Self-Dual Metrics,

Null Geodesics, &

Holomorphic Disks

(Lecture V)

Claude LeBrun
Stony Brook University

Autumn School on Holomorphic Disks
Schloss Rauischholzhausen, November 17, 2018

Joint work with

Lionel Mason
Oxford University

Joint work with

Lionel Mason
Oxford University

Nonlinear Gravitons, Null Geodesics,
and Holomorphic Disks,
Duke Math. J. 136 (2007) 205–273.

Twistor correspondences

Twistor correspondences (Penrose):

Twistor correspondences (Penrose):
geometries from moduli space

Twistor correspondences (Penrose):

geometries from moduli space of holomorphic curves

$$\mathbb{CP}_1 \hookrightarrow Z$$

Twistor correspondences (Penrose):
geometries from moduli space of holomorphic curves

$$\mathbb{CP}_1 \hookrightarrow Z$$

where Z complex manifold.

Twistor correspondences (Penrose):

geometries from moduli space of holomorphic curves

$$\mathbb{CP}_1 \hookrightarrow Z$$

where Z complex manifold.

Analogous pattern:

Twistor correspondences (Penrose):
geometries from moduli space of holomorphic curves

$$\mathbb{CP}_1 \hookrightarrow Z$$

where Z complex manifold.

Analogous pattern:
moduli of holomorphic disks

Twistor correspondences (Penrose):
geometries from moduli space of holomorphic curves

$$\mathbb{CP}_1 \hookrightarrow Z$$

where Z complex manifold.

Analogous pattern:
moduli of holomorphic disks

$$(D^2, S^1) \hookrightarrow (Z, P),$$

Twistor correspondences (Penrose):
geometries from moduli space of holomorphic curves

$$\mathbb{CP}_1 \hookrightarrow Z$$

where Z complex manifold.

Analogous pattern:
moduli of holomorphic disks

$$(D^2, S^1) \hookrightarrow (Z, P),$$

where Z complex manifold,

Twistor correspondences (Penrose):
geometries from moduli space of holomorphic curves

$$\mathbb{CP}_1 \hookrightarrow Z$$

where Z complex manifold.

Analogous pattern:
moduli of holomorphic disks

$$(D^2, S^1) \hookrightarrow (Z, P),$$

where Z complex manifold,
$P \subset Z$ totally real submanifold.

Twistor correspondences (Penrose):
geometries from moduli space of holomorphic curves

$$\mathbb{CP}_1 \hookrightarrow Z$$

where Z complex manifold.

Analogous pattern:
moduli of holomorphic disks

$$(D^2, S^1) \hookrightarrow (Z, P),$$

where Z complex manifold,
$P \subset Z$ totally real submanifold.

We have seen:

Twistor correspondences (Penrose):
geometries from moduli space of holomorphic curves

$$\mathbb{CP}_1 \hookrightarrow Z$$

where Z complex manifold.

Analogous pattern:
moduli of holomorphic disks

$$(D^2, S^1) \hookrightarrow (Z, P),$$

where Z complex manifold,
$P \subset Z$ totally real submanifold.

We have seen:
Holomorphic disks in $\mathbb{CP}_2 \leftrightsquigarrow$ Zoll surfaces.

Twistor correspondences (Penrose):
geometries from moduli space of holomorphic curves

$$\mathbb{CP}_1 \hookrightarrow Z$$

where Z complex manifold.

Analogous pattern:
moduli of holomorphic disks

$$(D^2, S^1) \hookrightarrow (Z, P),$$

where Z complex manifold,
$P \subset Z$ totally real submanifold.

We have seen:
Holomorphic disks in $\mathbb{CP}_2 \rightsquigarrow$ Zoll surfaces.

This lecture:

Twistor correspondences (Penrose):
geometries from moduli space of holomorphic curves

$$\mathbb{CP}_1 \hookrightarrow Z$$

where Z complex manifold.

Analogous pattern:
moduli of holomorphic disks

$$(D^2, S^1) \hookrightarrow (Z, P),$$

where Z complex manifold,
$P \subset Z$ totally real submanifold.

We have seen:
Holomorphic disks in $\mathbb{CP}_2 \leftrightsquigarrow$ Zoll surfaces.

This lecture:
Disks in $\mathbb{CP}_3$, with boundaries on deformed $\mathbb{RP}^3$.

Penrose

Penrose nonlinear graviton:

Penrose nonlinear graviton:

Local description of holomorphic self-dual 4-manifolds.

Penrose nonlinear graviton:

Local description of real-analytic self-dual 4-manifolds.

Penrose nonlinear graviton:

Local description of real-analytic self-dual 4-manifolds.

Atiyah-Hitchin-Singer:

Penrose nonlinear graviton:

Local description of real-analytic self-dual 4-manifolds.

Atiyah-Hitchin-Singer:

Global smooth Riemannian reformulation.

Penrose nonlinear graviton:

Local description of real-analytic self-dual 4-manifolds.

Atiyah-Hitchin-Singer:

Global smooth Riemannian reformulation.

This lecture:

Penrose nonlinear graviton:

Local description of real-analytic self-dual 4-manifolds.

Atiyah-Hitchin-Singer:

Global smooth Riemannian reformulation.

This lecture: split-signature metrics

Penrose nonlinear graviton:

Local description of real-analytic self-dual 4-manifolds.

Atiyah-Hitchin-Singer:

Global smooth Riemannian reformulation.

This lecture: split-signature metrics:

pseudo-Riemannian metrics

Penrose nonlinear graviton:

Local description of real-analytic self-dual 4-manifolds.

Atiyah-Hitchin-Singer:

Global smooth Riemannian reformulation.

This lecture: split-signature metrics:

pseudo-Riemannian metrics with components

$$\begin{bmatrix} +1 & & & \\ & +1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

in suitable local frame.

4-dimensional geometry is idiosyncratic.

4-dimensional geometry is idiosyncratic.

In split signature setting, this happens because. . .

The Lie group $SO(2,2)$ is *not simple*:

The Lie group $SO(2,2)$ is *not simple*:

$$\mathfrak{so}(2,2) \cong \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2).$$

The Lie group $SO(2,2)$ is *not simple*:

$$\mathfrak{so}(2,2) \cong \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2).$$

On oriented split-signature (M^4, g),

The Lie group $SO(2,2)$ is *not simple*:

$$\mathfrak{so}(2,2) \cong \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2).$$

On oriented split-signature (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

The Lie group $SO(2,2)$ is *not simple*:

$$\mathfrak{so}(2,2) \cong \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2).$$

On oriented split-signature (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

The Lie group $SO(2,2)$ is *not simple*:

$$\mathfrak{so}(2,2) \cong \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2).$$

On oriented split-signature (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

The Lie group $SO(2,2)$ is *not simple*:

$$\mathfrak{so}(2,2) \cong \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2).$$

On oriented split-signature (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.
Λ^- anti-self-dual 2-forms.

Why important?

The Lie group $SO(2,2)$ is *not simple*:

$$\mathfrak{so}(2,2) \cong \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2).$$

On oriented split-signature (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.
Λ^- anti-self-dual 2-forms.

Why important?

Curvature tensors are bundle-valued 2-forms!

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^{+}	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^{-}	$\mathring{r}$	$W_- + \frac{s}{12}$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^{+}	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^{-}	$\mathring{r}$	$W_- + \frac{s}{12}$

where

$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W_+ &= \text{self-dual Weyl curvature} \\ W_- &= \text{anti-self-dual Weyl curvature} \end{aligned}$$

$W_{\pm}$ are *conformally invariant*

$W_{\pm}$ are *conformally invariant*:

unchanged by conformal rescaling $g \rightsquigarrow fg$,

$W_{\pm}$ are *conformally invariant*:

unchanged by conformal rescaling $g \rightsquigarrow fg$,

$f : M \to \mathbb{R}^{\times}$ any non-zero function.

$W_\pm$ are *conformally invariant*:

unchanged by conformal rescaling $g \rightsquigarrow f g$,

$f : M \to \mathbb{R}^\times$ any non-zero function.

Definition. *Oriented split-signature* $(M^4, [g])$ *called* *self-dual* *iff satisfies* $W_- \equiv 0$.

$W_\pm$ are *conformally invariant*:

unchanged by conformal rescaling $g \rightsquigarrow f g$,

$f : M \to \mathbb{R}^\times$ any non-zero function.

Definition. *Oriented split-signature* $(M^4, [g])$ *called* *self-dual* *iff satisfies* $W_- \equiv 0$.

Here

$$[g] = \{f g \mid f \neq 0\}$$

denotes conformal class of split-signature metric.

Prototypical example:

Prototypical example:

standard indefinite product metric

$$g_0 = \pi_1^* h - \pi_2^* h$$

Prototypical example:
standard indefinite product metric

$$g_0 = \pi_1^* h - \pi_2^* h$$

on $S^2 \times S^2$, where h curvature 1 on S^2.

Prototypical example:

standard indefinite product metric

$$g_0 = \pi_1^* h - \pi_2^* h$$

on $S^2 \times S^2$, where h curvature 1 on S^2.

- conformally flat $\Longrightarrow$ self-dual.

Prototypical example:
standard indefinite product metric

$$g_0 = \pi_1^* h - \pi_2^* h$$

on $S^2 \times S^2$, where h curvature 1 on S^2.

- conformally flat $\Longrightarrow$ self-dual.
- null geodesics all *embedded circles*.

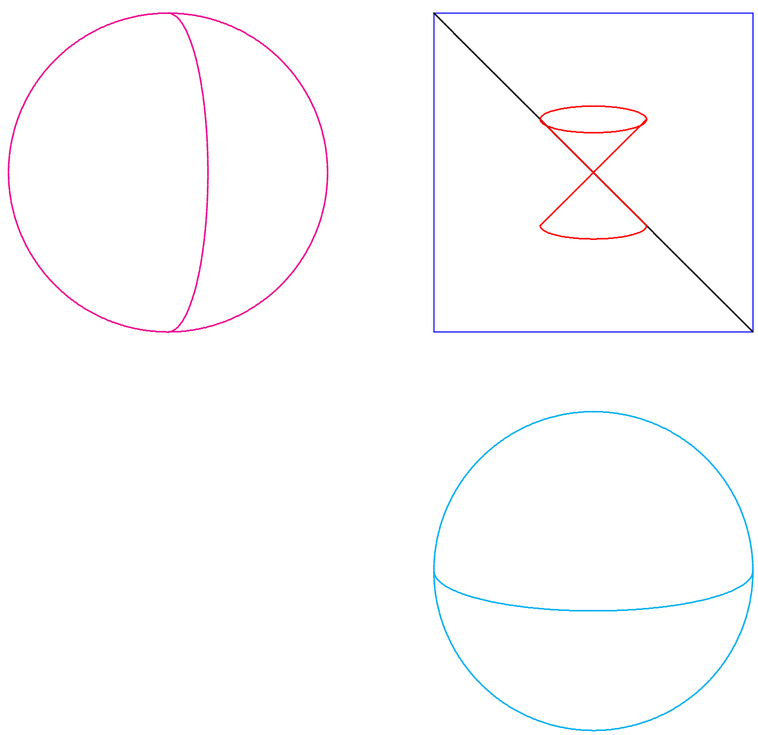
Prototypical example:

standard indefinite product metric

$$g_0 = \pi_1^* h - \pi_2^* h$$

on $S^2 \times S^2$, where h curvature 1 on S^2.

- conformally flat $\Longrightarrow$ self-dual.
- null geodesics all *embedded circles*.



Definition.

Definition (Guillemin).

Definition (Guillemin). $(M^n, [g])$

Definition (Guillemin). $(M^n, [g])$

(Indefinite pseudo-Riemannian manifold.)

Definition (Guillemin). $(M^n, [g])$

Definition (Guillemin)**.** $(M^n, [g])$ *called* Zollfrei

Definition (Guillemin). $(M^n, [g])$ *called* Zollfrei *if each null geodesic is embedded circle* $S^1 \subset M$.

Definition (Guillemin). $(M^n, [g])$ *called* Zollfrei *if each null geodesic is embedded circle* $S^1 \subset M$.

Surprising stability result:

Definition (Guillemin)**.** $(M^n, [g])$ *called* Zollfrei *if each null geodesic is embedded circle* $S^1 \subset M$*.*

Surprising stability result:

Theorem A.

Definition (Guillemin). $(M^n, [g])$ *called* Zollfrei *if each null geodesic is embedded circle* $S^1 \subset M$.

Surprising stability result:

Theorem A. *Let* (M, g)

Definition (Guillemin). $(M^n, [g])$ *called* Zollfrei *if each null geodesic is embedded circle* $S^1 \subset M$.

Surprising stability result:

Theorem A. *Let* (M, g) *be a self-dual*

Definition (Guillemin). $(M^n, [g])$ *called* Zollfrei *if each null geodesic is embedded circle* $S^1 \subset M$.

Surprising stability result:

Theorem A. *Let* (M, g) *be a self-dual Zollfrei*

Definition (Guillemin). $(M^n, [g])$ *called* Zollfrei *if each null geodesic is embedded circle* $S^1 \subset M$*.*

Surprising stability result:

Theorem A. *Let* (M, g) *be a self-dual Zollfrei* 4-*manifold. Then,*

Definition (Guillemin). $(M^n, [g])$ *called* Zollfrei *if each null geodesic is embedded circle* $S^1 \subset M$.

Surprising stability result:

Theorem A. *Let* (M, g) *be a self-dual Zollfrei* 4-*manifold. Then, with respect to the* C^2 *topology,*

Definition (Guillemin). $(M^n, [g])$ *called* Zollfrei *if each null geodesic is embedded circle* $S^1 \subset M$.

Surprising stability result:

Theorem A. *Let* (M, g) *be a self-dual Zollfrei* 4*-manifold. Then, with respect to the* C^2 *topology, there is an open neighborhood of* g

Definition (Guillemin). $(M^n, [g])$ *called* Zollfrei *if each null geodesic is embedded circle* $S^1 \subset M$.

Surprising stability result:

Theorem A. *Let* (M, g) *be a self-dual Zollfrei* 4*-manifold. Then, with respect to the* C^2 *topology, there is an open neighborhood of* g *in the space of pseudo-Riemannian metrics on* M

Definition (Guillemin). $(M^n, [g])$ *called* Zollfrei *if each null geodesic is embedded circle* $S^1 \subset M$*.*

Surprising stability result:

Theorem A. *Let* (M, g) *be a self-dual Zollfrei* 4*-manifold. Then, with respect to the* C^2 *topology, there is an open neighborhood of* g *in the space of pseudo-Riemannian metrics on* M *such that*

Definition (Guillemin). $(M^n, [g])$ *called* Zollfrei *if each null geodesic is embedded circle* $S^1 \subset M$.

Surprising stability result:

Theorem A. *Let* (M, g) *be a self-dual Zollfrei* 4*-manifold. Then, with respect to the* C^2 *topology, there is an open neighborhood of* g *in the space of pseudo-Riemannian metrics on* M *such that every self-dual metric*

Definition (Guillemin). $(M^n, [g])$ *called* Zollfrei *if each null geodesic is embedded circle* $S^1 \subset M$.

Surprising stability result:

Theorem A. *Let* (M, g) *be a self-dual Zollfrei* 4*-manifold. Then, with respect to the* C^2 *topology, there is an open neighborhood of* g *in the space of pseudo-Riemannian metrics on* M *such that every self-dual metric contained in this neighborhood*

Definition (Guillemin). $(M^n, [g])$ *called* Zollfrei *if each null geodesic is embedded circle* $S^1 \subset M$.

Surprising stability result:

Theorem A. *Let* (M, g) *be a* self-dual *Zollfrei* 4*-manifold. Then, with respect to the* C^2 *topology, there is an open neighborhood of* g *in the space of pseudo-Riemannian metrics on* M *such that every* self-dual *metric contained in this neighborhood is also Zollfrei.*

To study moduli of self-dual conformal structures, therefore reasonable to first focus on understanding self-dual metrics that are also Zollfrei.

To study moduli of self-dual conformal structures, therefore reasonable to first focus on understanding self-dual metrics that are also Zollfrei.

Which 4-manifolds *admit* self-dual Zollfrei metrics?

Second key example:

Second key example:

Real projective quadric

Second key example:

Real projective quadric

$$\mathbb{M}^{2,2} = \left\{ [x_1 : x_2 : x_3 : y_1 : y_2 : y_3] \in \mathbb{RP}^5 \;\middle|\; |\vec{x}|^2 - |\vec{y}|^2 = 0 \right\}$$

Second key example:

Real projective quadric

$$\mathbb{M}^{2,2} = \left\{ [x_1 : x_2 : x_3 : y_1 : y_2 : y_3] \in \mathbb{RP}^5 \;\middle|\; |\vec{x}|^2 - |\vec{y}|^2 = 0 \right\}$$
$$= (S^2 \times S^2)/\mathbb{Z}_2$$

since $\mathbb{Z}_2$-action

$$(\vec{x}, \vec{y}) \mapsto (-\vec{x}, -\vec{y})$$

preserves g_0.

Second key example:

Real projective quadric

$$\mathbb{M}^{2,2} = \left\{ [x_1 : x_2 : x_3 : y_1 : y_2 : y_3] \in \mathbb{RP}^5 \;\middle|\; |\vec{x}|^2 - |\vec{y}|^2 = 0 \right\}$$
$$= (S^2 \times S^2)/\mathbb{Z}_2$$

since $\mathbb{Z}_2$-action

$$(\vec{x}, \vec{y}) \mapsto (-\vec{x}, -\vec{y})$$

preserves g_0.

Exercise:

The affine chart $x_3 - y_3 = 1$ gives local coordinates (x_1, x_2, y_1, y_2) on $\mathbb{M}^{2,2}$ in which

$$g_0 \propto dx_1{}^2 + dx_2{}^2 - dy_1{}^2 - dy_2{}^2$$

Theorem B. *Let* $(M, [g])$ *be a connected oriented split-signature* 4*-manifold which is Zollfrei and self-dual. Then* M *is homeomorphic to either* $S^2 \times S^2$ *or* $\mathbb{M}^{2,2}$.

Theorem B. *Let* $(M, [g])$ *be a connected oriented split-signature* 4*-manifold which is Zollfrei and self-dual. Then* M *is homeomorphic to either* $S^2 \times S^2$ *or* $\mathbb{M}^{2,2}$*.*

Topological rigidity!

Theorem B. *Let $(M, [g])$ be a connected oriented split-signature 4-manifold which is Zollfrei and self-dual. Then M is homeomorphic to either $S^2 \times S^2$ or $\mathbb{M}^{2,2}$.*

Topological rigidity!

Geometric rigidity?

Flexibility!

Flexibility!

Theorem C. *There is a natural one-to-one correspondence between*

- *equivalence classes of smooth self-dual conformal structures* $[g]$ *on* $S^2 \times S^2$*; and*

Flexibility!

Theorem C. *There is a natural one-to-one correspondence between*

- *equivalence classes of smooth self-dual conformal structures* $[g]$ *on* $S^2 \times S^2$*; and*
- *equivalence classes of smooth embeddings*

$$\mathbb{RP}^3 \hookrightarrow \mathbb{CP}_3,$$

Flexibility!

Theorem C. *There is a natural one-to-one correspondence between*

- *equivalence classes of smooth* self-dual *conformal structures* $[g]$ *on* $S^2 \times S^2$*; and*
- *equivalence classes of smooth embeddings*

$$\mathbb{RP}^3 \hookrightarrow \mathbb{CP}_3,$$

at least in a neighborhood of the standard conformal metric $[g_0]$ *and the standard embedding of* $\mathbb{RP}^3$.

Actually, g_0 is indefinite scalar-flat Kähler metric

Actually, g_0 is indefinite scalar-flat Kähler metric on $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Actually, g_0 is indefinite scalar-flat Kähler metric on $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Observation of Gauduchon

Actually, g_0 is indefinite scalar-flat Kähler metric on $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Observation of Gauduchon

Any Riemannian scalar-flat Kähler metric g on a complex surfaces is automatically anti-self-dual.

Actually, g_0 is indefinite scalar-flat Kähler metric on $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Observation of Gauduchon

Actually, g_0 is indefinite scalar-flat Kähler metric on $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Observation of Gauduchon, stood on its head:

Actually, g_0 is indefinite scalar-flat Kähler metric on $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Observation of Gauduchon, stood on its head:

Any indefinite scalar-flat Kähler metric g on a complex surfaces is automatically self-dual.

Actually, g_0 is indefinite scalar-flat Kähler metric on $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Observation of Gauduchon, stood on its head:

Any indefinite scalar-flat Kähler metric g on a complex surfaces is automatically self-dual.

Complex orientation:
indefinite Kähler form ω is anti-self-dual...

Theorem D. *The only complex surface* (M, J) *admitting Zollfrei scalar-flat indef Kähler metrics is* $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem D. *The only complex surface* (M, J) *admitting Zollfrei scalar-flat indef Kähler metrics is* $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Every such metric arises from a family of analytic disks in $\mathbb{CP}_3$

Theorem D. *The only complex surface* (M, J) *admitting Zollfrei scalar-flat indef Kähler metrics is* $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Every such metric arises from a family of analytic disks in $\mathbb{CP}_3$ *with boundary on a totally real* $\mathbb{RP}^3$.

Theorem D. *The only complex surface* (M, J) *admitting Zollfrei scalar-flat indef Kähler metrics is* $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Every such metric arises from a family of analytic disks in $\mathbb{CP}_3$ *with boundary on a totally real* $\mathbb{RP}^3$.

Near the standard metric g_0 *scalar-flat Kähler metrics of fixed total volume*

Theorem D. *The only complex surface (M, J) admitting Zollfrei scalar-flat indef Kähler metrics is $\mathbb{CP}_1 \times \mathbb{CP}_1$.*

Every such metric arises from a family of analytic disks in $\mathbb{CP}_3$ with boundary on a totally real $\mathbb{RP}^3$.

Near the standard metric g_0 scalar-flat Kähler metrics of fixed total volume $\longleftrightarrow$ with those totally real embeddings

$$\mathbb{RP}^3 \hookrightarrow \mathbb{CP}^3 - Q$$

on which the pull-back of the 3-form

$$\phi = \Im m \frac{z_1 dz_2 \wedge dz_3 \wedge dz_4 - \cdots - z_4 dz_1 \wedge dz_2 \wedge dz_3}{({z_1}^2 + {z_2}^2 + {z_3}^2 + {z_4}^2)^2}$$

vanishes.

Theorem D. *The only complex surface* (M, J) *admitting Zollfrei scalar-flat indef Kähler metrics is* $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Every such metric arises from a family of analytic disks in $\mathbb{CP}_3$ *with boundary on a totally real* $\mathbb{RP}^3$.

Near the standard metric g_0 *scalar-flat Kähler metrics of fixed total volume* $\longleftrightarrow$ *with those totally real embeddings*

$$\mathbb{RP}^3 \hookrightarrow \mathbb{CP}^3 - Q$$

on which the pull-back of the 3-form

$$\phi = \Im m \frac{z_1 dz_2 \wedge dz_3 \wedge dz_4 - \cdots - z_4 dz_1 \wedge dz_2 \wedge dz_3}{({z_1}^2 + {z_2}^2 + {z_3}^2 + {z_4}^2)^2}$$

vanishes. Here Q *denotes the quadric surface*

$${z_1}^2 + {z_2}^2 + {z_3}^2 + {z_4}^2 = 0.$$

$\Longrightarrow$ Near g_0,

$$\{\text{self-dual } [g] \text{ on } S^2 \times S^2\}/\{\textit{based diffeomorphisms}\}$$

$$\longleftrightarrow \{\text{vector fields } v \text{ on } \mathbb{RP}^3\},$$

$\Longrightarrow$ Near g_0,

$$\{\text{self-dual } [g] \text{ on } S^2 \times S^2\}/\{\textit{based diffeomorphisms}\}$$

$$\longleftrightarrow \{\text{vector fields } v \text{ on } \mathbb{RP}^3\},$$

while

$$\{\text{scalar-flat Kähler } g\}/\{\textit{diffeomorphisms, rescaling}\}$$

$$\longleftrightarrow \{\text{vector fields } v \mid \nabla \cdot v = 0\}.$$

$\Longrightarrow$ Near g_0,

$$\{\text{self-dual } [g] \text{ on } S^2 \times S^2\}/\{\textit{based diffeomorphisms}\}$$

$$\longleftrightarrow \{\text{vector fields } v \text{ on } \mathbb{RP}^3\},$$

while

$$\{\text{scalar-flat Kähler } g\}/\{\textit{diffeomorphisms, rescaling}\}$$

$$\longleftrightarrow \{\text{vector fields } v \mid \nabla \cdot v = 0\}.$$

$\Longrightarrow$ Both moduli spaces are infinite-dimensional.

End, Part IV