Mass in

Kähler Geometry

Claude LeBrun
Stony Brook University

Differential Geometry in the Large
Firenze, Italia, 2016/07/15

Joint work with

Joint work with

Hans-Joachim Hein
University of Maryland

Joint work with

Hans-Joachim Hein
Fordham University

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e-print: arXiv:1507.08885 [math.DG]

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Now on-line in Comm. Math. Phys.

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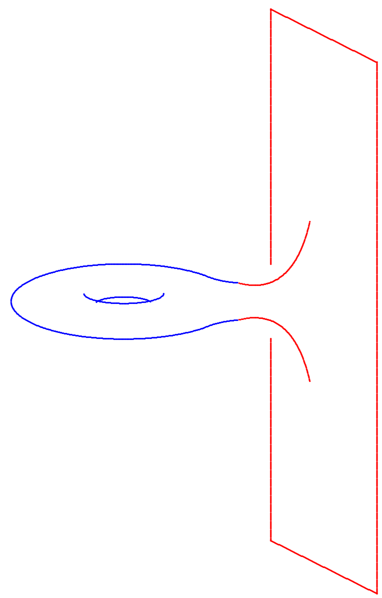
On-line first: Comm. Math. Phys.
doi: 10.1007/200220-016-2661-4

Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g)



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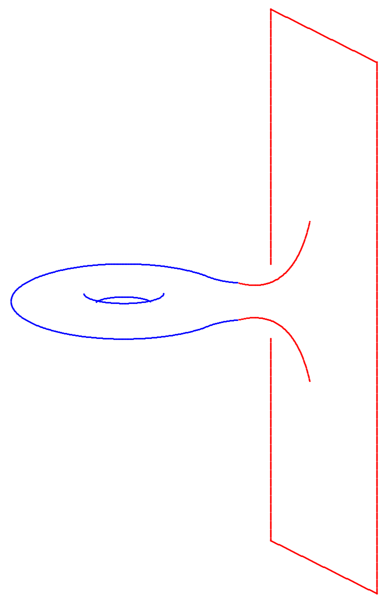

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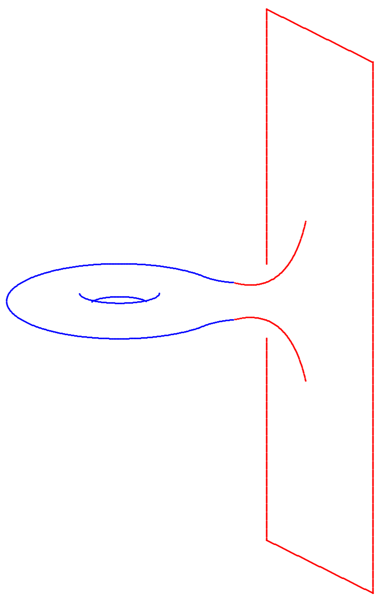

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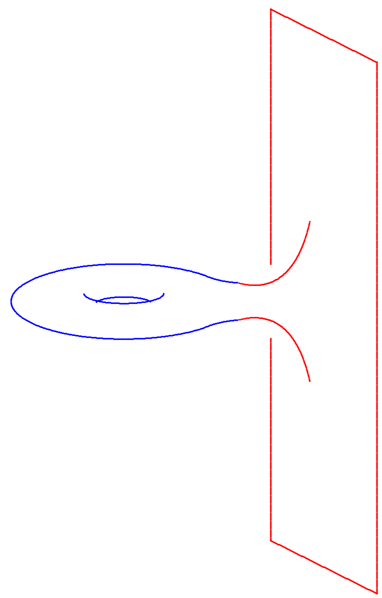

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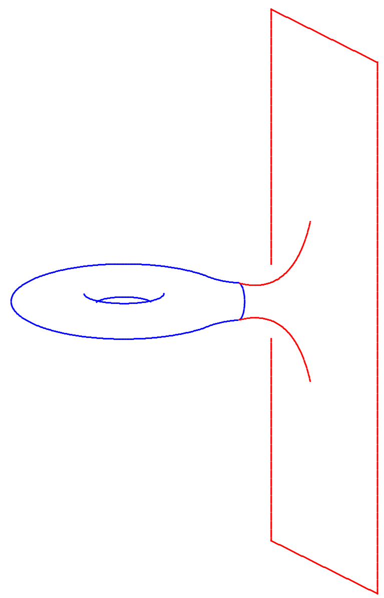

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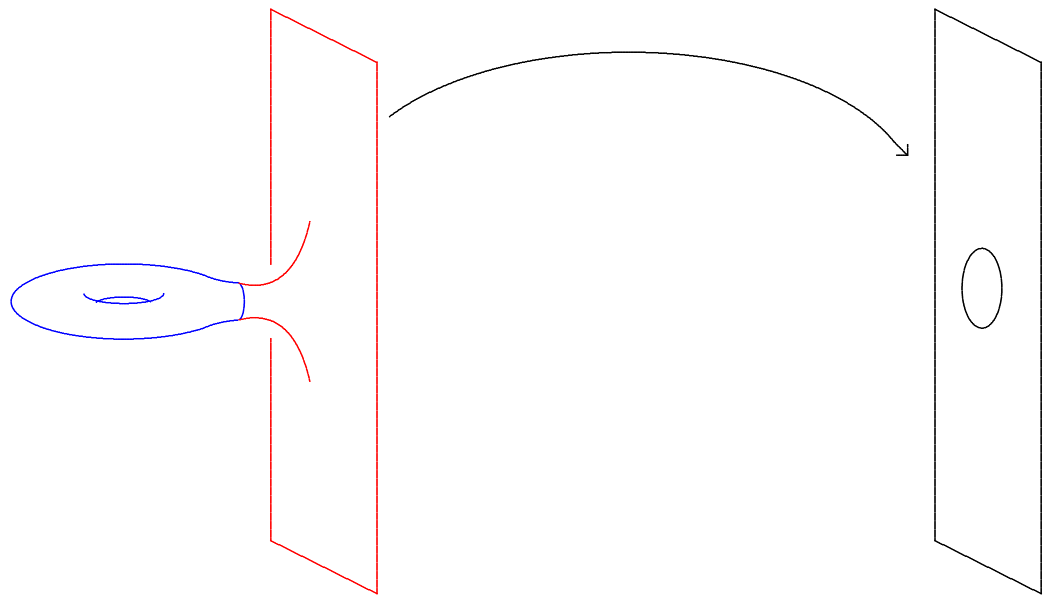

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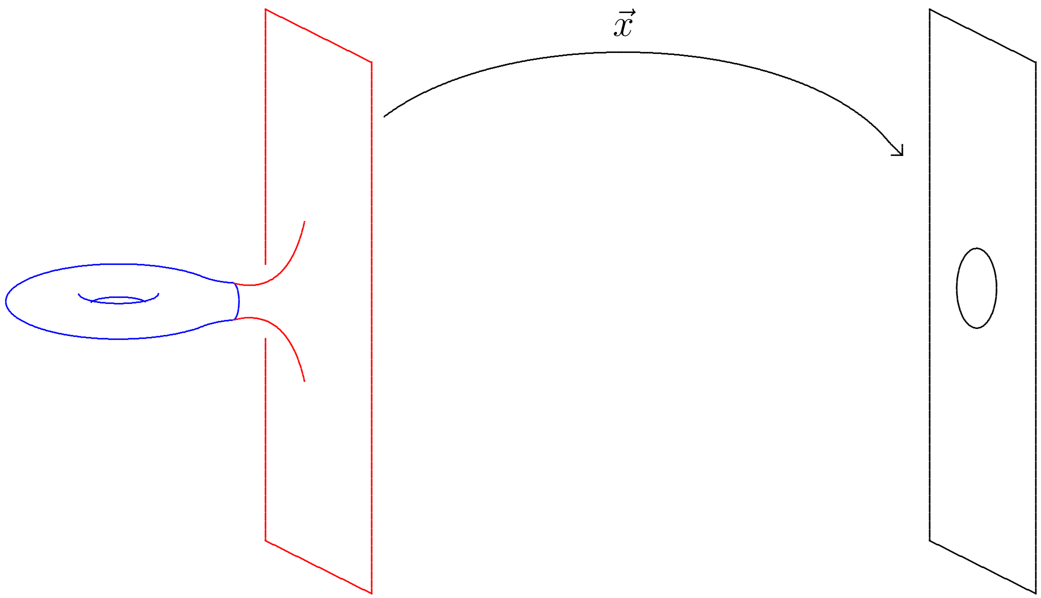

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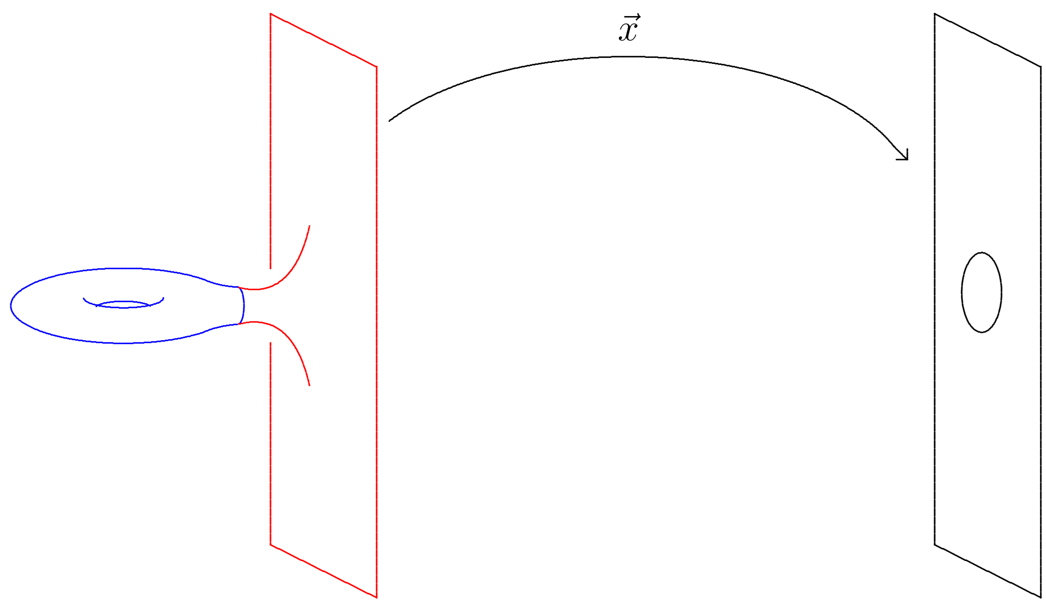

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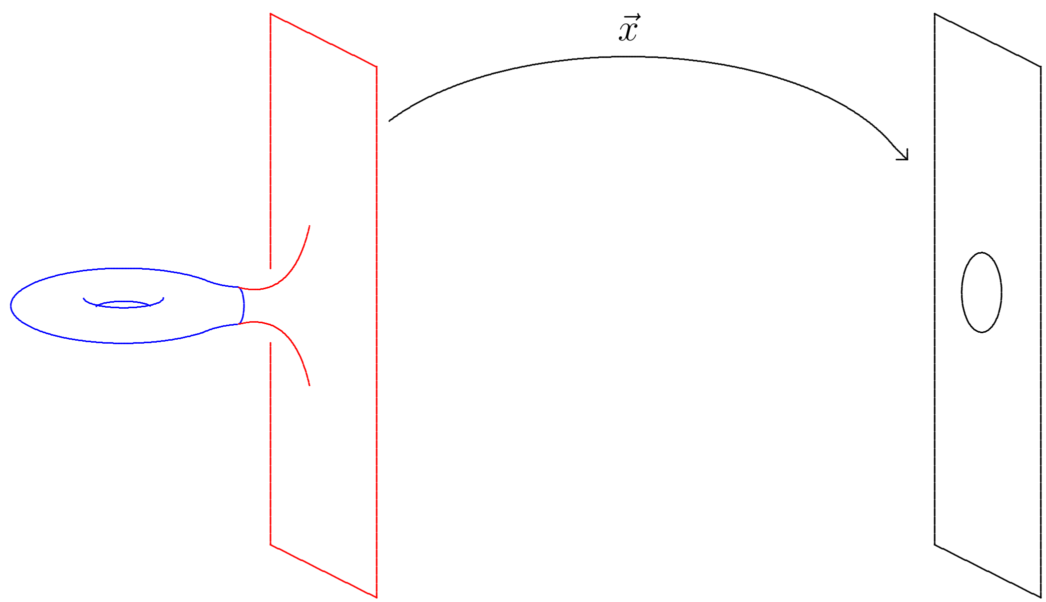

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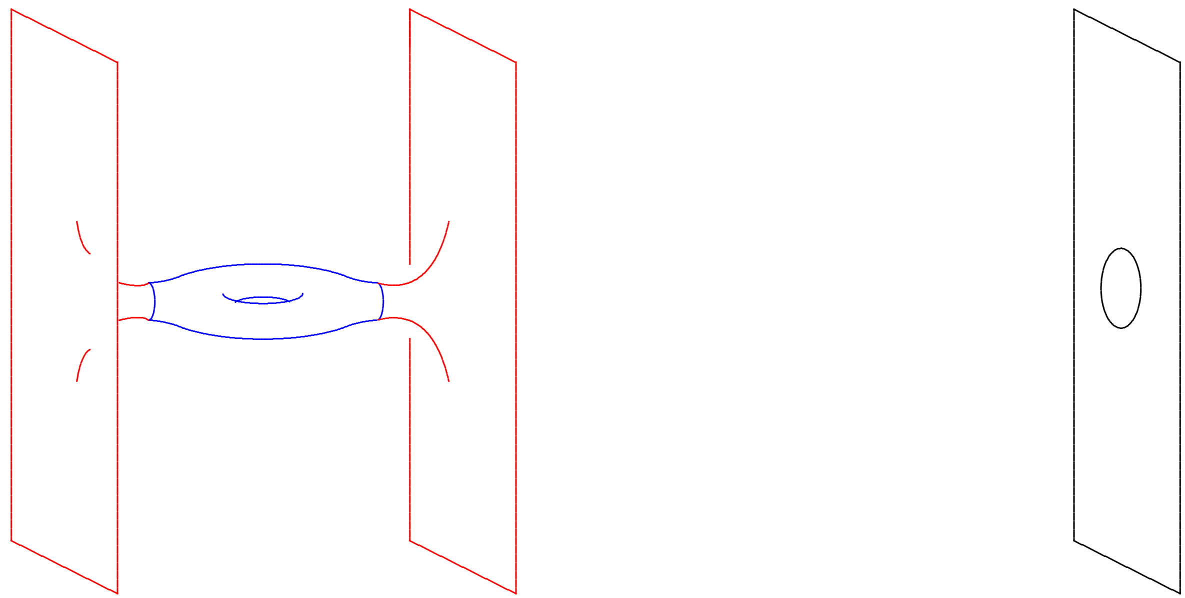

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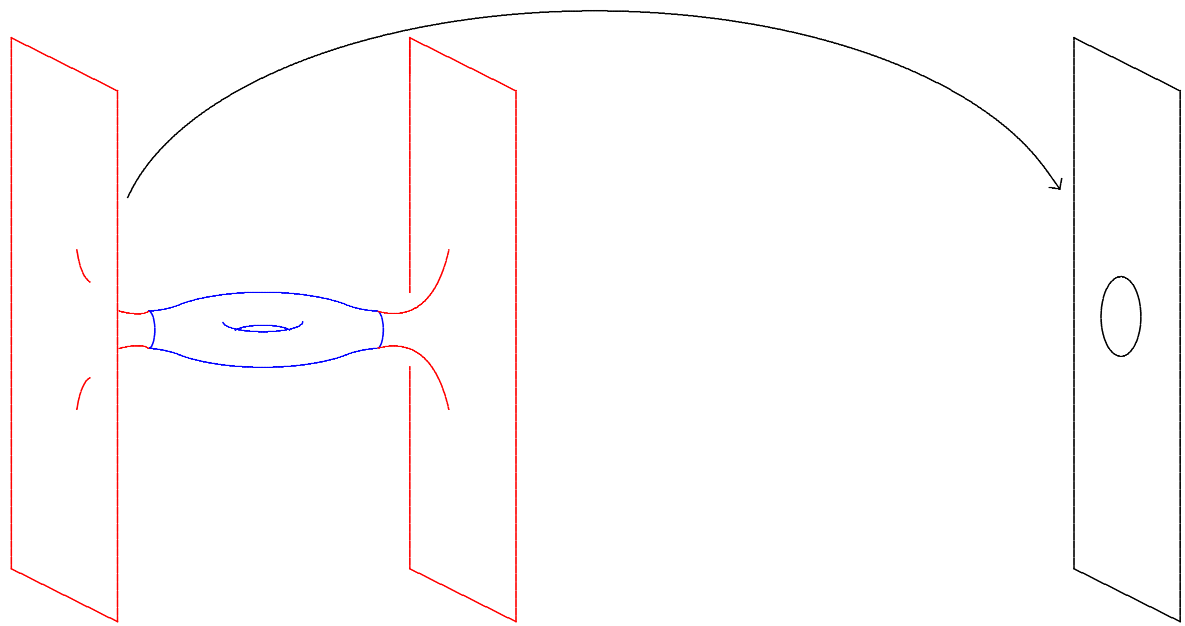

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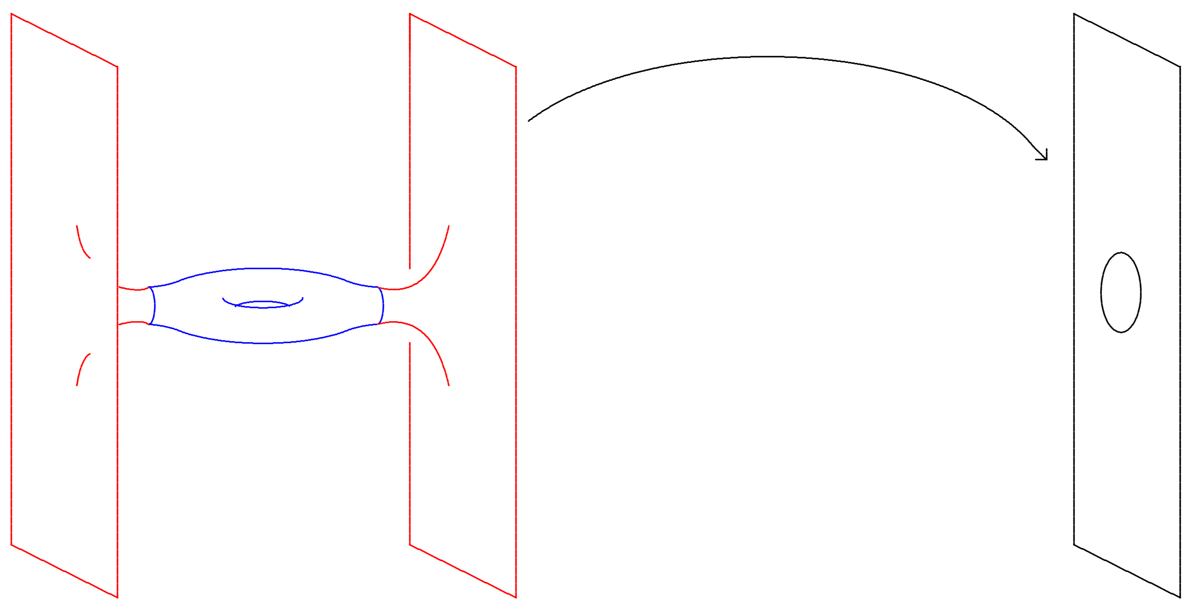

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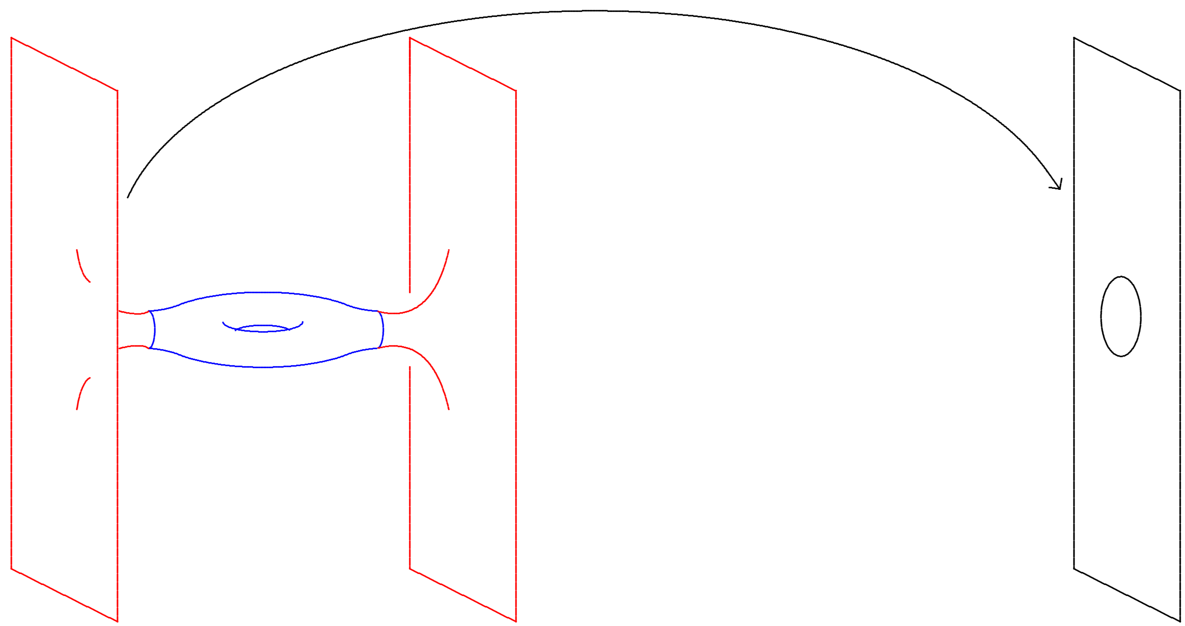

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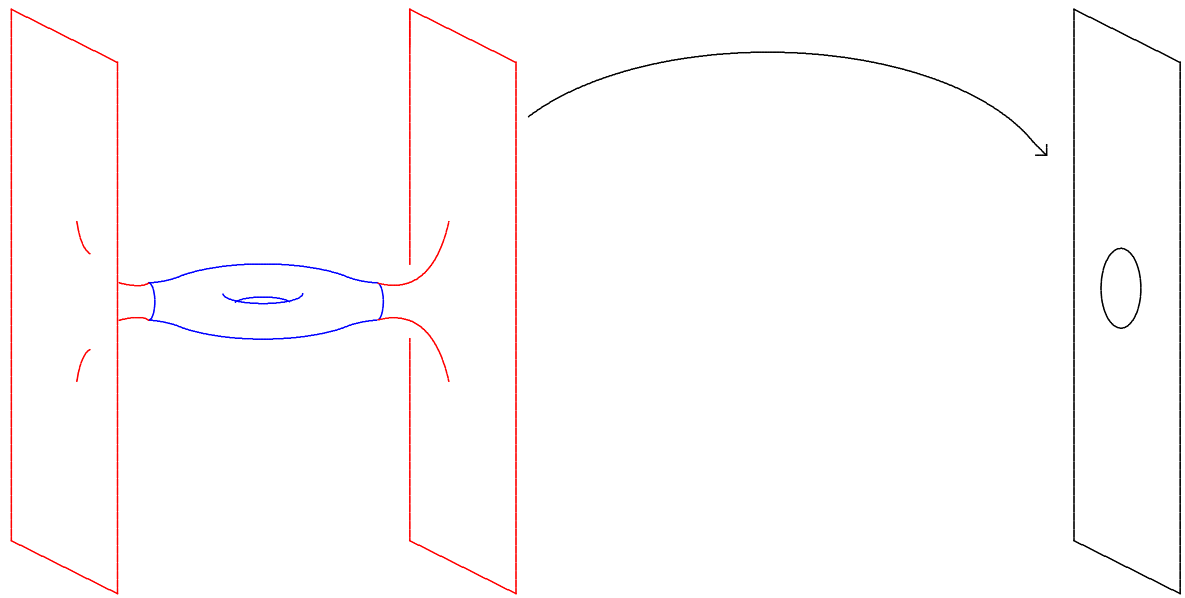

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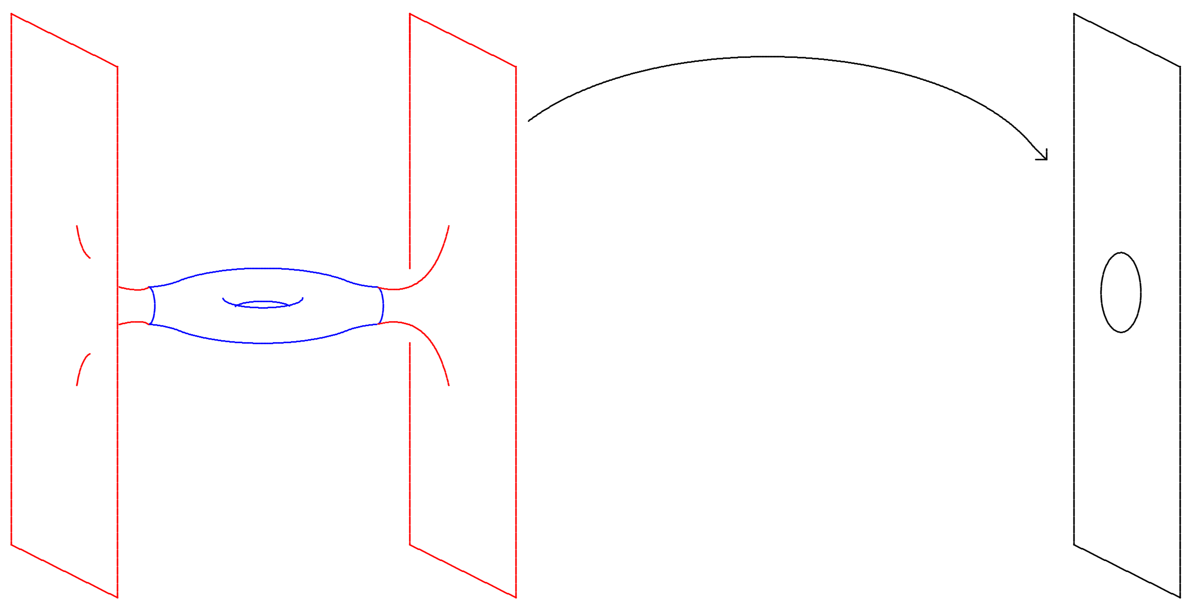

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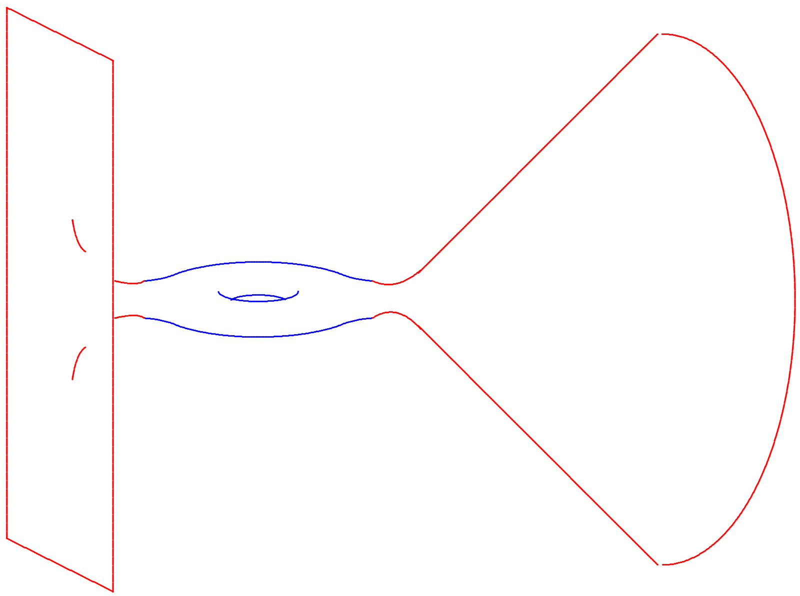

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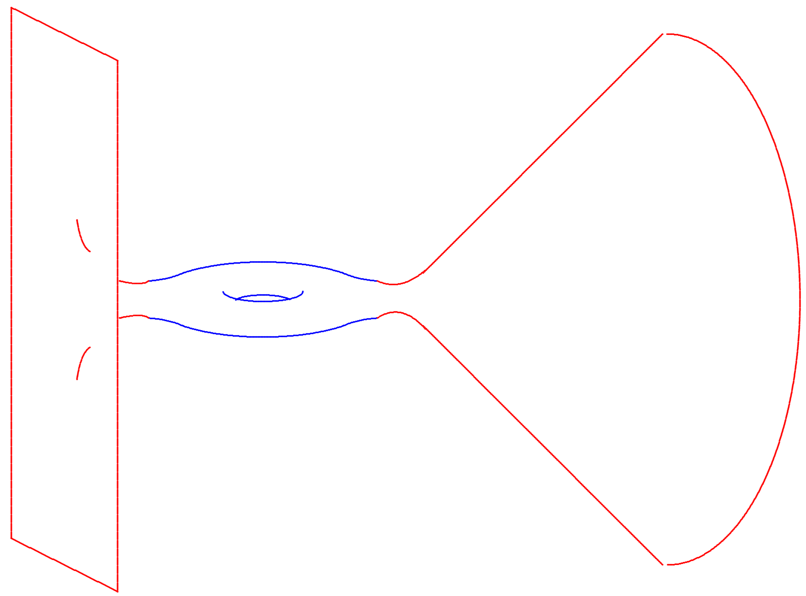

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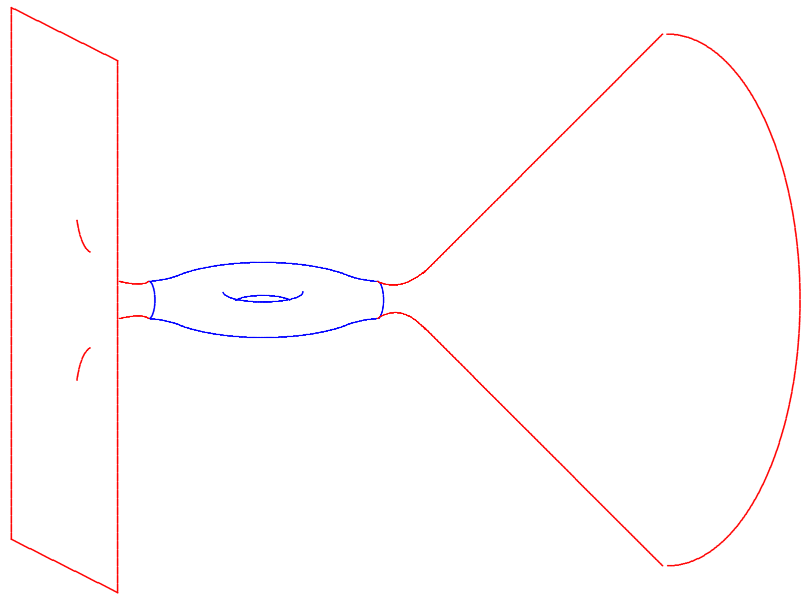

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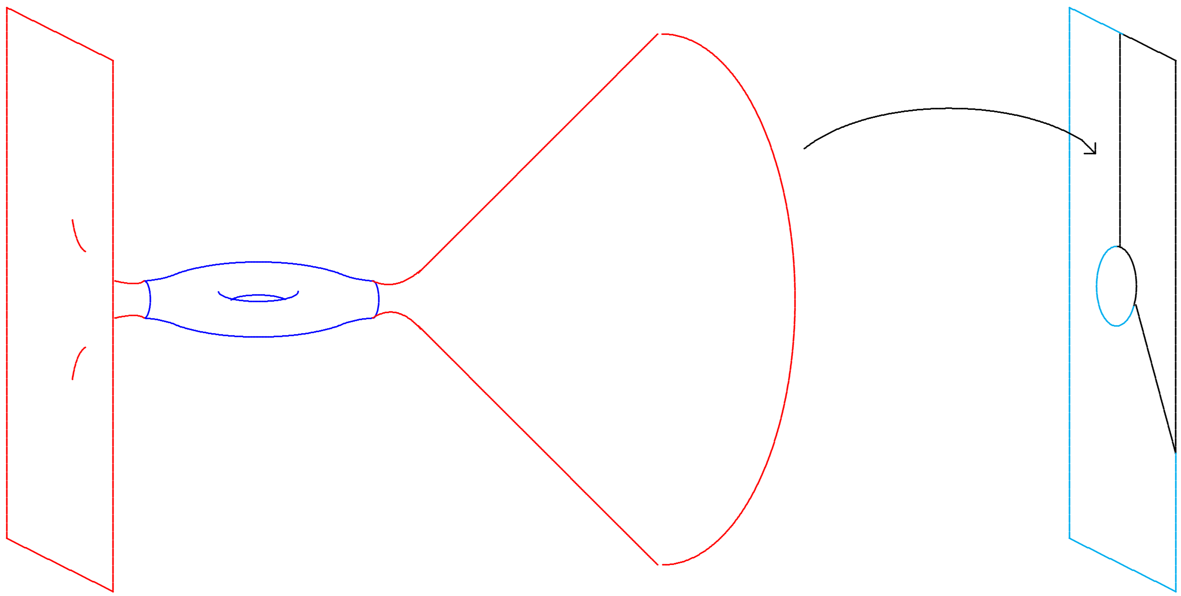

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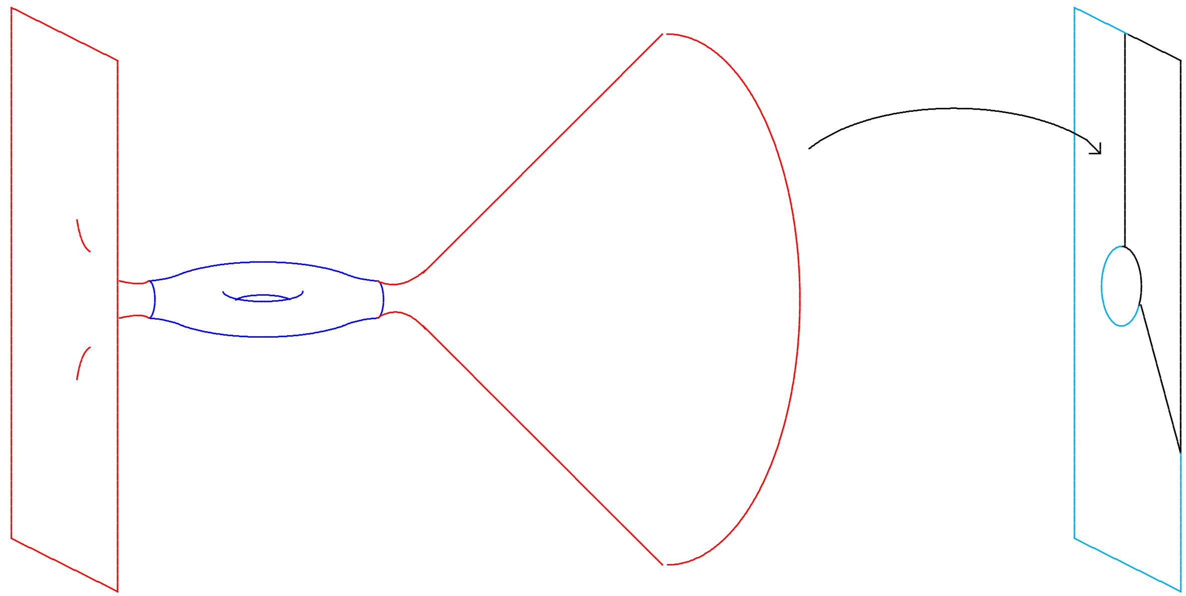

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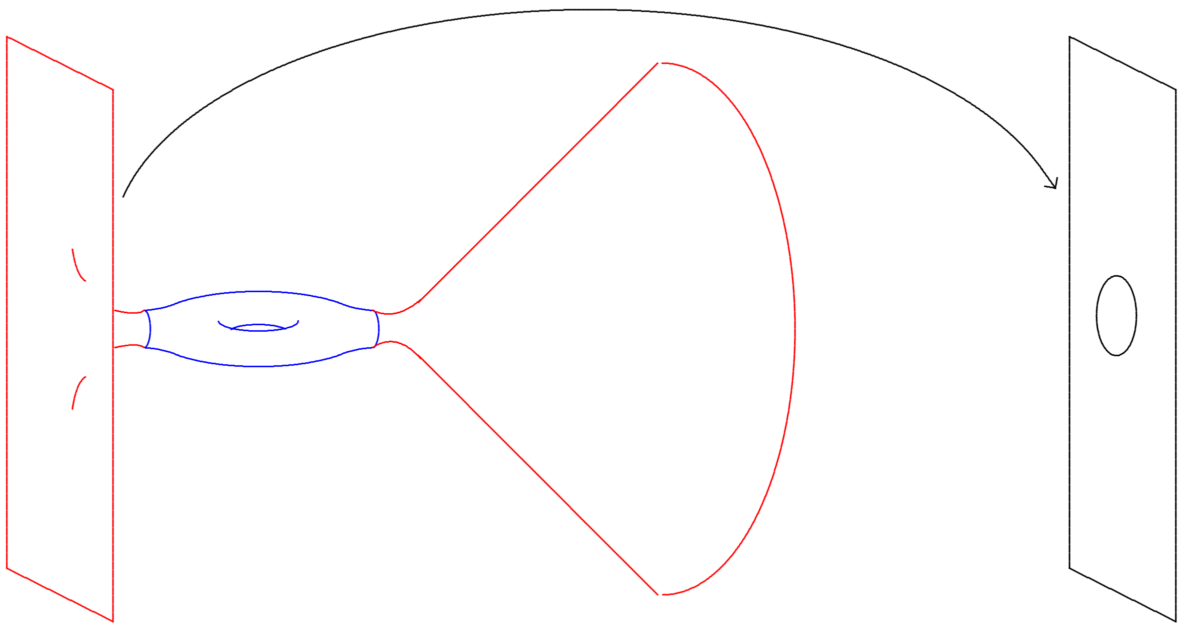

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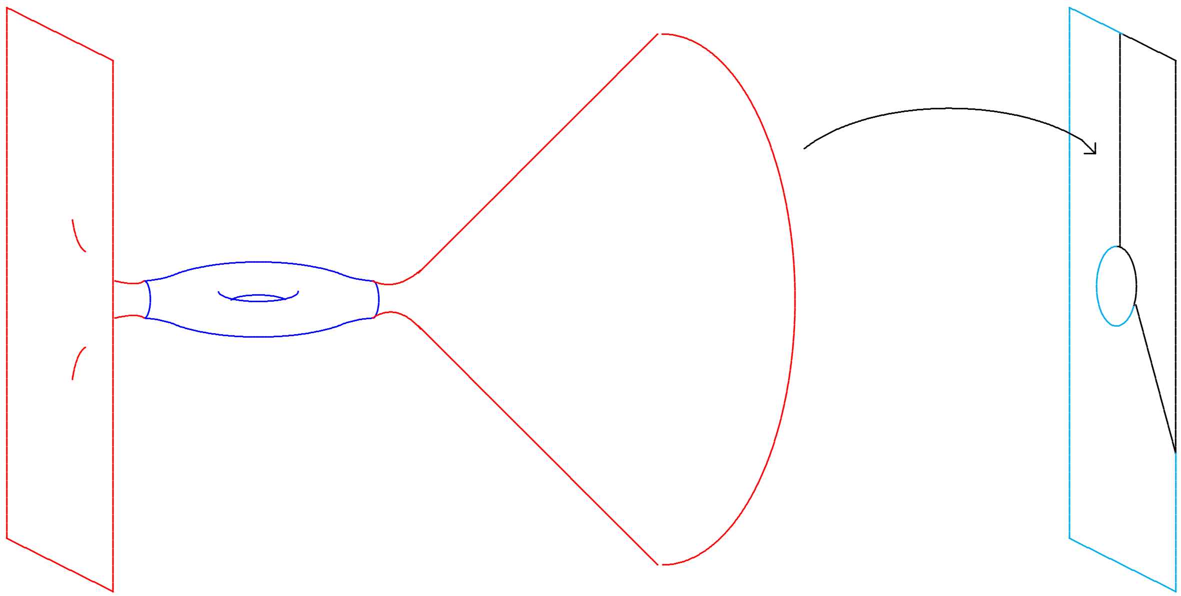

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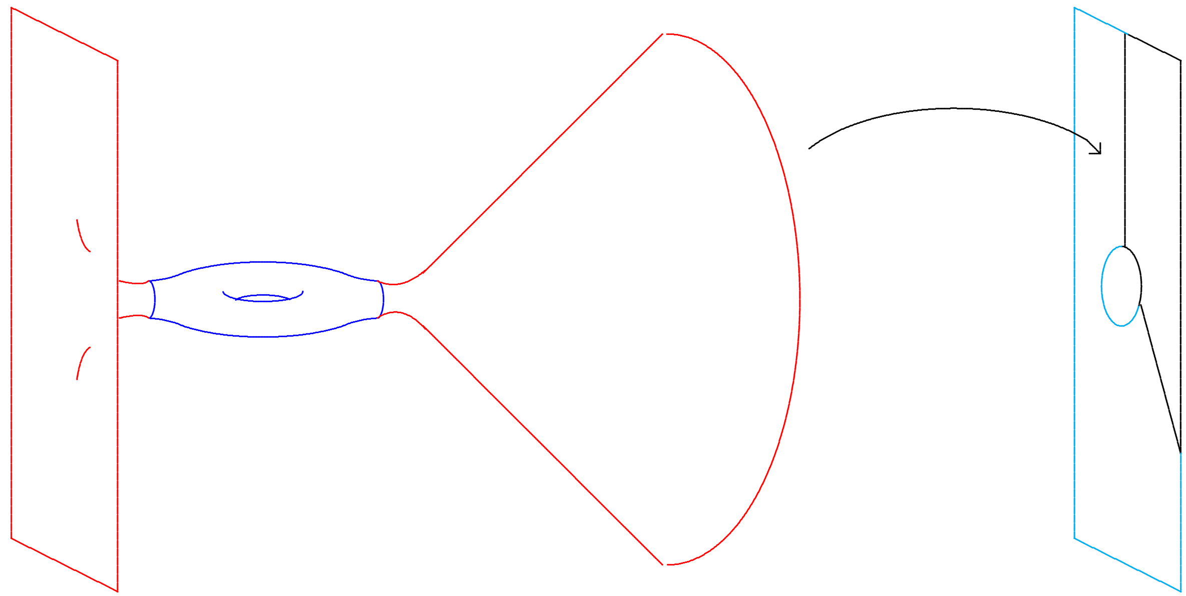

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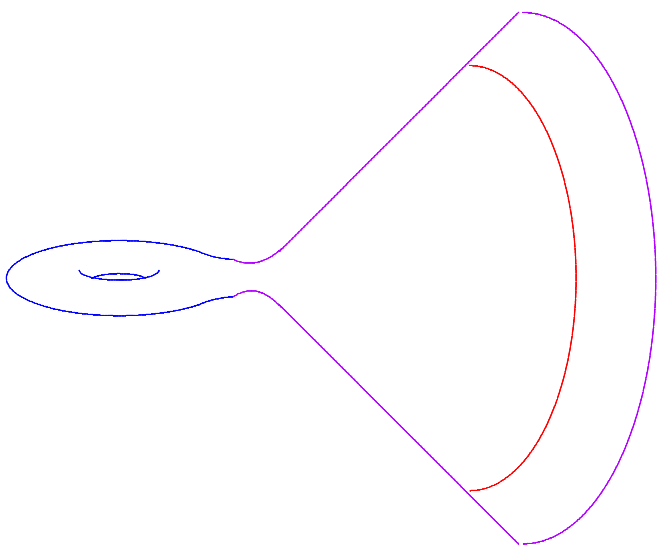

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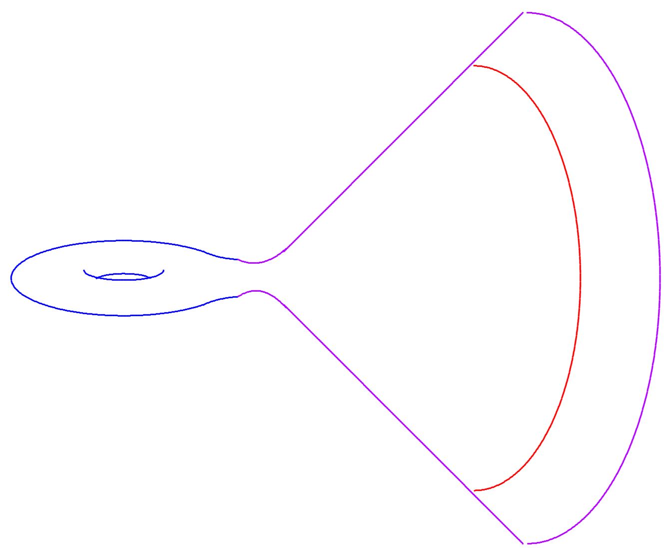

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Seems to depend on choice of coordinates!

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Chruściel-type fall-off:

$$g_{jk} - \delta_{jk} \in C^1_{-\tau}, \quad \tau > \frac{n-2}{2}$$

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- ν *is the outpointing Euclidean unit normal; and*
- α_E *is the volume* $(n-1)$*-from induced by the Euclidean metric.*

Bartnik/Chruściel (1986): With weak fall-off conditions, the mass is well-defined & coordinate independent.

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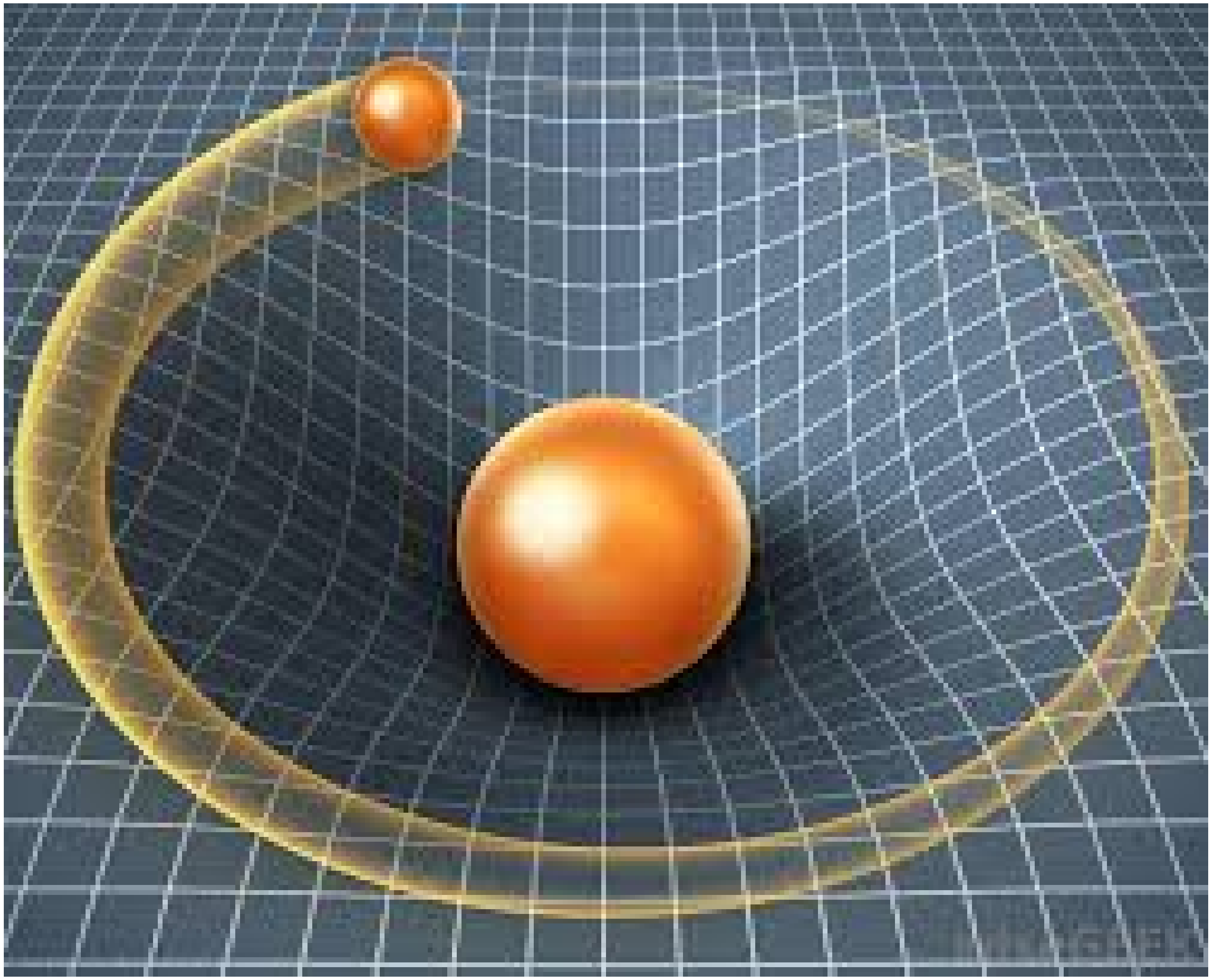

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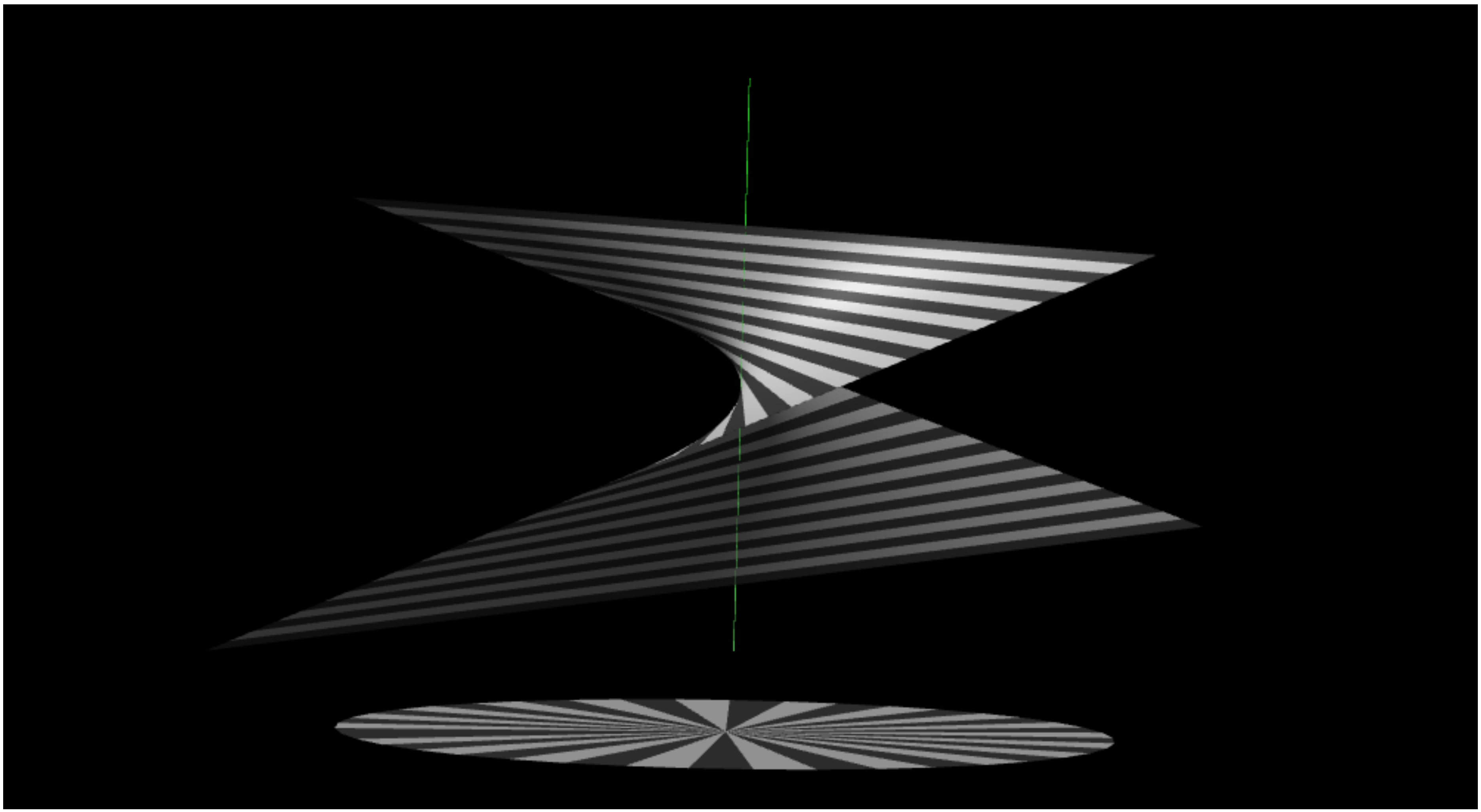

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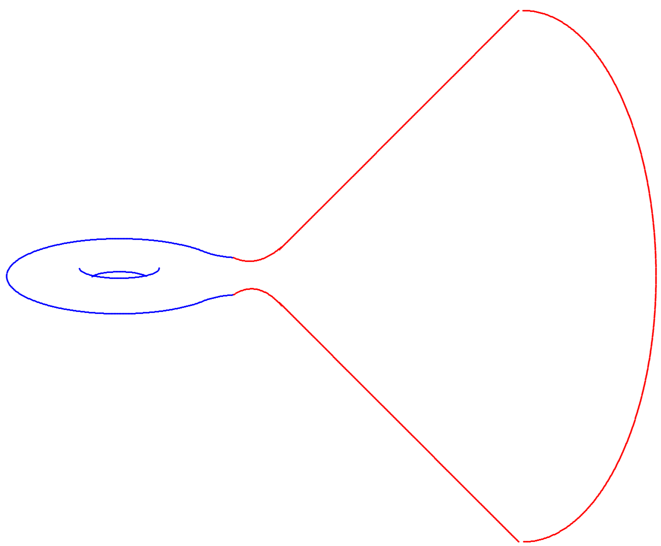

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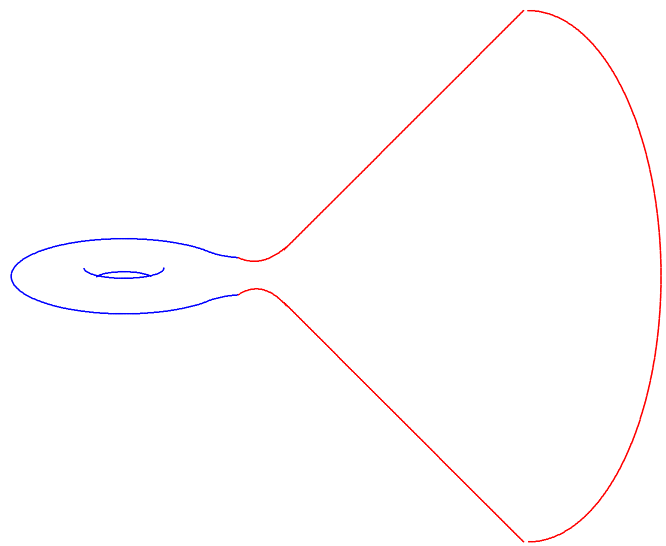

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Does not depend on the choice of an end!

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Theorem A.

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

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(Discovered independently by Rollin, Singer, & Şuvaina, using different methods.)

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So the mass is a "boundary correction" to the topological formula for the total scalar curvature.

Theorem C. *Any ALE Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

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Corollary. *Any ALE scalar-flat Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

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So **Theorem A** is an immediate consequence!

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$$g^{jk}\left(g_{j\ell,k} - g_{jk,\ell}\right)\nu^\ell \alpha_E = -\star d \log\left(\sqrt{\det g}\right) + O(\varrho^{-3-\varepsilon}).$$

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$$d(\theta\wedge\omega) = \rho\wedge\omega = \frac{s}{4}\omega^2 = 0.$$

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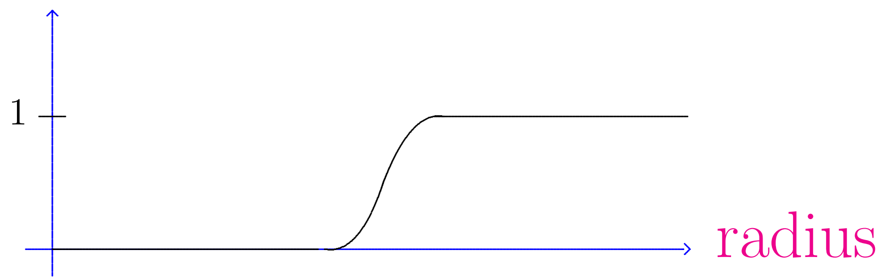

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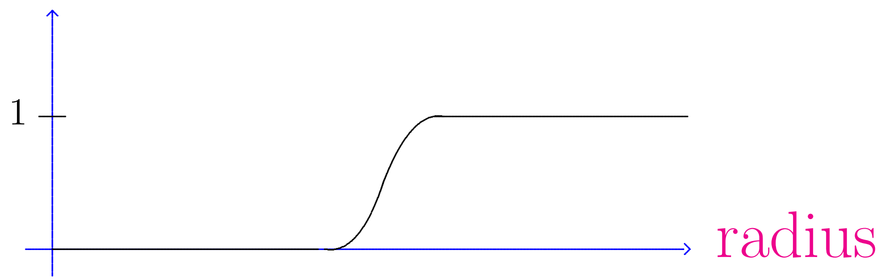

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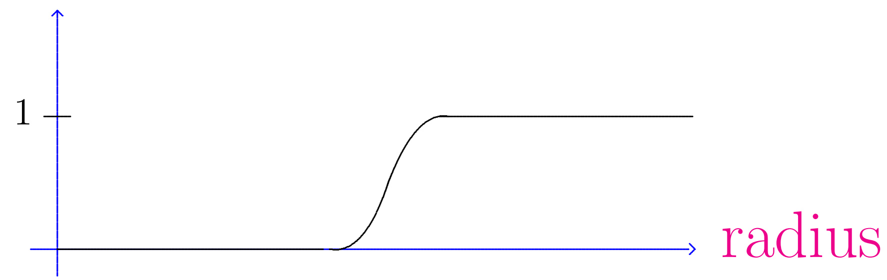

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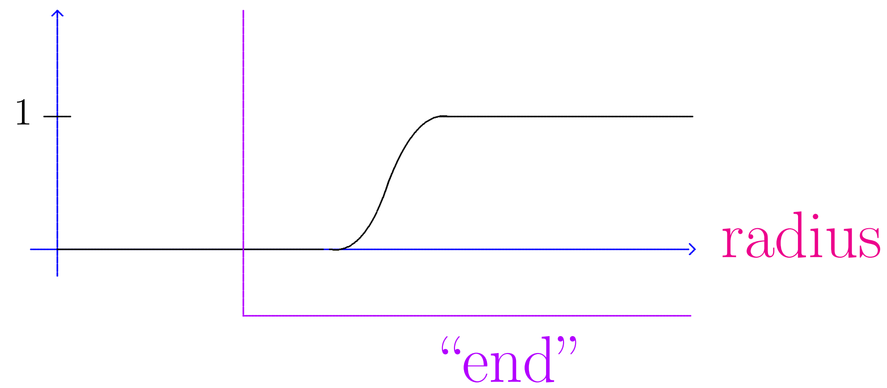

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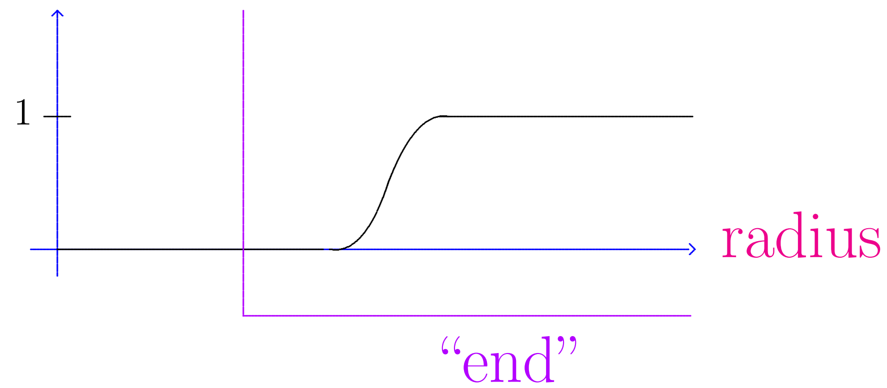

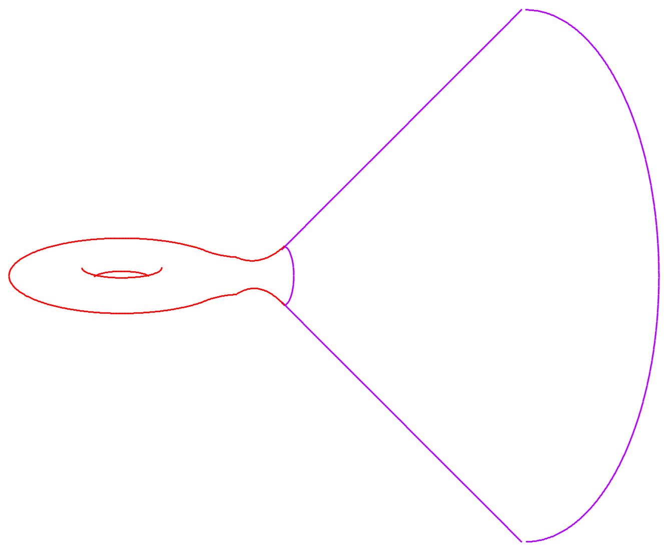

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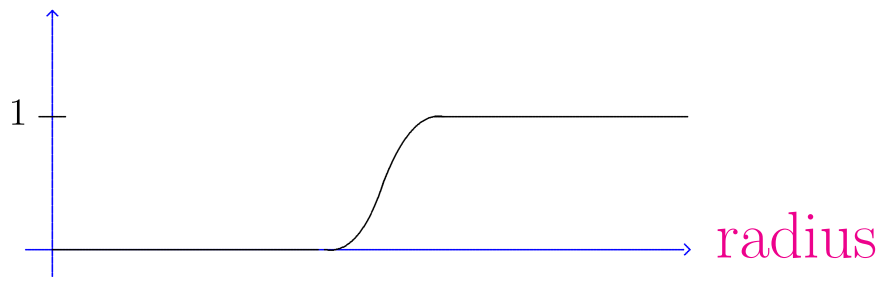

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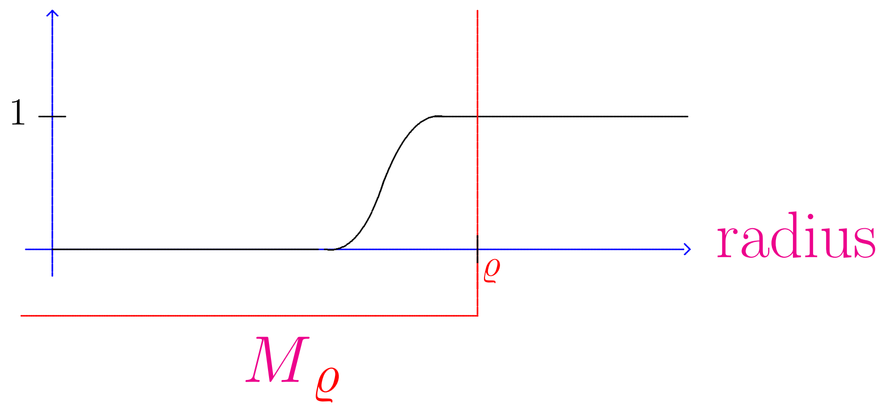

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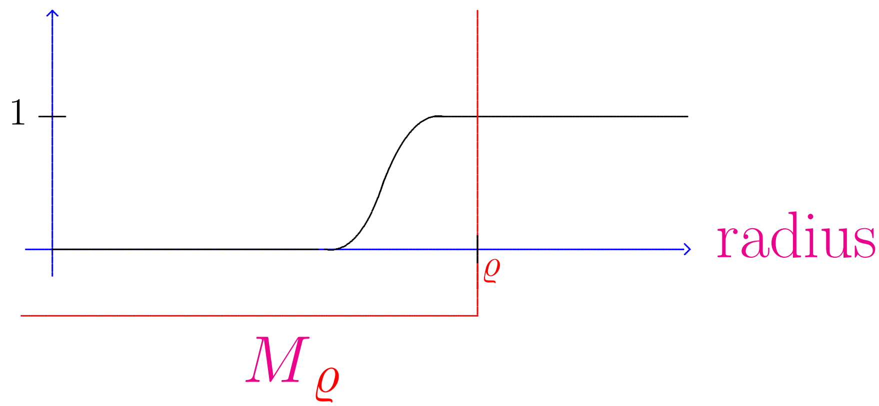

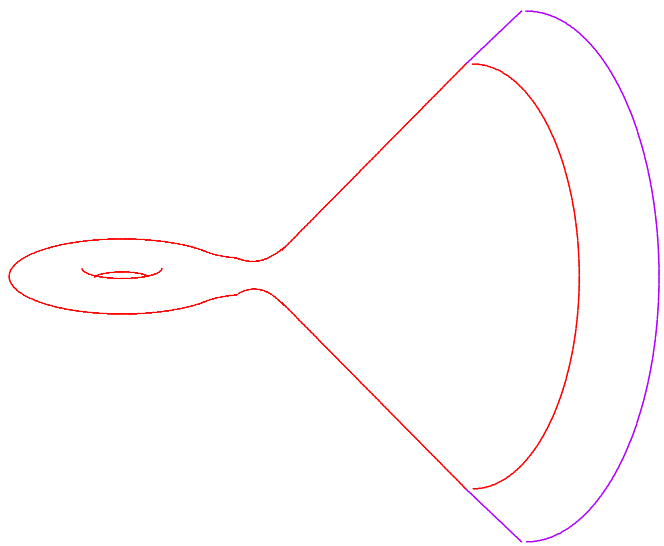

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Compactly supported, because $d\theta = \rho$ near infinity.

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$$[\psi] = \clubsuit([\rho]) = 2\pi\clubsuit(c_1) \in H^2_c(M)$$

$$\langle 2\pi\clubsuit(c_1), \omega\rangle = -\int_{S_\varrho/\Gamma} \theta \wedge \omega$$

by Stokes' theorem.

So

$$m(M, g) = -\frac{1}{3\pi}\langle\clubsuit(c_1), [\omega]\rangle$$

Let $f : M \to \mathbb{R}$ be smooth cut-off function:

$\equiv 0$ away from end,

$\equiv 1$ near infinity.

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as claimed.

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- $m = 2$;
- $s \equiv 0$; and
- Complex structure J standard at infinity.

General case:

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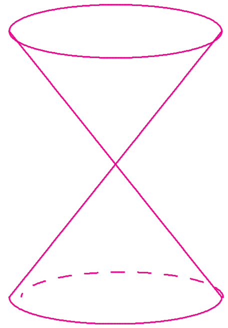

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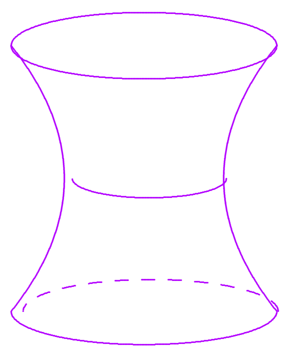

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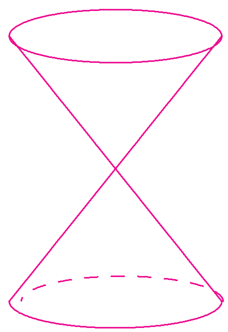

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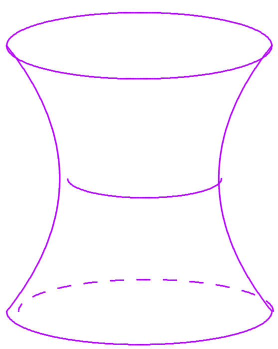

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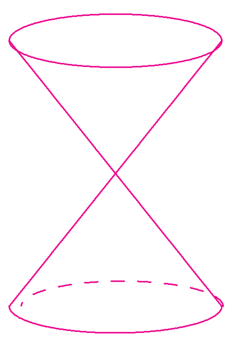

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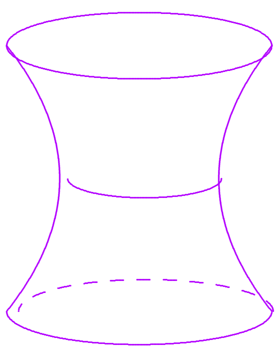
Example: Honda metrics. Scalar-flat Kähler.



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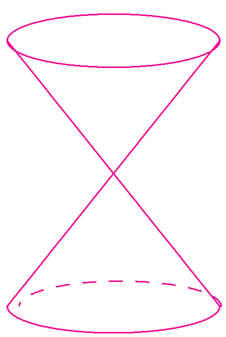

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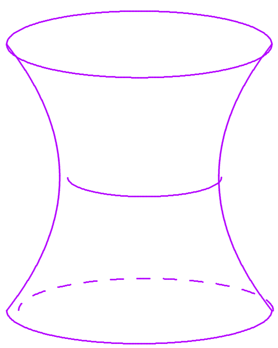
Example: Honda metrics. Deform $\mathcal{O}(-3) \to \mathbb{CP}_1$.



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One argument proceeds by osculation:

$$J = J_0 + O(\varrho^{-3}), \qquad \nabla J = O(\varrho^{-4})$$

in suitable asymptotic coordinates adapted to g.

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This has some interesting consequences...

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Proof actually shows something stronger!

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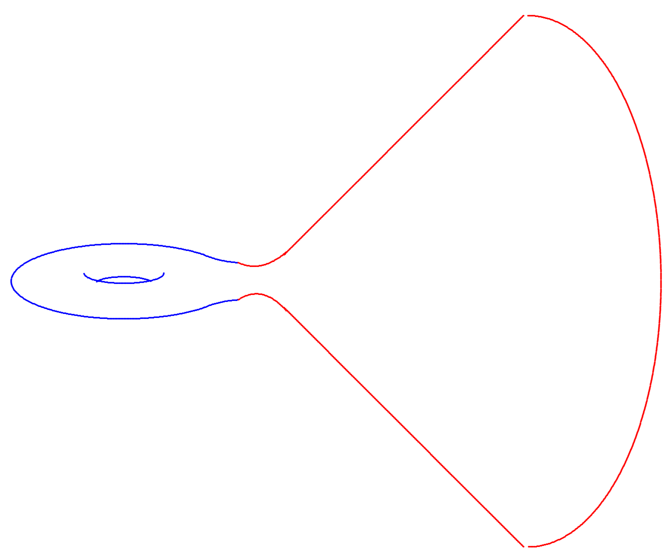
so the mass formula implies the claim.

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g \, d\mu_g$$



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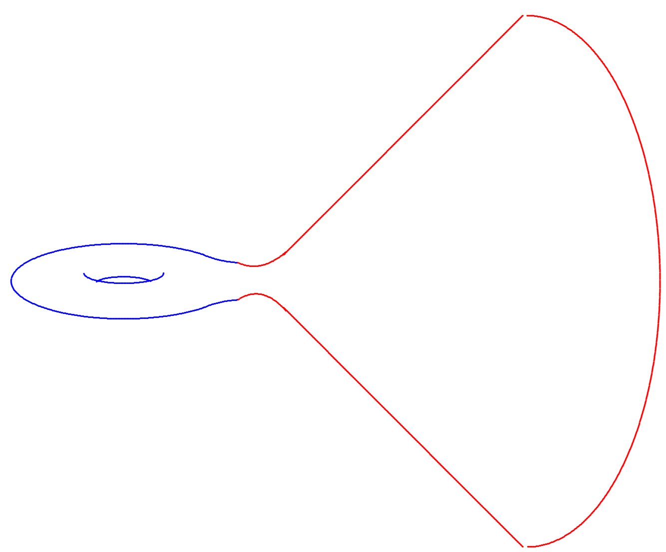

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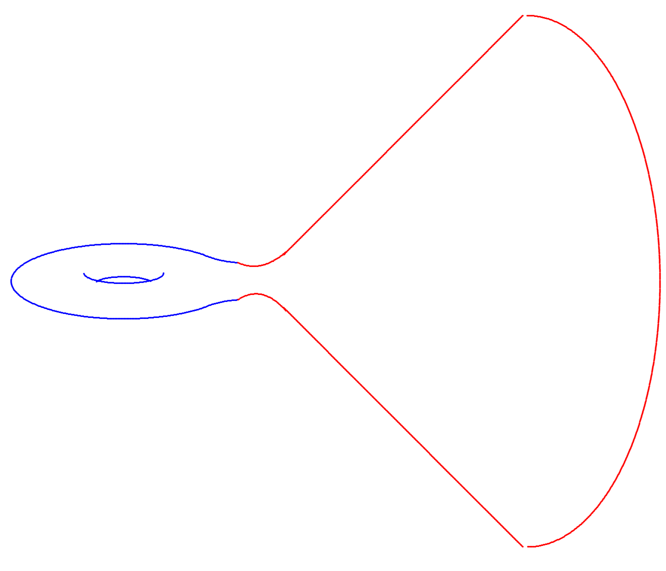

Before ending,

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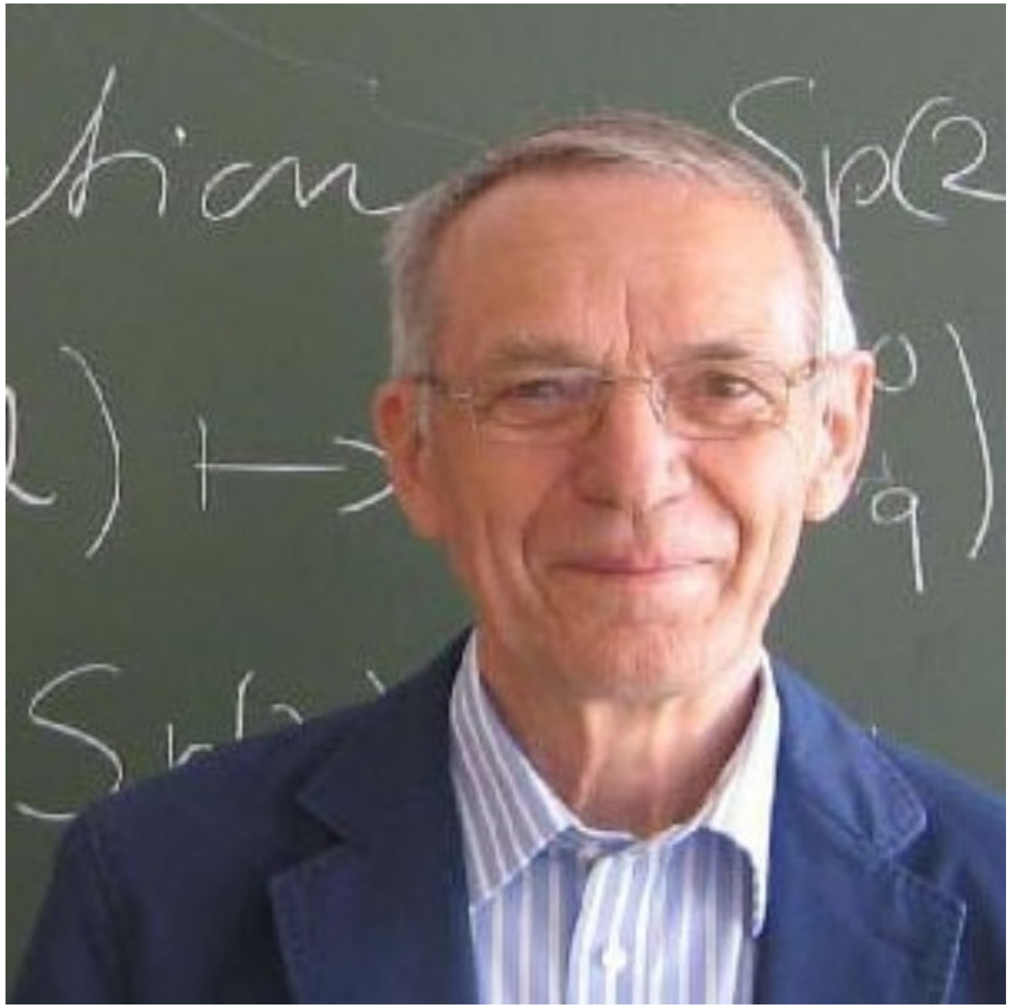
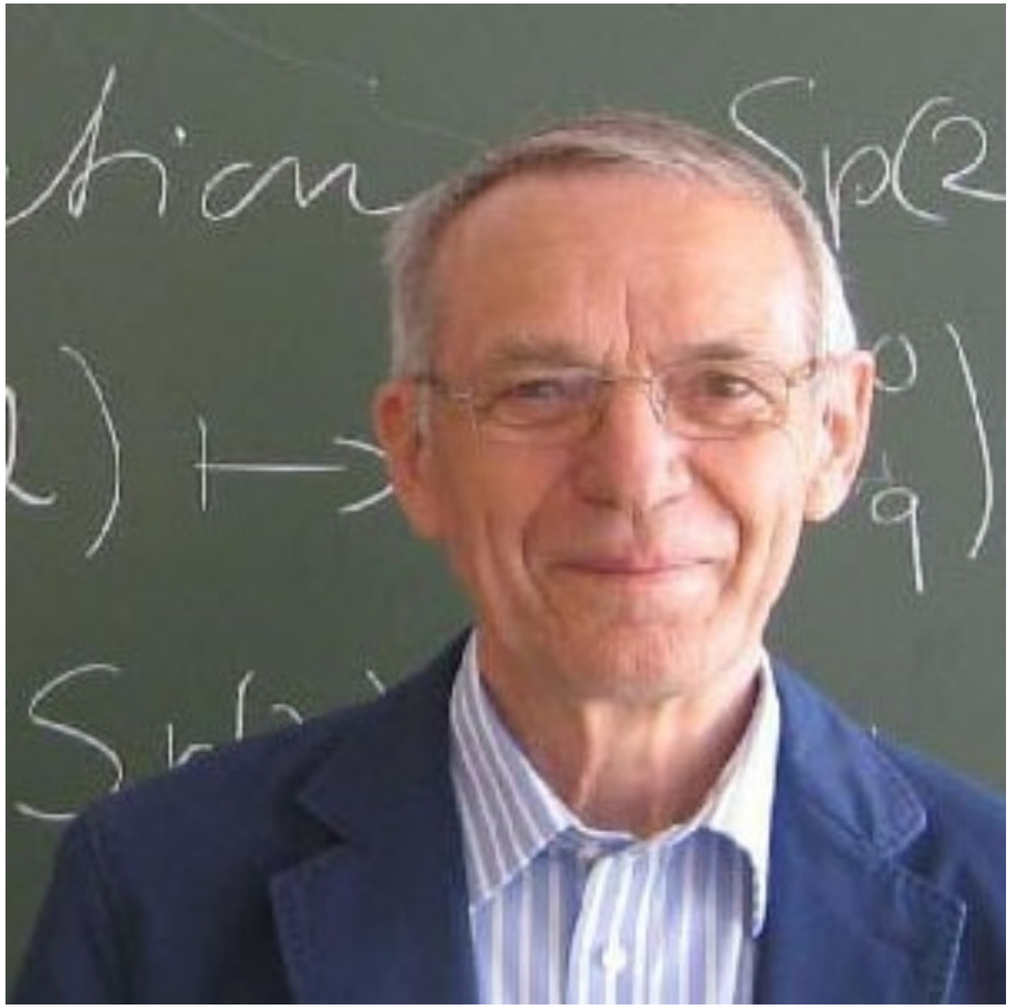


Herzlichen Glückwunsch



zum Geburtstag, Wolfgang!

E grazie a tutti gli organizatori

di questo bellissimo convegno!