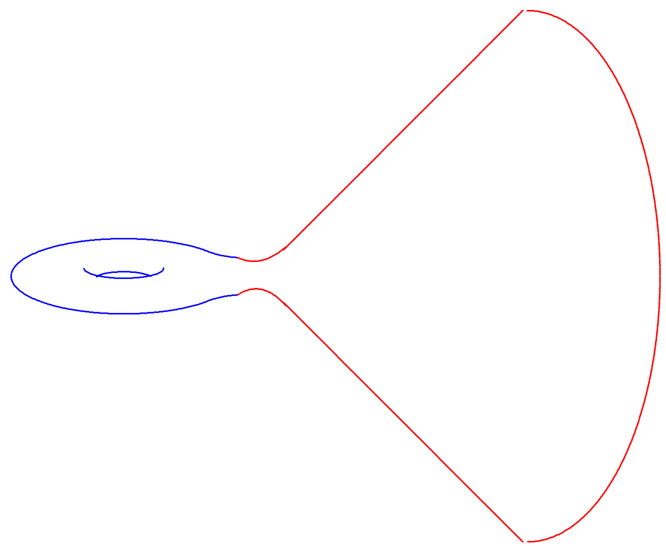

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Mass of an ALE Kähler manifold is unambiguous.

Does not depend on the choice of an end!

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

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(Discovered independently by Rollin, Singer, & Şuvaina, using different methods.)

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$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g d\mu_g$$

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So the mass is a "boundary correction" to the topological formula for the total scalar curvature.

Theorem C. *Any ALE Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

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Corollary. *Any ALE scalar-flat Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

$$m(M, g) = -\frac{\langle ♣(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}}.$$

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So **Theorem A** is an immediate consequence!

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Proof actually shows something stronger!

Theorem E.

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