Einstein Manifolds,

Self-Dual Weyl Curvature, &

Conformally Kähler Geometry

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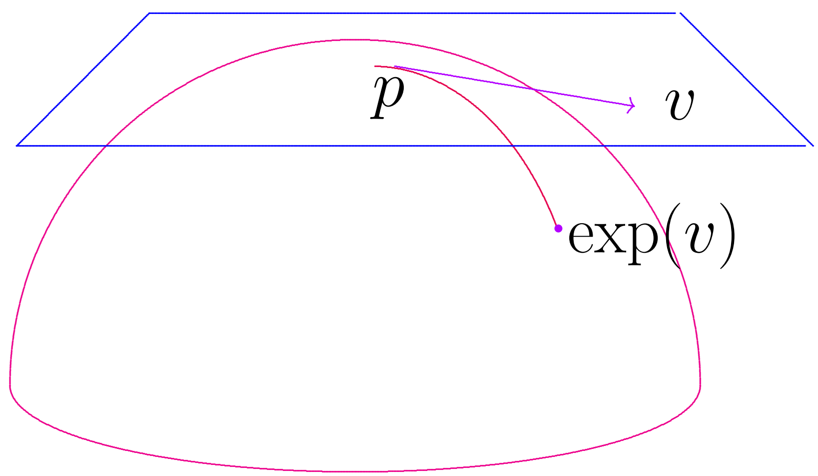

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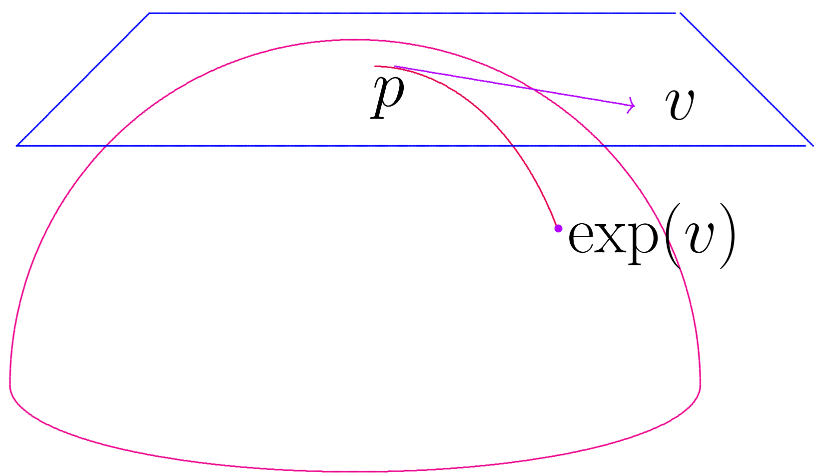

Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

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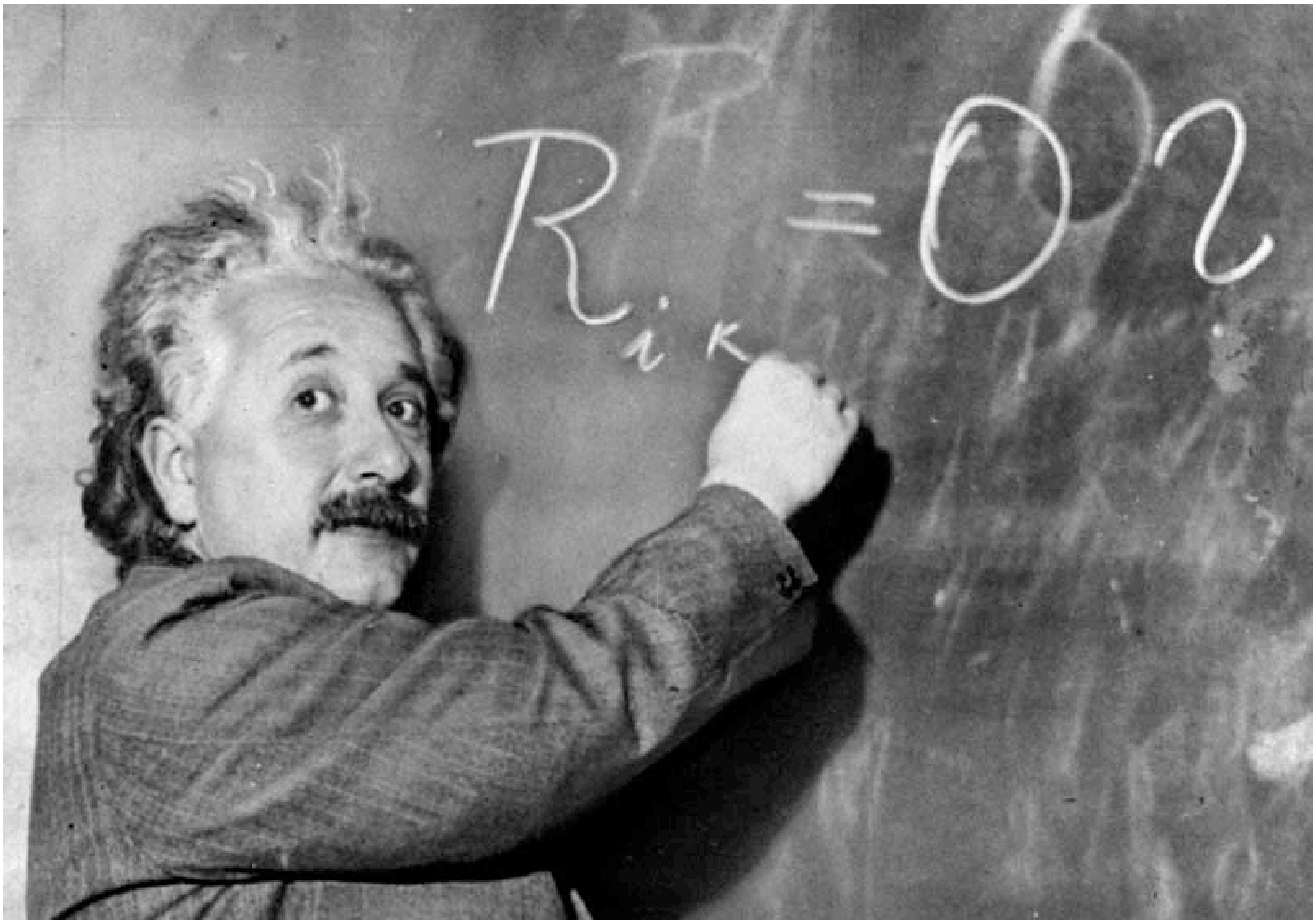

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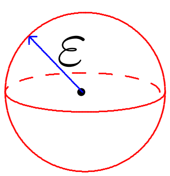

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$$\Delta x^j = 0 \Longrightarrow r_{jk} = \frac{1}{2}\Delta g_{jk} + \ell ots.$$

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- When $n \geq 6$, wide open. Maybe???

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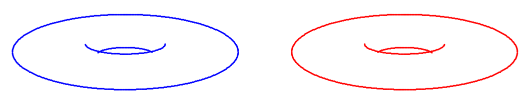

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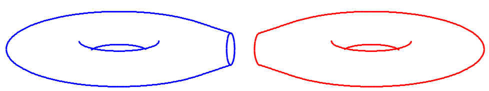

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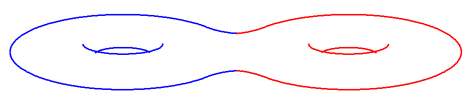

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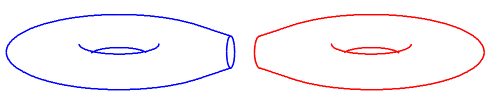

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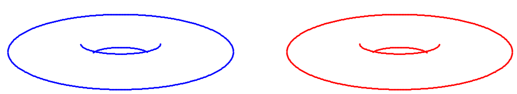

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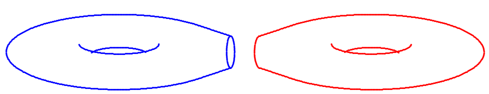

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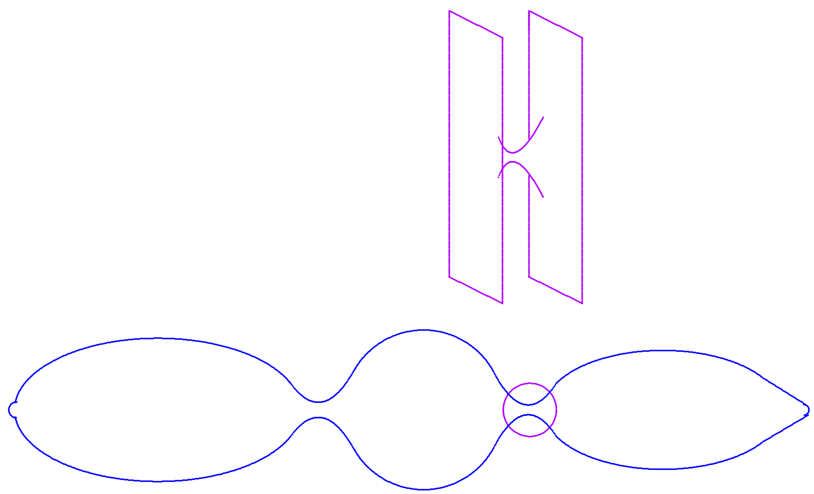

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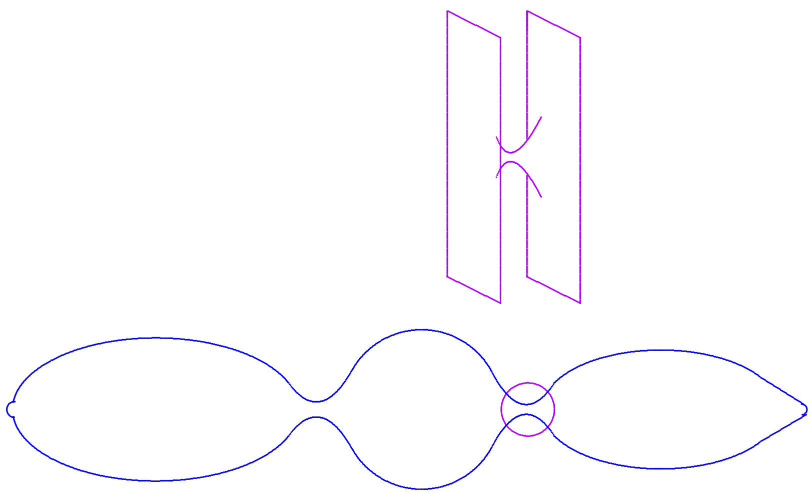

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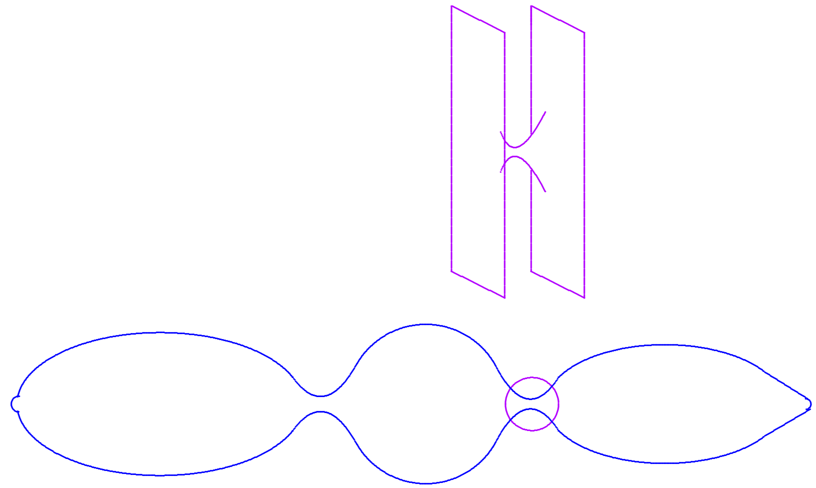

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Similar results for most simply connected spin 5-manifolds. (Boyer, Galicki, Kollár, et al.)

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(Terminology to be explained later!)

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By contrast, high-dimensional Einstein metrics too common; have little to do with geometrization.

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$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.
Λ^- anti-self-dual 2-forms.

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W_+ = self-dual Weyl curvature *(conformally invariant)*

W_- = anti-self-dual Weyl curvature ″

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

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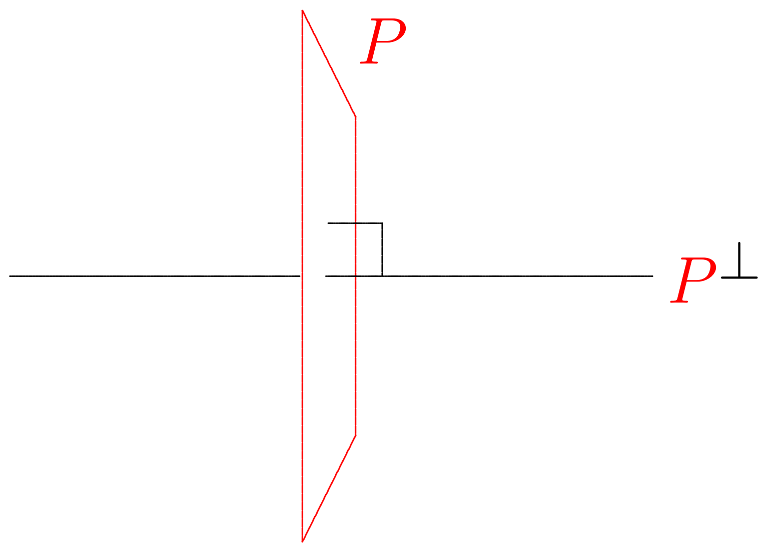

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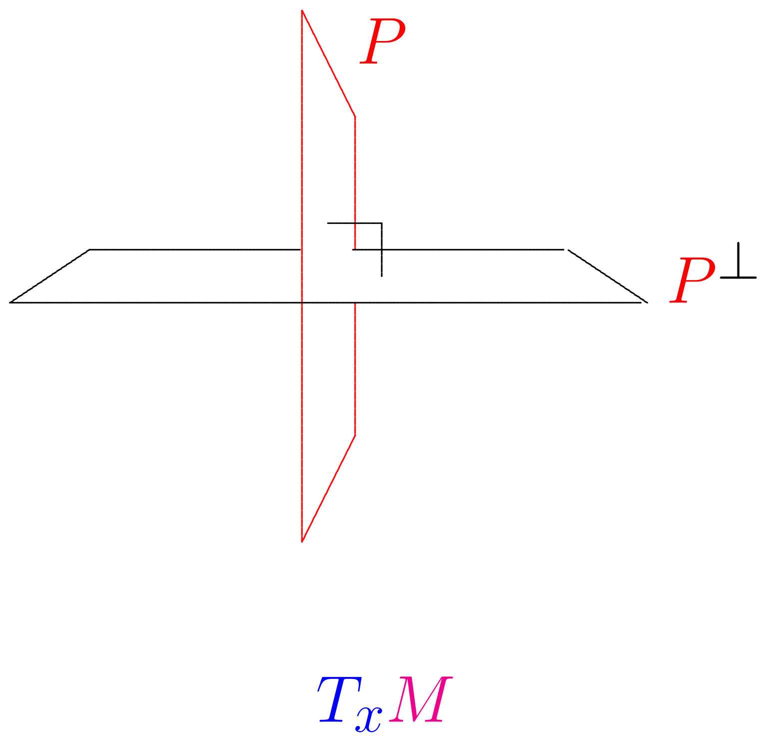

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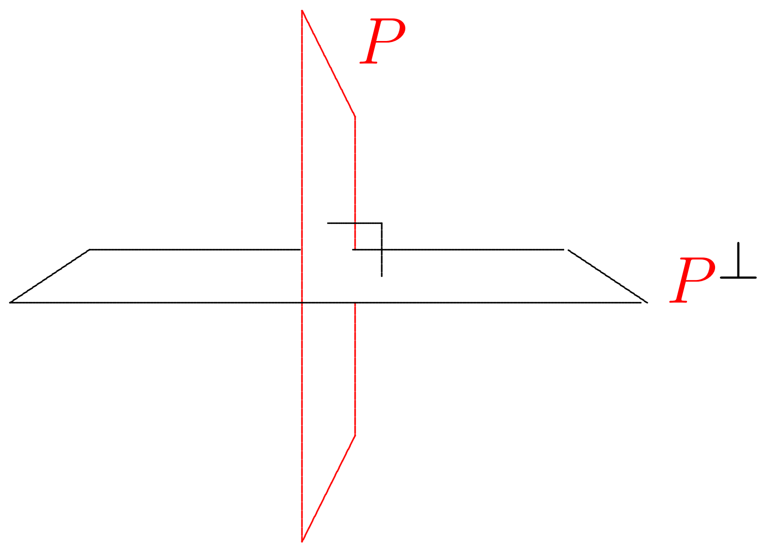

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(Terminology to be explained latter)

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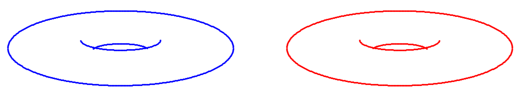

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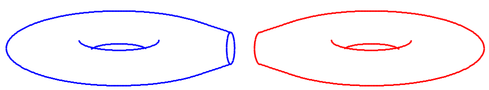

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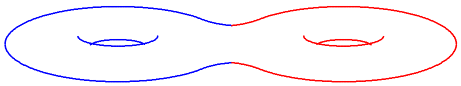

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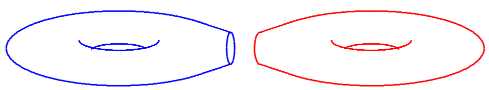

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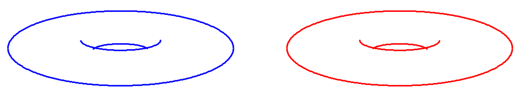

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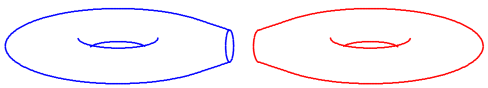

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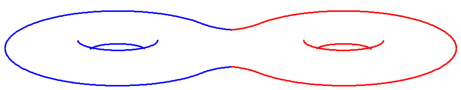

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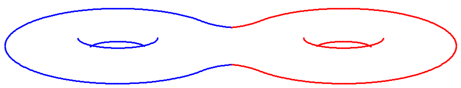

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$K3$ manifold...

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Spin, $\chi = 24$, $\tau = -16$.

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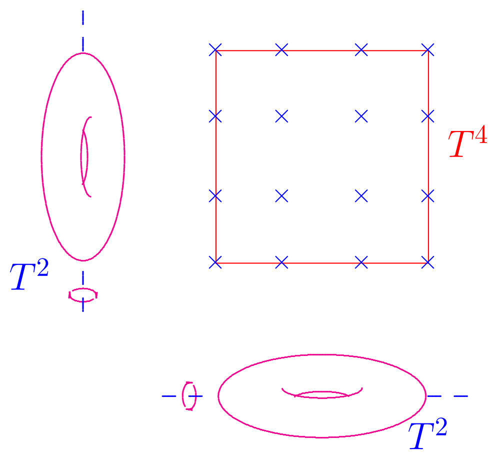

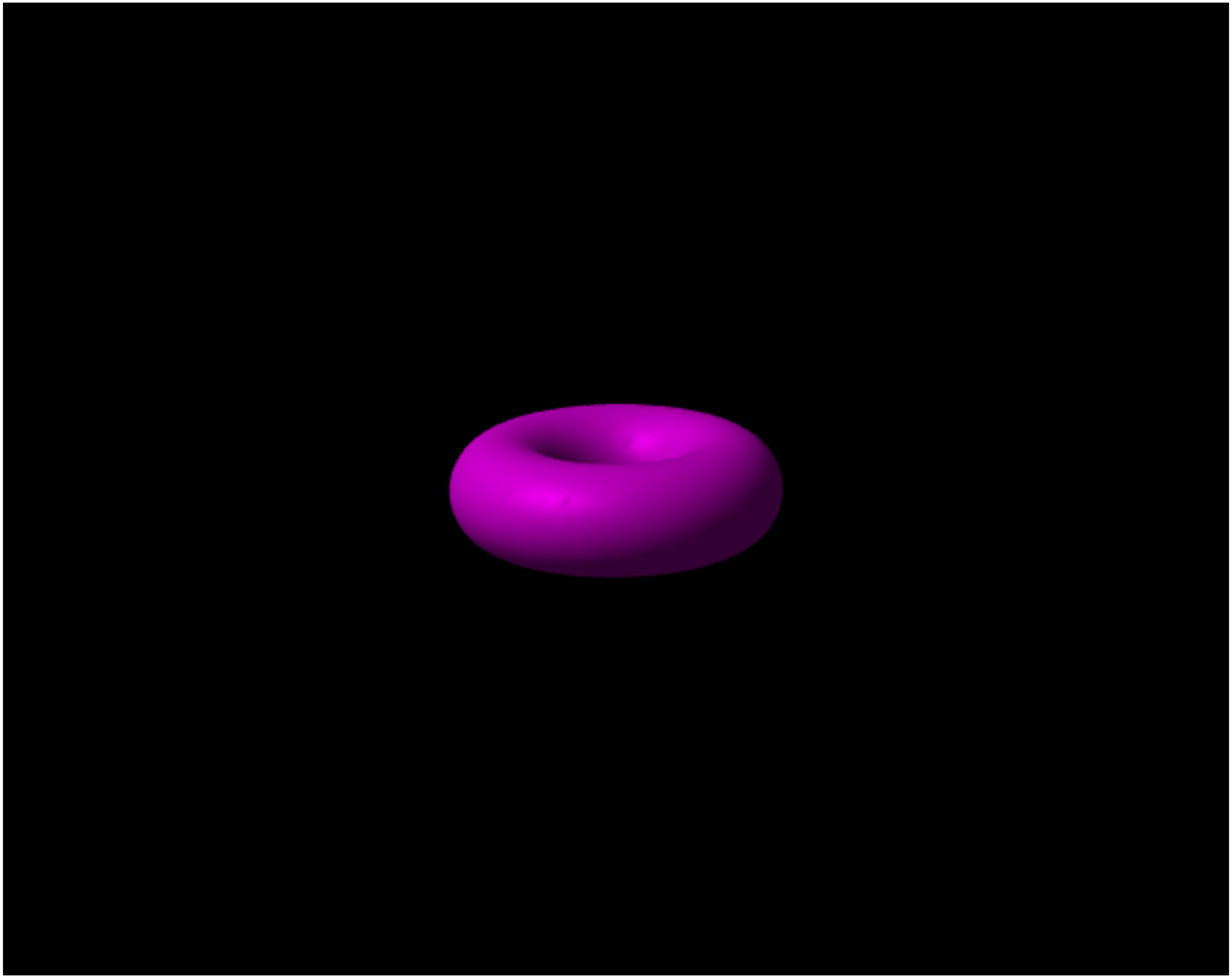

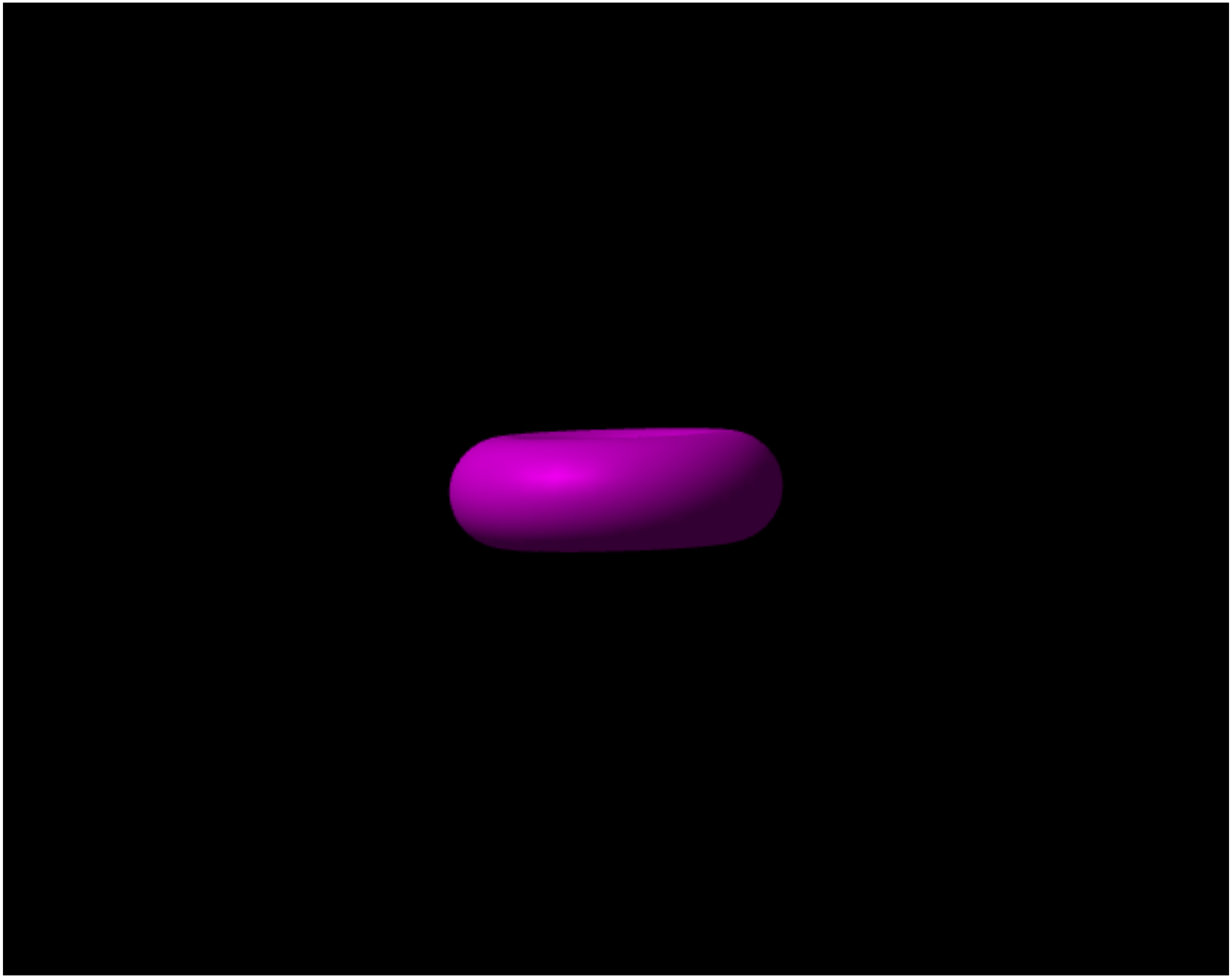

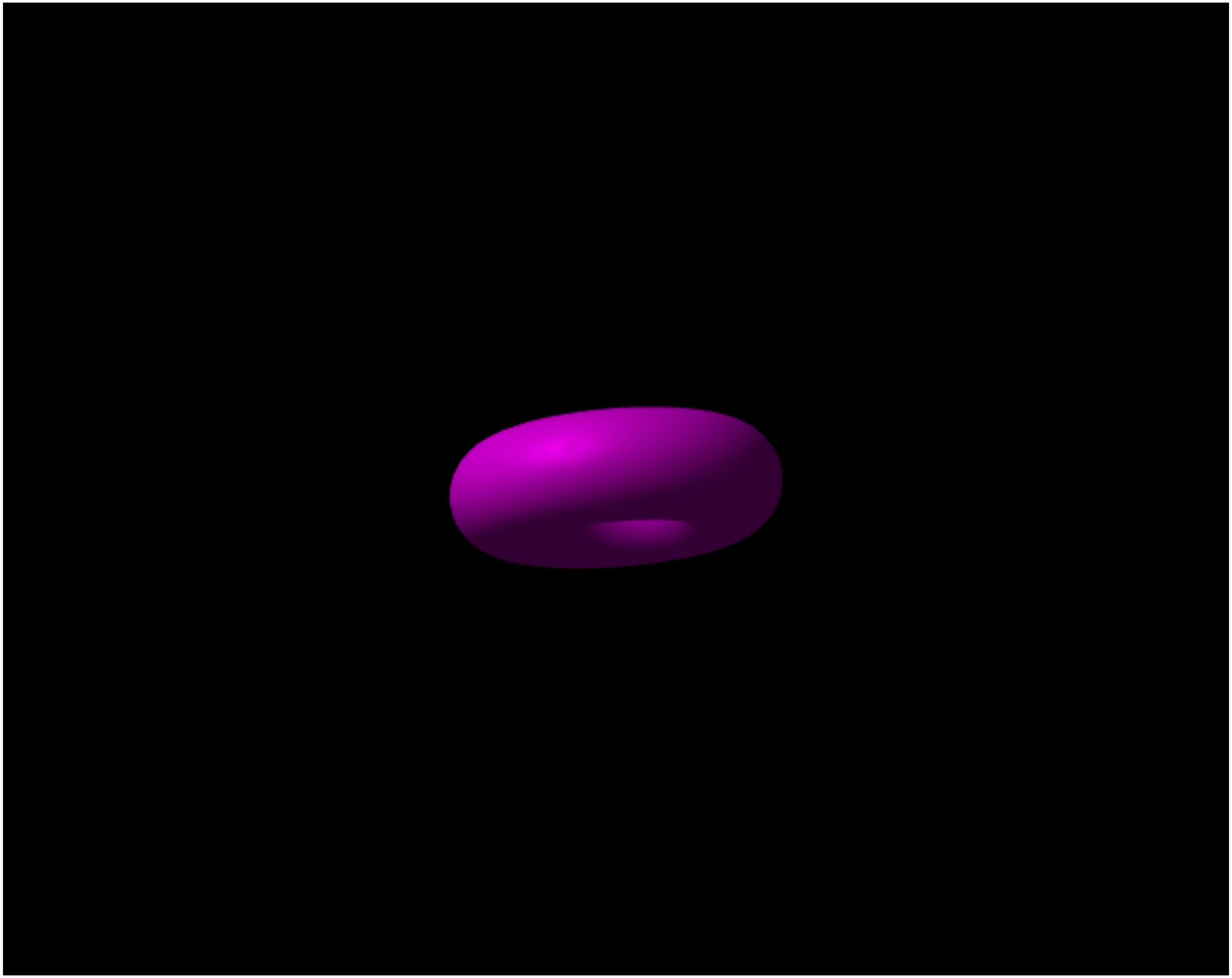

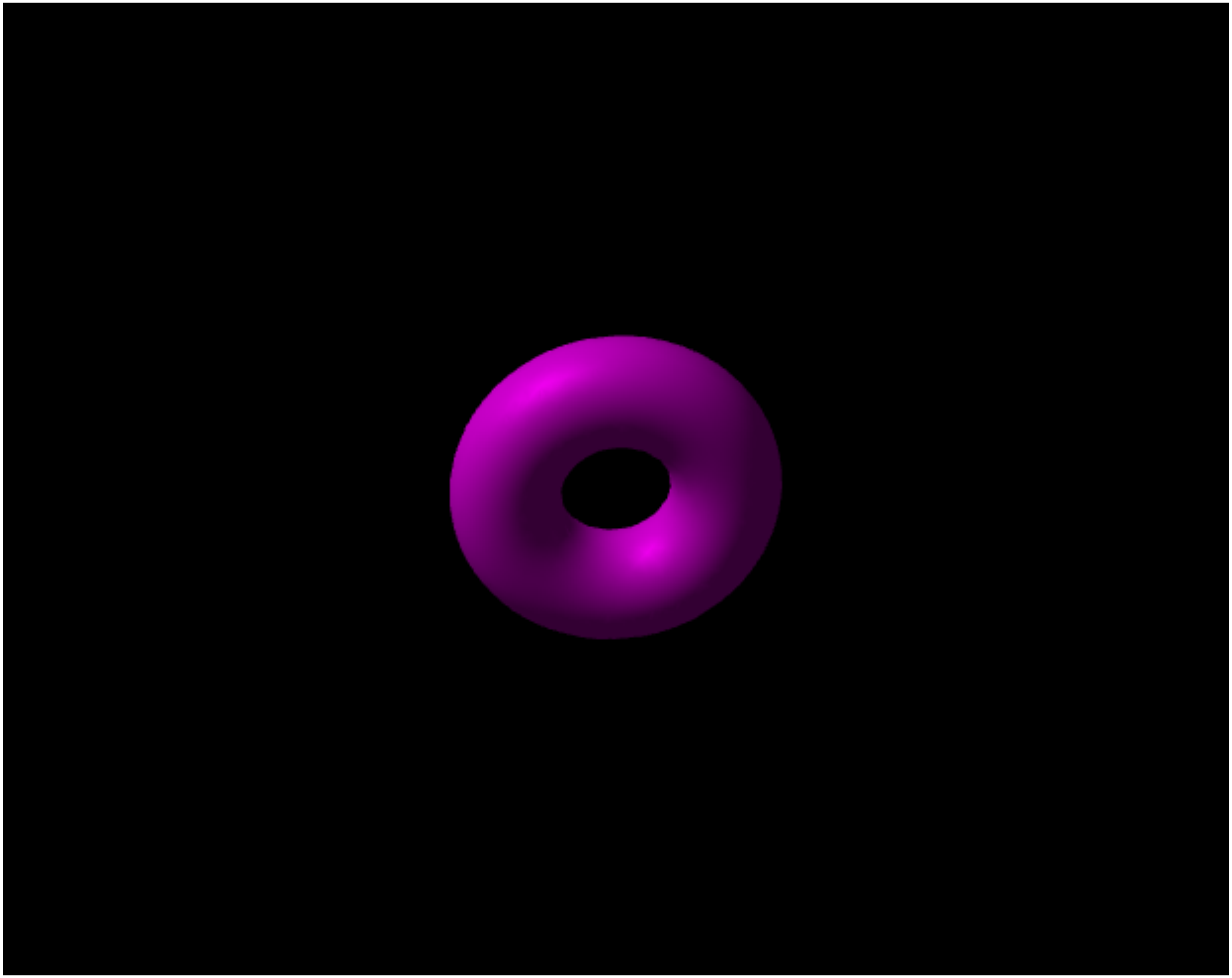

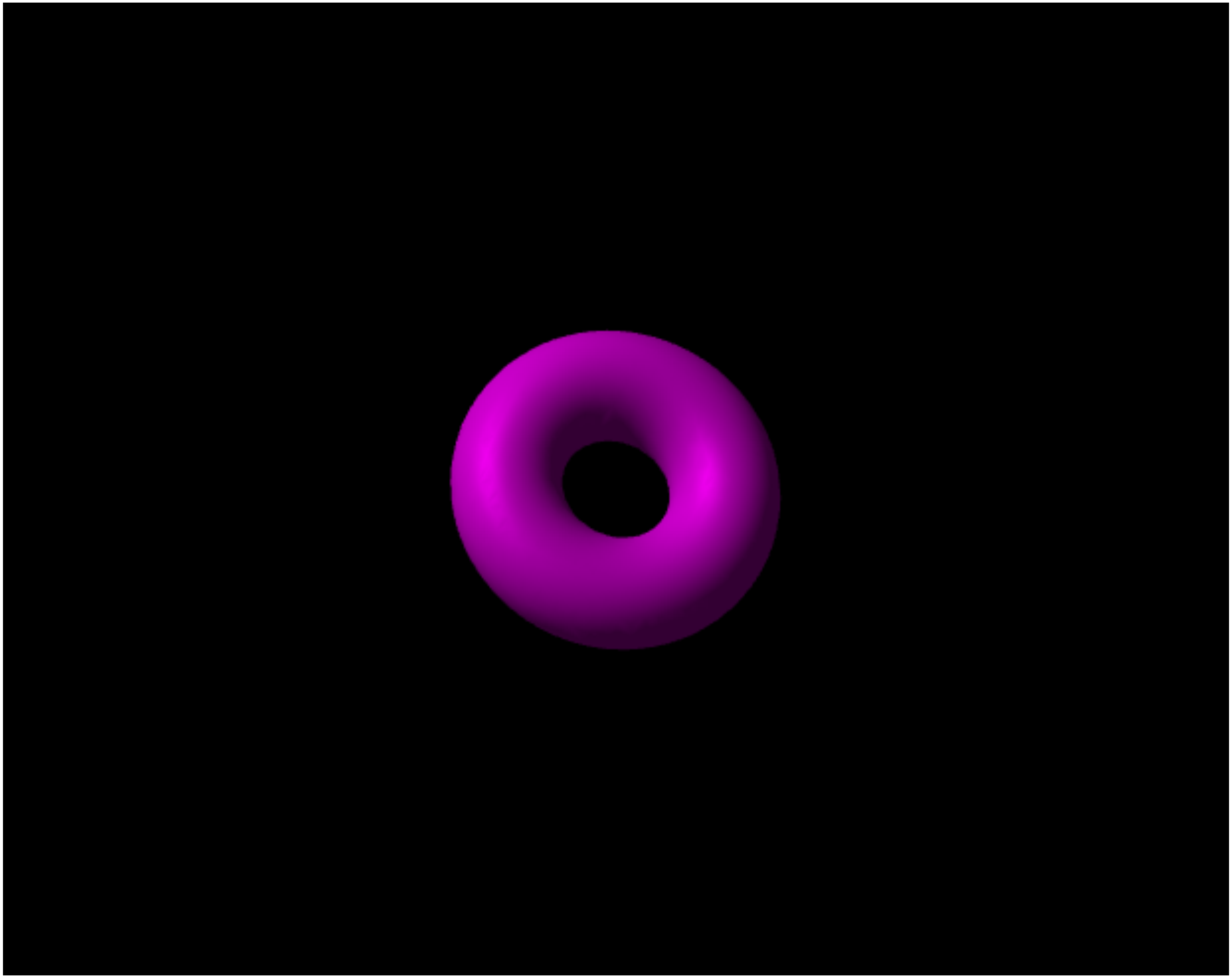

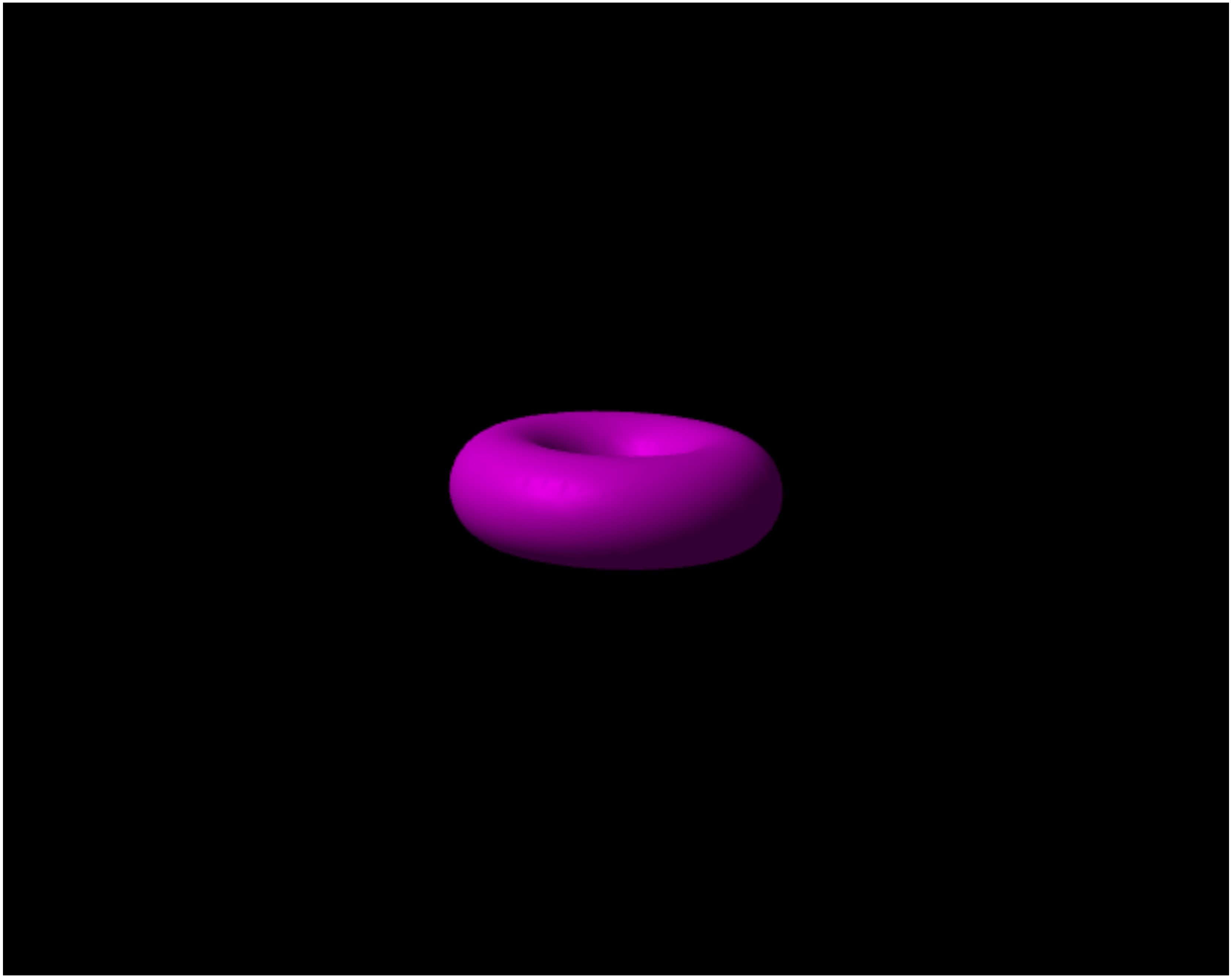

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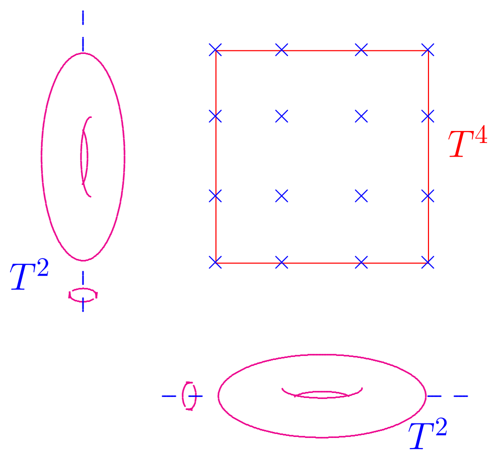

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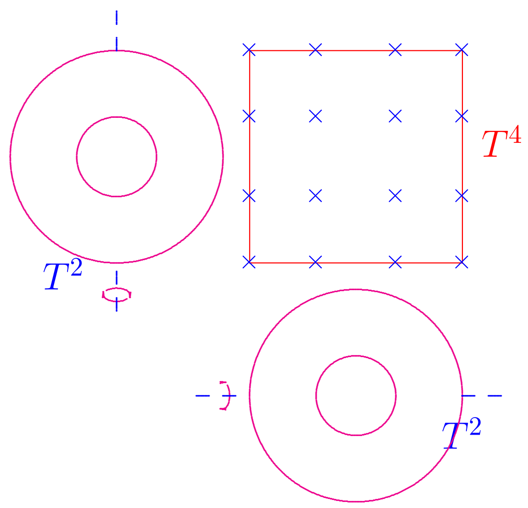

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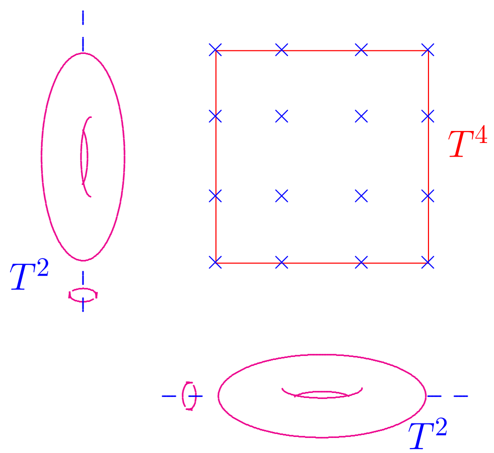

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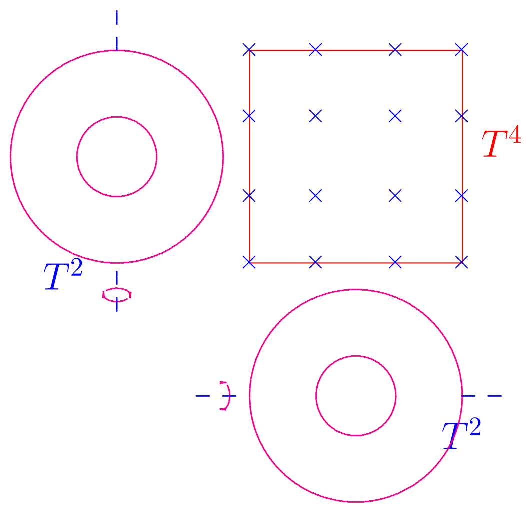

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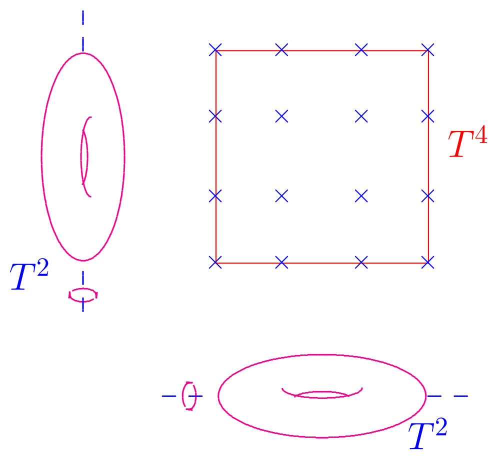

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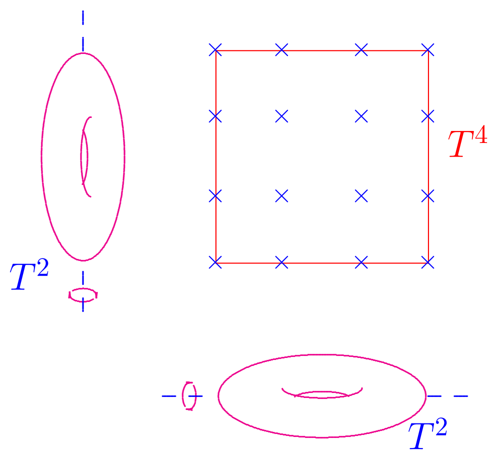

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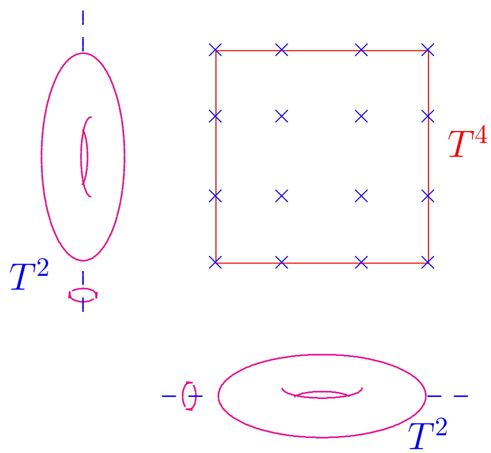

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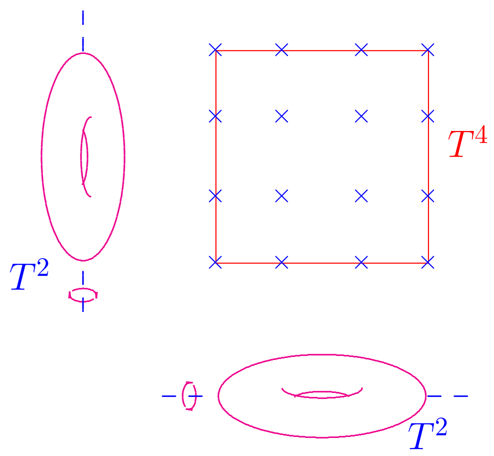

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T^4 = Picard torus of curve of genus 2.

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Kummer construction:

Begin with $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$.

Remove singularities by deforming equation.

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Theorem (Freedman/Donaldson). *Two smooth compact* simply connected *oriented* 4-*manifolds are orientedly homeomorphic if and only if*

- *they have the same Euler characteristic* χ;
- *they have the same signature* τ; *and*
- *both are spin, or both are non-spin.*

Corollary. *Any smooth compact* simply connected *non-spin* 4*-manifold* M *is homeomorphic to a connect sum* $j\mathbb{CP}_2 \# k\overline{\mathbb{CP}_2}$.

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Conjecture (11/8 Conjecture)**.** *Any smooth compact* simply connected *spin* 4*-manifold* M *is (un-orientedly) homeomorphic to either* S^4 *or a connected sum* $jK3\#k(S^2 \times S^2)$.

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Certainly true of all examples in this lecture!

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Narrower Question. *If* (M^4, ω) *is a compact symplectic manifold, when does* M^4 *admit an Einstein metric* g *(unrelated to* ω*) with Einstein constant* $\lambda \geq 0$*?*

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Del Pezzo surfaces,
K3 surface, Enriques surface,
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No others: Hitchin-Thorpe, Seiberg-Witten, . . .

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Every Einstein metric is Ricci-flat Kähler.

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(M^{2m}, g) has holonomy $\subset \mathbf{U}(m)$.

(M^n, g):

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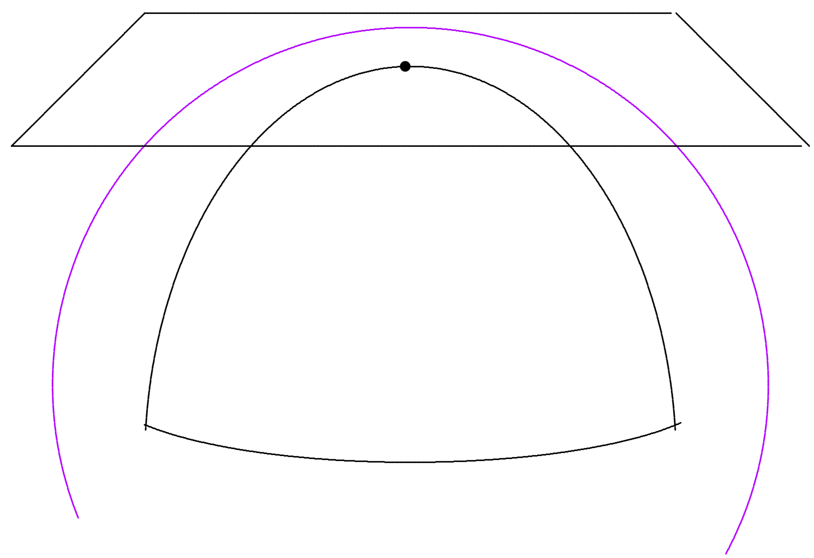

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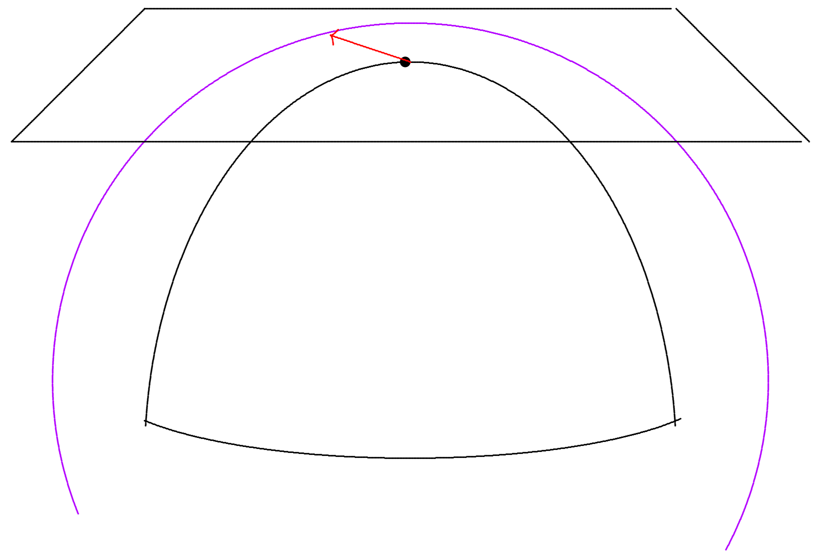

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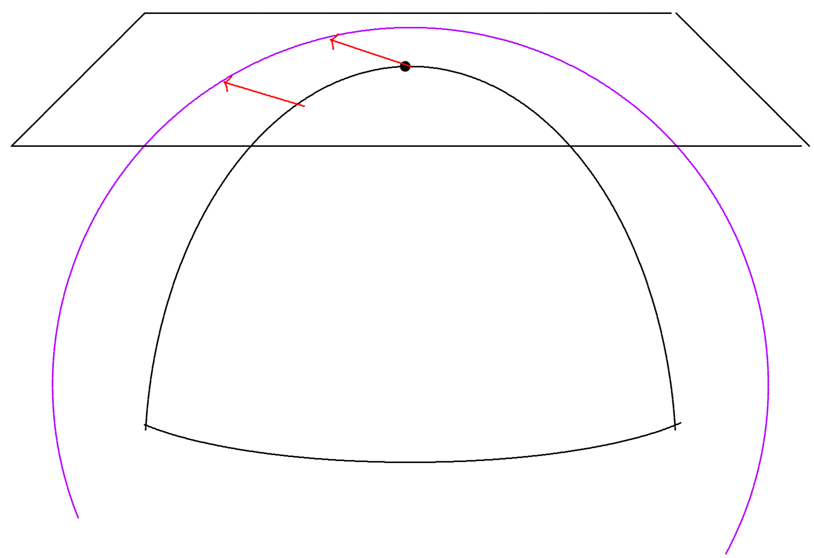

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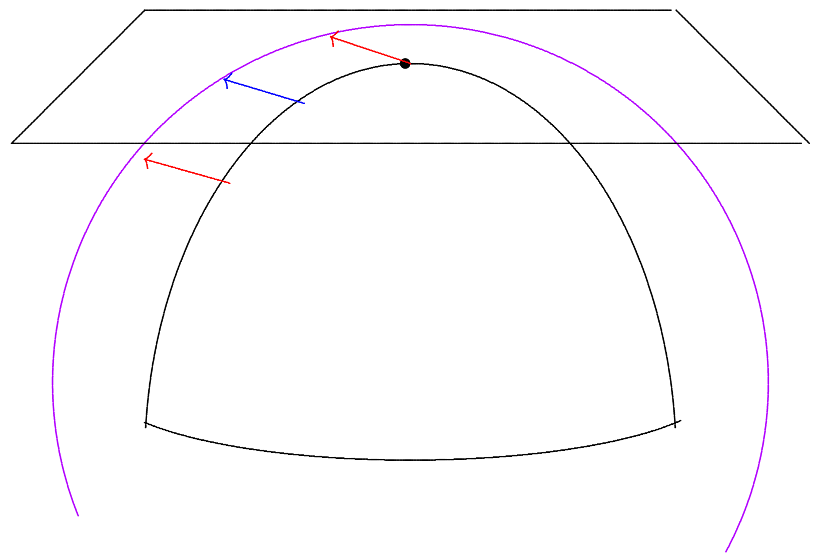

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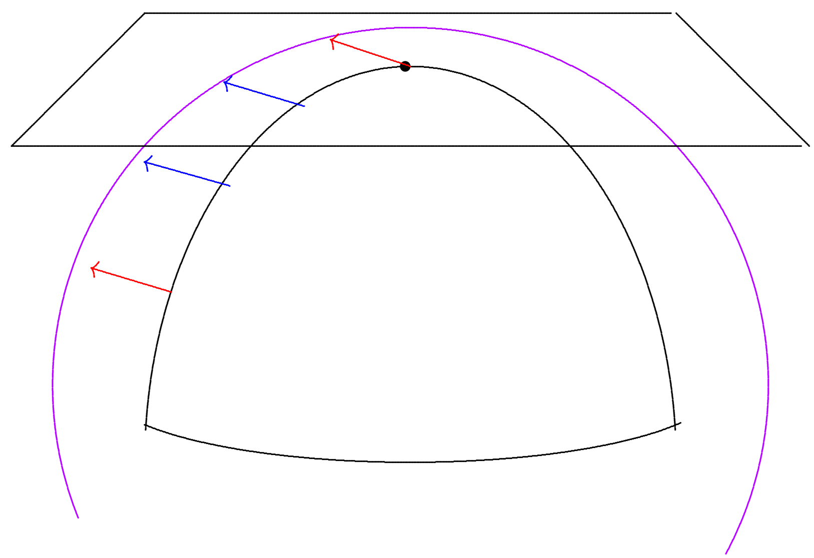

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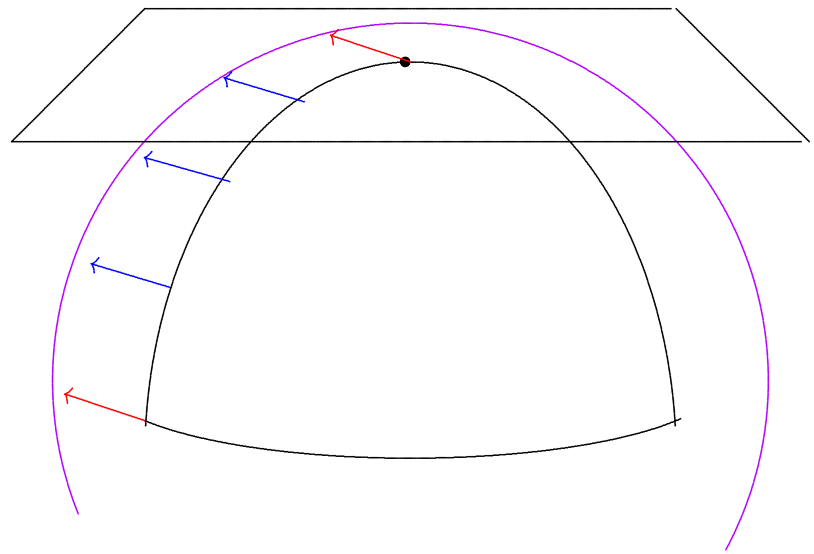

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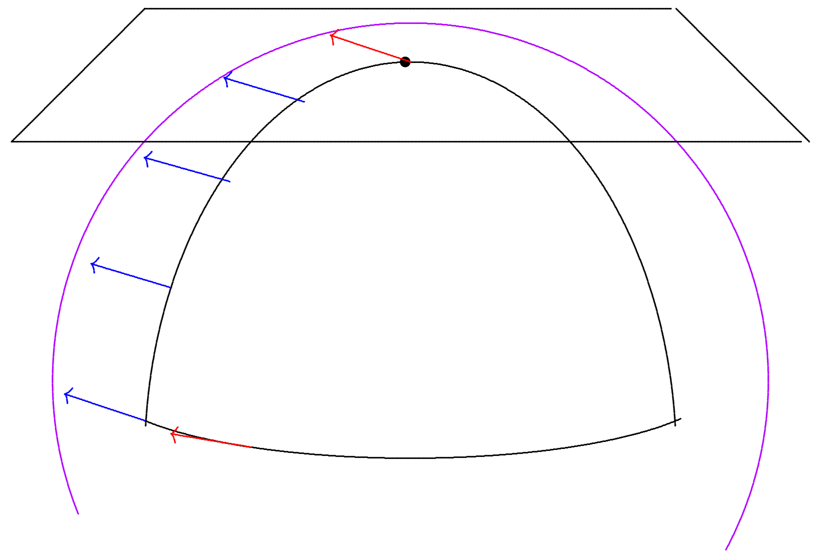

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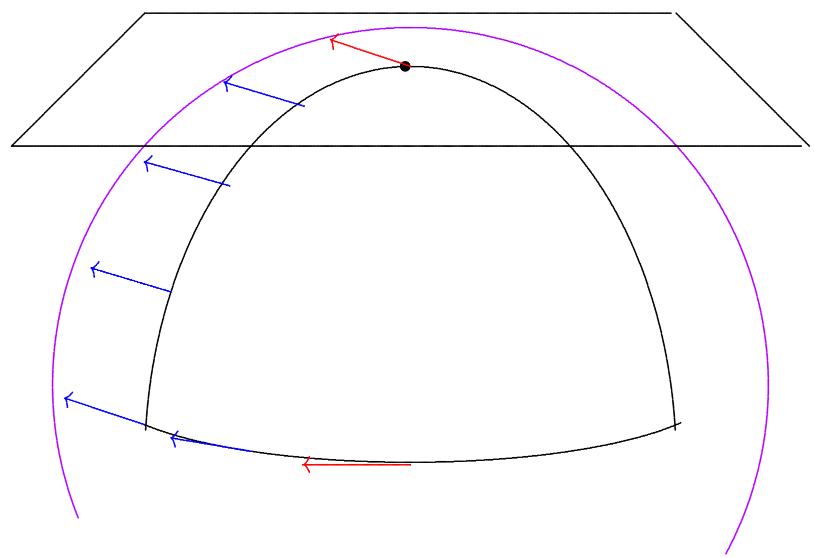

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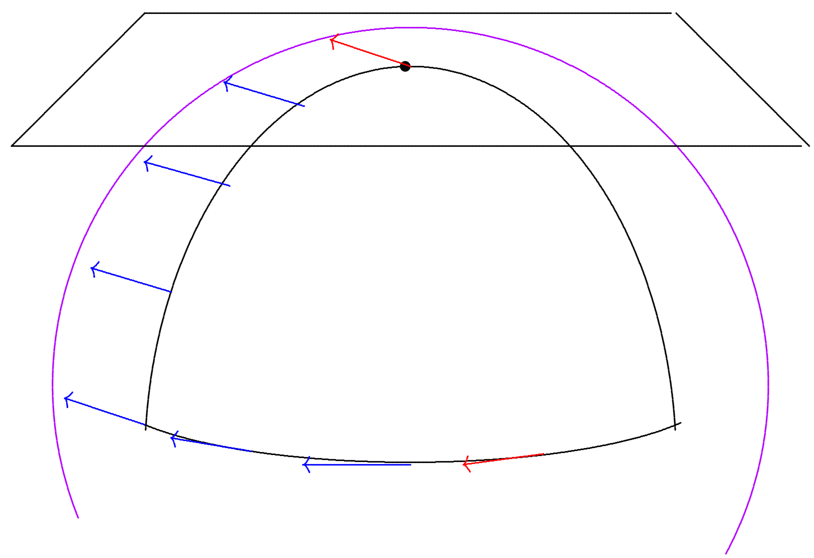

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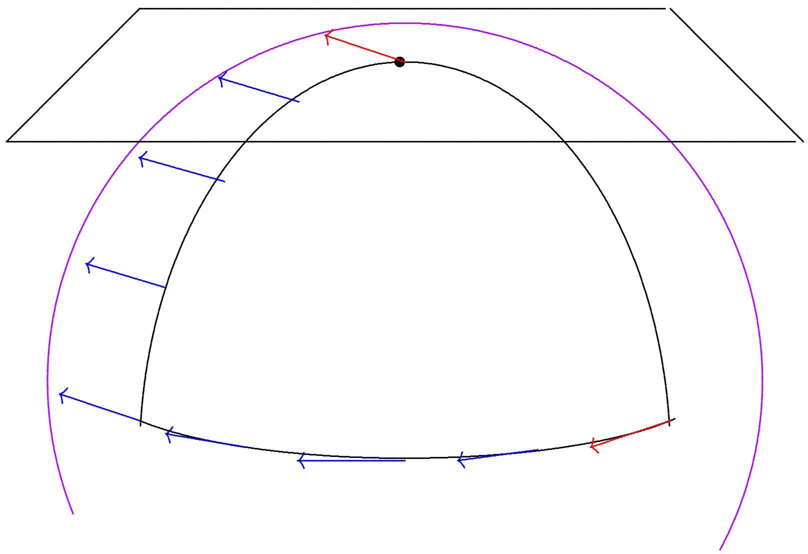

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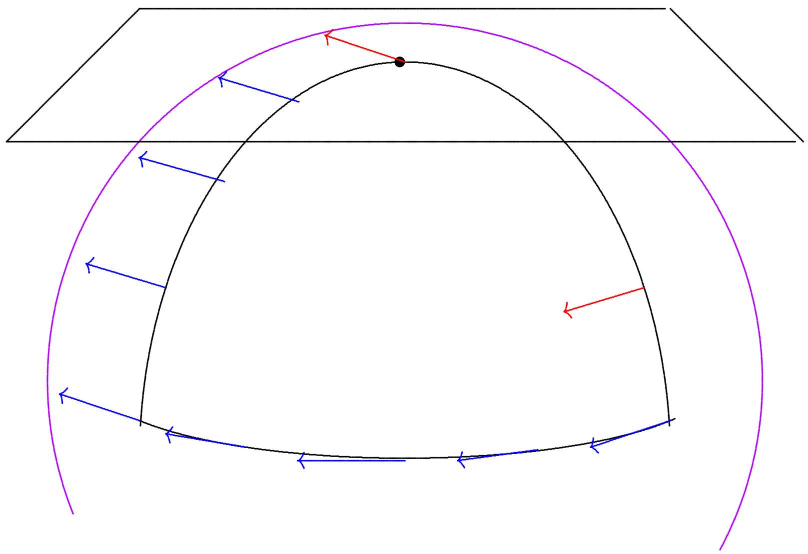

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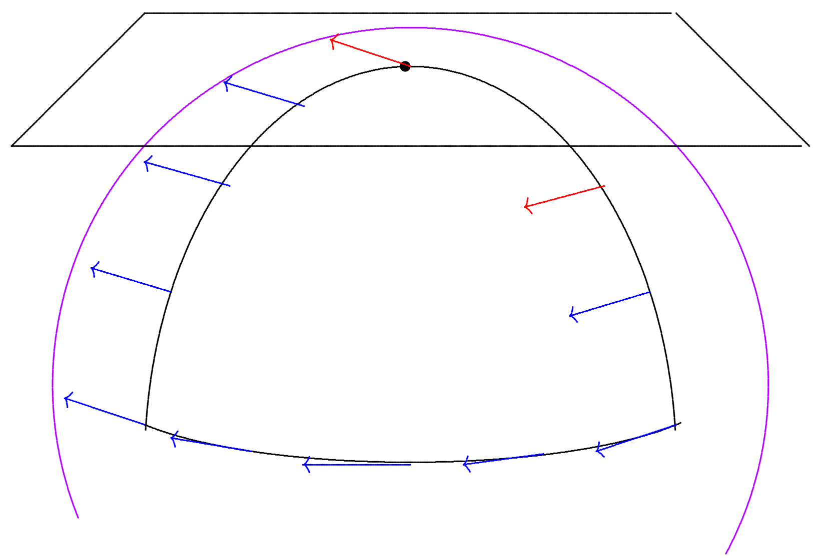

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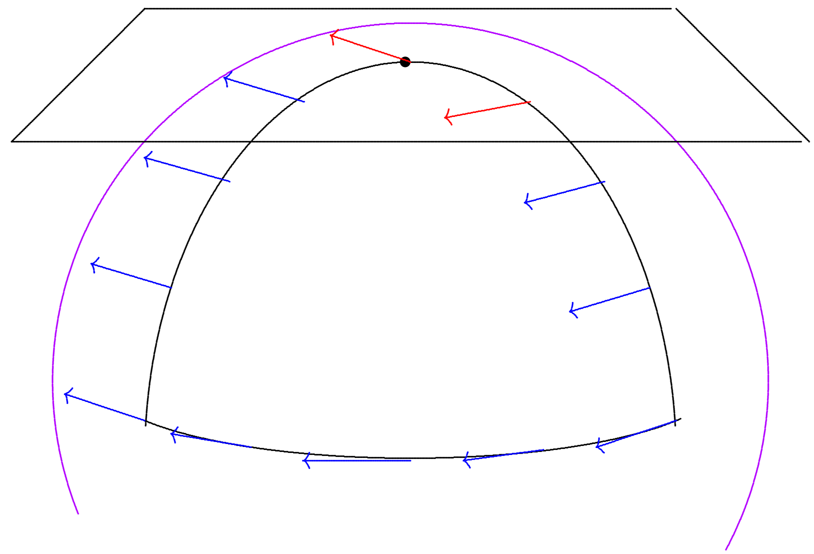

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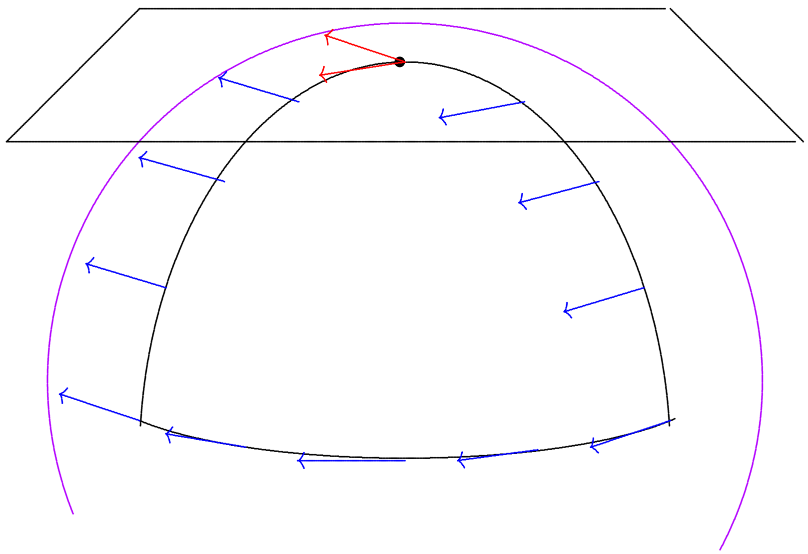

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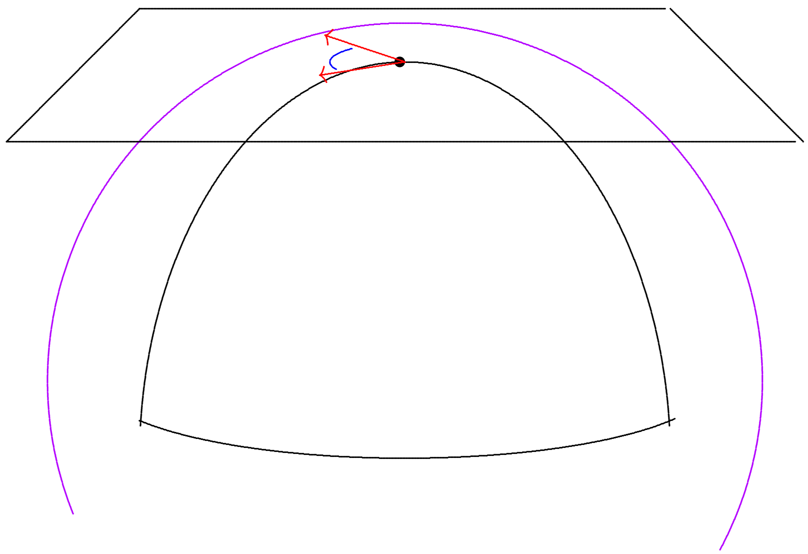
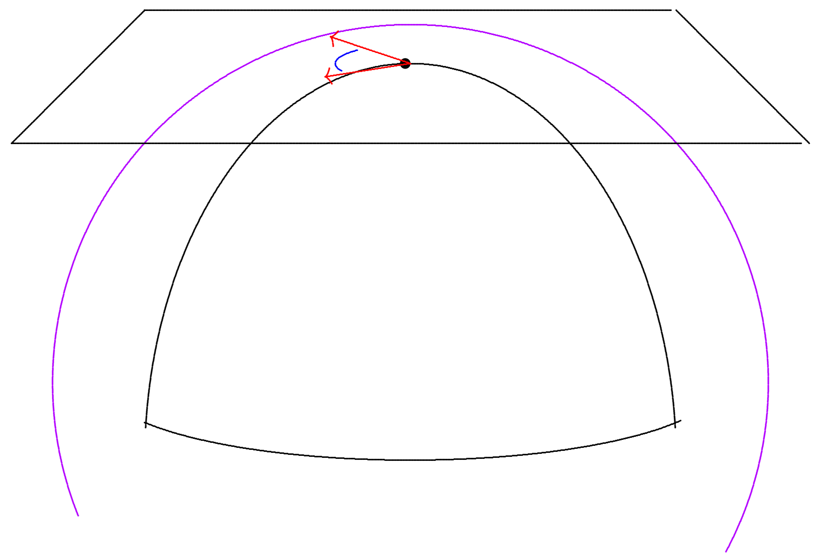


(M^n, g): holonomy $\subset \mathbf{O}(n)$



Kähler metrics:

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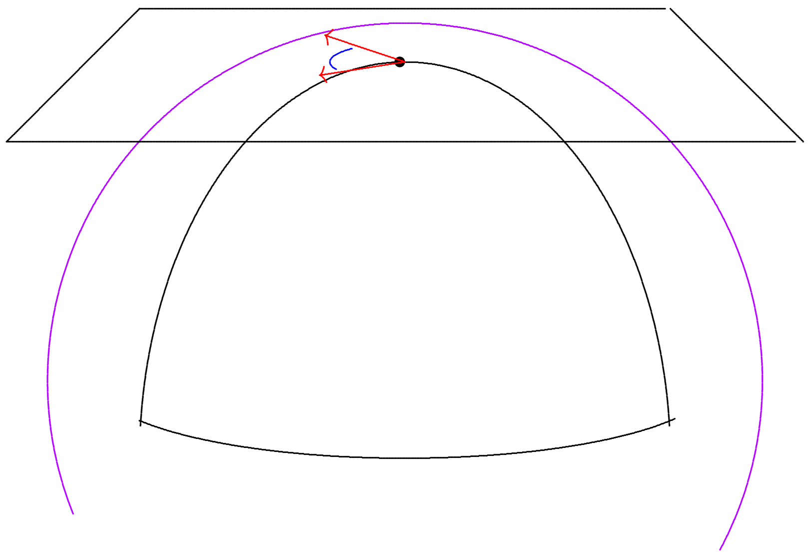

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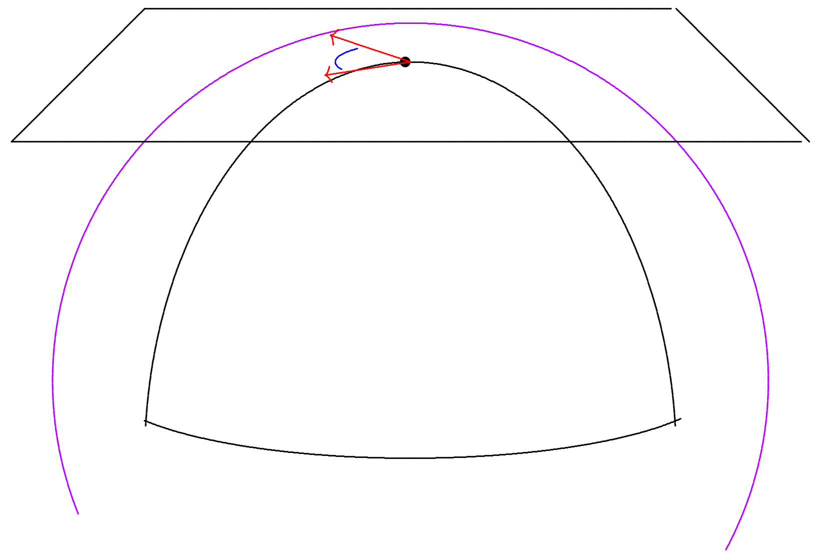

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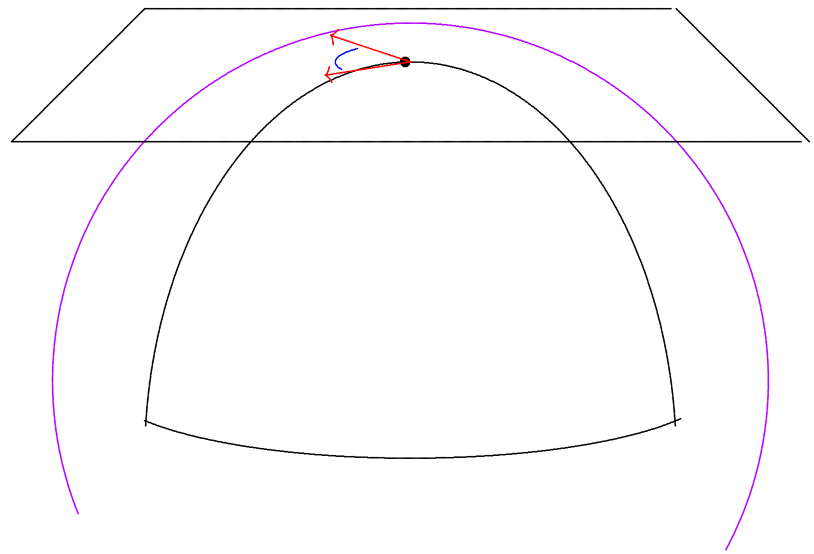

$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

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$$g = \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right]$$

for a locally defined function f.

Modern definition:

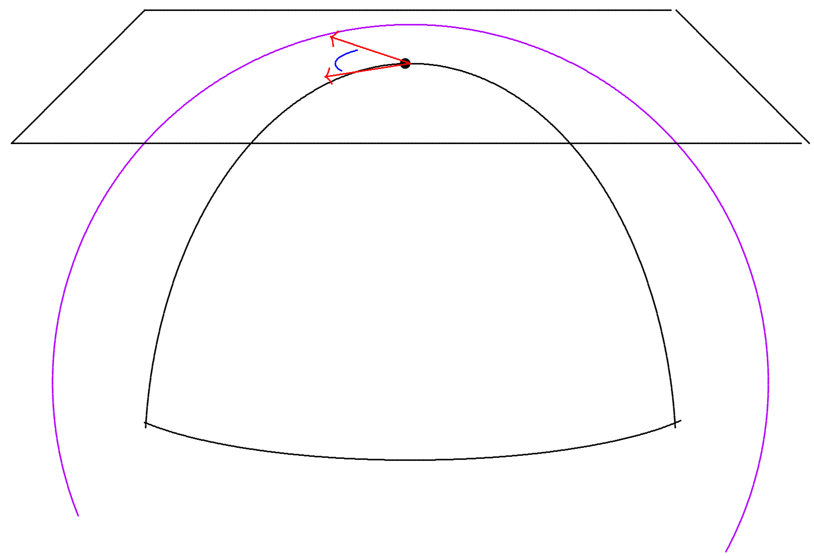
(M^{2m}, g) has holonomy $\subset \mathbf{U}(m)$.

Ricci-flat Kähler:

$(\widetilde{M}^{2m}, g)$ has holonomy $\subset \mathbf{SU}(m)$.

Ricci-flat Kähler metrics:

(M^{2m}, g): holonomy



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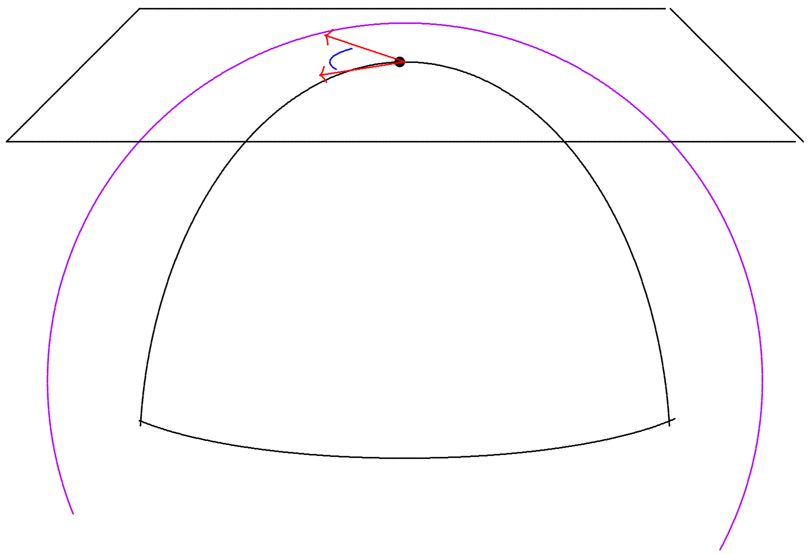

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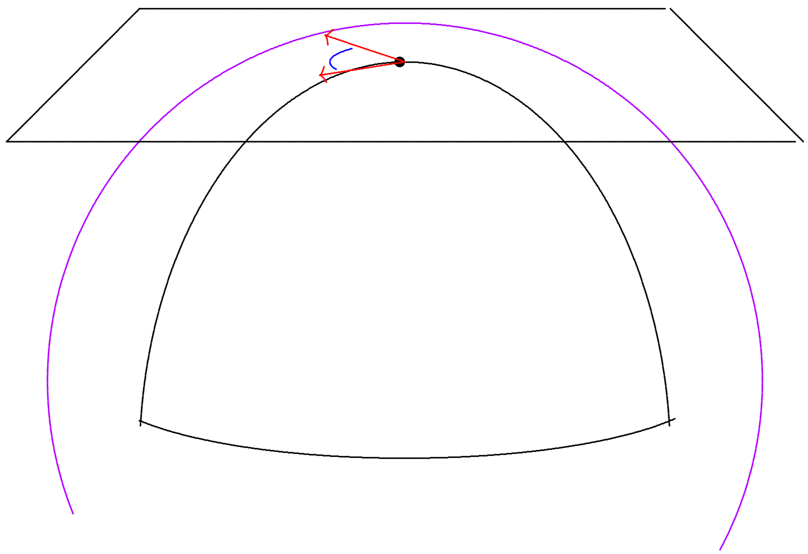

$\mathbf{SU}(m) \subset \mathbf{U}(m)$: $\{A \mid \det A = 1\}$

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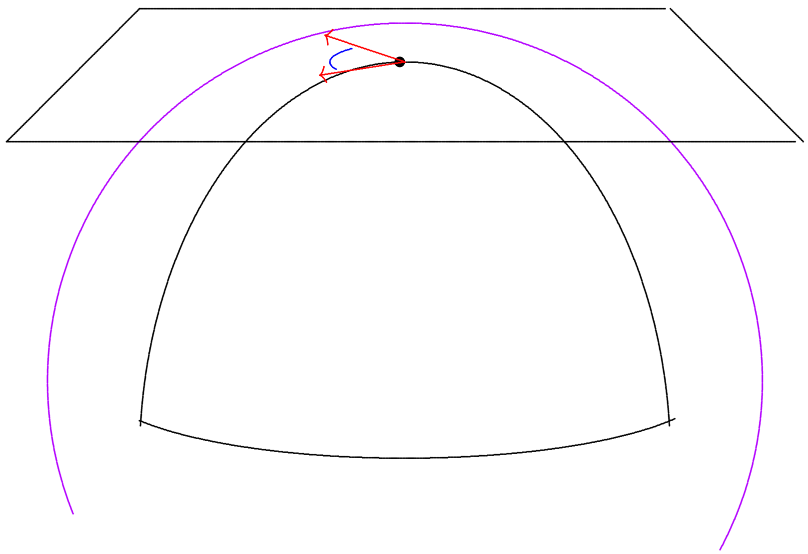

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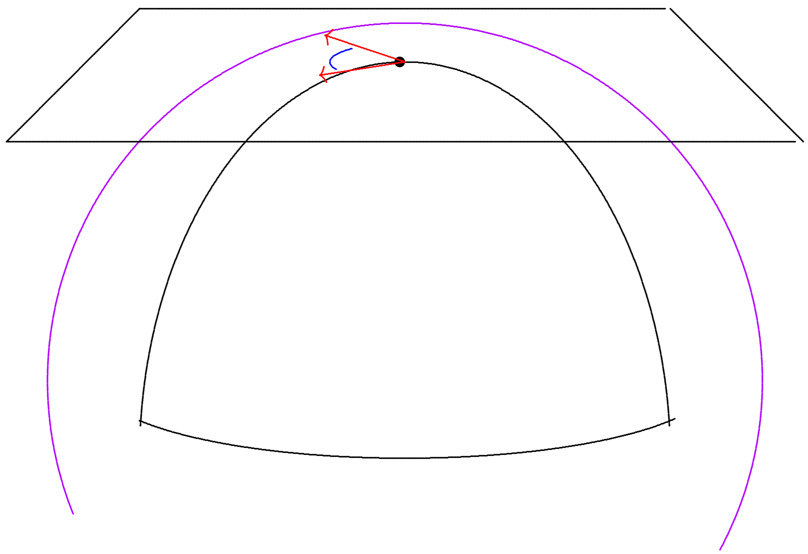

if M is simply connected.

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(Kähler form $\omega = g(J\cdot, \cdot)$ is closed 2-form.)

Corollary. $\exists$ $\lambda = 0$ *Einstein metrics on* $K3$.

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Theorem (L '09). *Suppose that* M *is a smooth compact oriented* 4*-manifold which admits a symplectic form* ω. *Then* M *also admits an Einstein metric* g *with* $\lambda \geq 0$ *if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

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Know an Einstein metric on each manifold.

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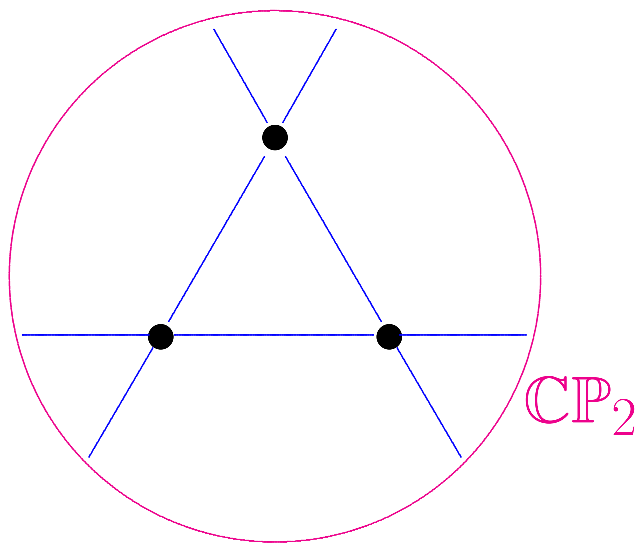

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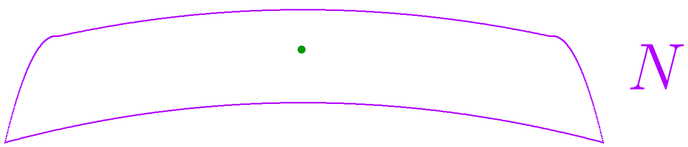

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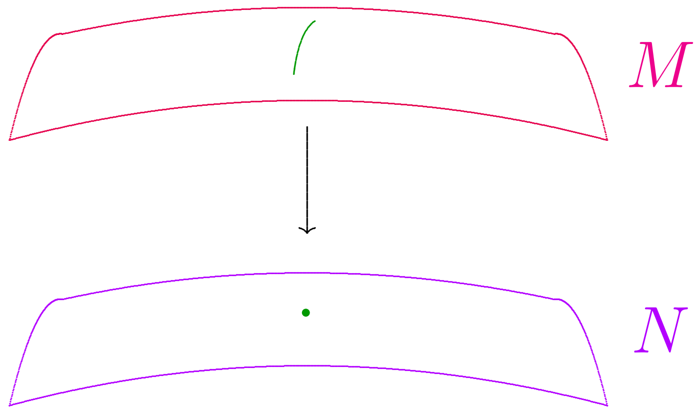

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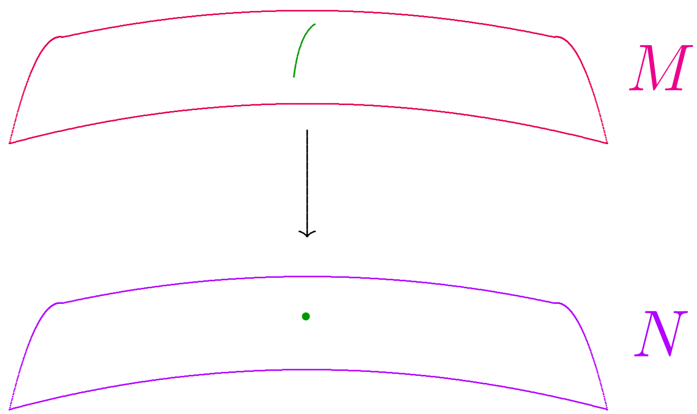

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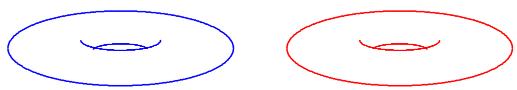

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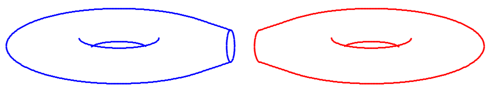

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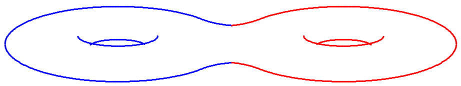

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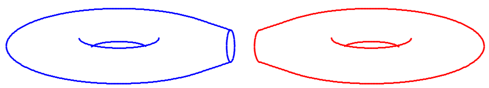

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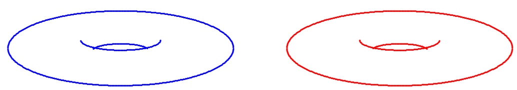

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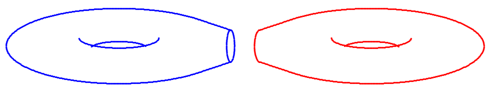

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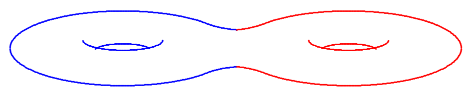

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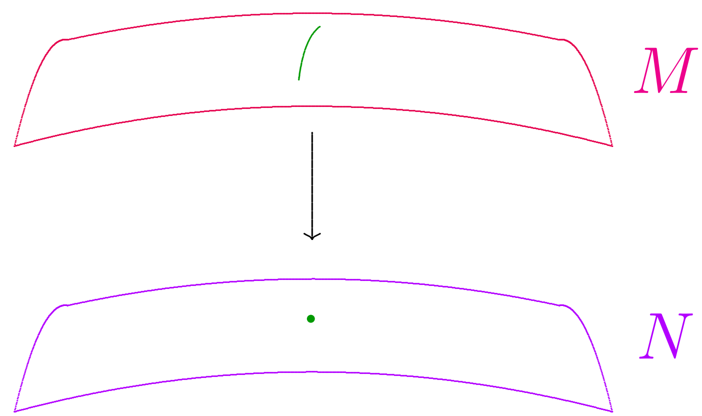

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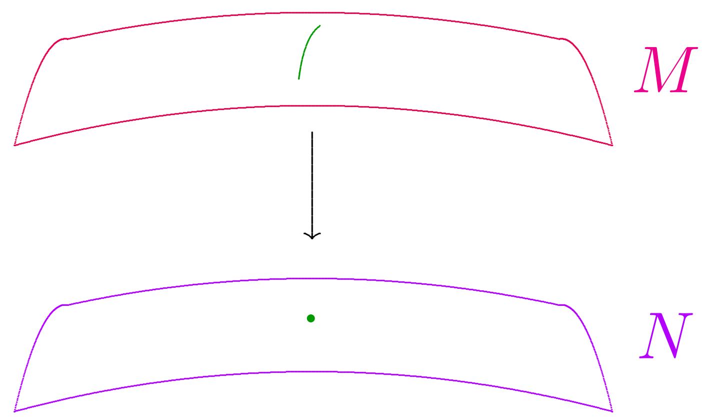

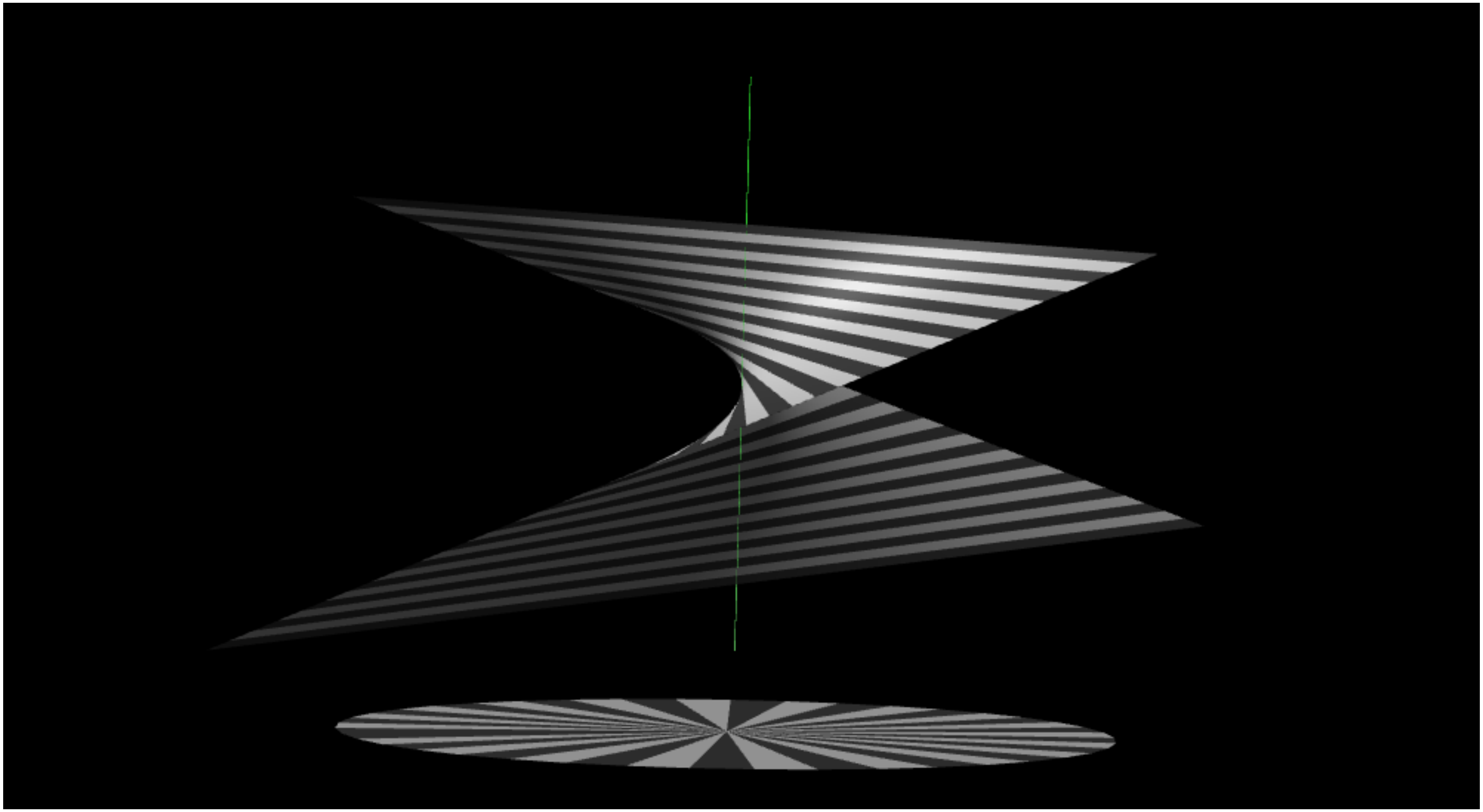

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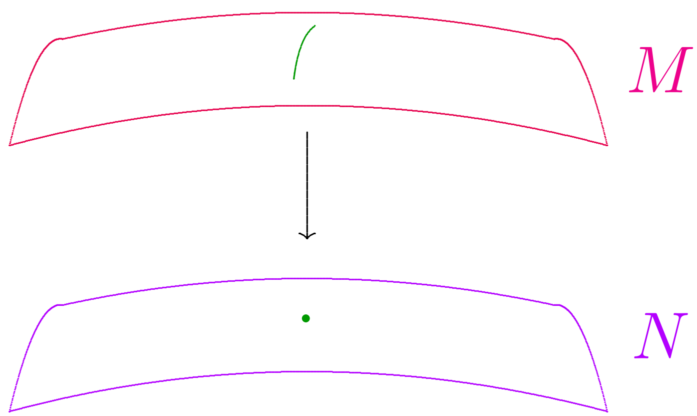

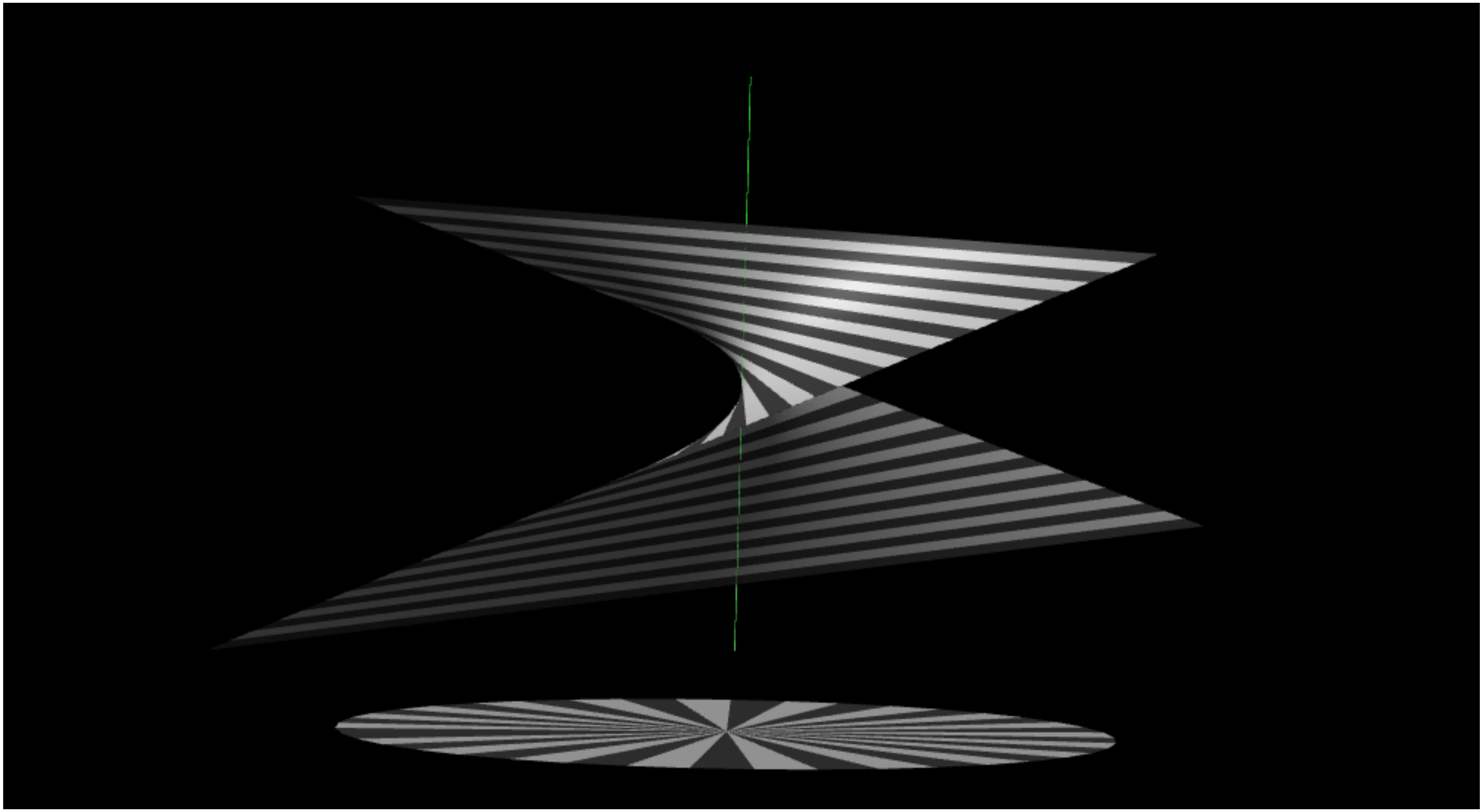

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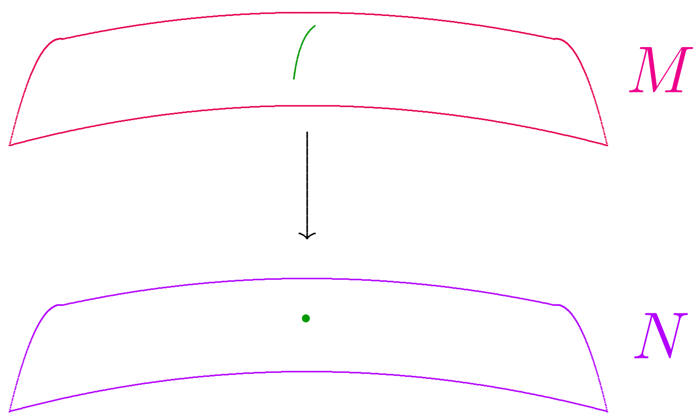

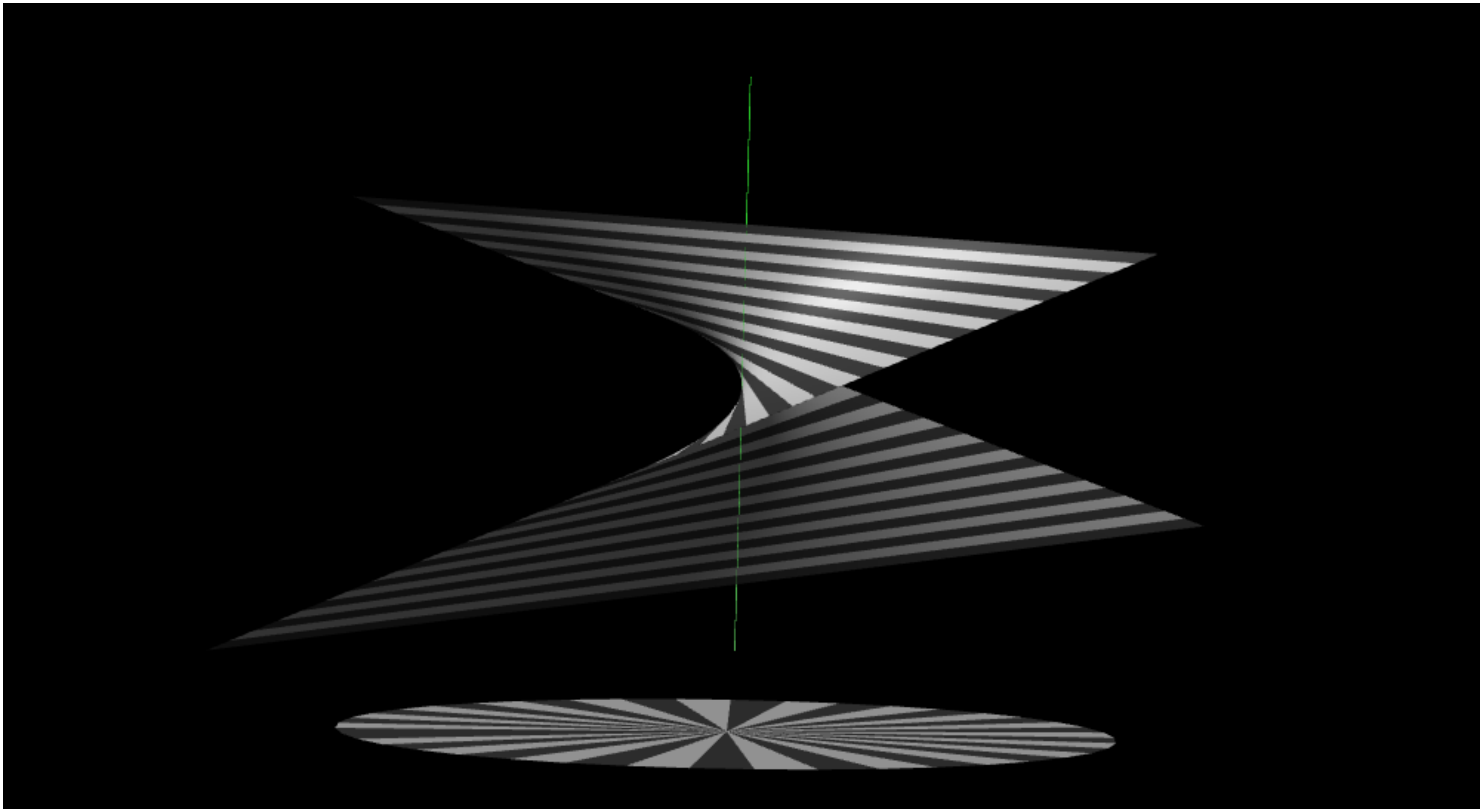

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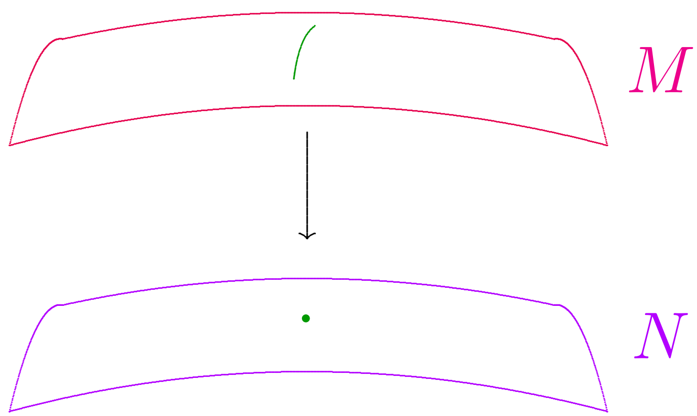

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

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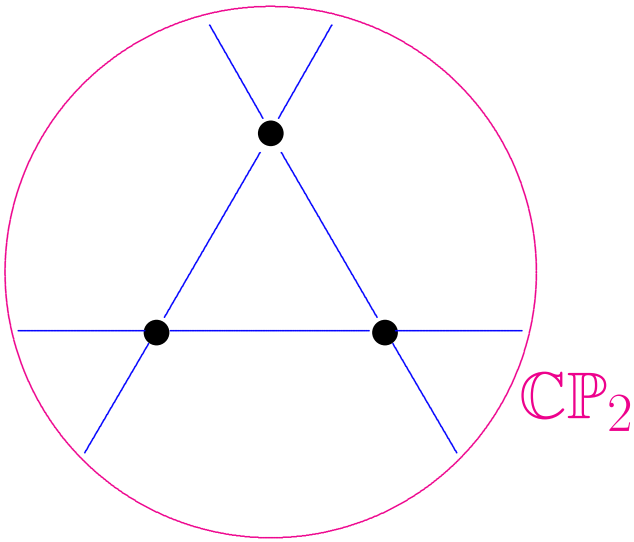

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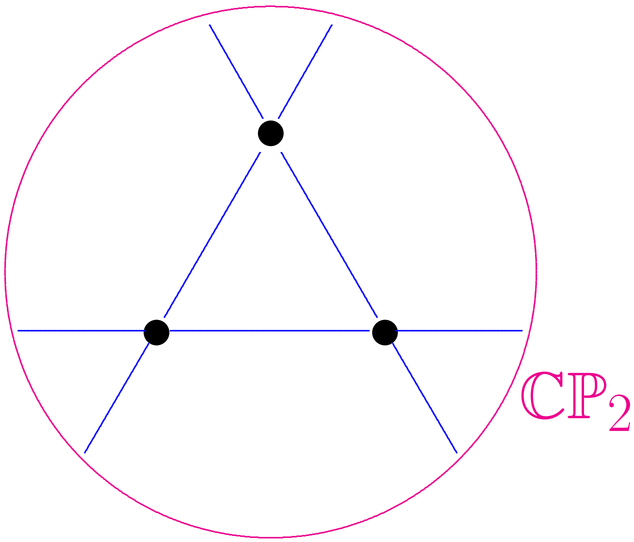

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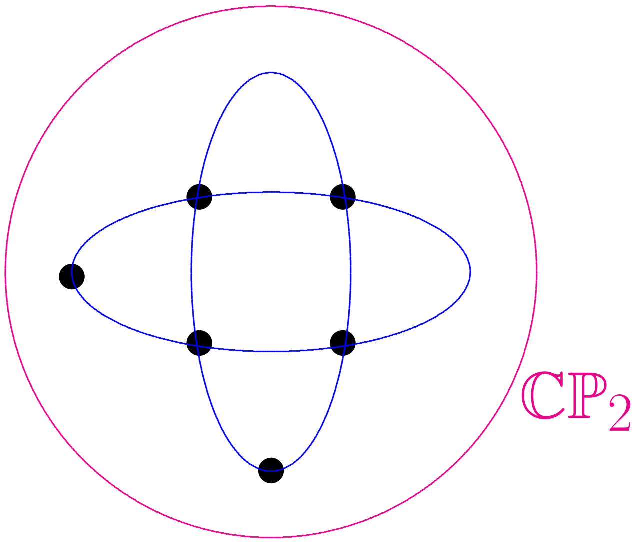

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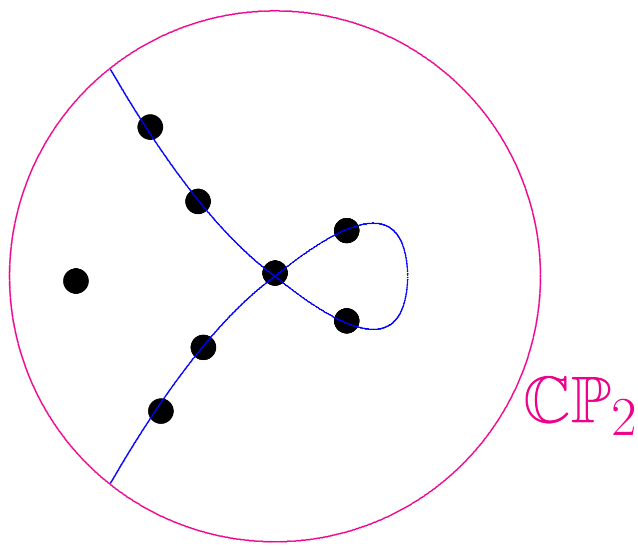

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Existence: Page

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian,

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either on M or double cover $\widetilde{M}$.

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Claim: (M, g) compact Einstein $\Longrightarrow$ J integrable.

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Integrability proof based on Weitzenböck formula

$$\boxed{0 = \nabla^*\nabla W^+ + \frac{s}{2}W^+ - 6W^+ \circ W^+ + 2|W^+|^2 I}$$

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

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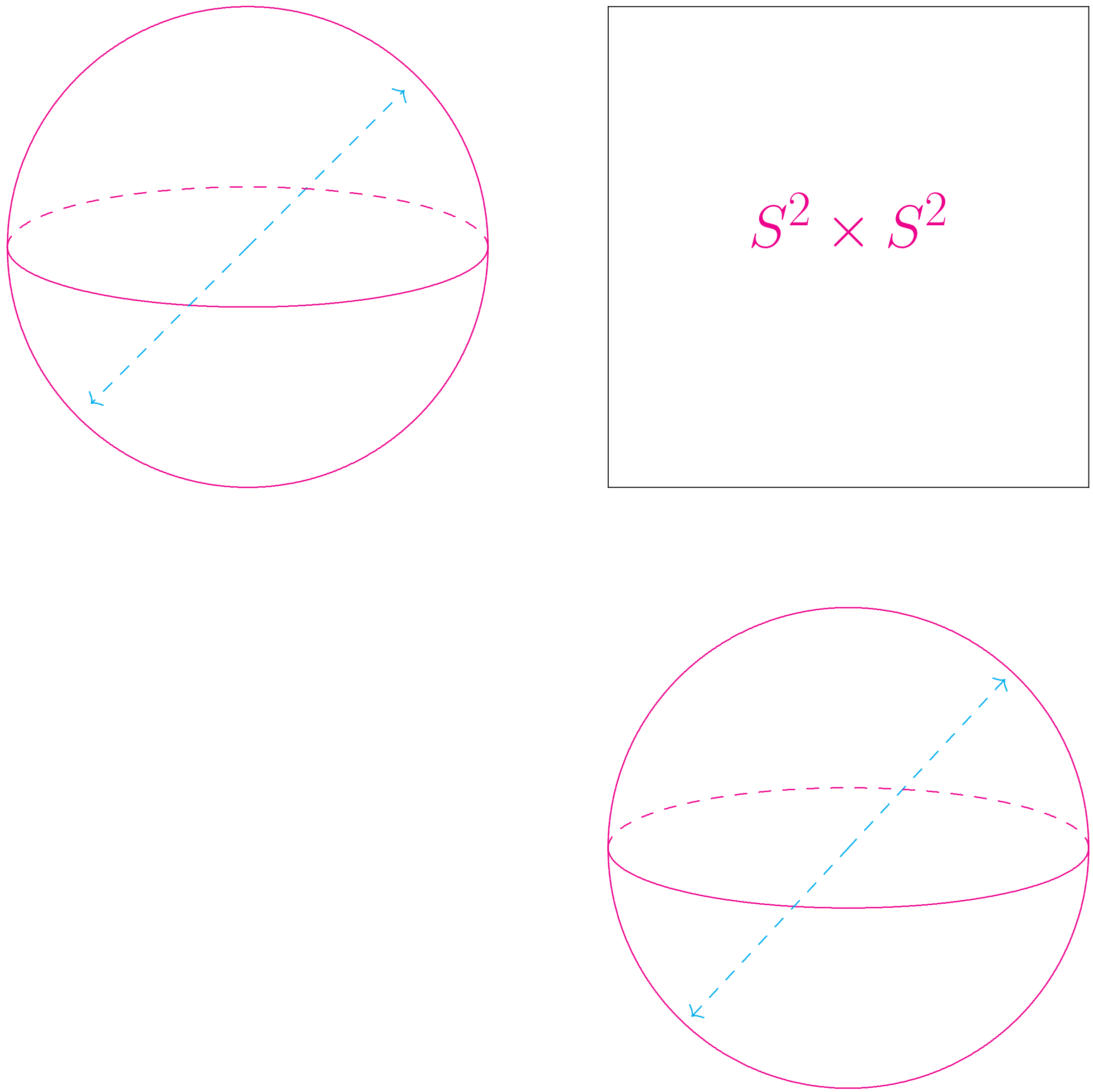
Excludes 5 types with $\pi_1 = \mathbb{Z}_2$ and $b_+(M) = 0.$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Can also understand these by same methods.

Thanks for the invitation!

It’s a pleasure to be here!