Zoll Manifolds,

Complex Surfaces, &

Holomorphic Disks, III

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Stony Brook University

Autumn School on Holomorphic Disks
Schloss Rauischholzhausen, November 16, 2018

Joint work with

Joint work with

Lionel Mason
Oxford University

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Zoll Manifolds and Complex Surfaces

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J. Diff. Geom. 347 (2002) 453–535.

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Definition. *A Zoll projective structure $[\nabla]$ on M is the projective equivalence class of some Zoll connection ∇.*

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Theorem. *If a compact surface* M^2 *admits a Zoll projective connection* $[\nabla]$*, then*

$$|\pi_1(M)| < \infty,$$

and hence

$$M \approx S^2 \text{ or } \mathbb{RP}^2.$$

Proposition. *For any Zoll* $[\nabla]$ *on a compact surface* M^2*, we have a double fibration*

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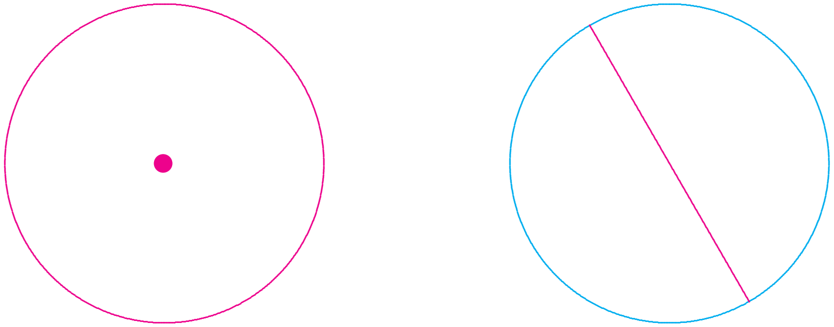
Proto-type:

Poncelet duality:



$\mathbb{P}(\mathbb{V})$ $\mathbb{P}(\mathbb{V}^*)$

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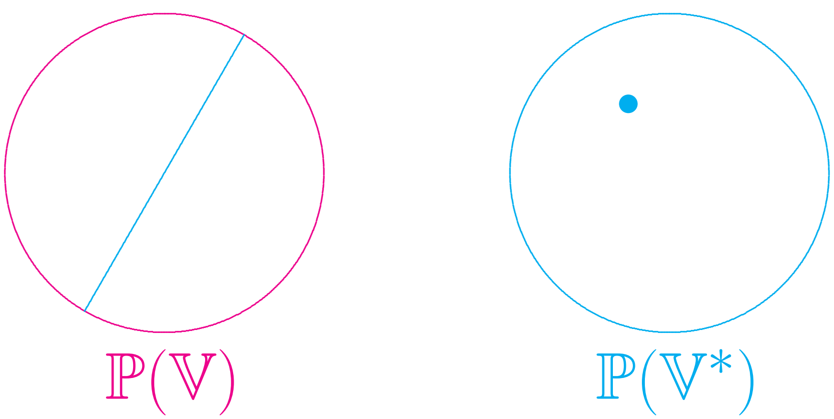

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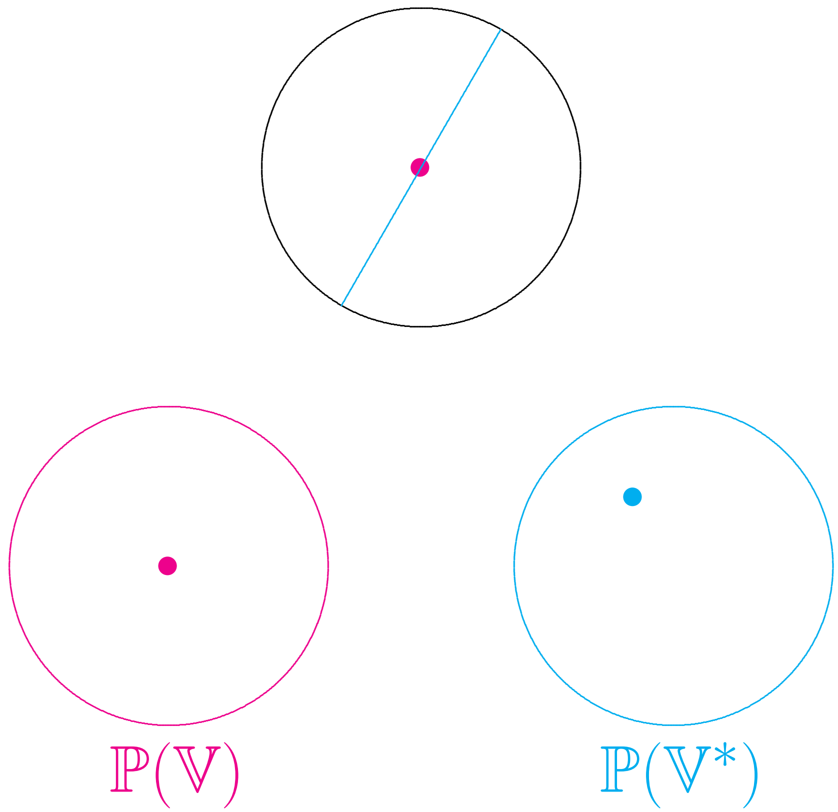

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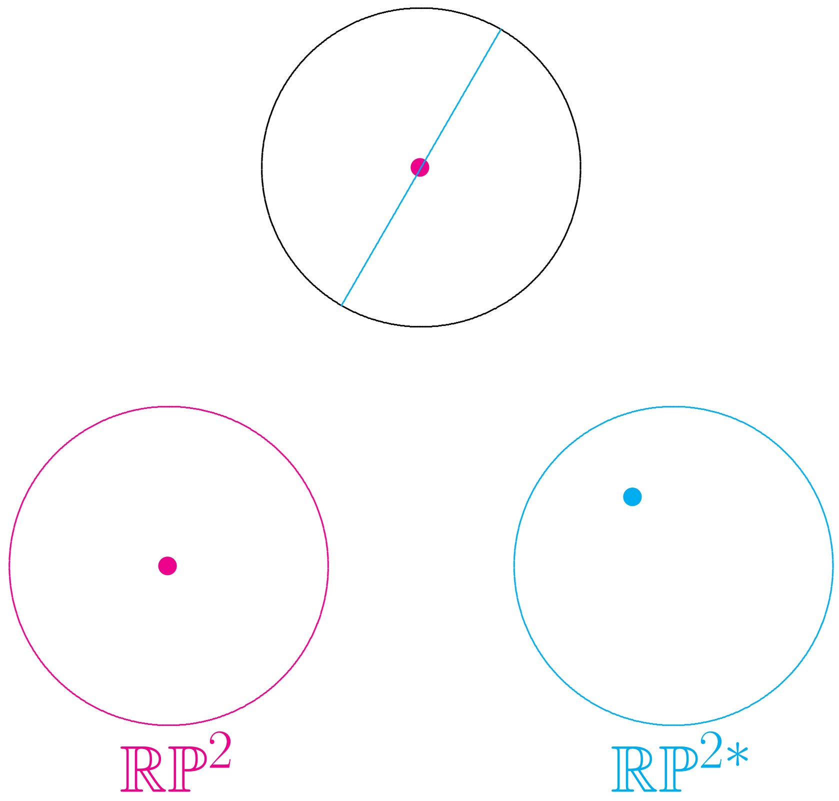

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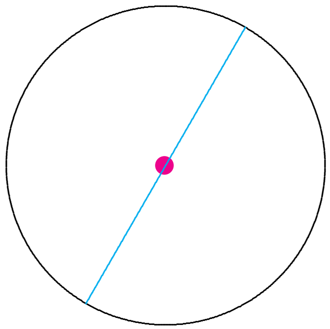
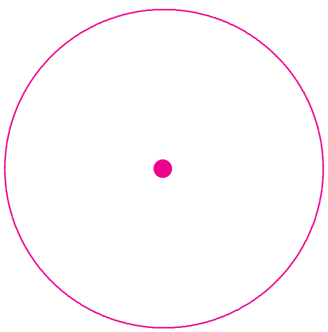
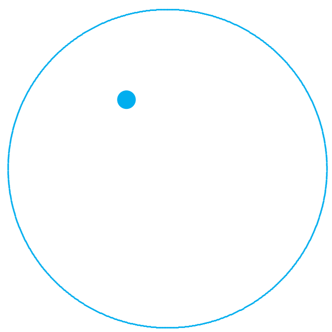




$\mathbb{CP}_2$



$\mathbb{CP}_2^*$

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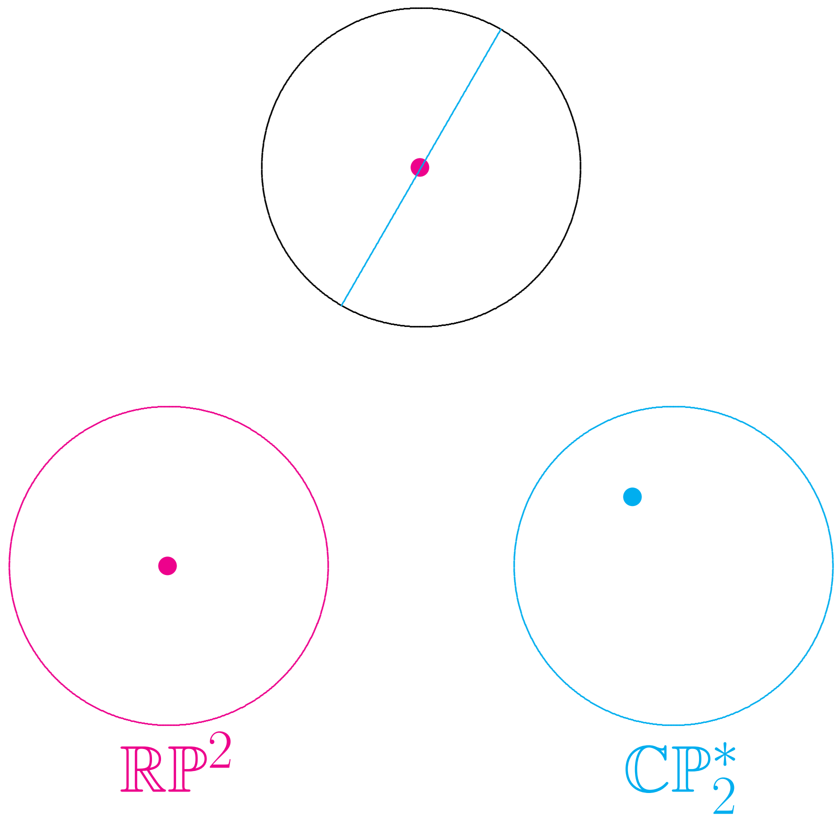
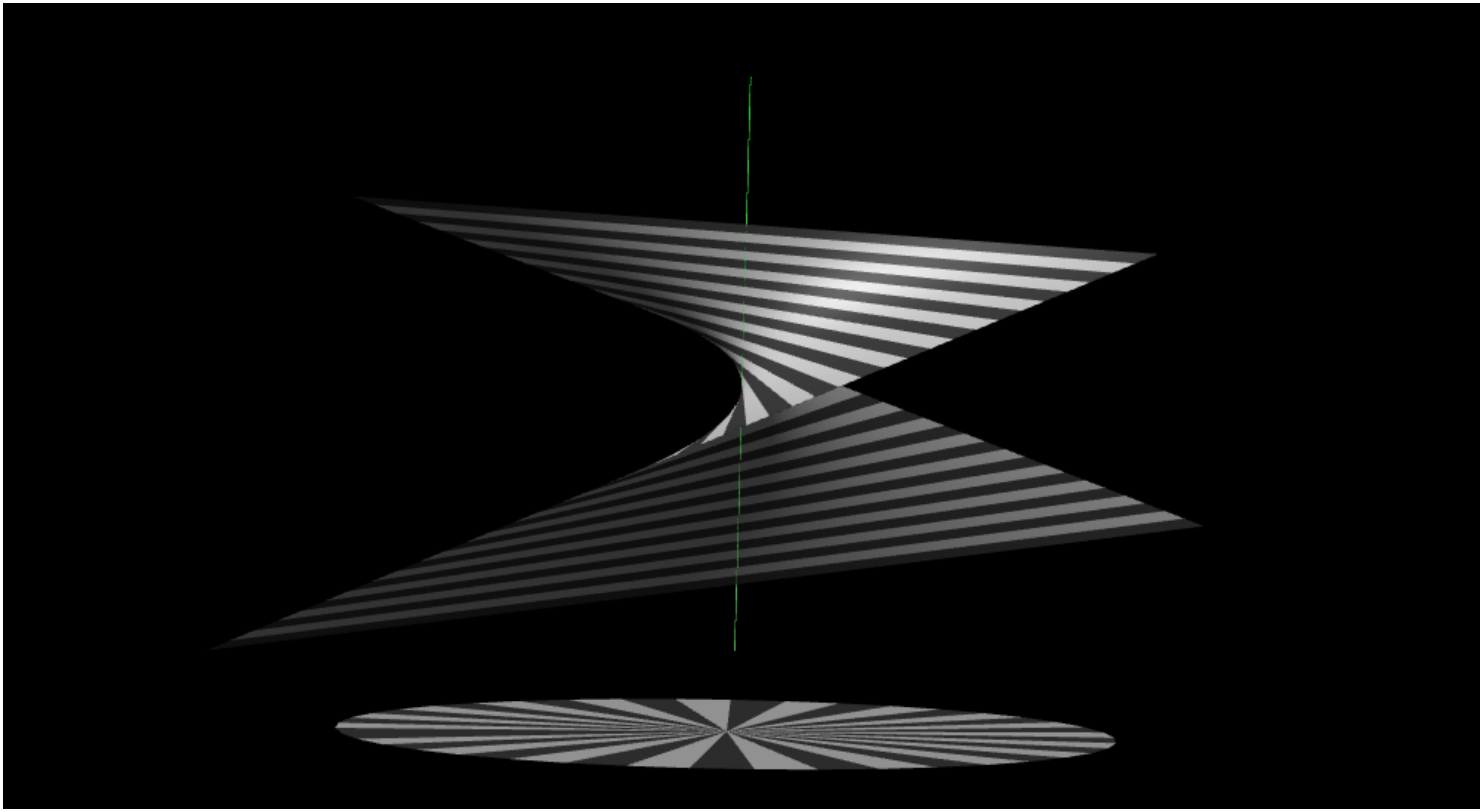


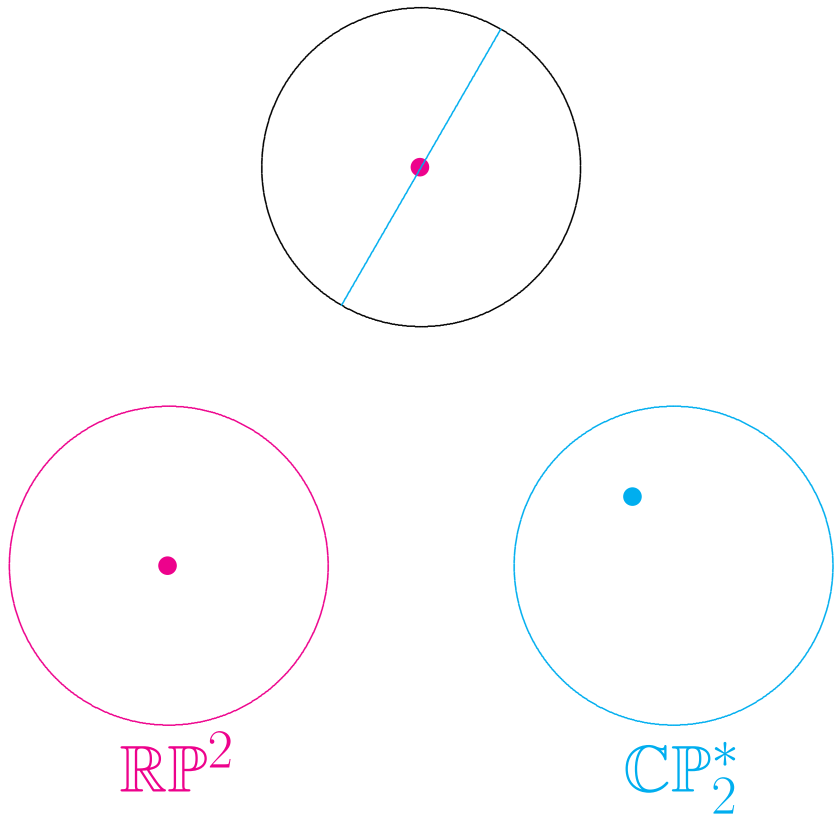
Proto-type:

Blowing up:



Proto-type:

Real blow-up of $\mathbb{CP}_2^*$ along $\mathbb{RP}^{2^*}$:



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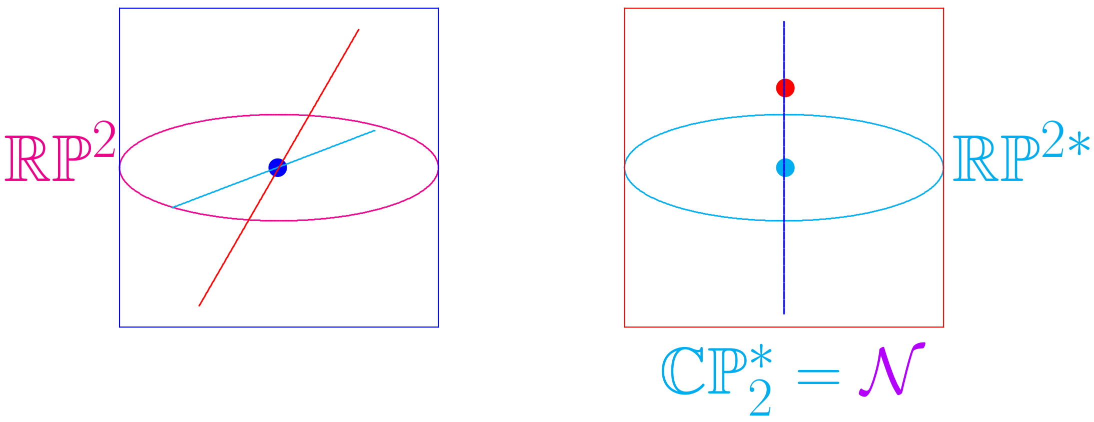

Proposition. *Let M^2 be any surface, and let $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ be its projectivized complexified tangent bundle.*

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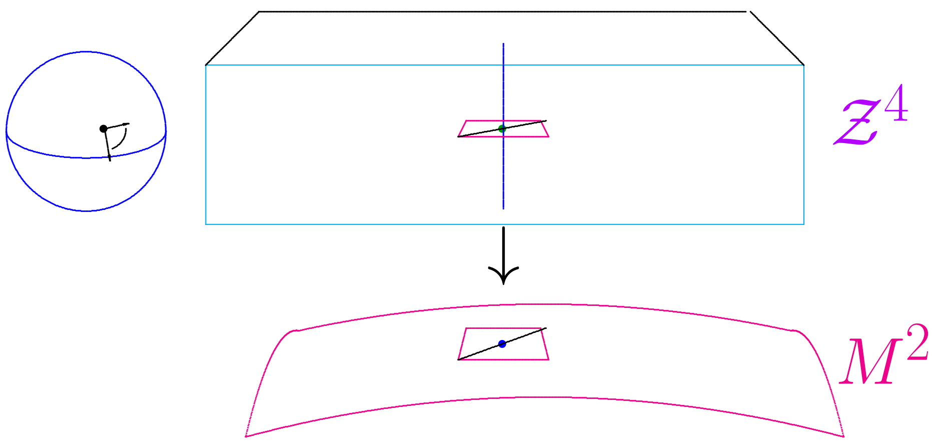

$$\mathbf{D} = \mathbf{L}_1 \oplus \mathbf{L}_2 \subset T_{\mathbb{C}}\mathcal{Z}$$

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$$\mathbf{D} = \operatorname{span}\left\{ \Xi \, , \quad \frac{\partial}{\partial \bar{\zeta}} \right\}$$

$$\Xi = \frac{\partial}{\partial x^1} + \zeta \frac{\partial}{\partial x^2} + P(x, \zeta) \frac{\partial}{\partial \zeta},$$

$$P = -\Gamma_{11}^2 + \left[\Gamma_{11}^1 - 2\Gamma_{12}^2\right] \zeta + \left[2\Gamma_{12}^1 - \Gamma_{22}^2\right] \zeta^2 + \Gamma_{22}^1 \zeta^3$$

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Moreover, two connections ∇ *and* $\hat{\nabla}$ *give rise to the same* $\mathbf{D}$ *iff they are projectively equivalent.*

Corollary. *For any* $(M^2, [\nabla])$,

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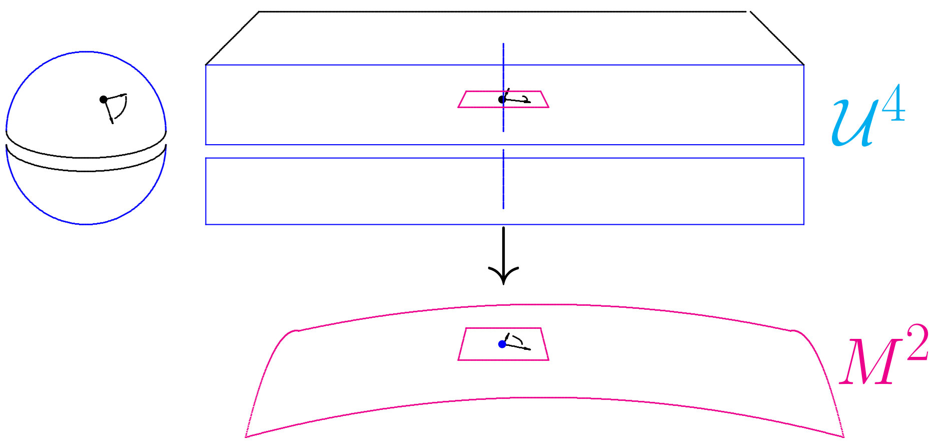
is a complex manifold.



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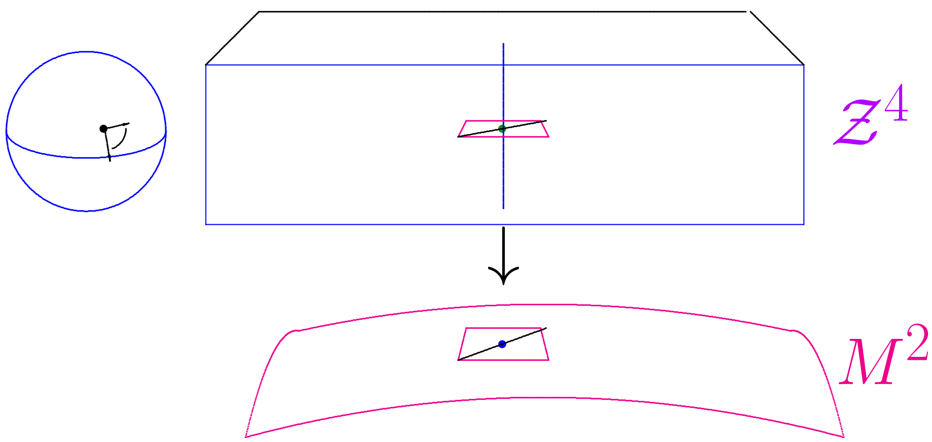

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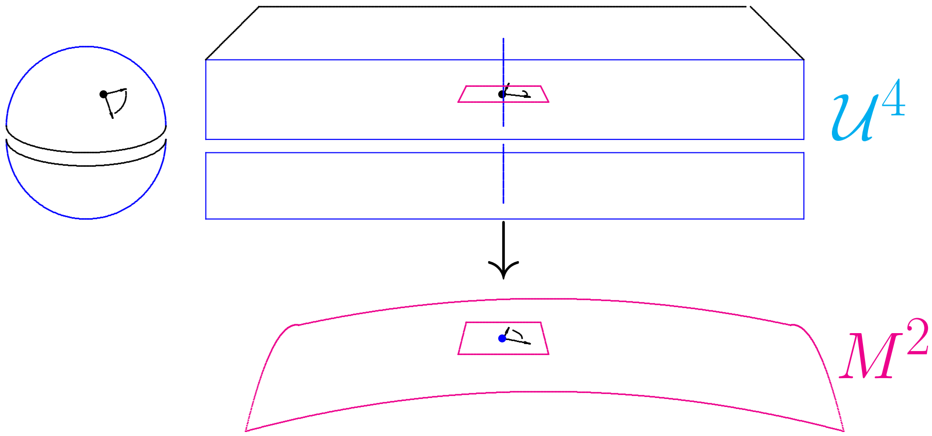

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Functions killed by $\mathbf{D}$ across real slice?

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$$\Psi : \mathcal{Z} \to \mathcal{N}$$

be the tautological blowing down map.

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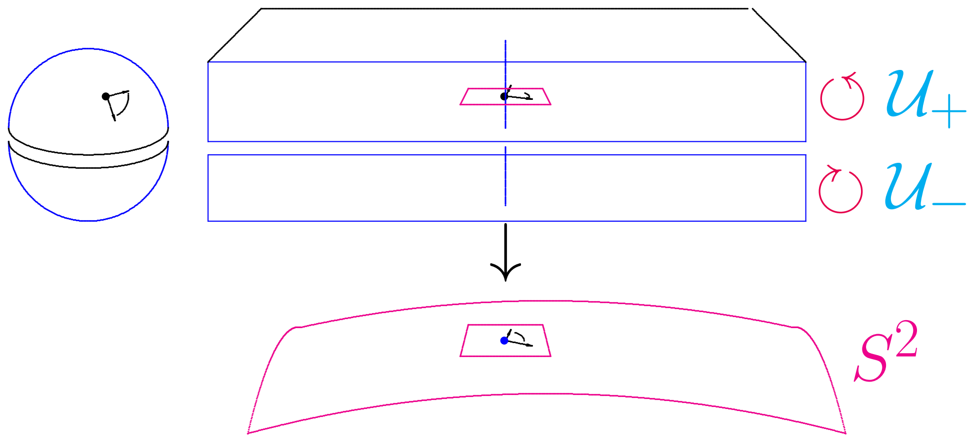

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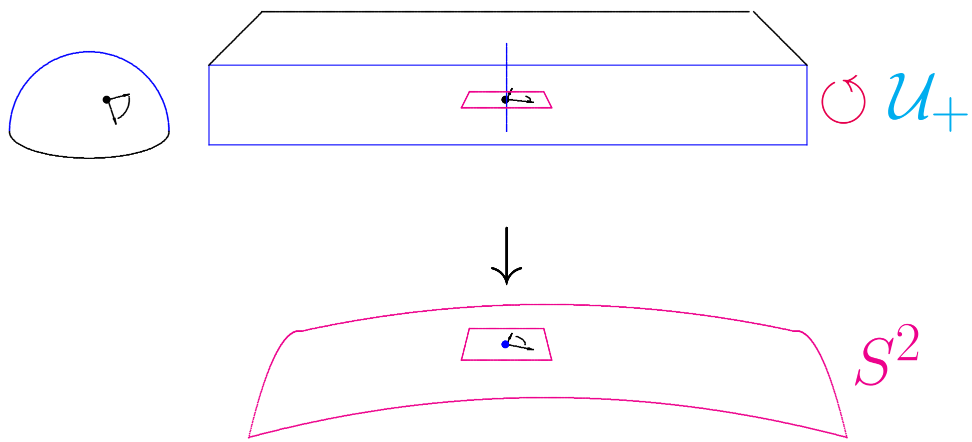

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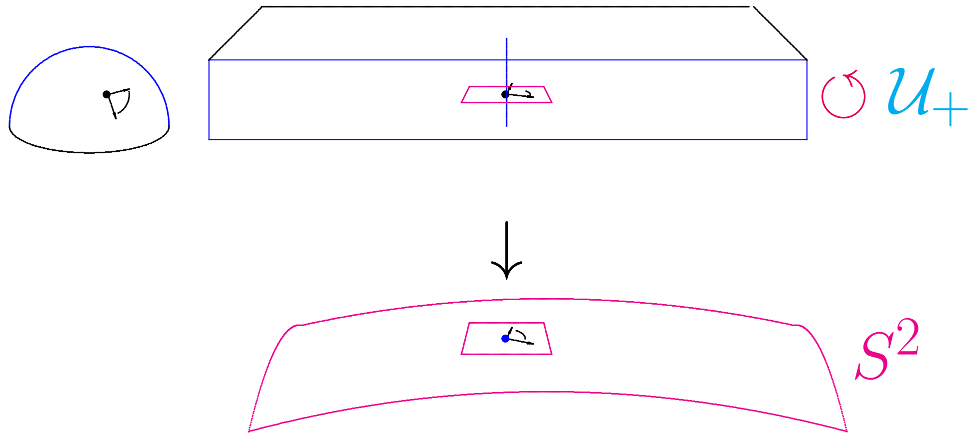

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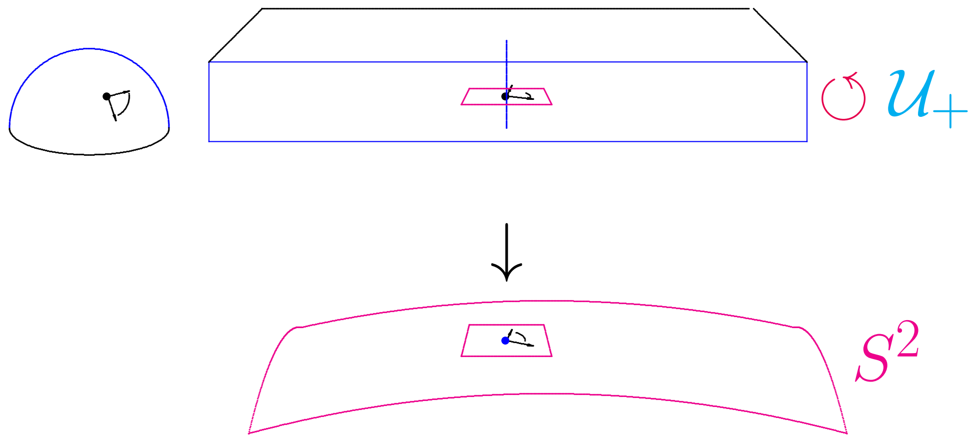

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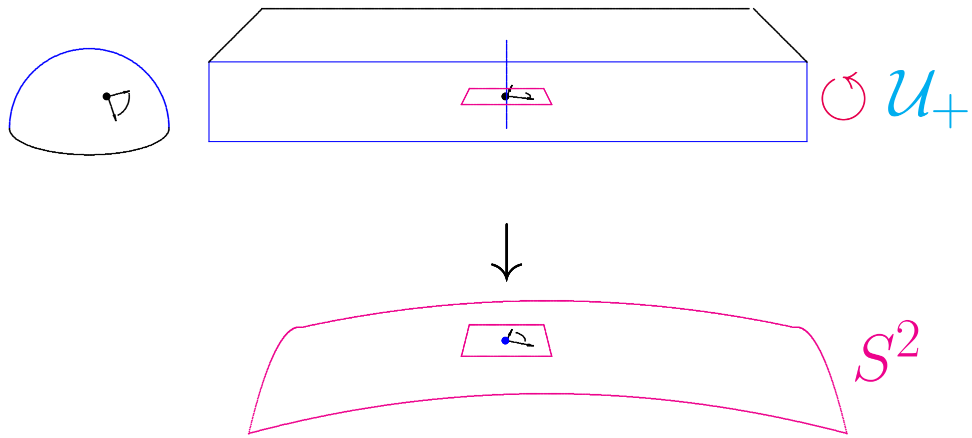

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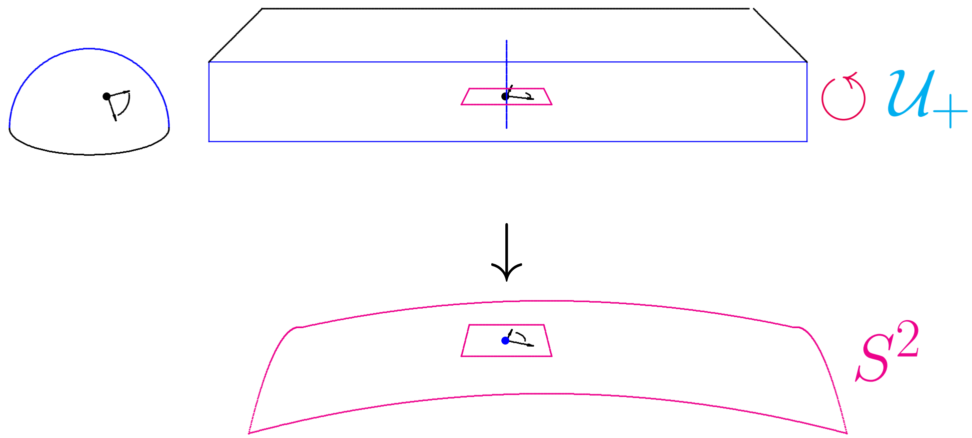

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$[\nabla]$	J	Integrability Theorem
C^{14}	C^4	Newlander-Nirenberg (1957)
C^{10}	C^2	Malgrange (1968)
C^3	Lipschitz	Hill-Taylor (2002)

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- $\pi_1(\mathcal{N}) = 0$, $b_2(\mathcal{N}) = 1$.
- $N^2 \subset \mathcal{N}^4$ totally real.
- If $M \approx S^2$, family of holomorphic disks D^2 with $\partial D^2 \subset N$. Interiors foliate $\mathcal{N} - N$.
- If $M \approx \mathbb{RP}^2$, family of genus 0 compact complex curves Σ. Anti-holomorphic map $\sigma : \mathcal{N} \to \mathcal{N}$, $\sigma^2 = \mathbf{I}$, fixing N, preserving each Σ.

Lemma (Yau)**.**

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Proof quite deep! Kähler-Einstein metrics, etc.

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Blaschke case: may use low-tech substitute…

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Lemma. *Let* $\mathcal{S}$ *be a compact complex surface, equipped with a fixed homology class* $\mathbf{a} \in H_2(\mathcal{S}, \mathbb{Z})$ *such that* $\mathbf{a} \cdot \mathbf{a} = 1$. *For every* $p \in \mathcal{S}$, *suppose that there exists a non-singular, embedded complex curve* $\Sigma \subset \mathcal{S}$ *of genus* 0 *passing through* p, *with homology class* $[\Sigma] = \mathbf{a}$. *Then there is a biholomorphism*

$$F : \mathcal{S} \xrightarrow{\cong} \mathbb{CP}_2,$$

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Proof particularly easy if assume $b_1(\mathcal{S}) = 0$. Ideas due to Castelnuovo, Enriques & Kodaira.

Theorem A. *Let* $[\nabla]$ *be Zoll projective structure on a compact surface* M^2. *If*

$$\pi_1(M) \neq 0,$$

there is a diffeomorphism

$$\Phi : M \xrightarrow{\approx} \mathbb{RP}^2$$

such that $[\nabla] = [\Phi^* \triangledown]$, *where* $\triangledown$ *is the Levi-Civita connection of the standard, constant curvature Riemannian metric* h *on* $\mathbb{RP}^2$.

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Zoll metric case: If ∇ is metric connection of Zoll g on $M \approx \mathbb{RP}^2$, also get a complex curve $\mathcal{Q} \subset \mathcal{N}$. Can arrange for F to also send to standard conic.

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Classical Blaschke conjecture follows because Φ preserves both geodesics and conformal structure, and hence $\Phi^* h = cg$.

End, Part III