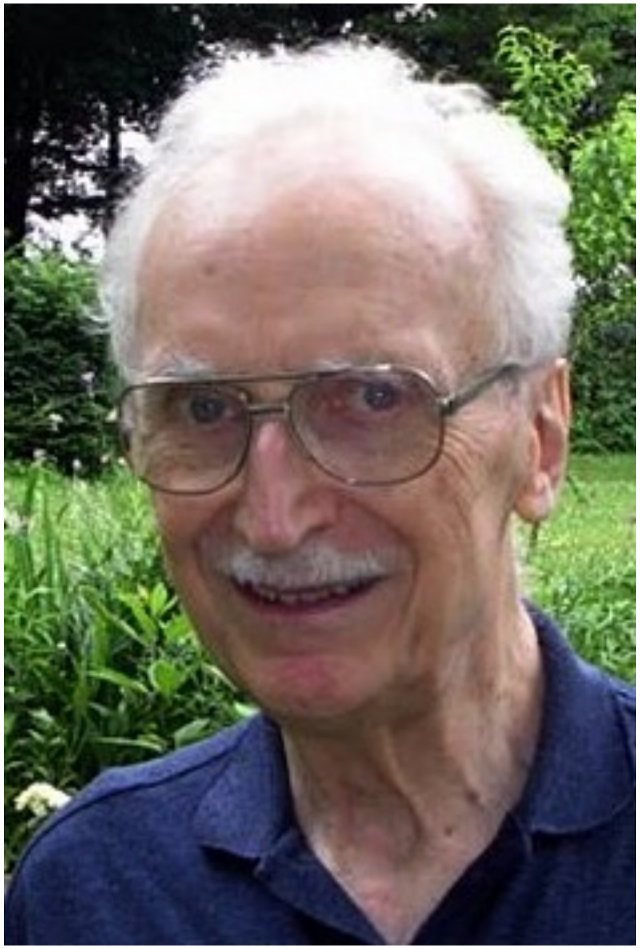
On

Hermitian, Einstein

4-Manifolds

Claude LeBrun
Stony Brook University

For Eugenio Calabi



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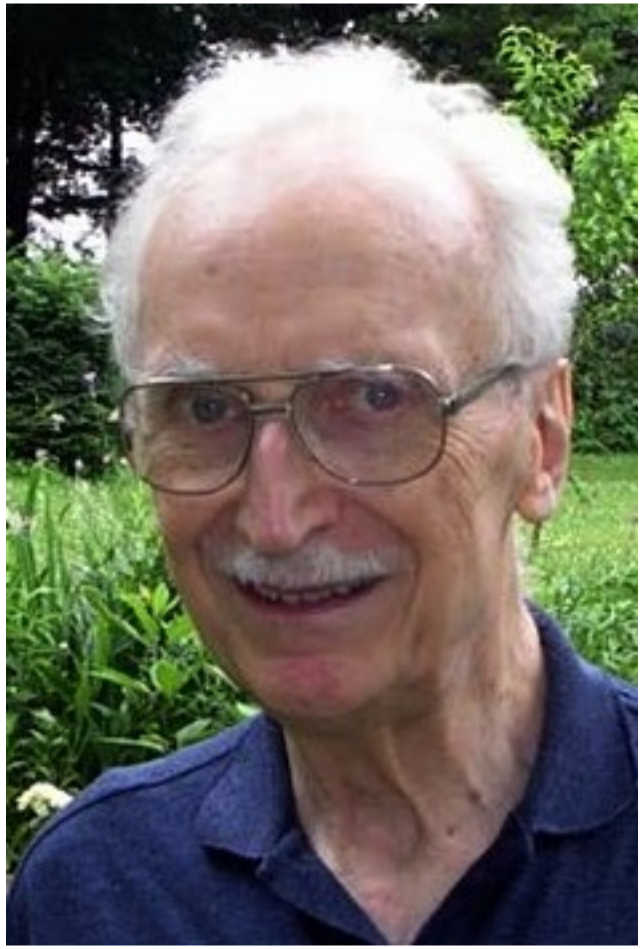

who discovered the magic link between
Einstein manifolds and complex geometry.

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"...the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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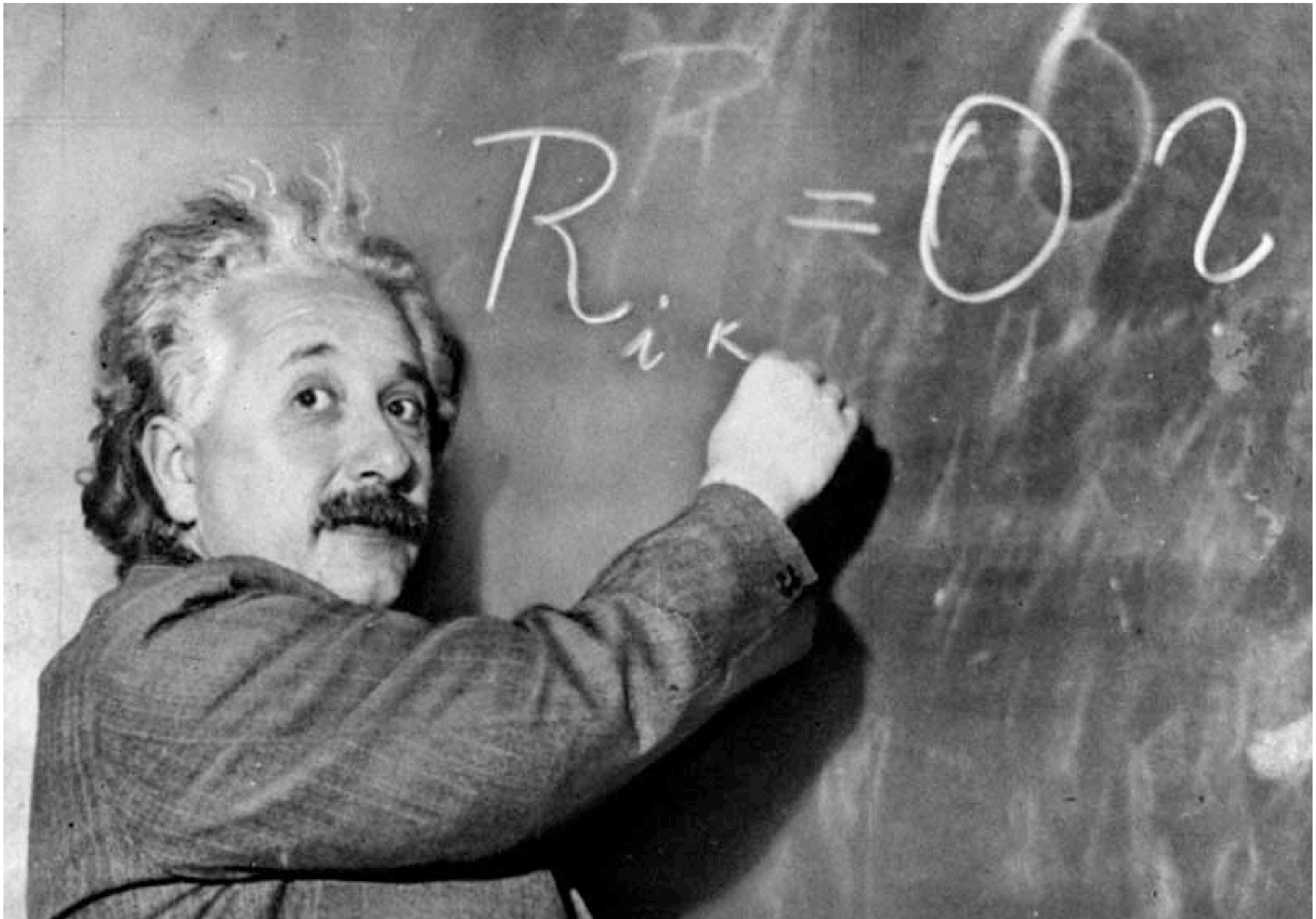

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where $[h_{j\bar{k}}]$ Hermitian matrix at each point.

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$[\omega] \in H^2(M, \mathbb{R})$ called the Kähler class.

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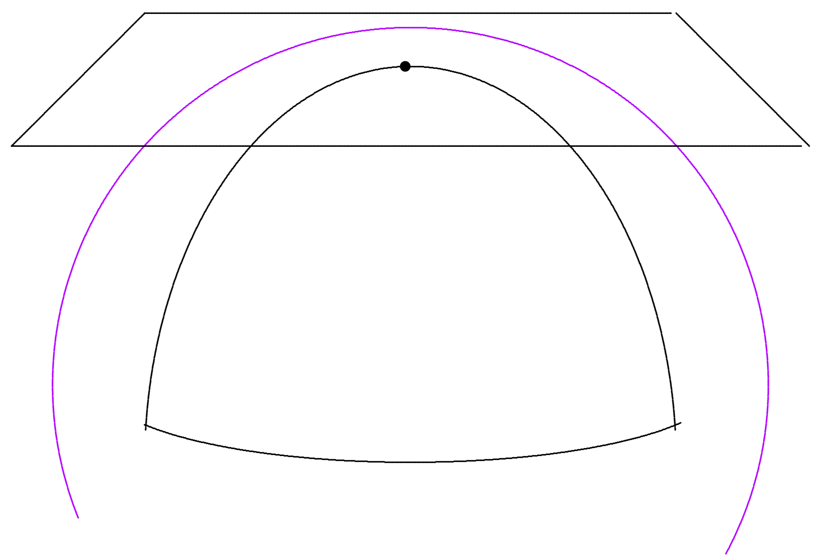

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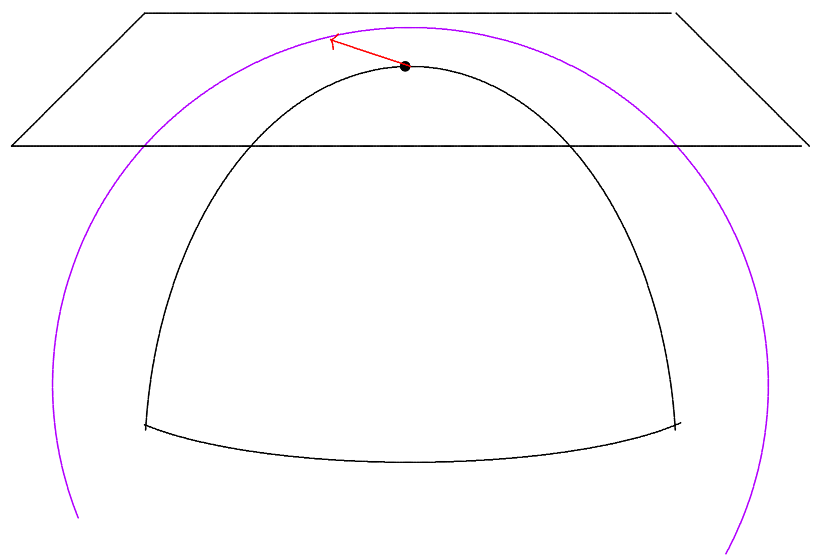

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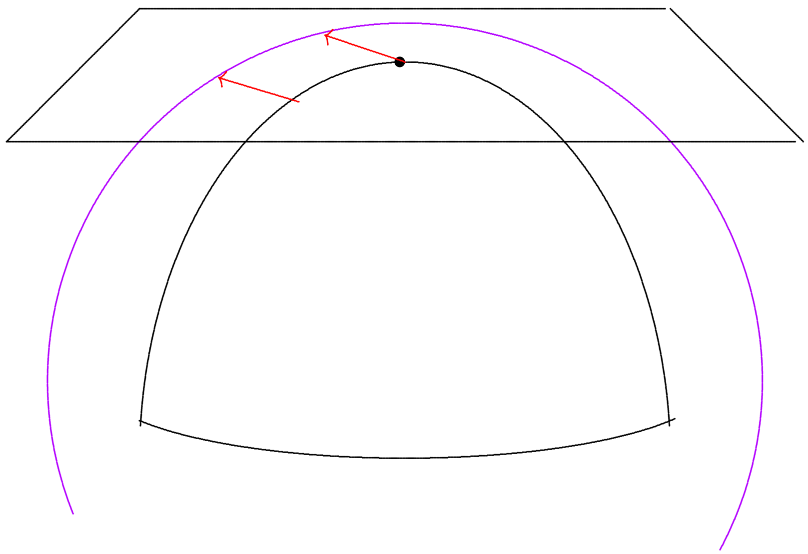

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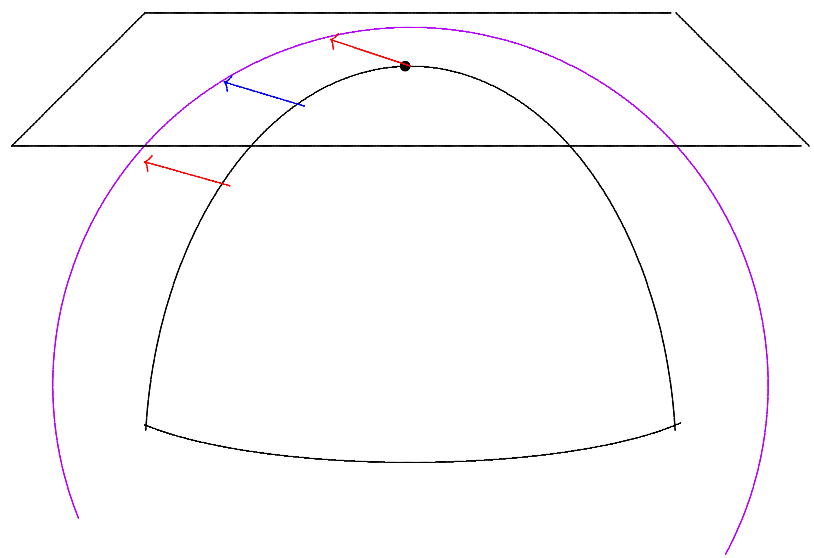

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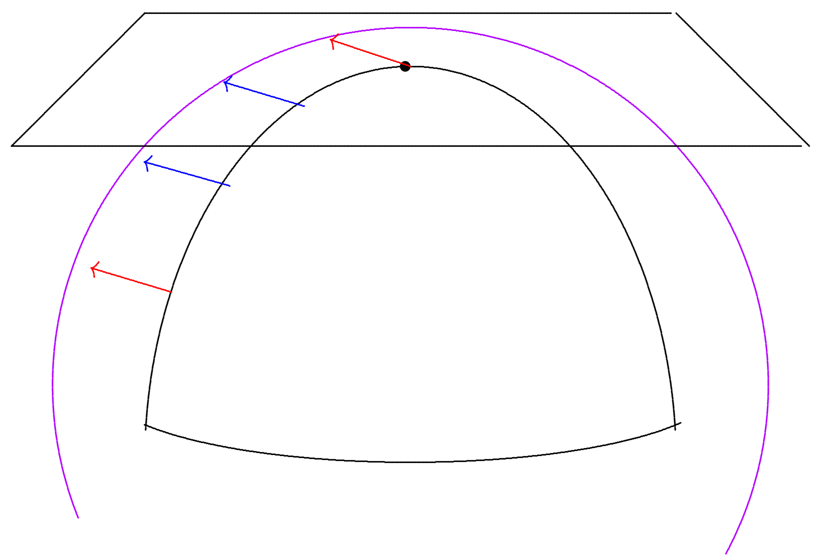

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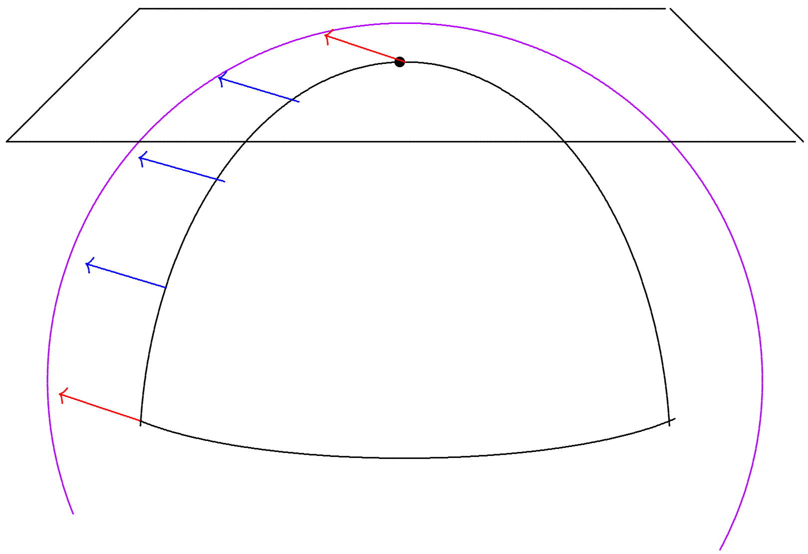

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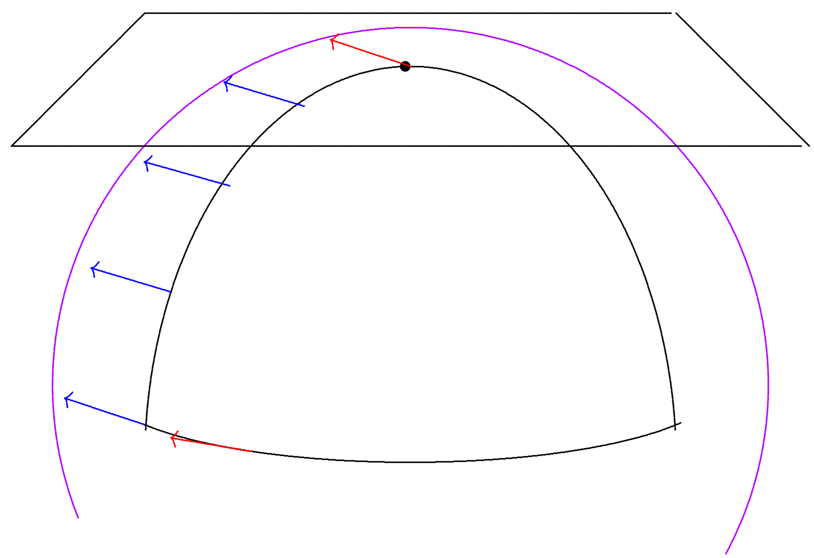

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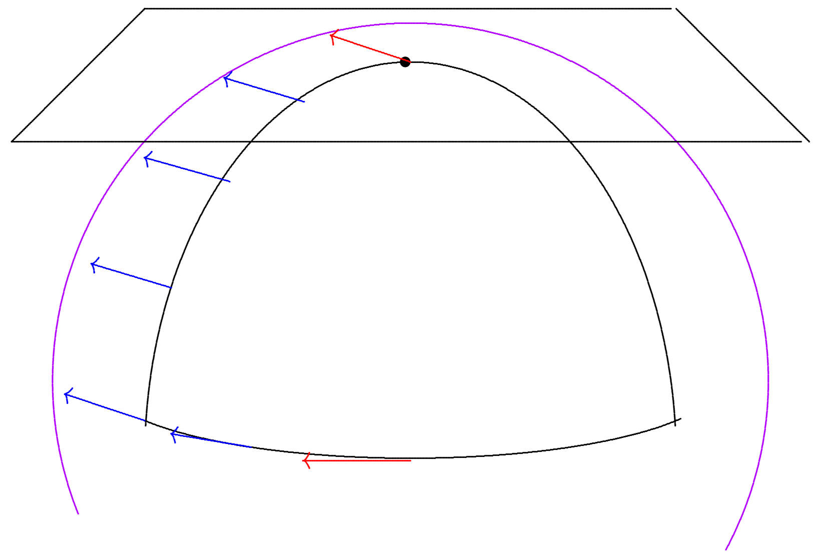

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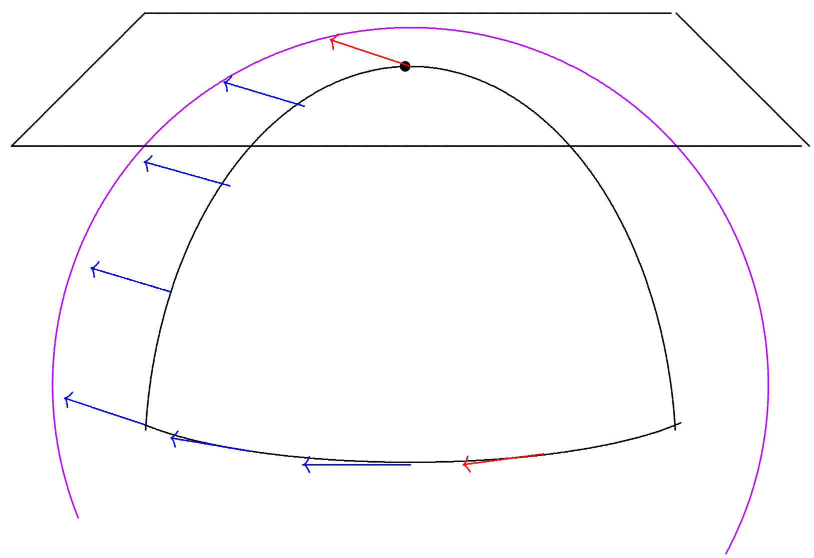

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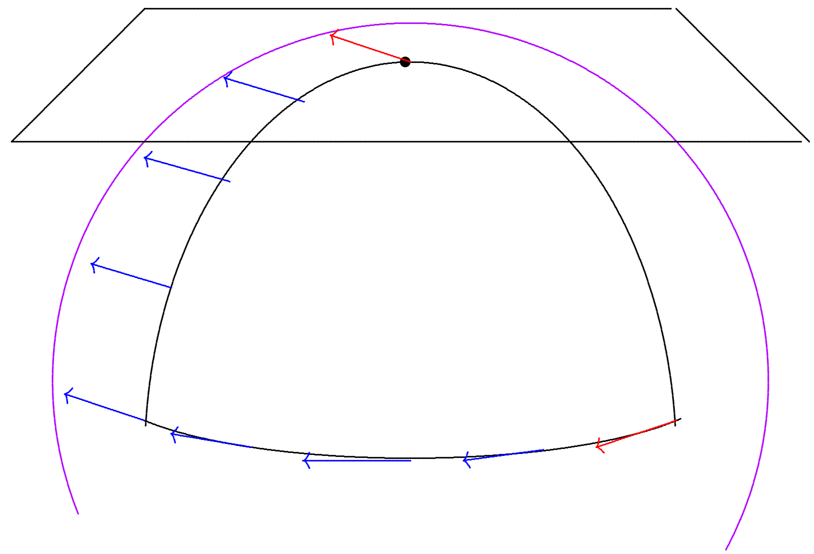

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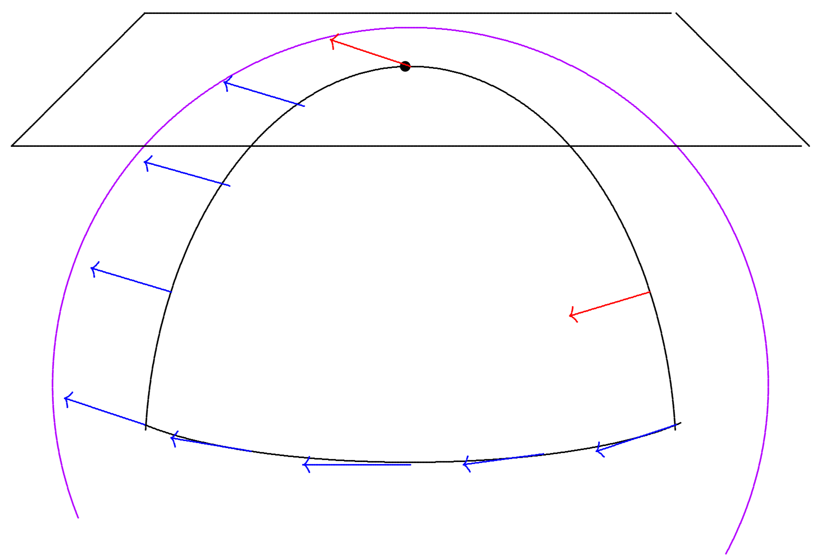

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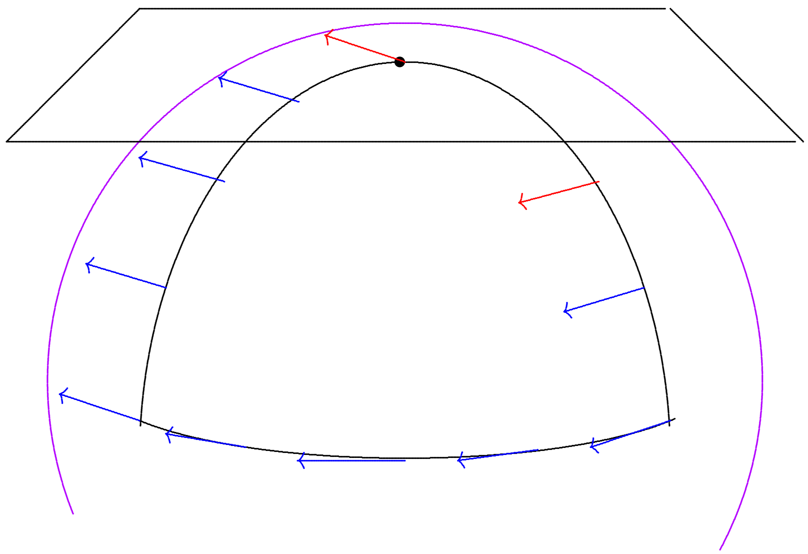

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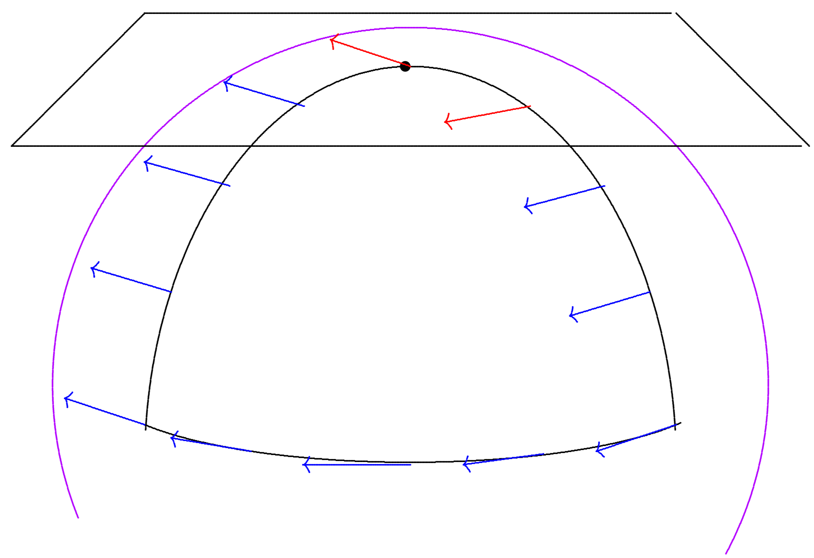

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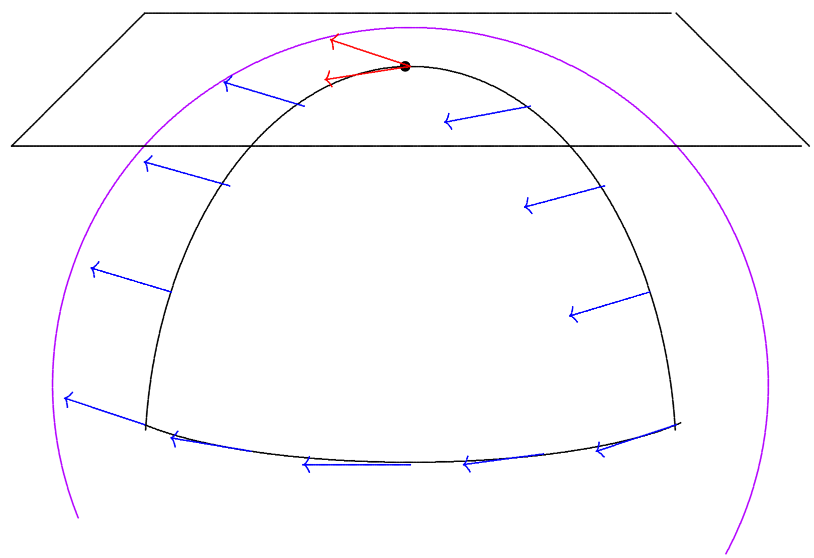

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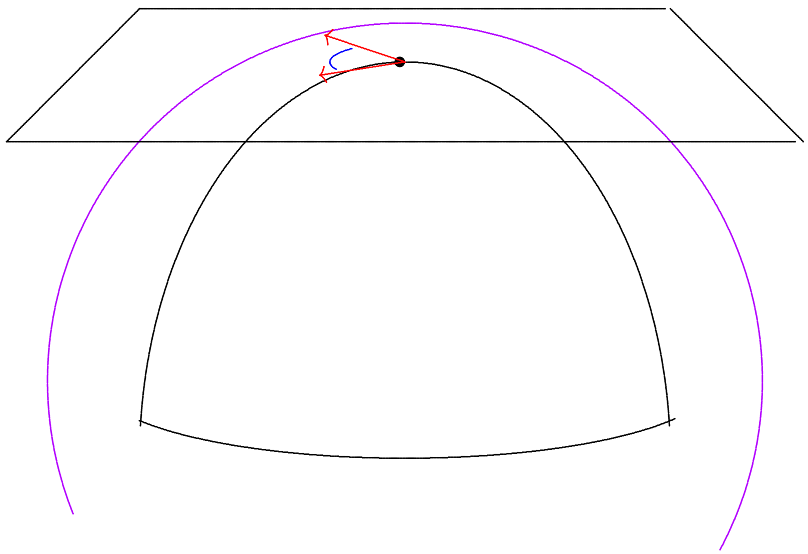
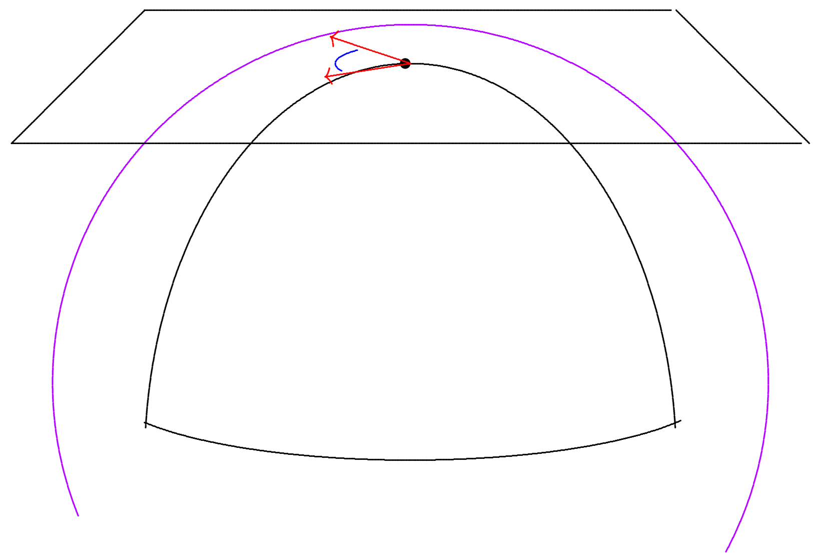


(M^n, g): holonomy $\subset O(n)$



Kähler metrics:

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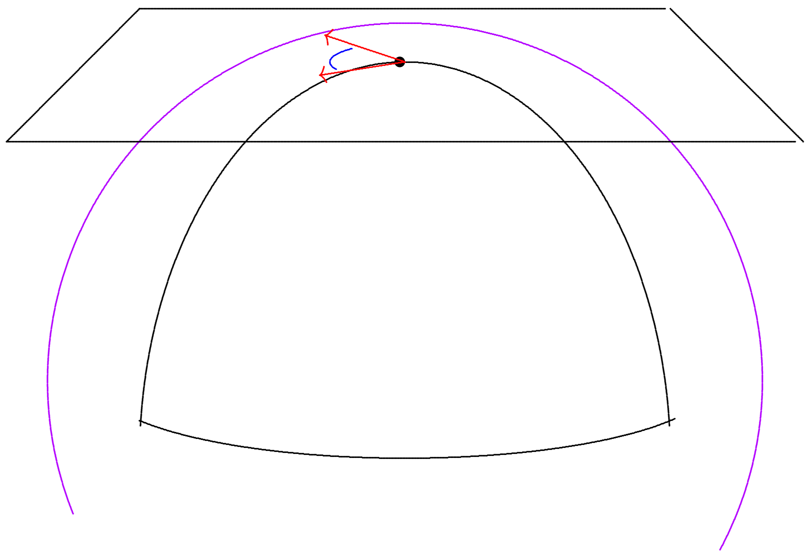

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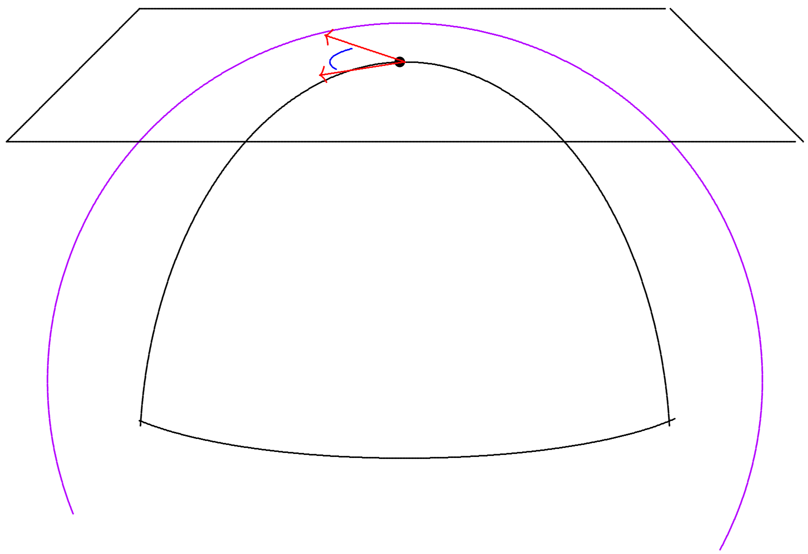

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is curvature of canonical line bundle $K = \Lambda^{m,0}$.

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In local complex coordinates

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Both are actually Hermitian.

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- $M \approx \mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$ *and* h *is a constant times the Chen-LeBrun-Weber metric.*

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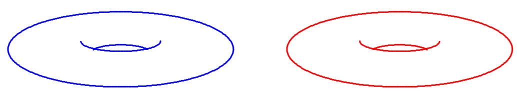
Connected sum #:



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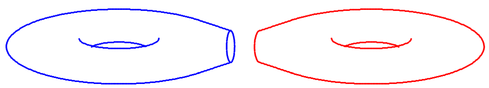

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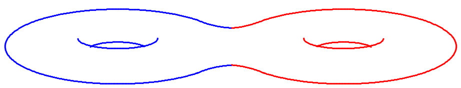

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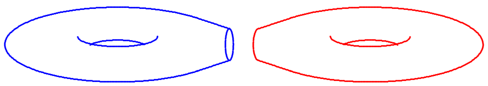

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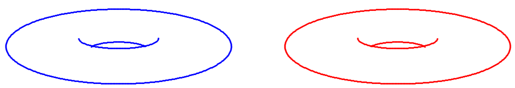

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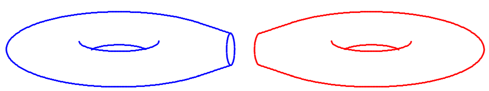

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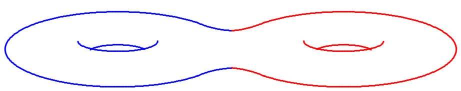

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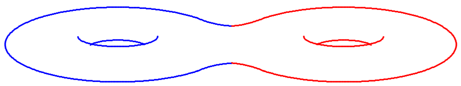

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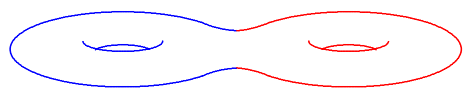

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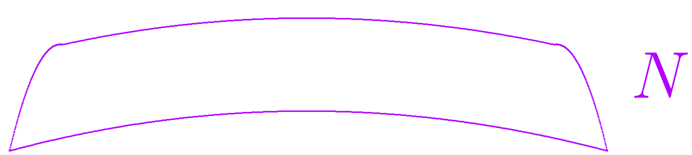

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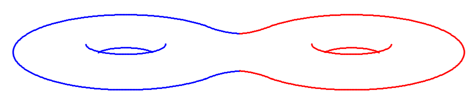
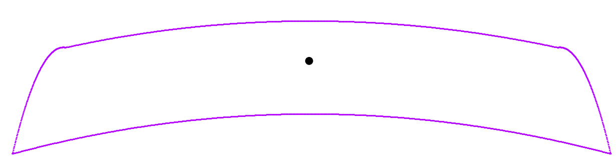


Blowing up:

If N is a complex surface, may replace $p \in N$



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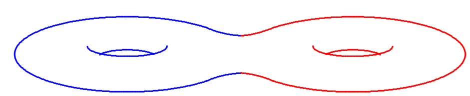

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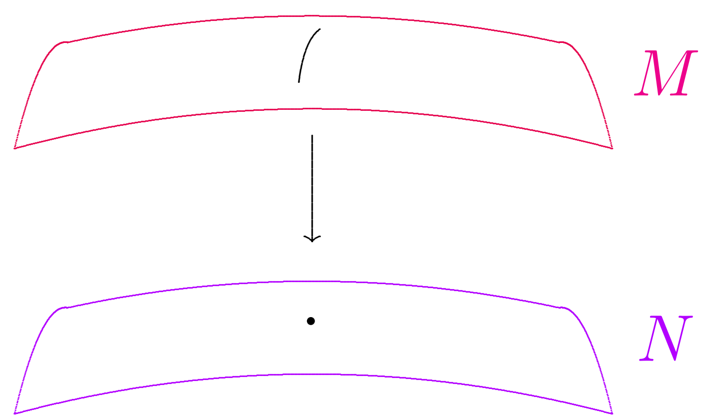

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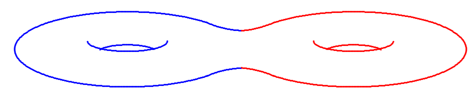

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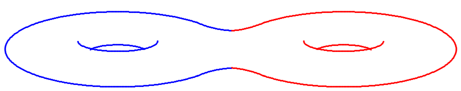

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in which new $\mathbb{CP}_1$ has self-intersection -1.

Theorem A. *Let* (M^4, J) *be a compact complex surface, and suppose that* h *is an Einstein metric on* M *which is Hermitian with respect to* J*:*

$$h(J\cdot, J\cdot) = h.$$

Then either

- (M, J, h) *is Kähler-Einstein; or*
- $M \approx \mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$*, and* h *is a constant times the Page metric; or*
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Warning: when h is non-Kähler, its relation to ω is surprisingly complicated!

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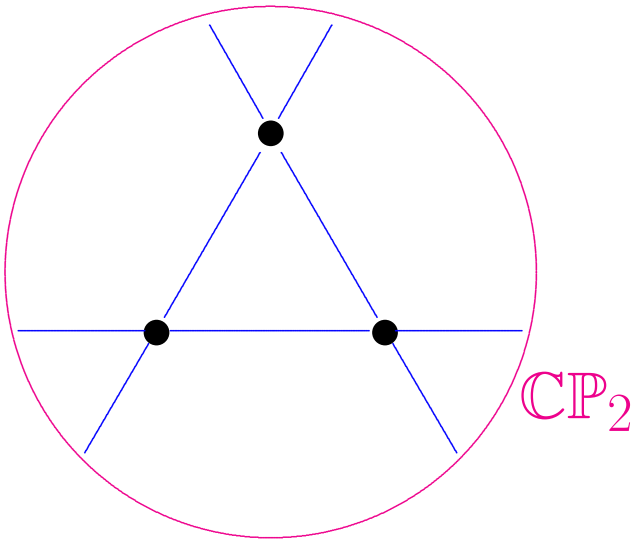

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In other words,

$$h = fg$$

$\exists$ Kähler metric g, smooth function $f : M \to \mathbb{R}^+$.

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Similarly for $S^{2n+1} \times S^{2m+1}$.

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X.X. Chen: always minimizers.

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Donaldson/Mabuchi/Chen-Tian:
unique in Kähler class, modulo bihomorphisms.

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Λ^+ self-dual 2-forms.
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Notice that W_{+} has a repeated eigenvalue.

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$\nabla \cdot W_+ = 0$, while $T^{1,0}M$ isotropic & involutive.

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When $W_+ \equiv 0$, use global results of Boyer et al.

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$$|W_{+}|^{2} = \frac{s^{2}}{24}$$

The Bach Tensor

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Conformally invariant Riemannian functional:

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$$\mathcal{W}_+(g) = 2\int_M |W_+|_g^2 \, d\mu_g.$$

1-parameter family of metrics

$$g_t := g + t\dot{g} + O(t^2)$$

First variation

$$\frac{d}{dt}\mathcal{W}_+(g_t)\Big|_{t=0} = -\int \dot{g}^{ab} B_{ab} \, d\mu_g$$

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$$\nabla^a B_{ab} = 0$$

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Conformally Einstein $\Longrightarrow$ $B = 0$

Restriction of $\mathcal{W}_+$ to Kähler metrics?

On Kähler metrics,

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In fact, for Kähler metrics,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\Big]$$

where Hess_0 denotes trace-free part of $\nabla\nabla$.

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So the critical metrics of restriction of $\mathcal{W}_+$ to $\{$Kähler metrics$\}$ are Bach-flat Kähler metrics.

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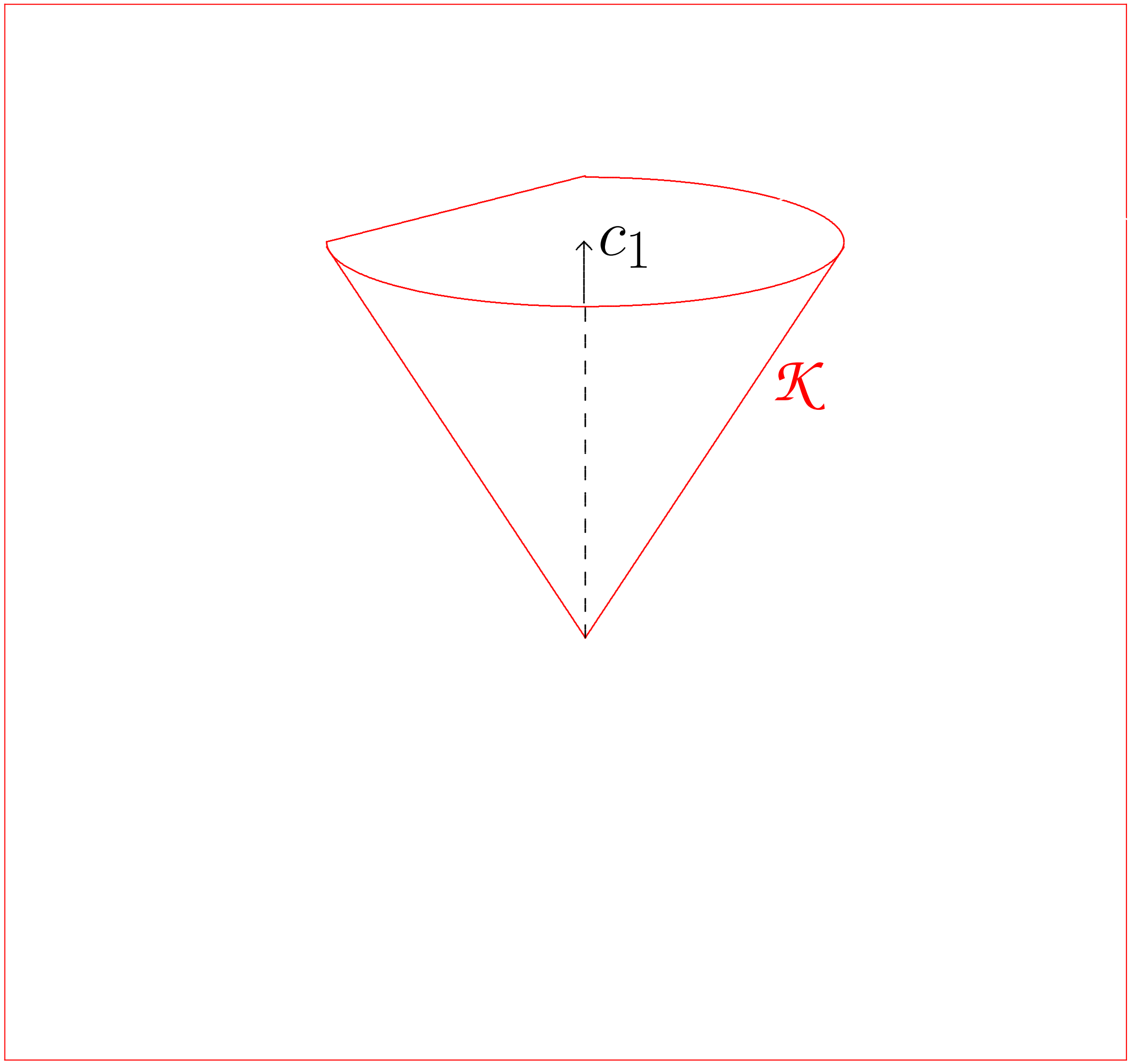

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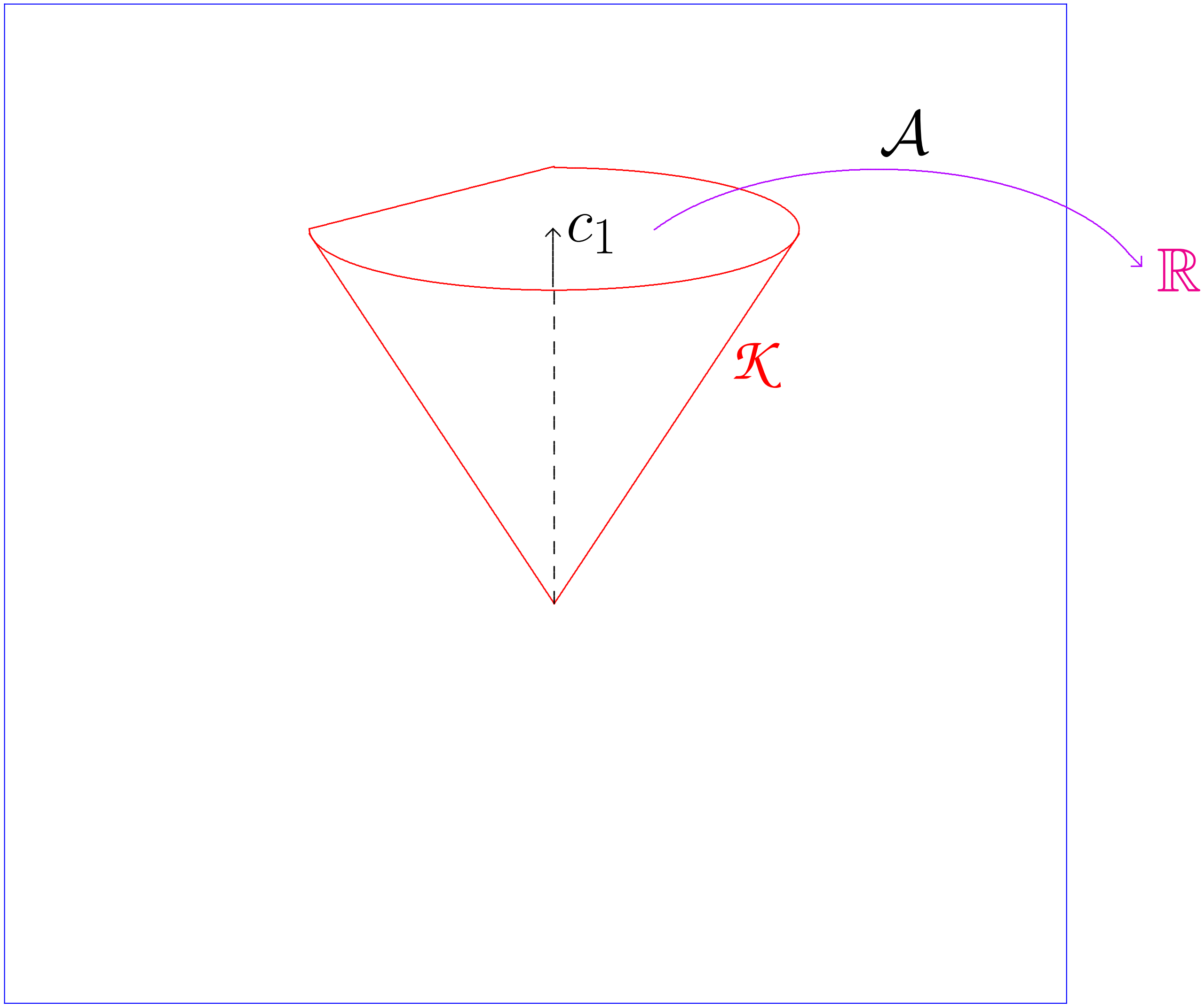

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$$0 = 6s^{-1}B = \mathring{r} + 2s^{-1}\mathrm{Hess}_0(s)$$

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$$\rho + 2i\partial\bar{\partial}\log s > 0.$$

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and uniqueness Theorem A follows.

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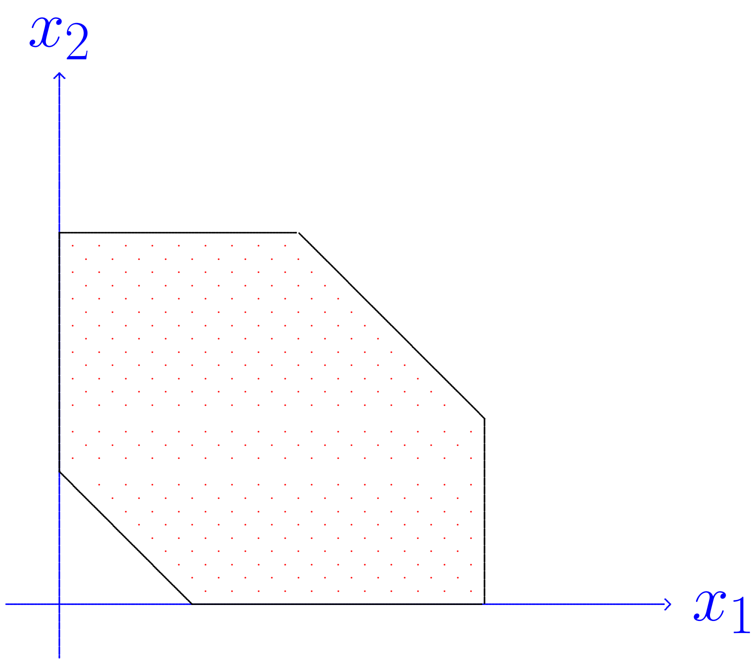
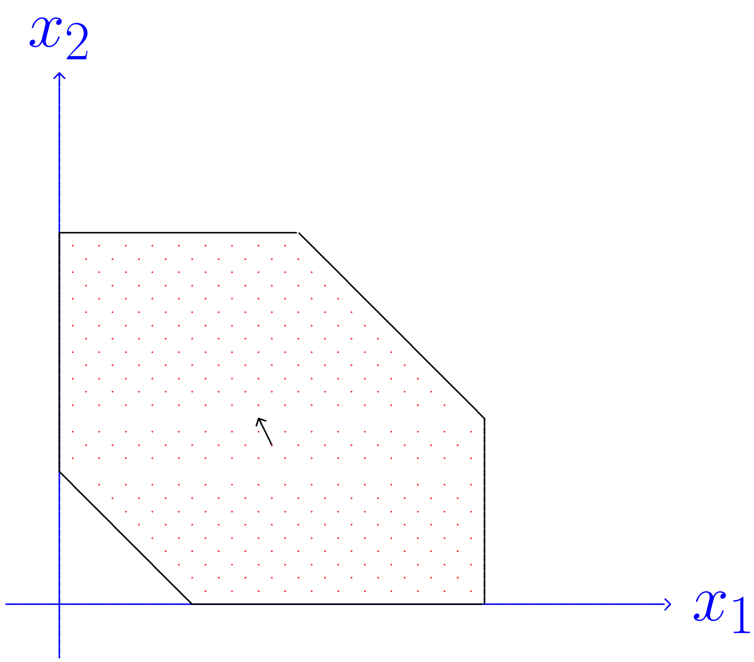
Only three cases are non-trivial:

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \qquad k = 1, 2, 3.$$

The non-trivial cases are toric, and the action $\mathcal{A}$ can be directly computed from moment polygon.



The non-trivial cases are toric, and the action $\mathcal{A}$ can be directly computed from moment polygon. Formula involves barycenters, moments of inertia.



$$\mathcal{A}([\omega]) = \frac{|\partial P|^2}{2}\left(\frac{1}{|P|} + \vec{\mathfrak{D}} \cdot \Pi^{-1}\vec{\mathfrak{D}}\right)$$

To prove Theorem, show that

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$\mathcal{A}$ is explicit rational function —

$$3\big[3+28\gamma+96\gamma^2+168\gamma^3+164\gamma^4+80\gamma^5+16\gamma^6+16\beta^6(1+\gamma)^4+16\alpha^6(1+\beta+\gamma)^4+16\beta^5(5+24\gamma+43\gamma^2+37\gamma^3+15\gamma^4+2\gamma^5)+4\beta^4(41+228\gamma+478\gamma^2+496\gamma^3+263\gamma^4+$$

$$60\gamma^5+4\gamma^6)+8\beta^3(21+135\gamma+326\gamma^2+392\gamma^3+248\gamma^4+74\gamma^5+8\gamma^6)+4\beta(7+58\gamma+176\gamma^2+270\gamma^3+228\gamma^4+96\gamma^5+16\gamma^6)+4\beta^2(24+176\gamma+479\gamma^2+652\gamma^3+478\gamma^4+$$

$$172\gamma^5+24\gamma^6)+16\alpha^5(5+2\beta^5+24\gamma+43\gamma^2+37\gamma^3+15\gamma^4+2\gamma^5+\beta^4(15+14\gamma)+\beta^3(37+70\gamma+30\gamma^2)+\beta^2(43+123\gamma+108\gamma^2+30\gamma^3)+\beta(24+92\gamma+123\gamma^2+70\gamma^3+$$

$$14\gamma^4))+4\alpha^4(41+4\beta^6+228\gamma+478\gamma^2+496\gamma^3+263\gamma^4+60\gamma^5+4\gamma^6+\beta^5(60+56\gamma)+\beta^4(263+476\gamma+196\gamma^2)+8\beta^3(62+169\gamma+139\gamma^2+35\gamma^3)+2\beta^2(239+876\gamma+1089\gamma^2+$$

$$556\gamma^3+98\gamma^4)+4\beta(57+263\gamma+438\gamma^2+338\gamma^3+119\gamma^4+14\gamma^5))+8\alpha^3(21+135\gamma+326\gamma^2+392\gamma^3+248\gamma^4+74\gamma^5+8\gamma^6+8\beta^6(1+\gamma)+2\beta^5(37+70\gamma+30\gamma^2)+4\beta^4(62+$$

$$169\gamma+139\gamma^2+35\gamma^3)+4\beta^3(98+353\gamma+428\gamma^2+210\gamma^3+35\gamma^4)+2\beta^2(163+735\gamma+1179\gamma^2+856\gamma^3+278\gamma^4+30\gamma^5)+\beta(135+736\gamma+1470\gamma^2+1412\gamma^3+676\gamma^4+140\gamma^5+$$

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$$24\gamma^6))+4\alpha^2(24+176\gamma+479\gamma^2+652\gamma^3+478\gamma^4+172\gamma^5+24\gamma^6+24\beta^6(1+\gamma)^2+4\beta^5(43+123\gamma+108\gamma^2+30\gamma^3)+2\beta^4(239+876\gamma+1089\gamma^2+556\gamma^3+98\gamma^4)+4\beta^3(163+$$

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$$\big[1+10\gamma+36\gamma^2+64\gamma^3+60\gamma^4+24\gamma^5+24\beta^5(1+\gamma)^5+24\alpha^5(1+\beta+\gamma)^5+12\beta^4(1+\gamma)^2(5+20\gamma+23\gamma^2+10\gamma^3)+16\beta^3(4+28\gamma+72\gamma^2+90\gamma^3+57\gamma^4+15\gamma^5)+$$

$$12\beta^2(3+24\gamma+69\gamma^2+96\gamma^3+68\gamma^4+20\gamma^5)+2\beta(5+45\gamma+144\gamma^2+224\gamma^3+180\gamma^4+60\gamma^5)+12\alpha^4(1+\beta+\gamma)^2(5+20\gamma+23\gamma^2+10\gamma^3+10\beta^3(1+\gamma)+\beta^2(23+46\gamma+$$

$$16\gamma^2)+2\beta(10+30\gamma+23\gamma^2+5\gamma^3))+16\alpha^3(4+28\gamma+72\gamma^2+90\gamma^3+57\gamma^4+15\gamma^5+15\beta^5(1+\gamma)^2+3\beta^4(19+57\gamma+50\gamma^2+13\gamma^3)+3\beta^3(30+120\gamma+155\gamma^2+78\gamma^3+$$

$$13\gamma^4)+3\beta^2(24+120\gamma+206\gamma^2+155\gamma^3+50\gamma^4+5\gamma^5)+\beta(28+168\gamma+360\gamma^2+360\gamma^3+171\gamma^4+30\gamma^5))+12\alpha^2(3+24\gamma+69\gamma^2+96\gamma^3+68\gamma^4+20\gamma^5+20\beta^5(1+\gamma)^3+$$

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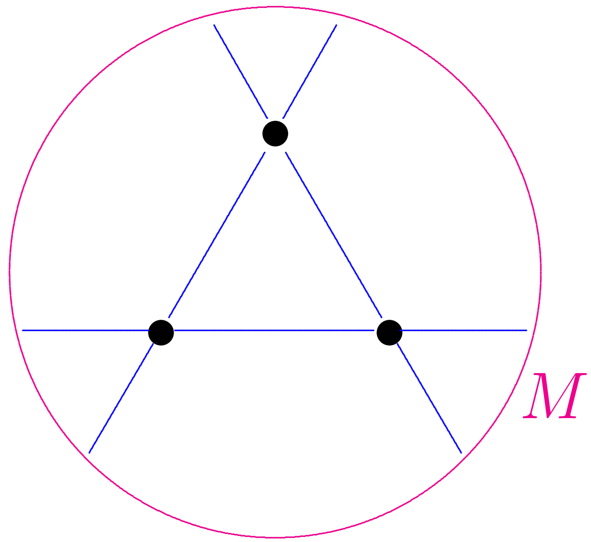
Proof proceeds by showing critical point invariant under certain discrete automorphisms of M.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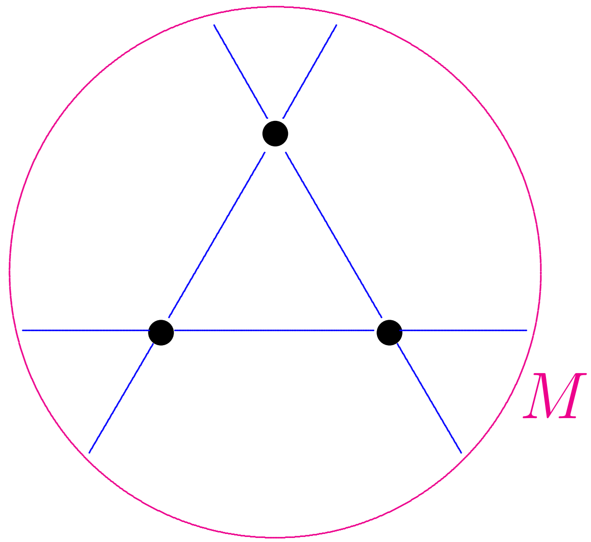
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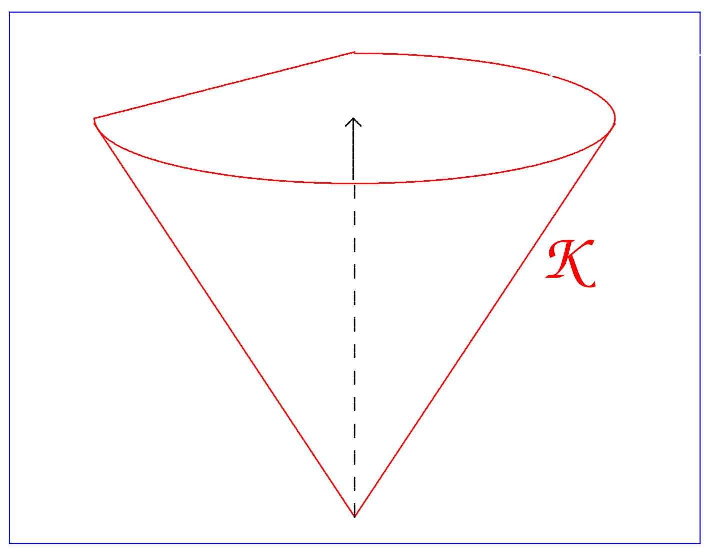
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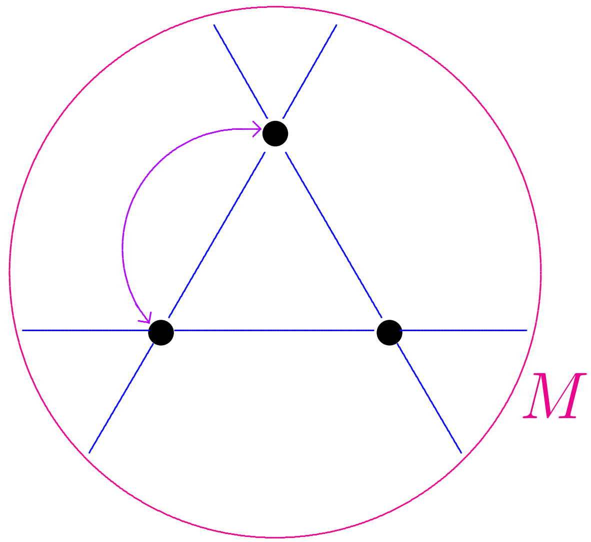
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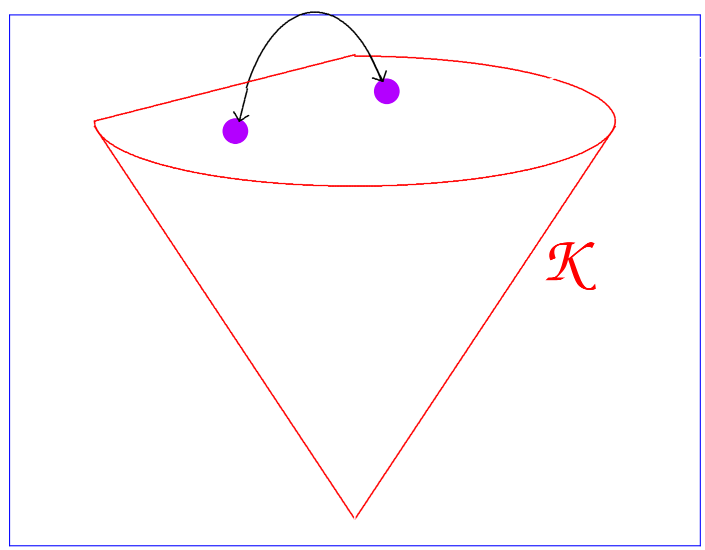
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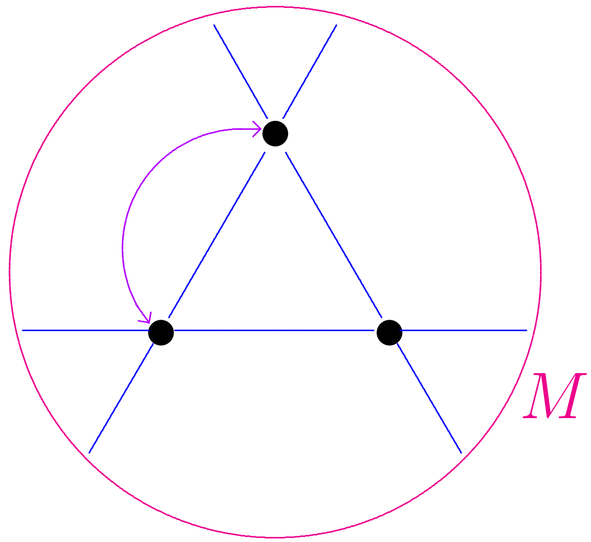
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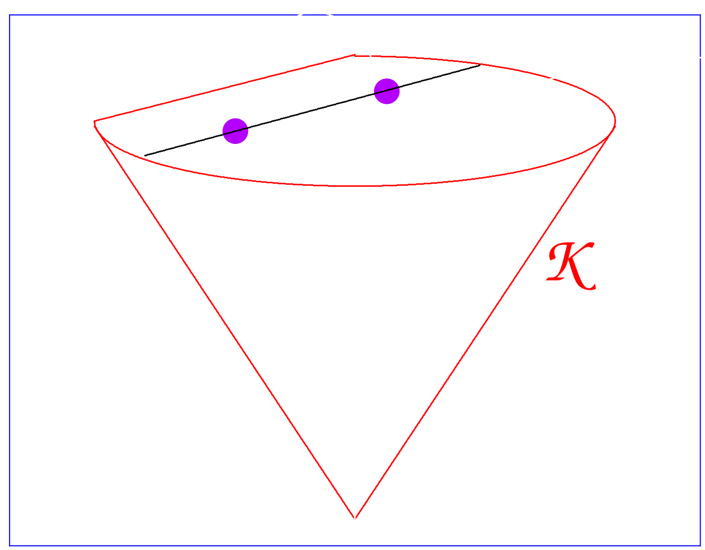
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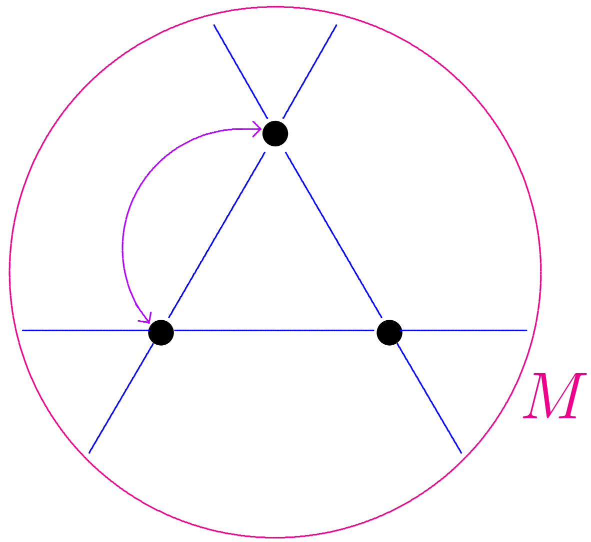

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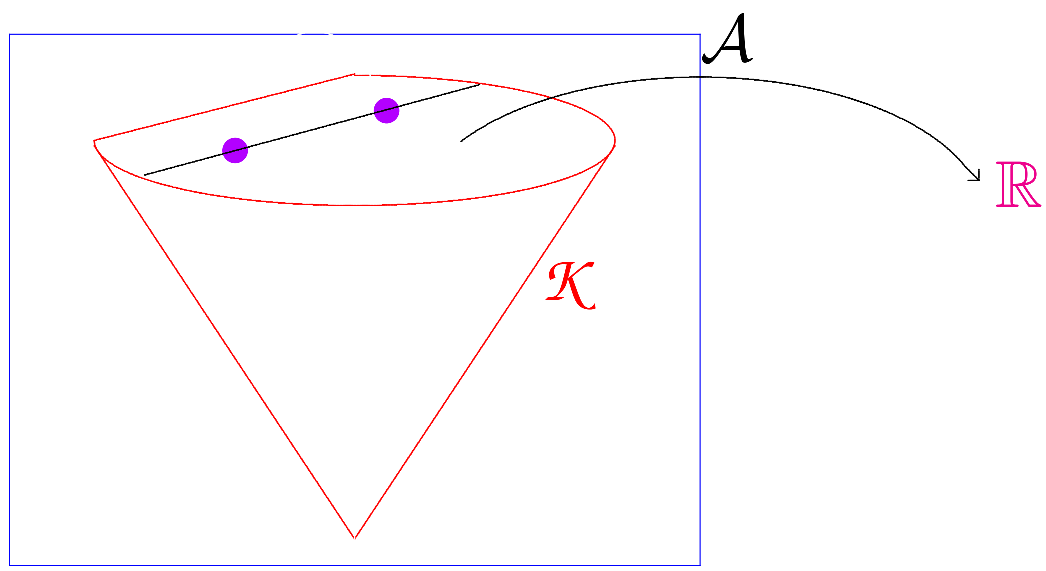

$\mathcal{A}$
$\mathbb{R}$
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To prove Theorem, show that

$$\mathcal{A} : \check{\mathcal{K}} \to \mathbb{R}$$

has unique critical point for relevant M.

Here $\check{\mathcal{K}} = \mathcal{K}/\mathbb{R}^{+}$.

$\mathcal{A}$ is explicit rational function —
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Proof proceeds by showing critical point invariant under certain discrete automorphisms of M.

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Final step then just calculus in one variable...

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Similar calculations also led to new existence proof. . .

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Could also reconstruct Page metric this way. . .

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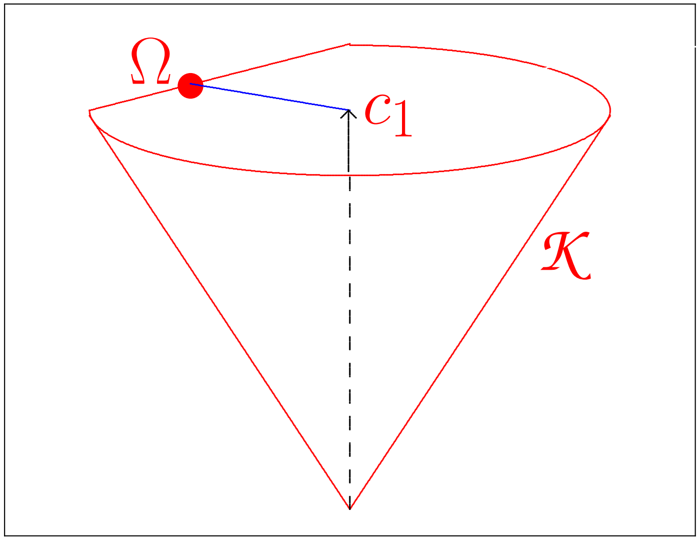

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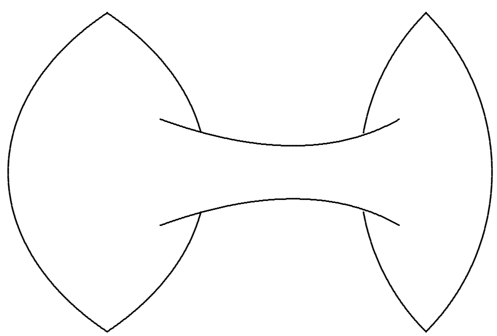

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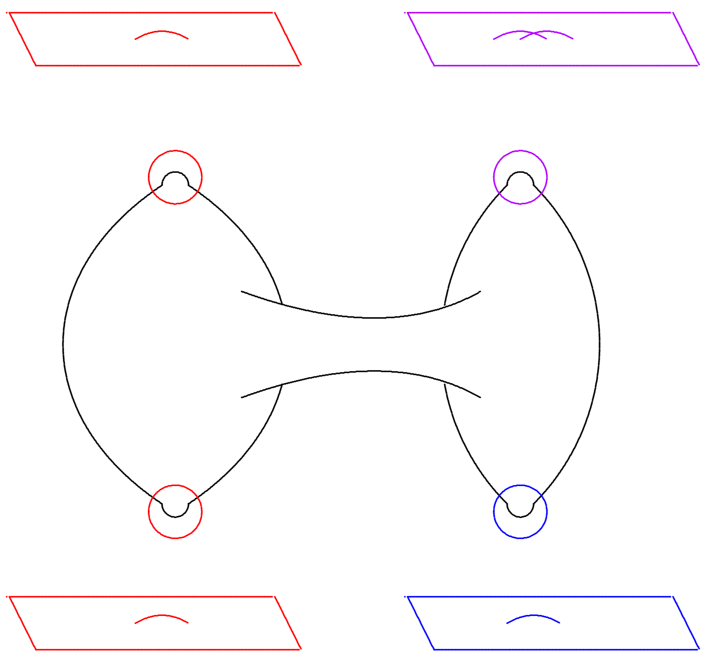

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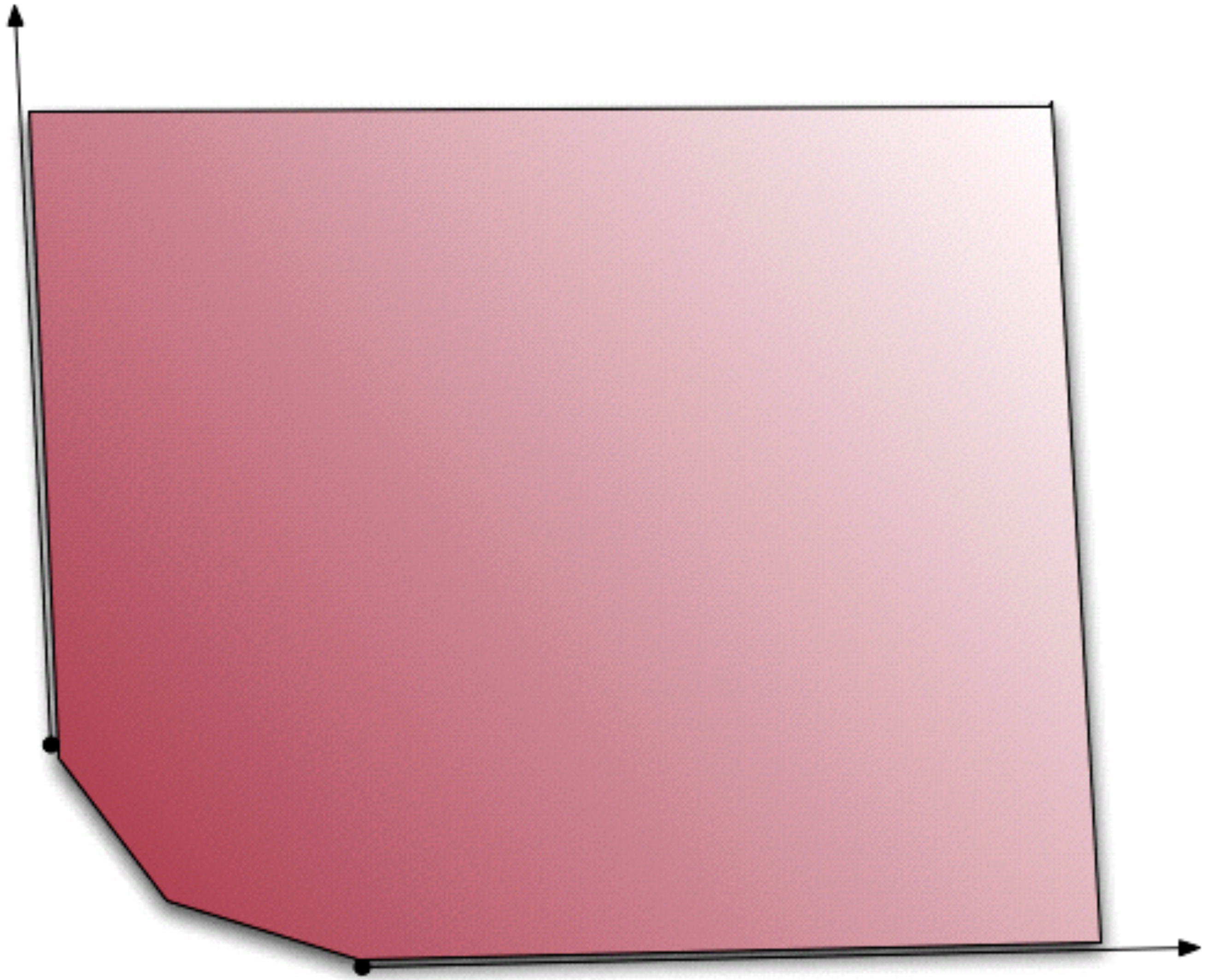
Any bubble must be toric, scalar-flat Kähler, ALE:



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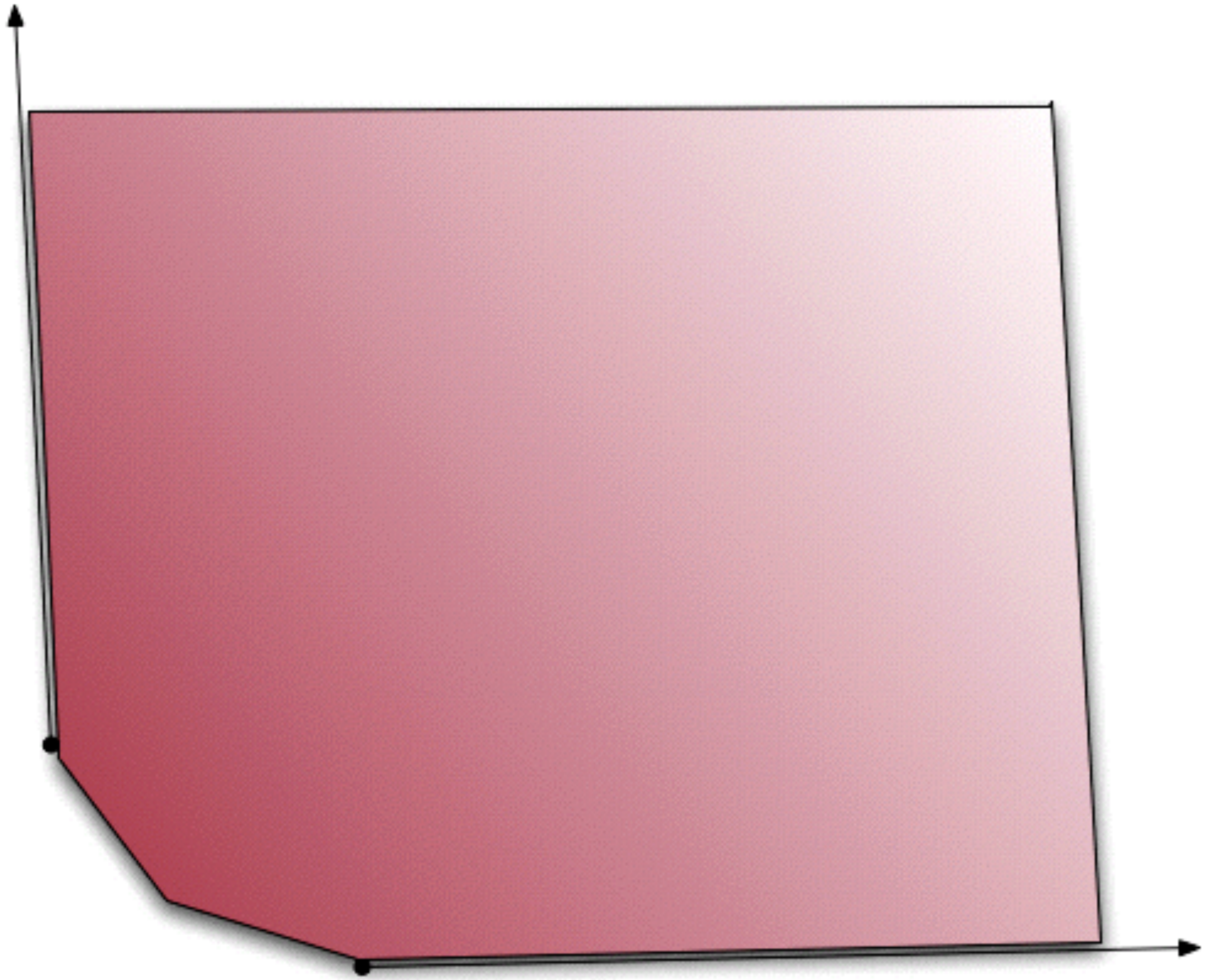

Must contain a holomorphic 2-sphere S with

$$[S] \cdot [S] < 0.$$

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  - Symplectic 2-spheres $\rightsquigarrow$ Lagrangian 2-spheres

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But every such class in M represented by a holomorphic curve! So $\Omega_{t_\infty} = \Omega_1$, and we have just bubbled off a (-1)-curve, as desired!

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