

Einstein Metrics,
Harmonic Forms, &
Conformally Kähler Geometry

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Séminaire de Géométrie
Institut Fourier, 30 avril, 2019

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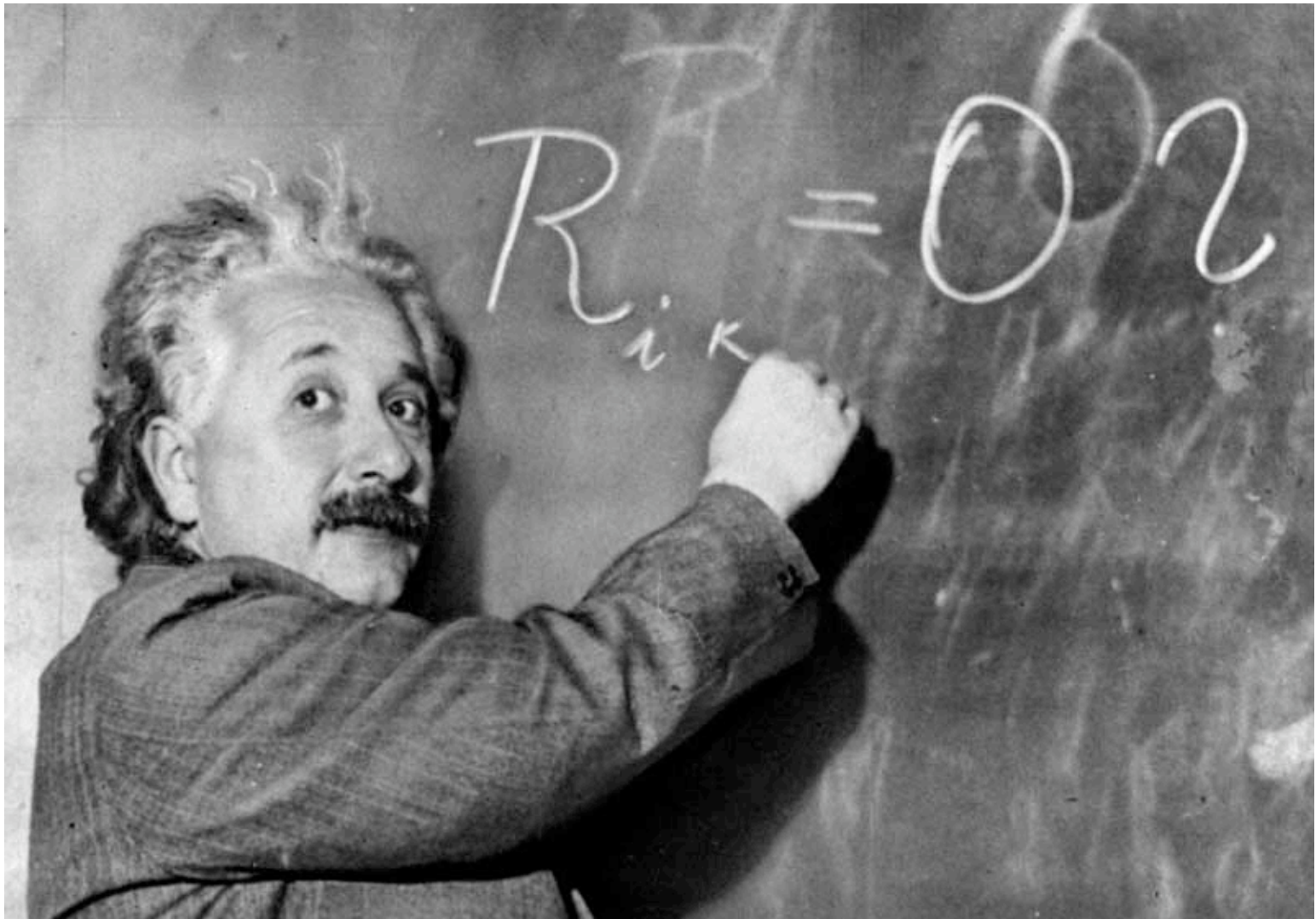
“... the greatest blunder of my life!”

— A. Einstein, to G. Gamow

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Has same sign as the *scalar curvature*

$$s = r_j^j = \mathcal{R}^{ij}_{ij}.$$

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When $n = 4$, situation is more encouraging...

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One key question:

Does enough rigidity really hold in dimension four to make this a genuine geometrization?

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Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric h (unrelated to ω)? What if we also require $\lambda \geq 0$?*

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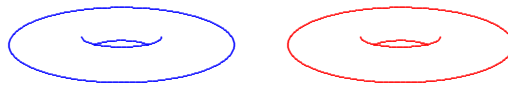
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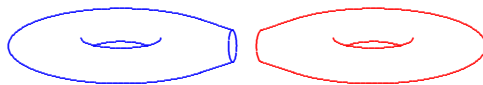
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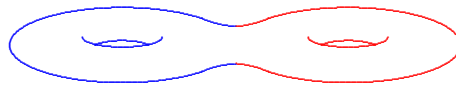
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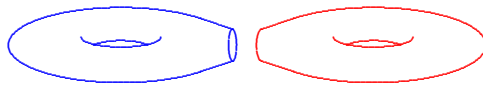
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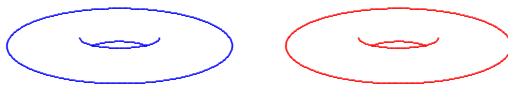
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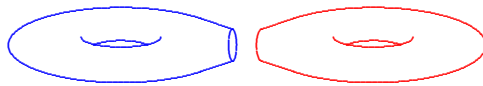
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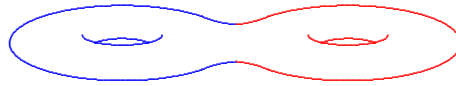
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Calabi/Yau: Admits Ricci-flat Kähler metrics.

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Know an Einstein metric on each manifold.

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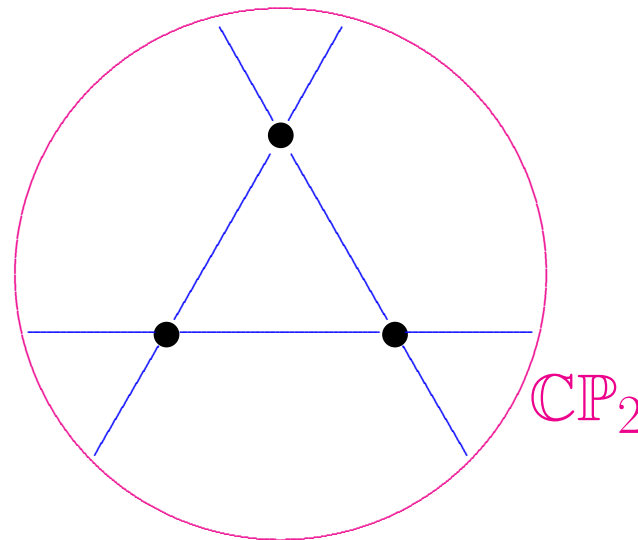
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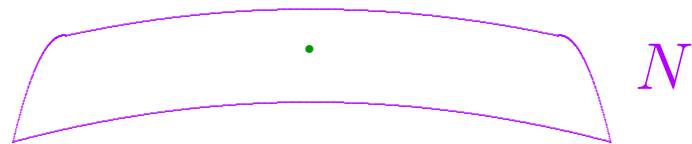
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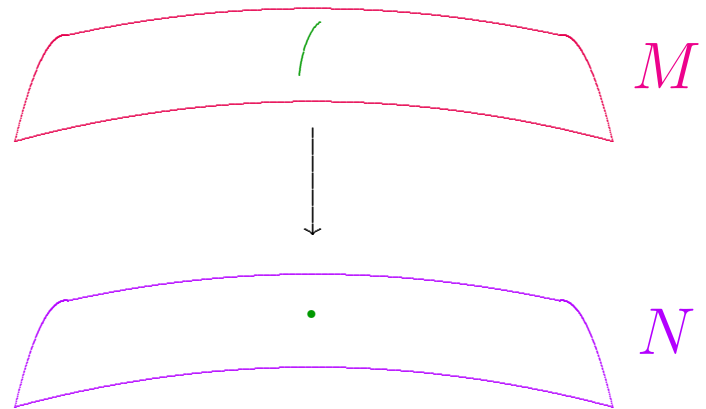
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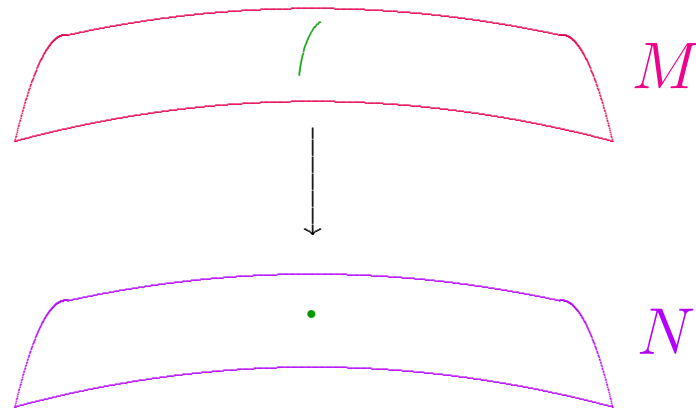


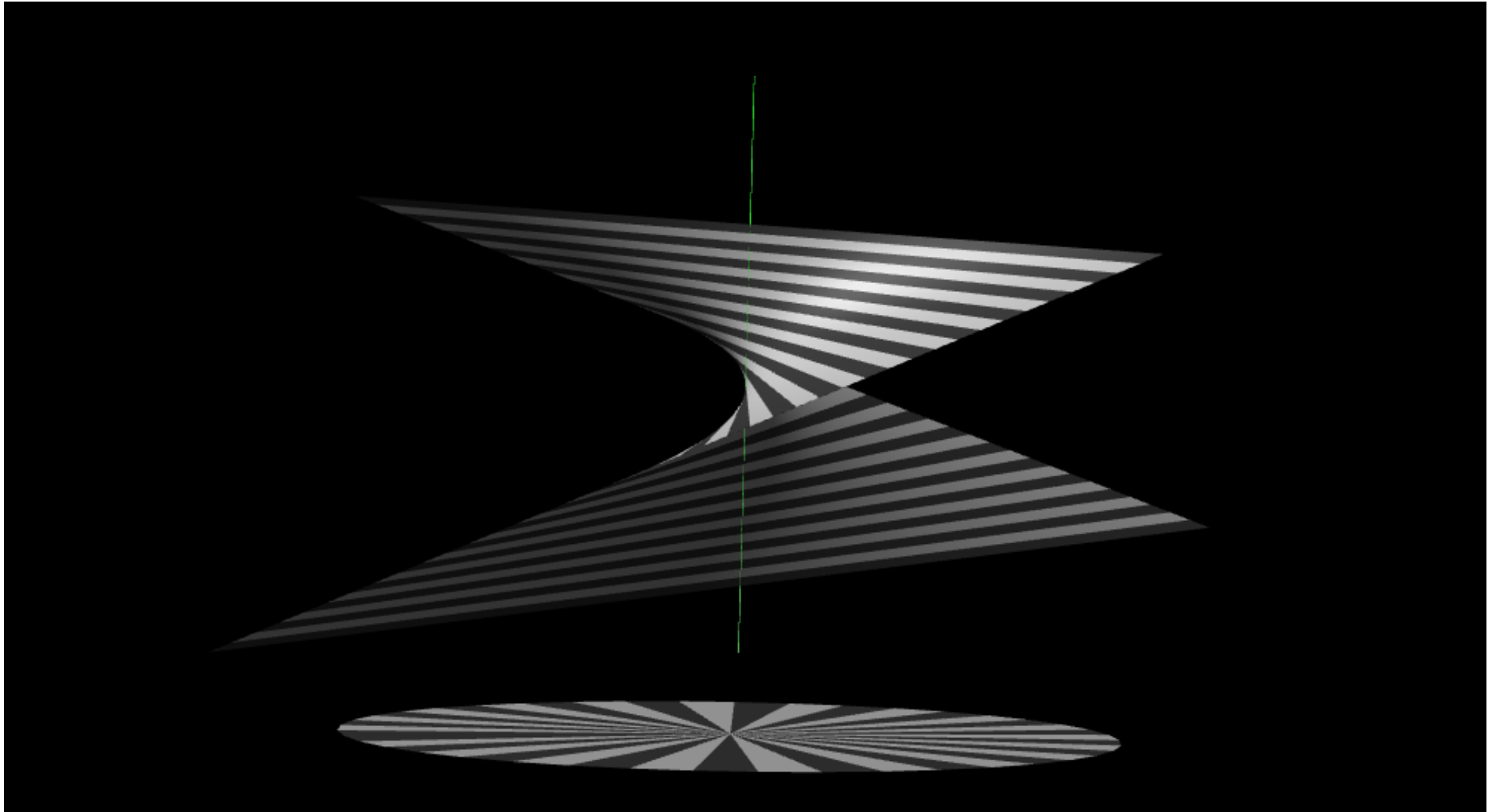
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{C}P_1$ to obtain **blow-up**

$$M \approx N \# \overline{\mathbb{C}P_2}$$

in which added $\mathbb{C}P_1$ has normal bundle $\mathcal{O}(-1)$.



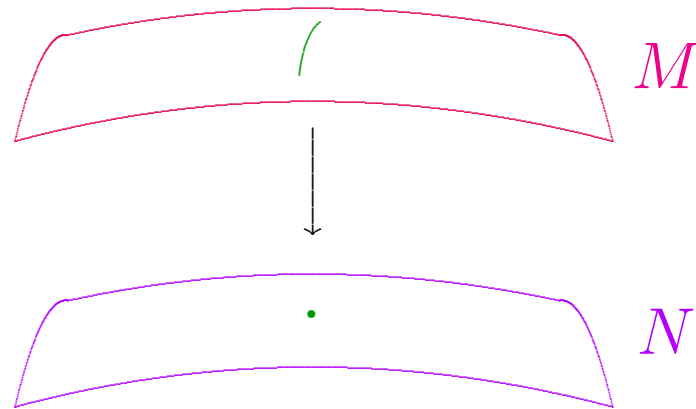


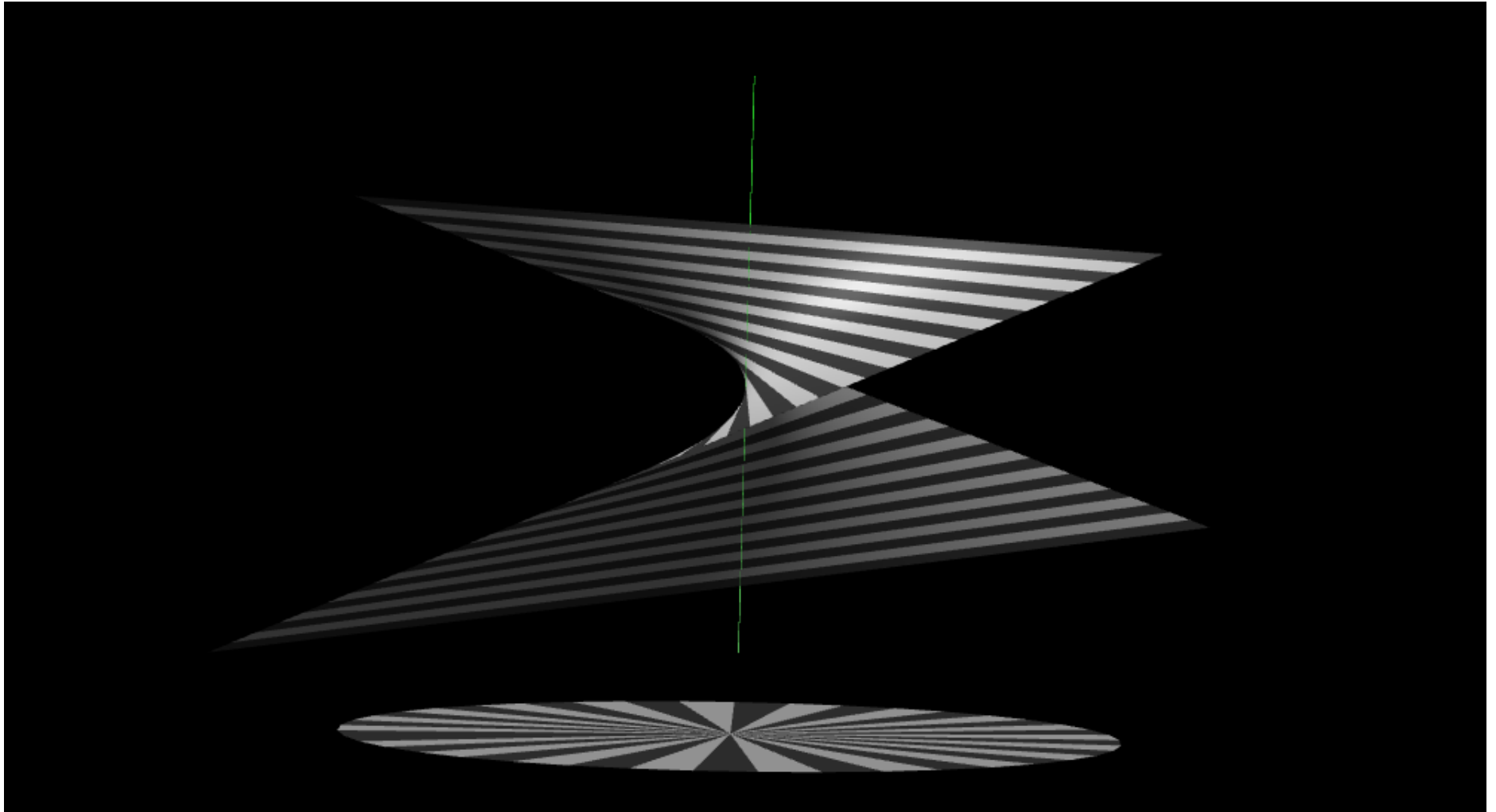
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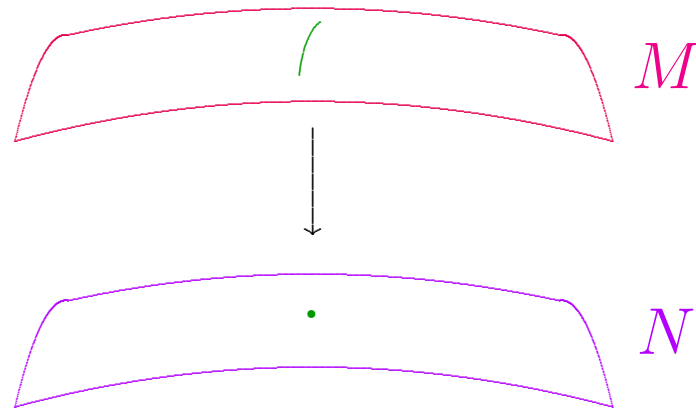


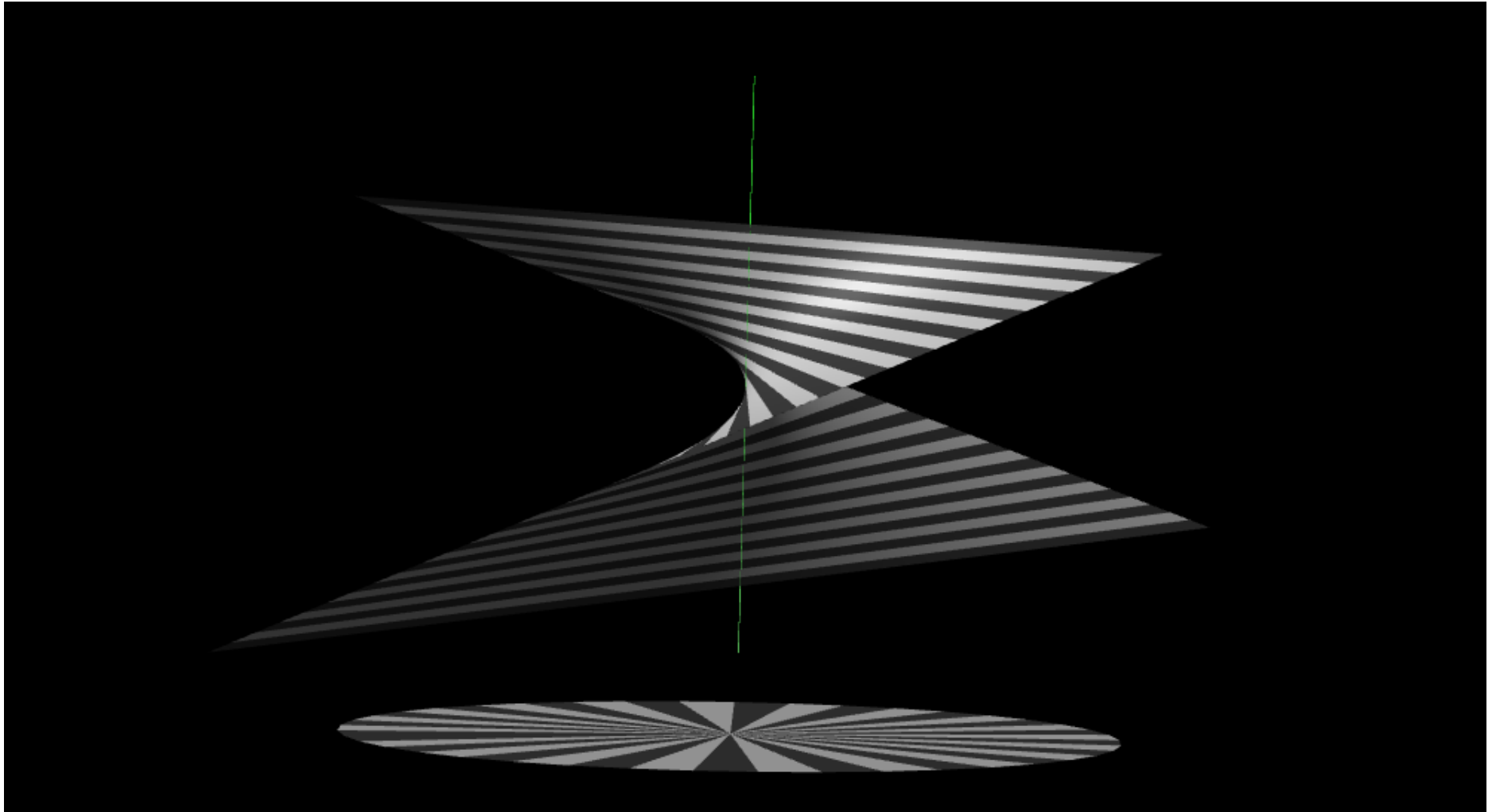
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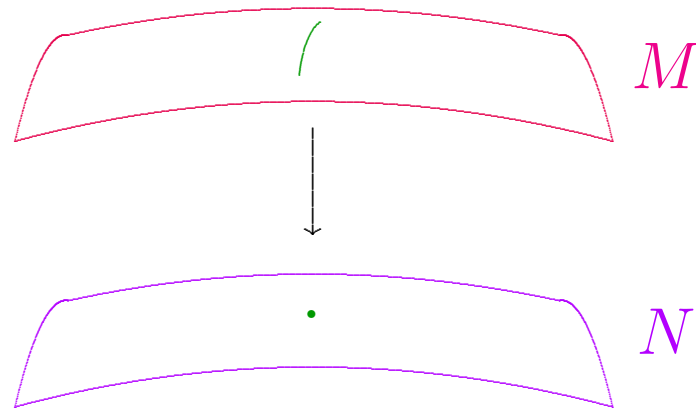


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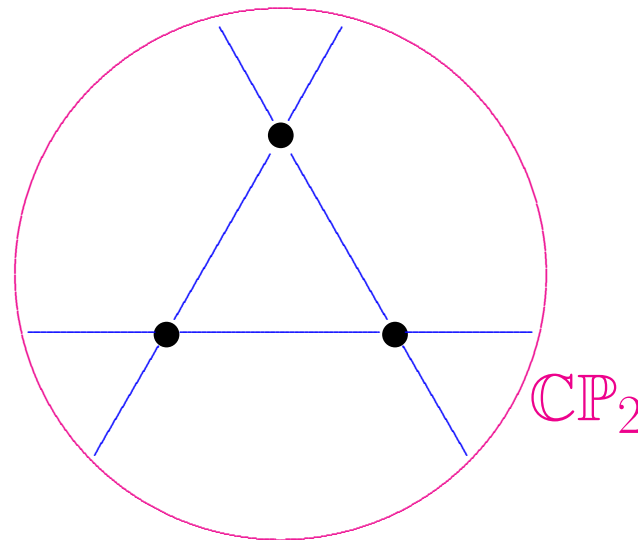


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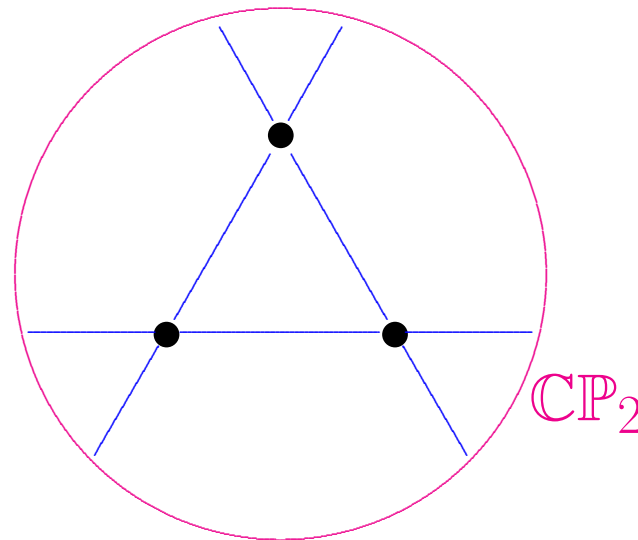
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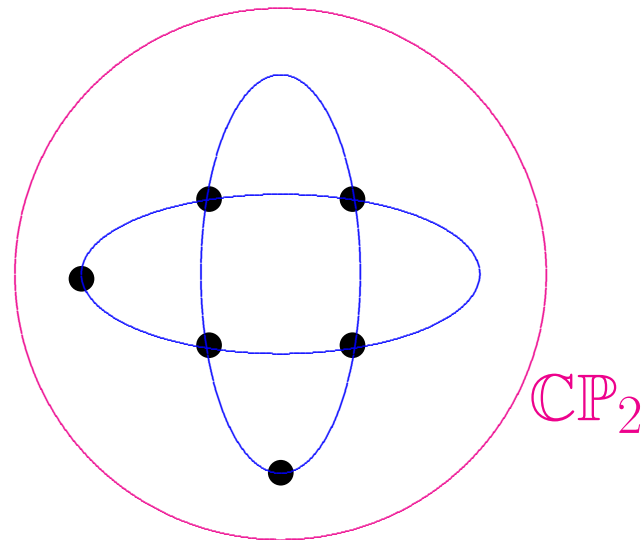


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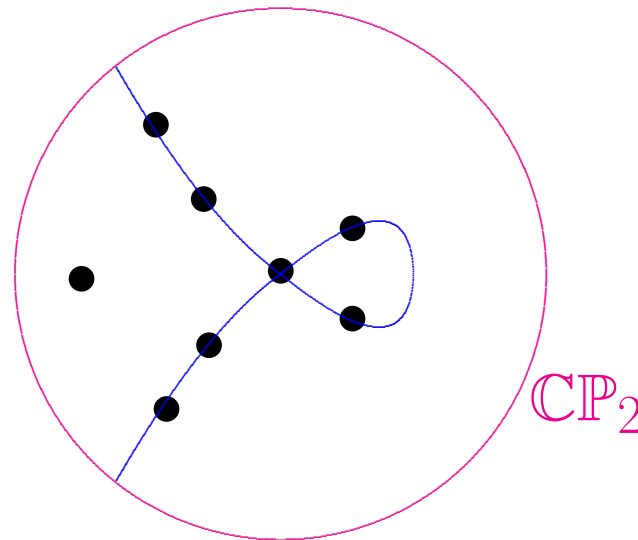


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Uniqueness: Bando-Mabuchi '87, L '12.

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Formulation depends on . . .

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More generally, their dimensions

$$b_\pm(M) = \dim \mathcal{H}_h^\pm$$

are completely metric-independent, and are oriented homotopy invariants of M .

Key background result:

Theorem (L '15).

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Yes — with a reasonable extra hypothesis on ω ...

Definition.

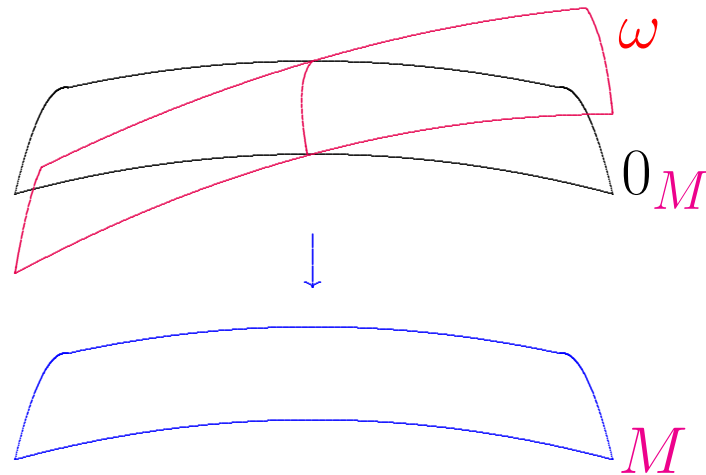
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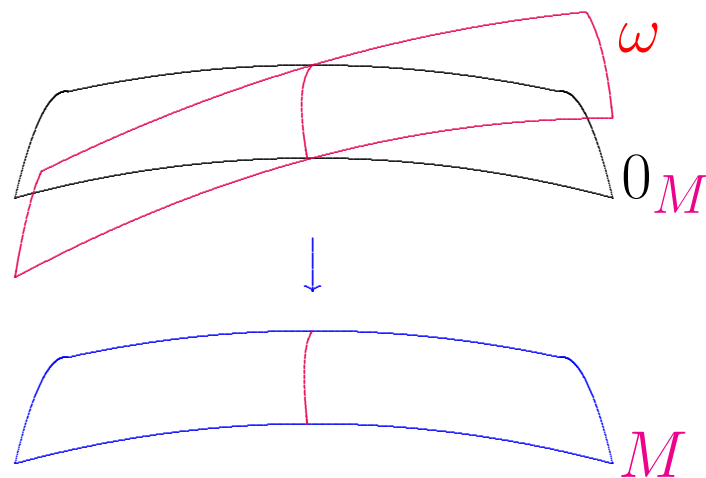
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\implies Zero set Z of ω has codimension 3:

$$Z \approx \sqcup_{j=1}^n S^1.$$

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Theorem (Taubes, et al). If $b_+(M) \neq 0$, such forms exist for an open dense set of metrics h on M .

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Moral: Taubes' genericity result does not guarantee genericity among metrics solving an equation!

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Before discussing **Theorems A & B**,
consider simpler case when $W^+(\omega, \omega) > 0$.

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$\therefore h \propto s^{-2}g$ globally on M .

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