Mass, Scalar Curvature, &

Kähler Geometry, III

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Extremal Metrics & Relative K-Stability
Institut Mathématiques de Jussieu
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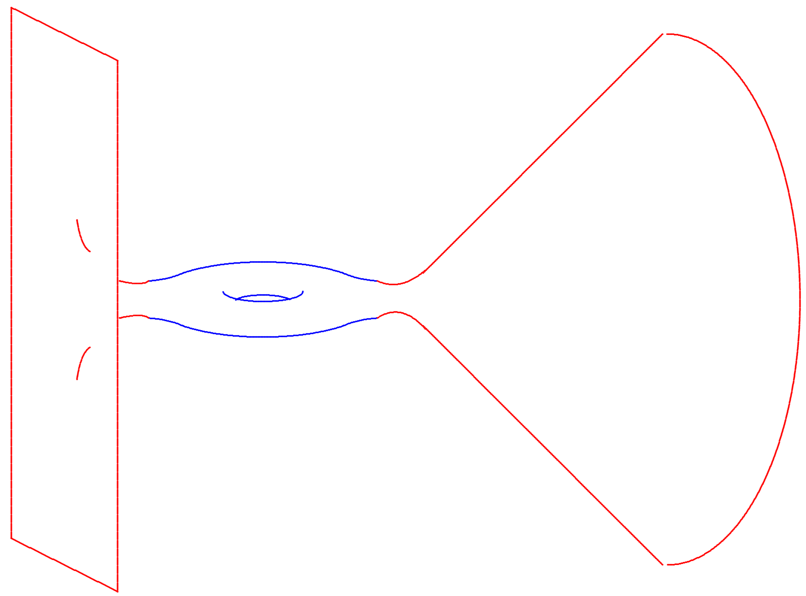
Definition. *Complete, non-compact* n*-manifold* (M^n, g) *is* asymptotically locally Euclidean



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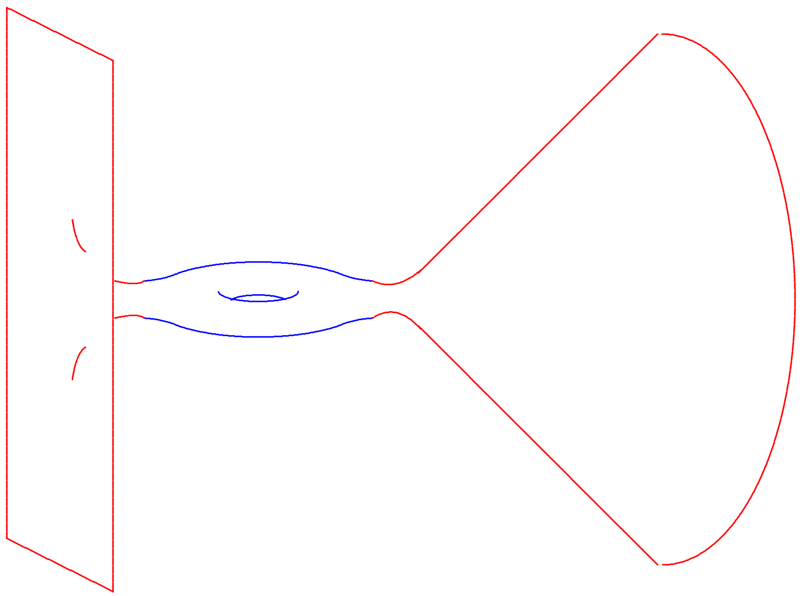

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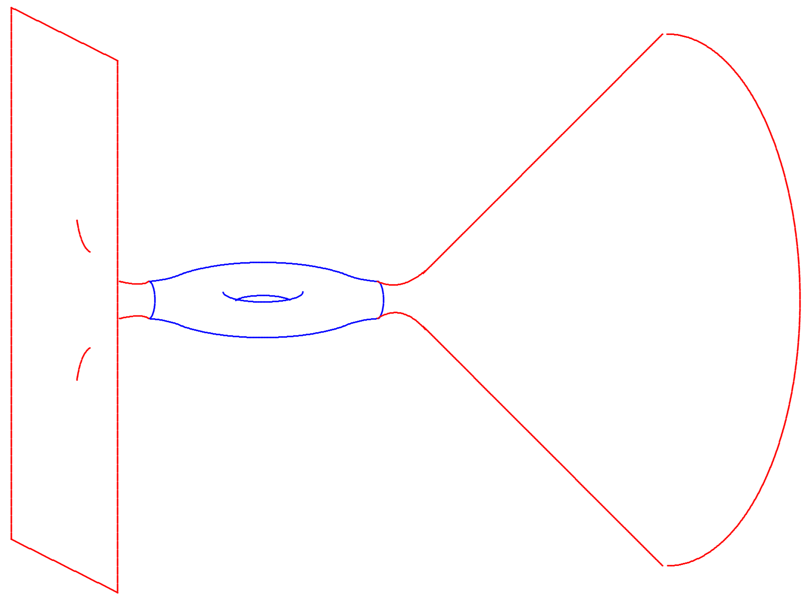

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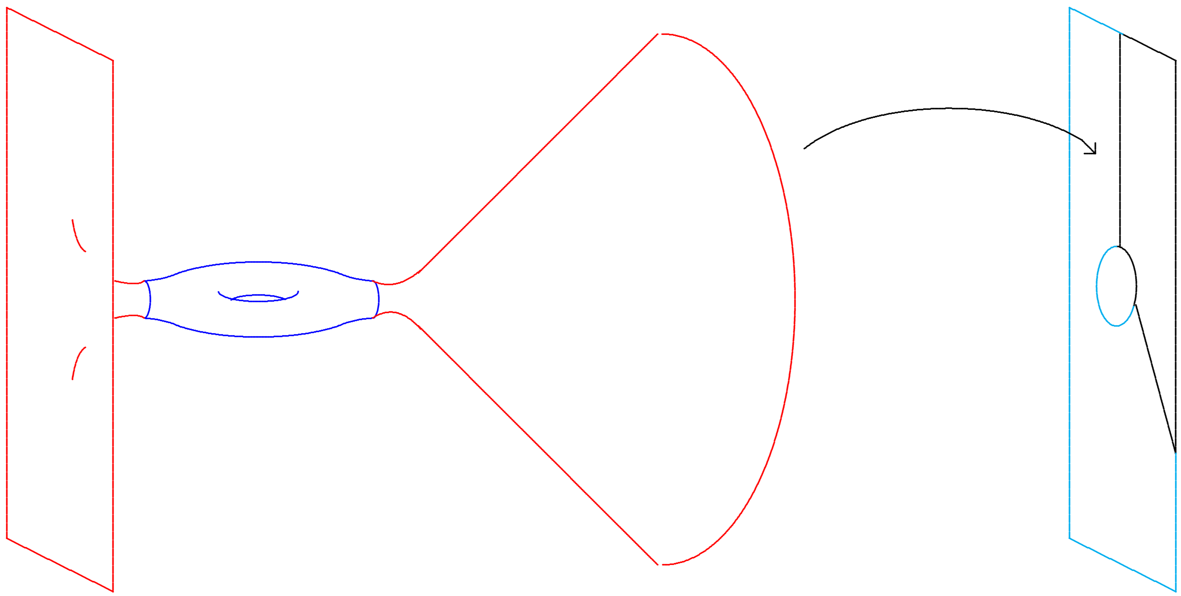

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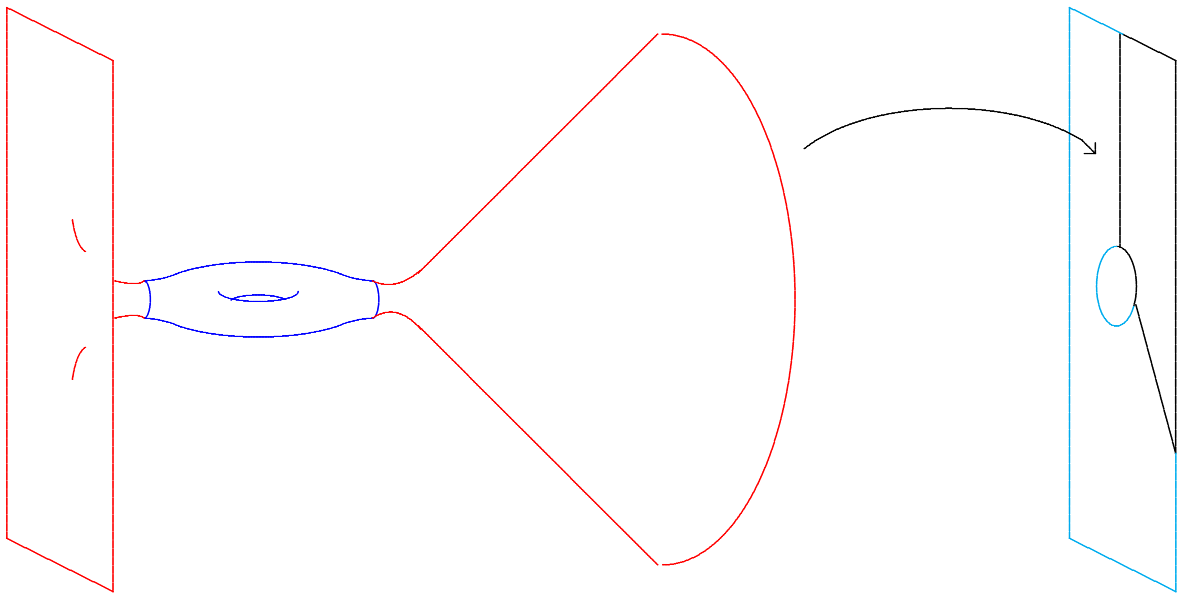

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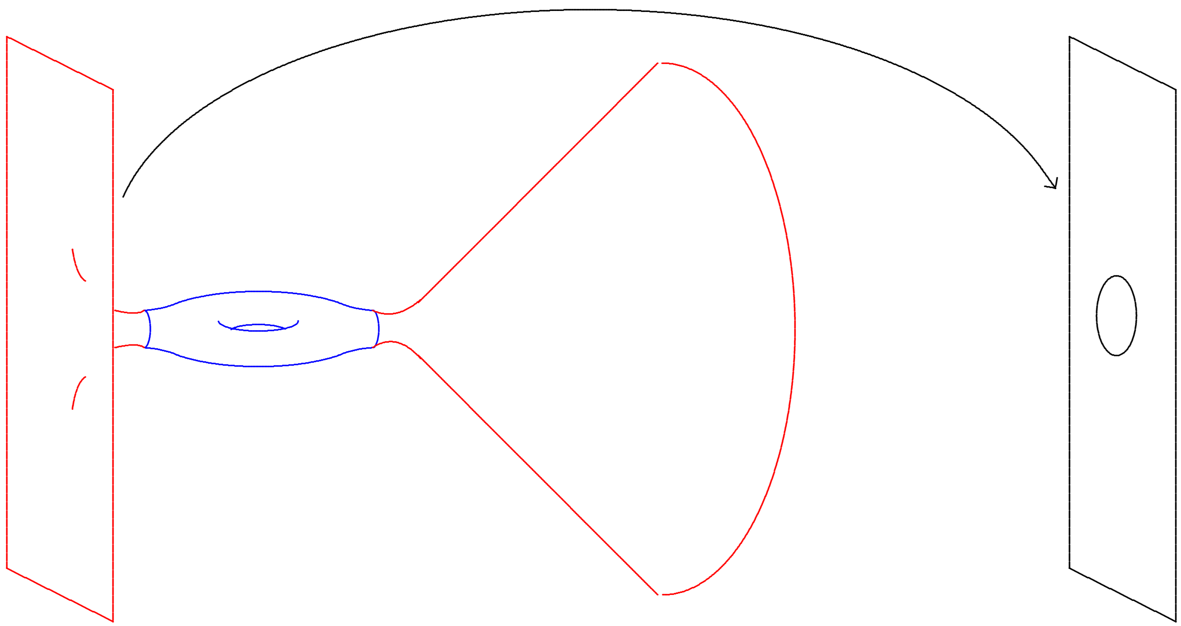

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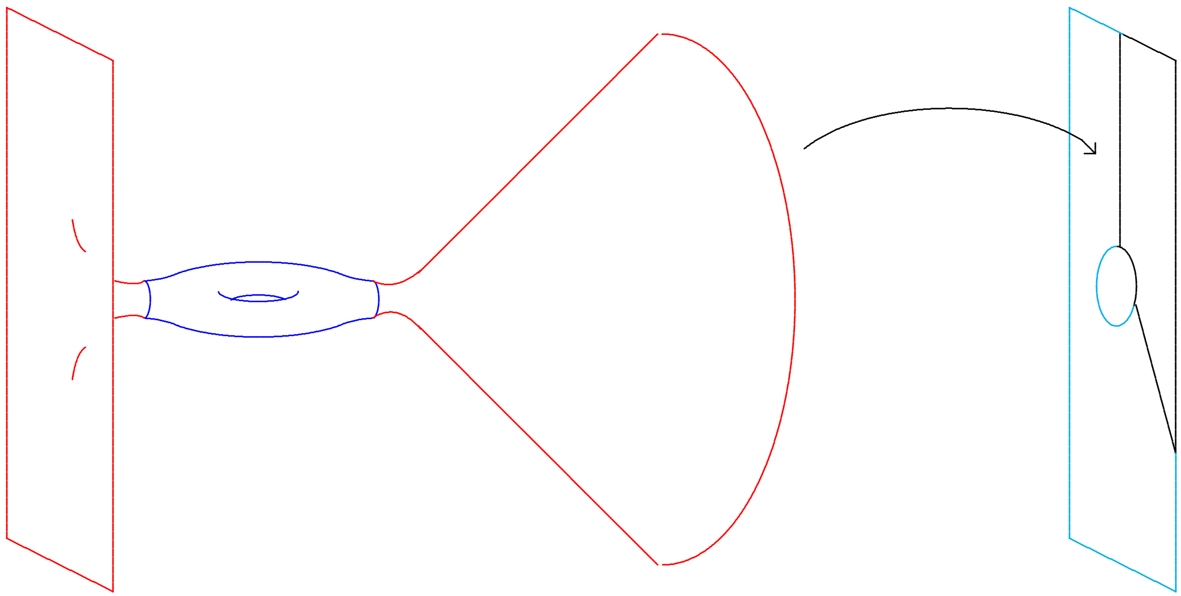

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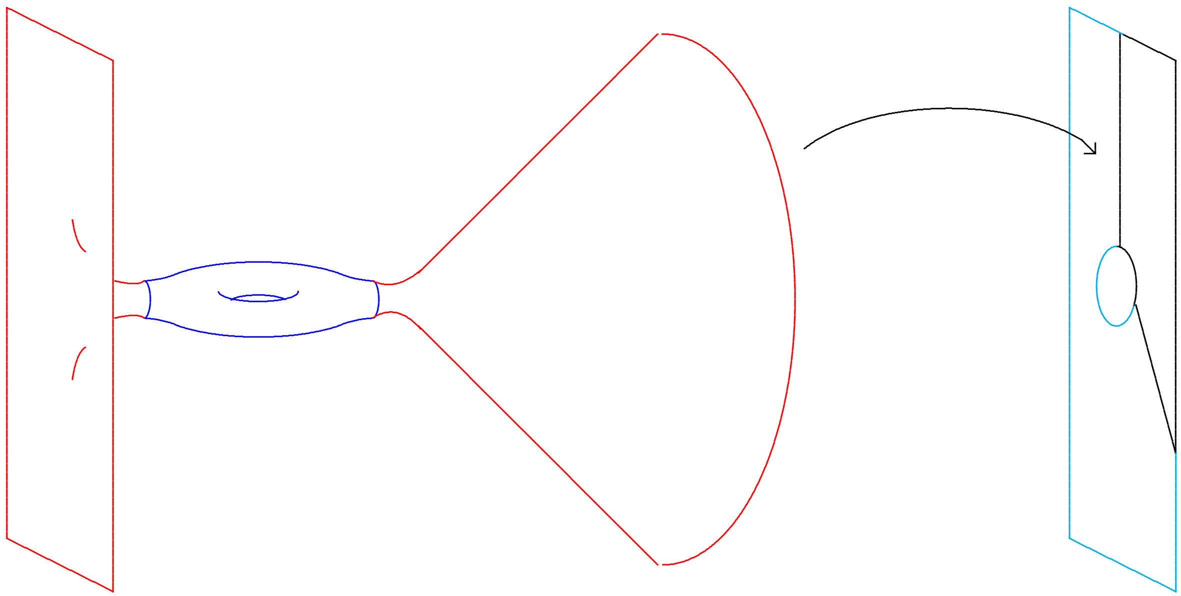

$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

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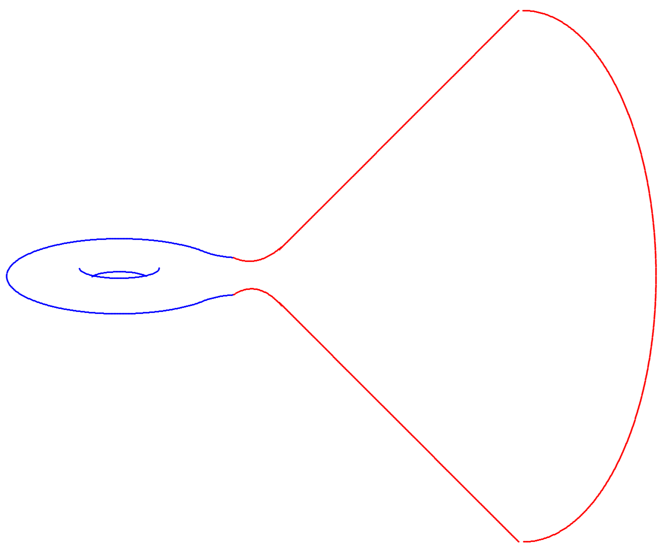
Lemma. *Any ALE Kähler manifold has only one end.*



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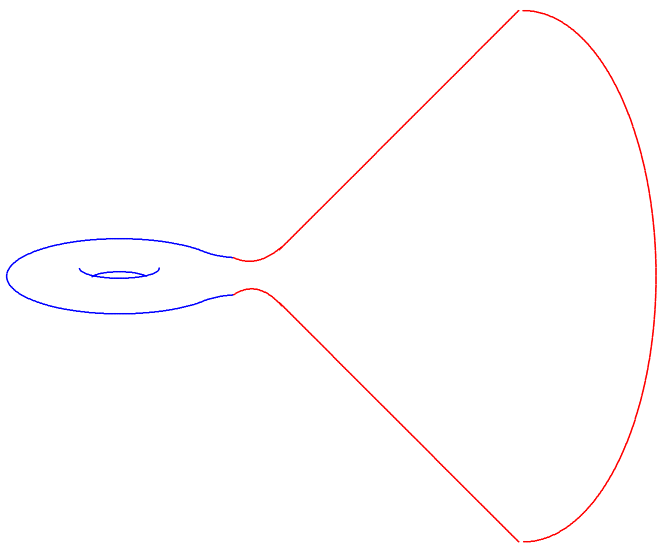

Proof later today!

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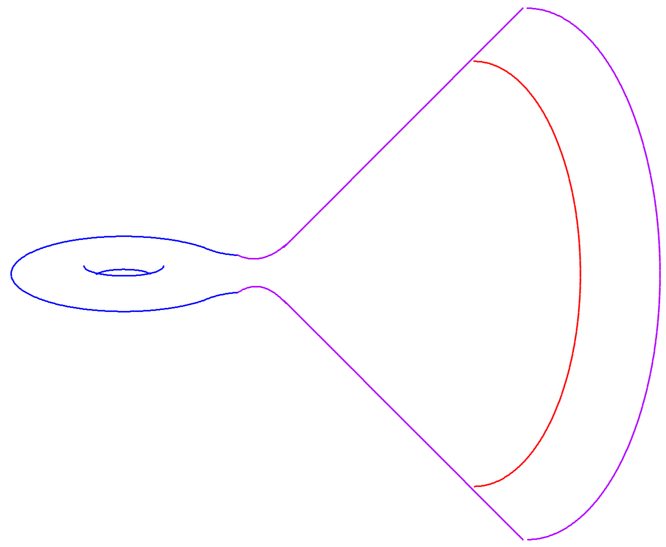

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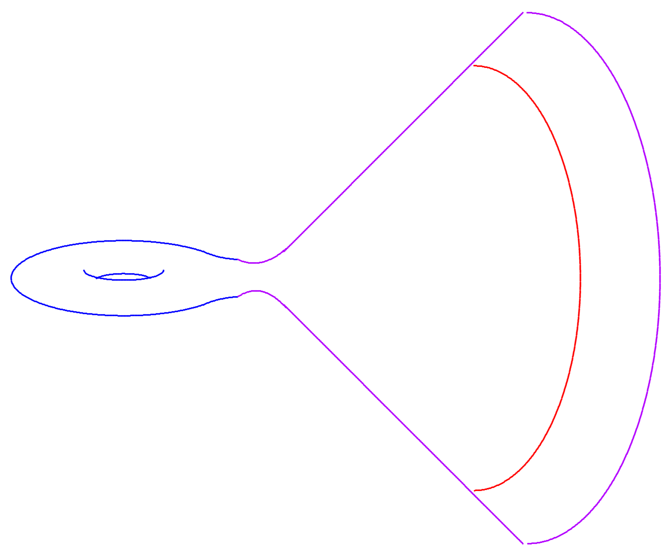

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- $\clubsuit : H^2(M) \xrightarrow{\cong} H^2_c(M)$ *inverse of natural map.*

Scalar-flat Kähler surface:

$$m(M, g) = -\frac{1}{3\pi}\langle \clubsuit(c_1), [\omega]\rangle$$

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$$d(\theta \wedge \omega) = \rho \wedge \omega = \frac{s}{4}\omega^2 = 0.$$

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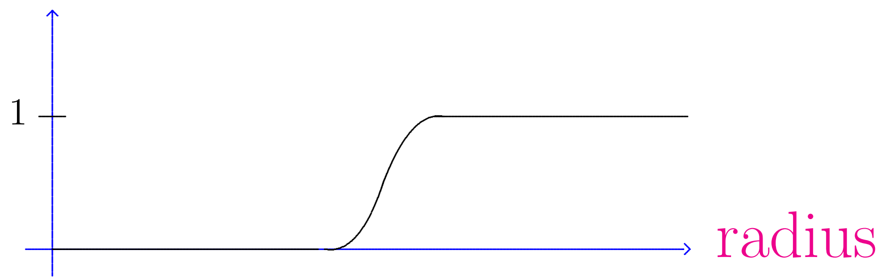

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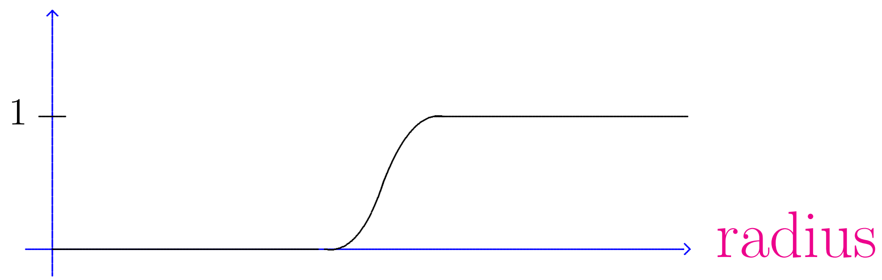

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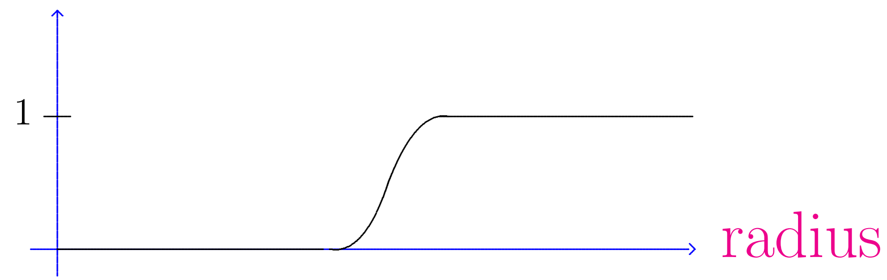

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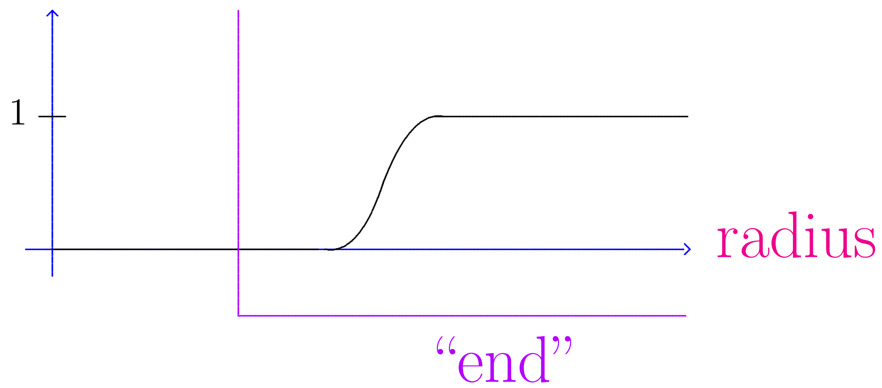

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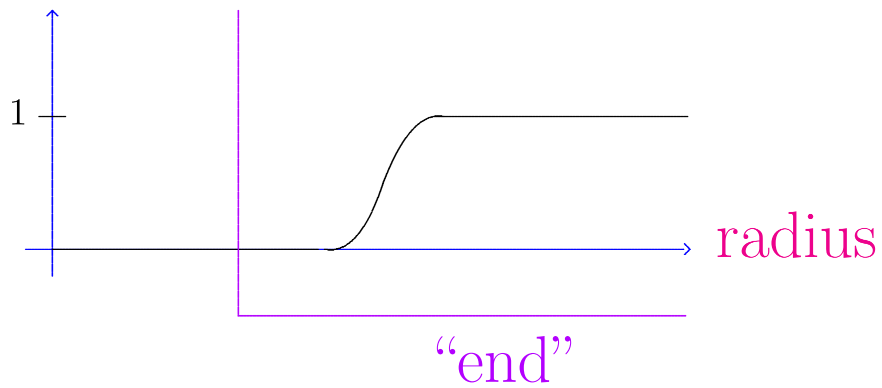

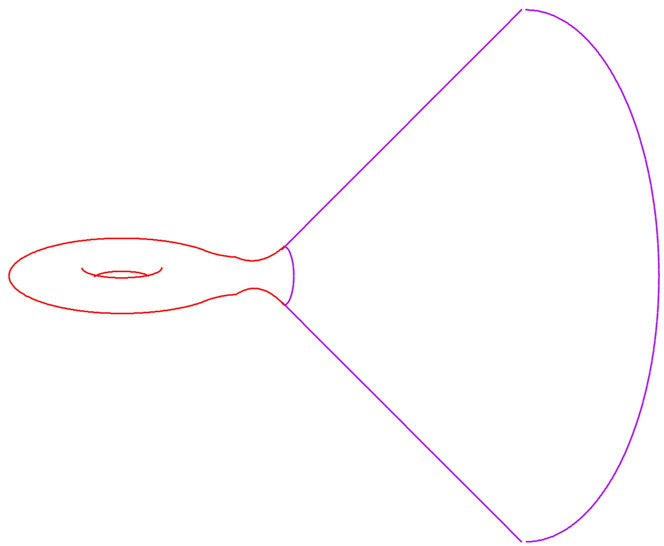

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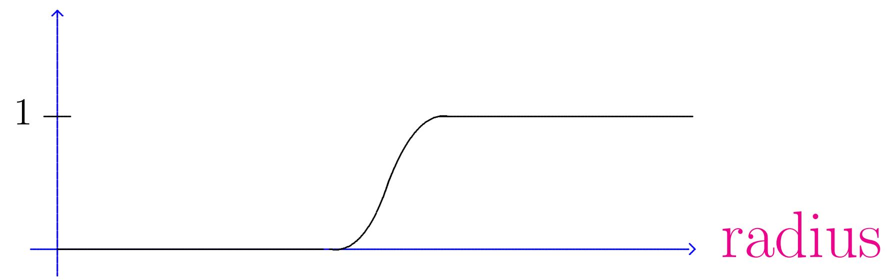

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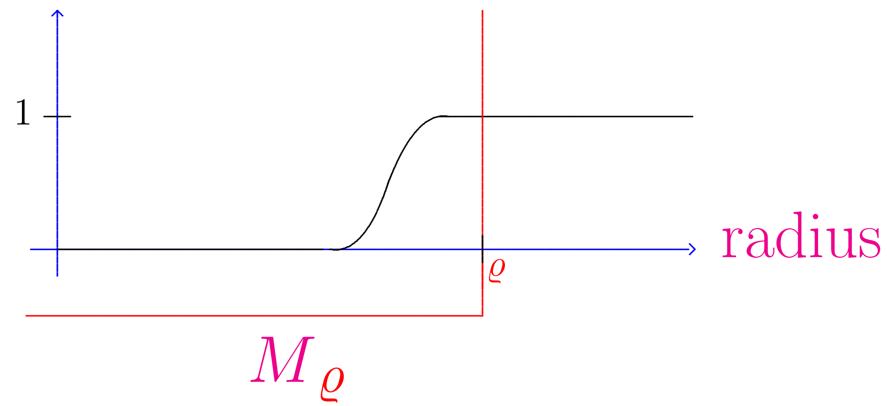

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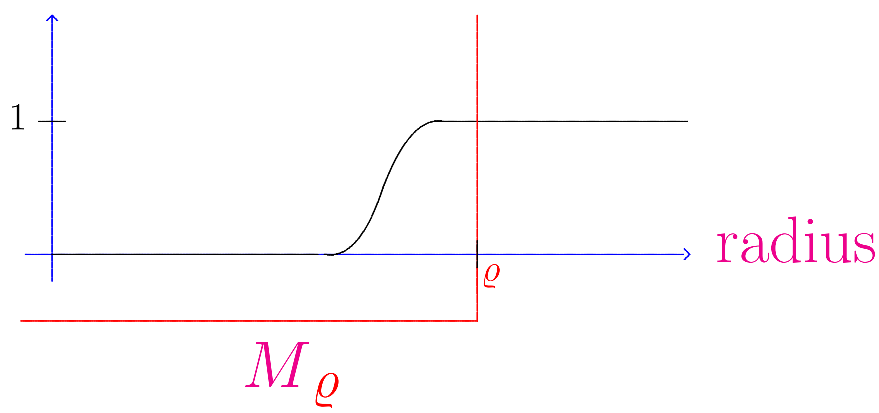

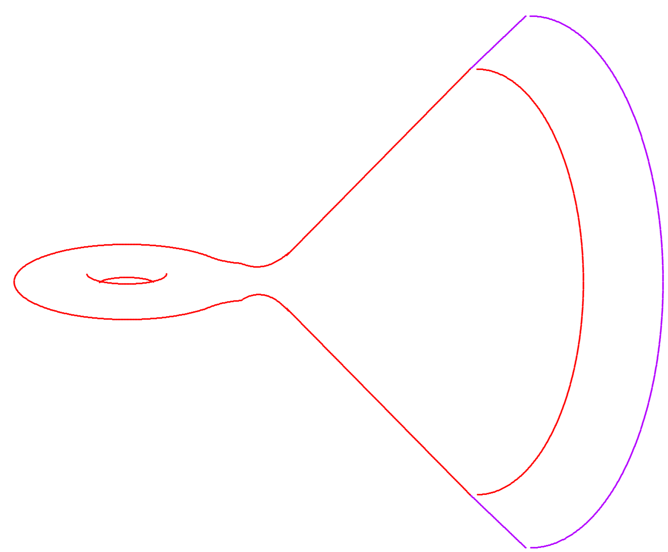

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where M_ϱ defined by radius $\leq \varrho$.

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scalar-flat $\Longrightarrow \rho \wedge \omega = 0.$

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Let $f : M \to \mathbb{R}$ be smooth cut-off function:

$\equiv 0$ away from end,

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But were our assumptions justified?

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- $m = 2$;
- $s \equiv 0$; and
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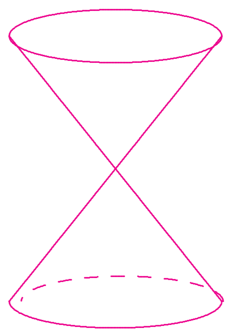

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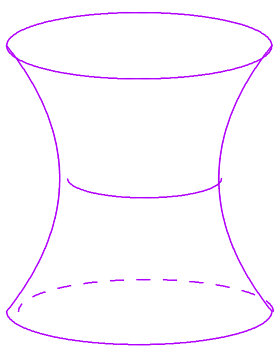

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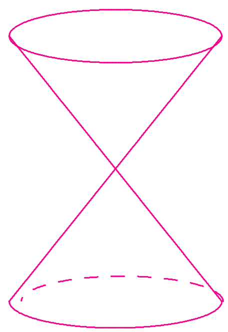

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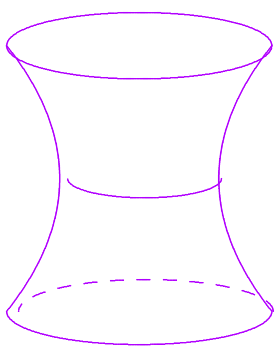

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In this case, compactified end $\cong_{\text{bih}} \mathbb{CP}_m - B^{2m}$.

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When $m = 2$, can still show $\widetilde{M}$ of Kähler type.

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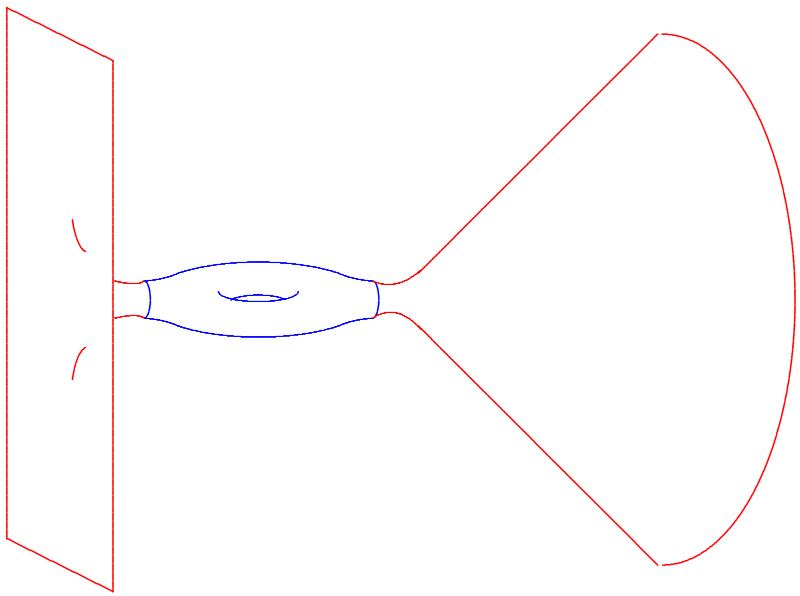

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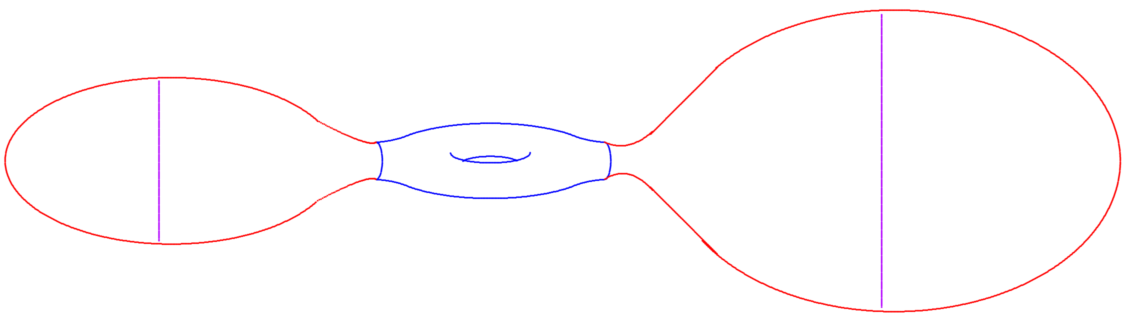

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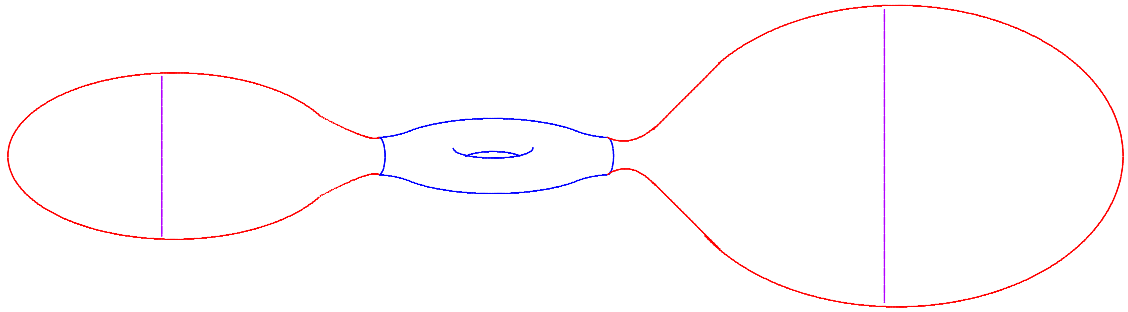

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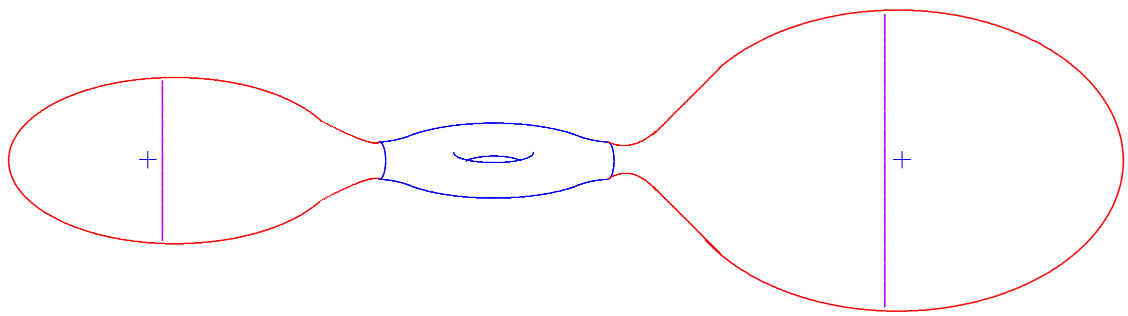

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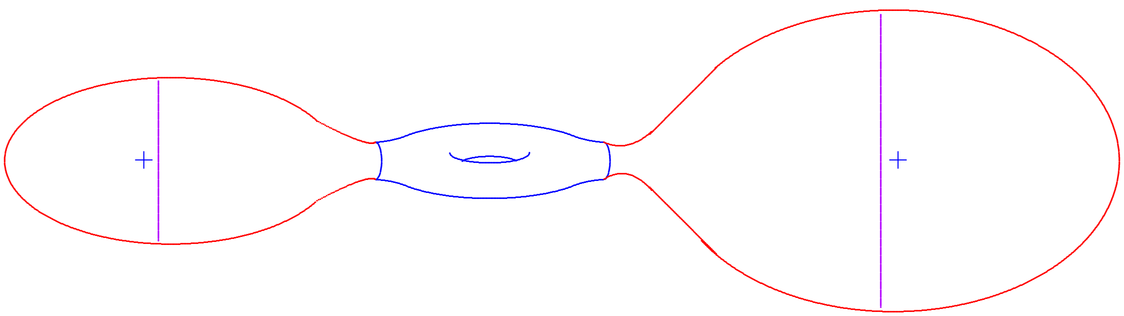

Intersection form

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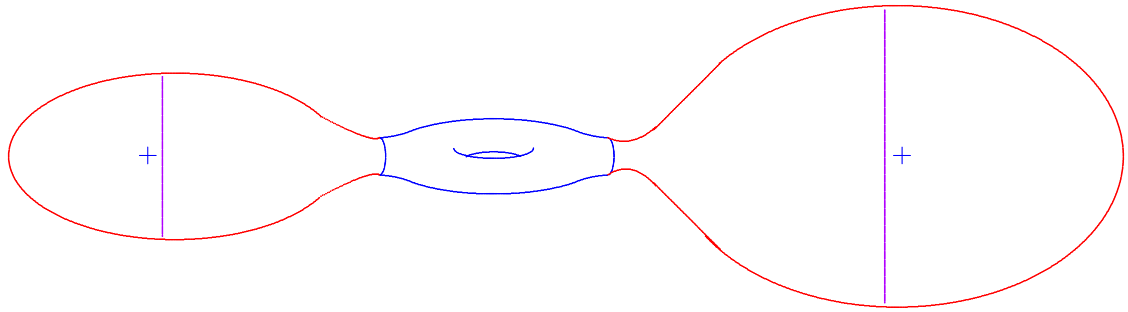

Intersection form

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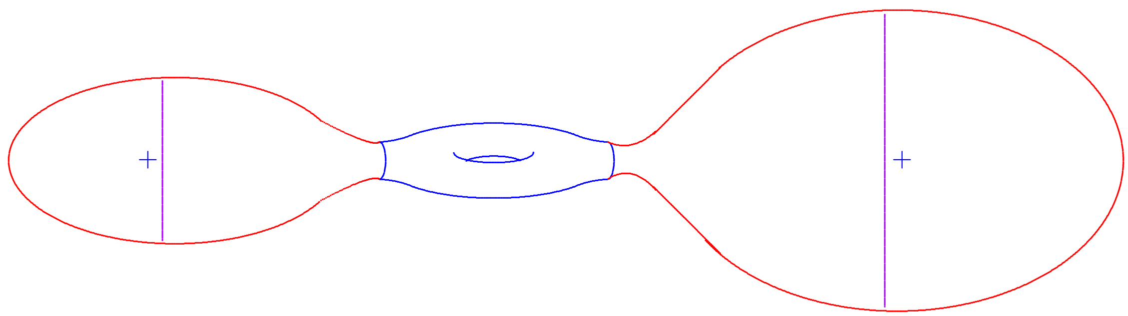

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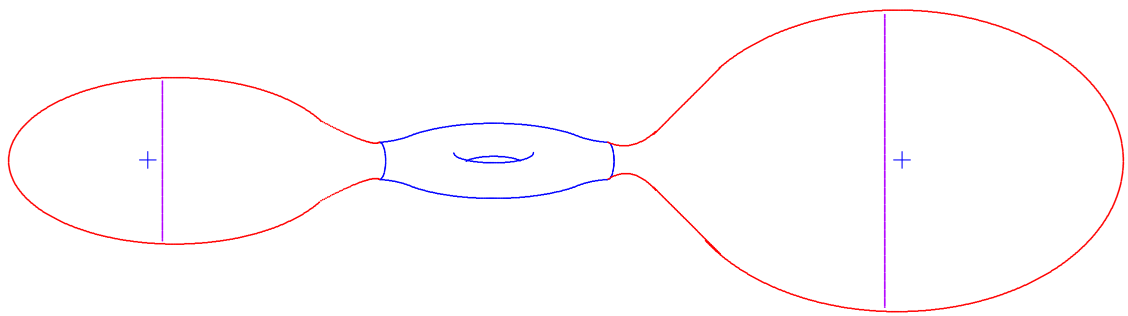

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thus has one positive direction for each end.

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Hodge theorem on intersection form

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Form has only one positive direction in $H^{1,1}(\widehat{M}, \mathbb{R})$:

$$(+- \cdots -)$$

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So Hodge theorem on intersection form implies:

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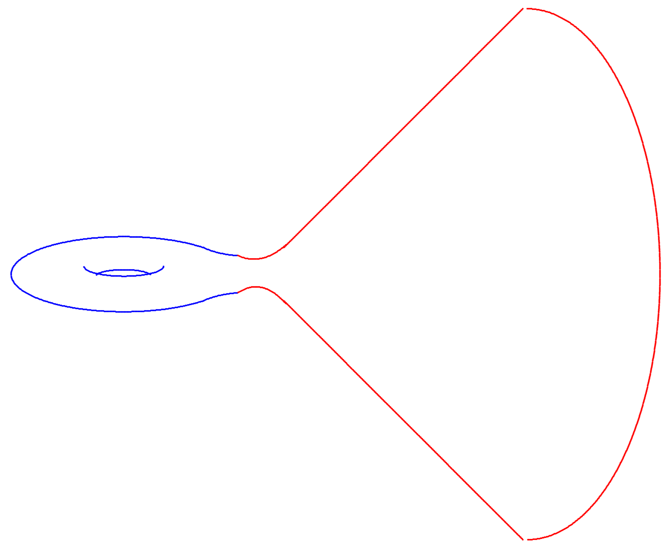
So Lefschetz theorem on intersection form implies:

Lemma. *Any ALE Kähler manifold has only one end.*



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and such that $d\theta = \rho$, where ρ is the Ricci form of g with respect to a given compatible integrable almost-complex structure J.

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Proof is heavily computational. Idea:

First rotate coordinates so that $J \to J_0$ at ∞, where J_0 standard complex structure on $\mathbb{C}^m$.

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Integrating on S_ϱ/Γ therefore yields:

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The mass formula then follows, much as before.

Theorem C. *Any ALE Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

$$\mathfrak{m}(M,g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g d\mu_g$$

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This has some interesting consequences. . .

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Proof actually shows something stronger!

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with $=$ $\Longleftrightarrow$ (M, g, J) *is scalar-flat Kähler.*

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so the mass formula implies the claim.

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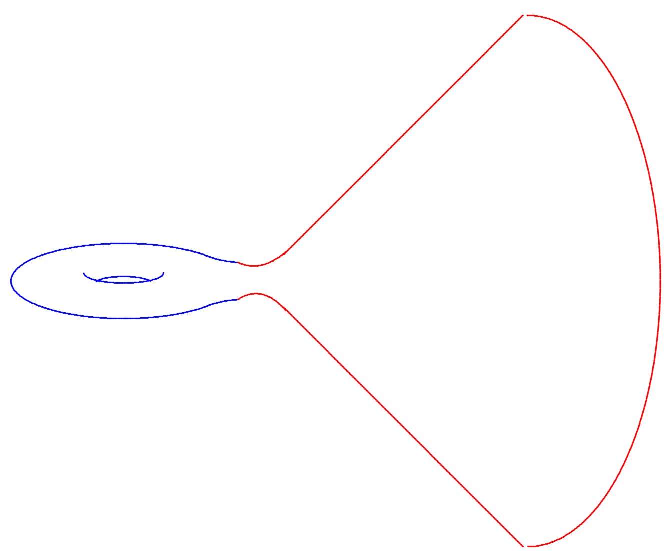

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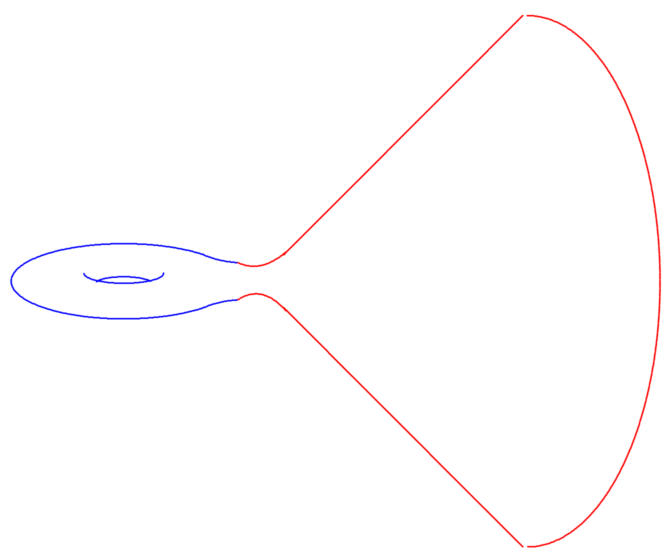

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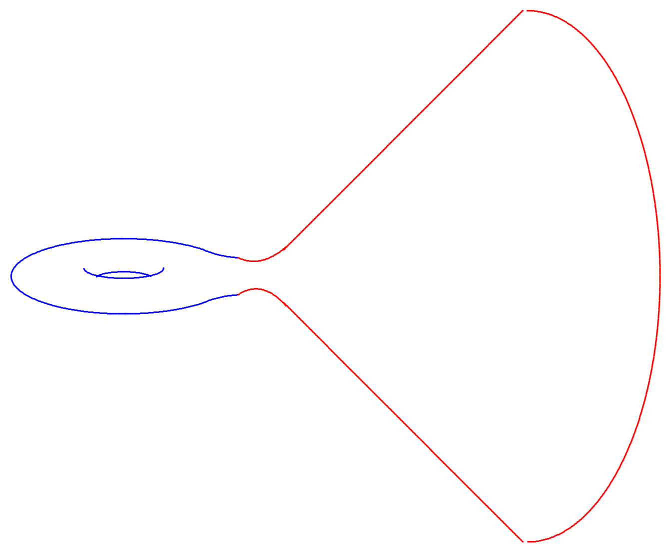

End, Part III

Merci aux organisatrices,
et à l'École Normale Supérieure,
de m'avoir invité!