Mass, Kähler Manifolds, &

Symplectic Geometry

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Stony Brook University

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Workshop on Differential Geometry in the Large.
Creswick, Victoria, Australia. February 7, 2019.

Mass, Kähler Manifolds, and Symplectic Geometry

arXiv: 1810.11417 [math.DG]

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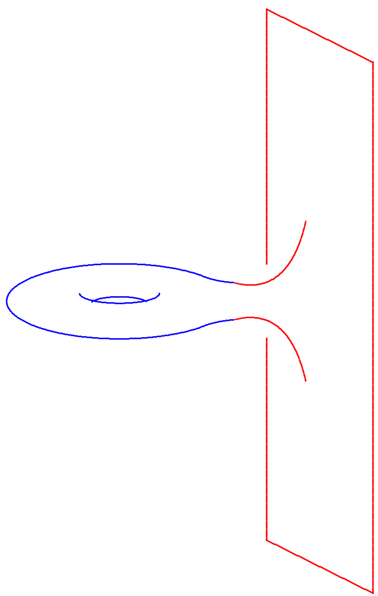
Builds on previous paper

Mass in Kähler Geometry
Comm. Math. Phys. 347 (2016) 621–653.

(Joint with Hans-Joachim Hein)

Definition. *A complete, non-compact Riemannian n-manifold (M^n, g)*



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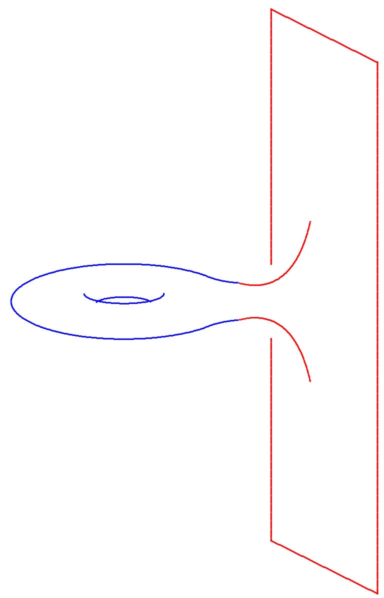

$$n \geq 3$$

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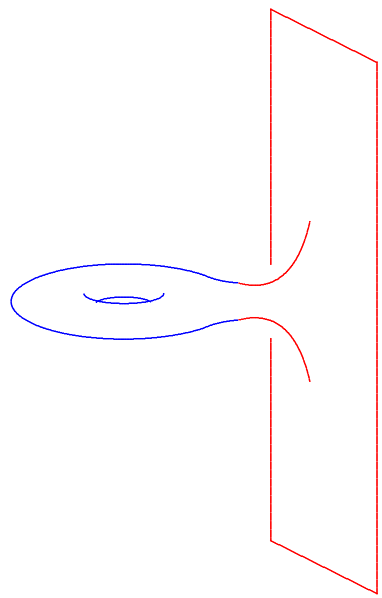

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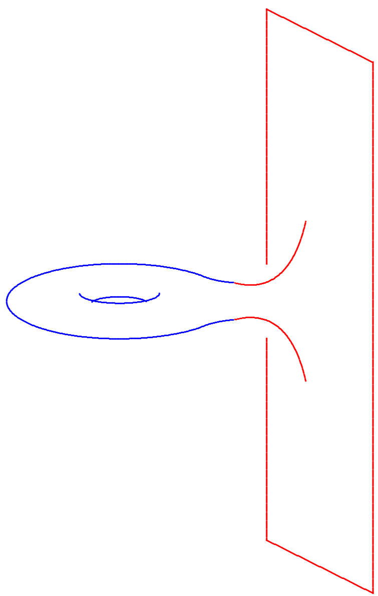

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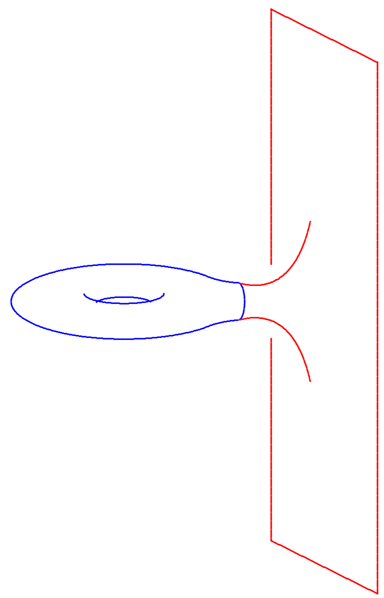
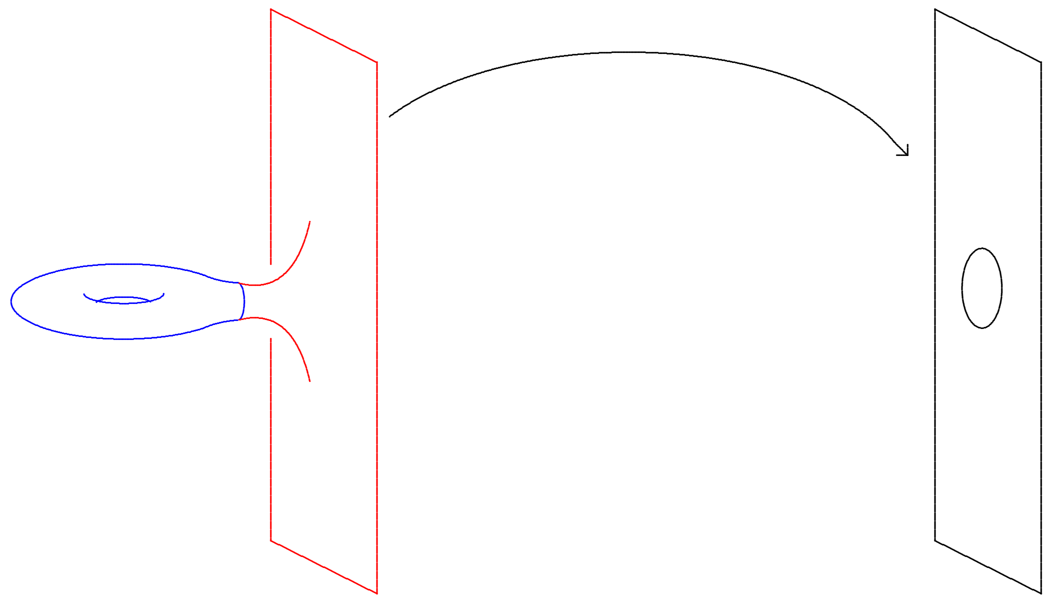


Definition. *A complete, non-compact Riemannian n-manifold (M^n, g) is called* asymptotically Euclidean *(AE) if there is a compact set $K \subset M$ such that $M - K$ is diffeomorphic to $\mathbb{R}^n - D^n$*



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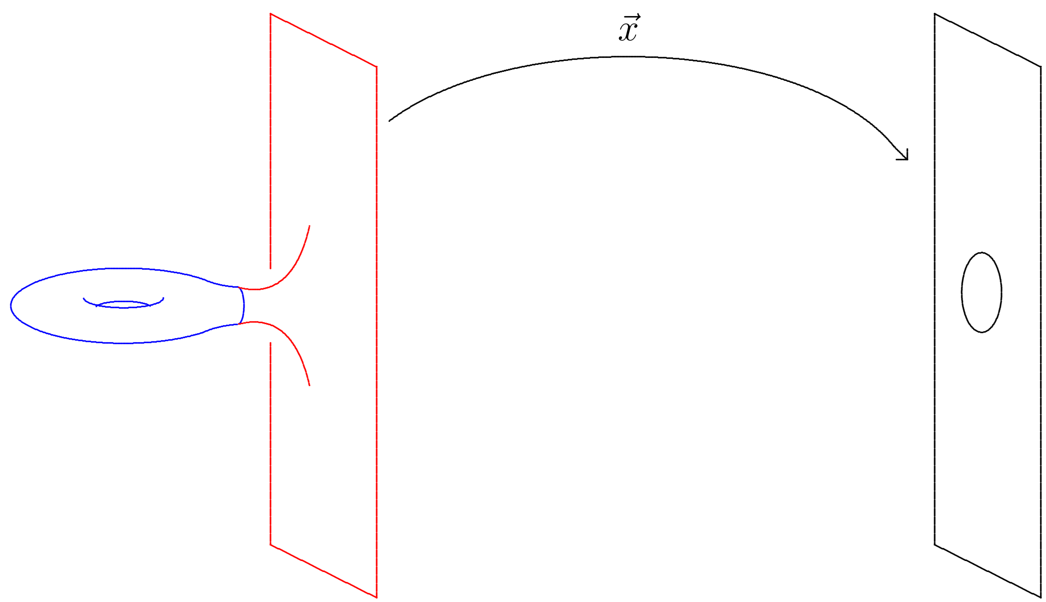

$$g_{jk} = \delta_{jk} + \text{terms that fall-off at infinity}$$

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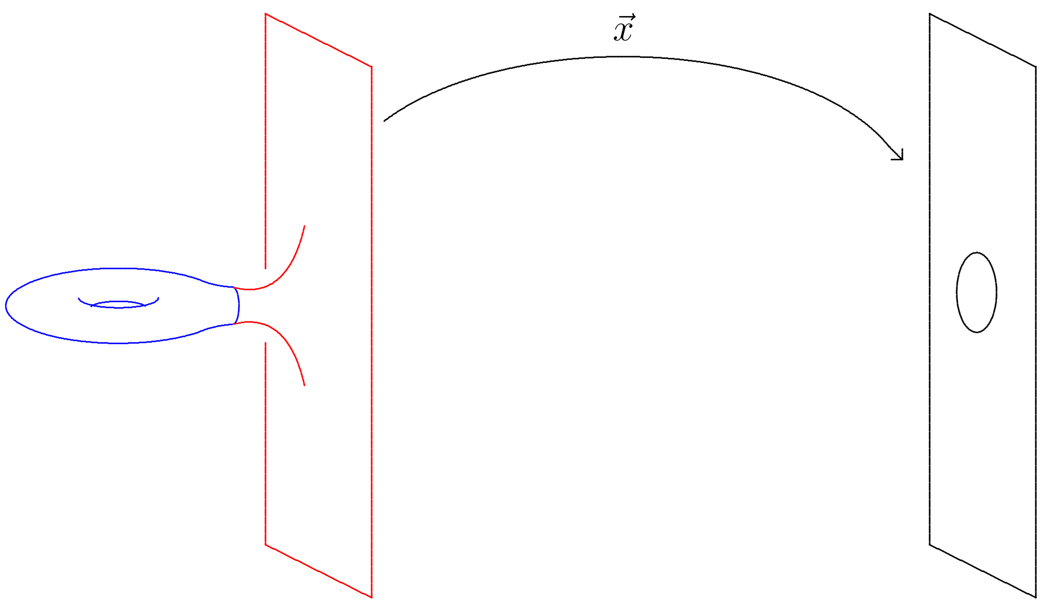

Weakest reasonable assumption:

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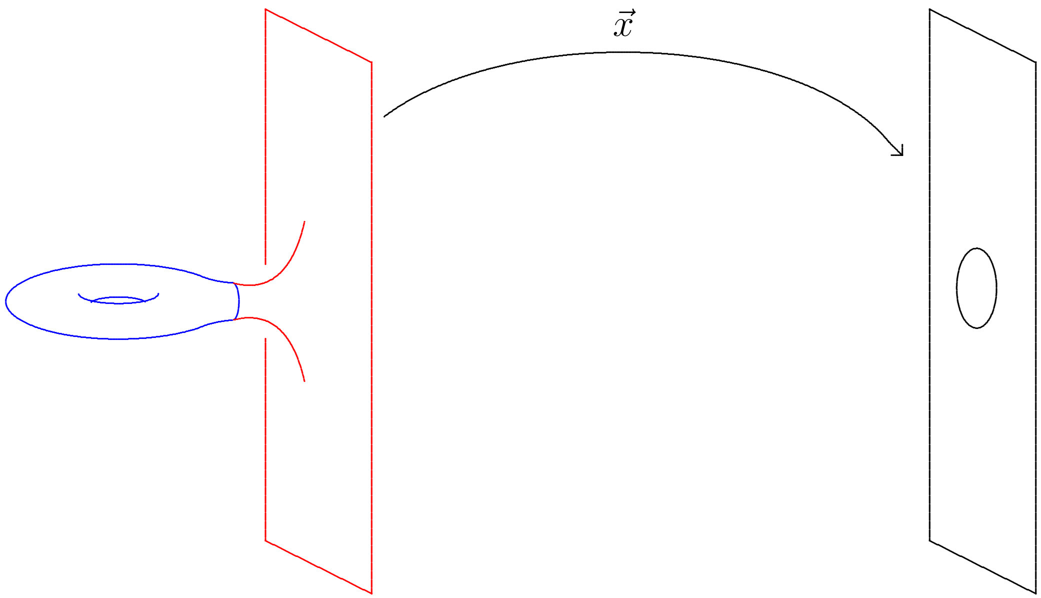

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Chruściel-type fall-off:

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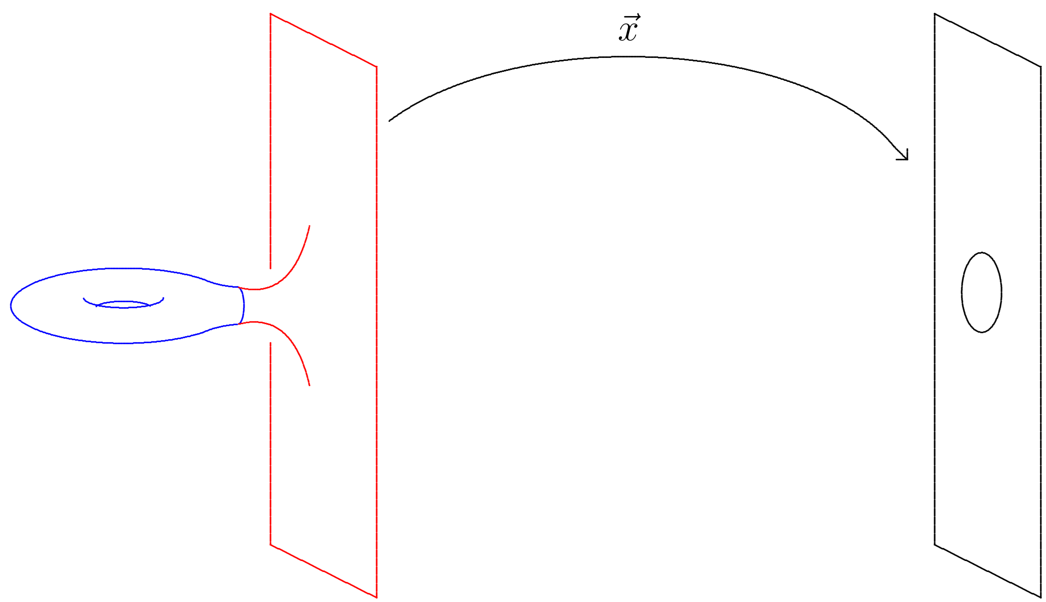

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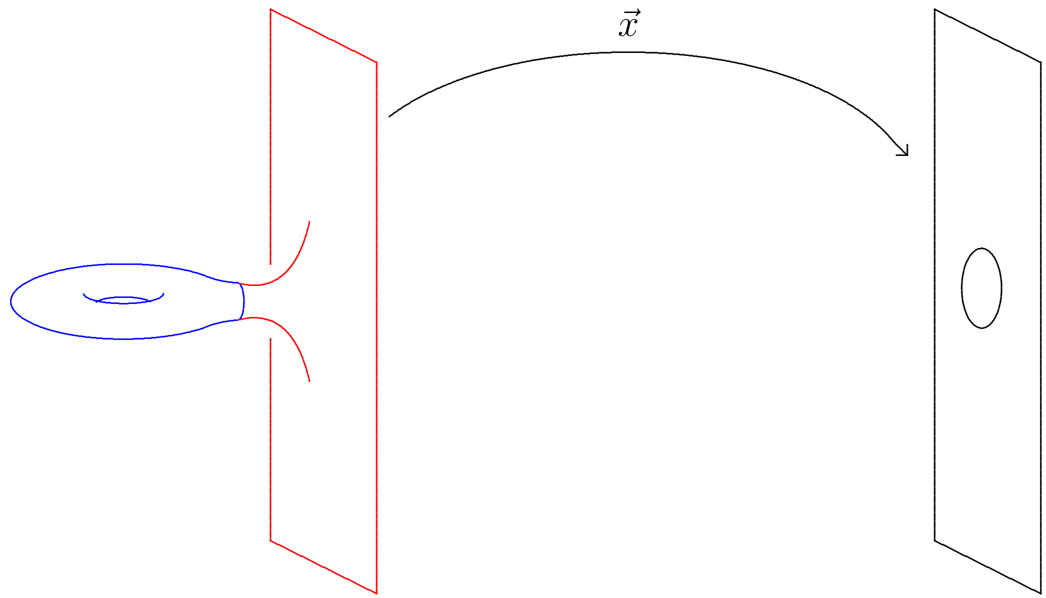

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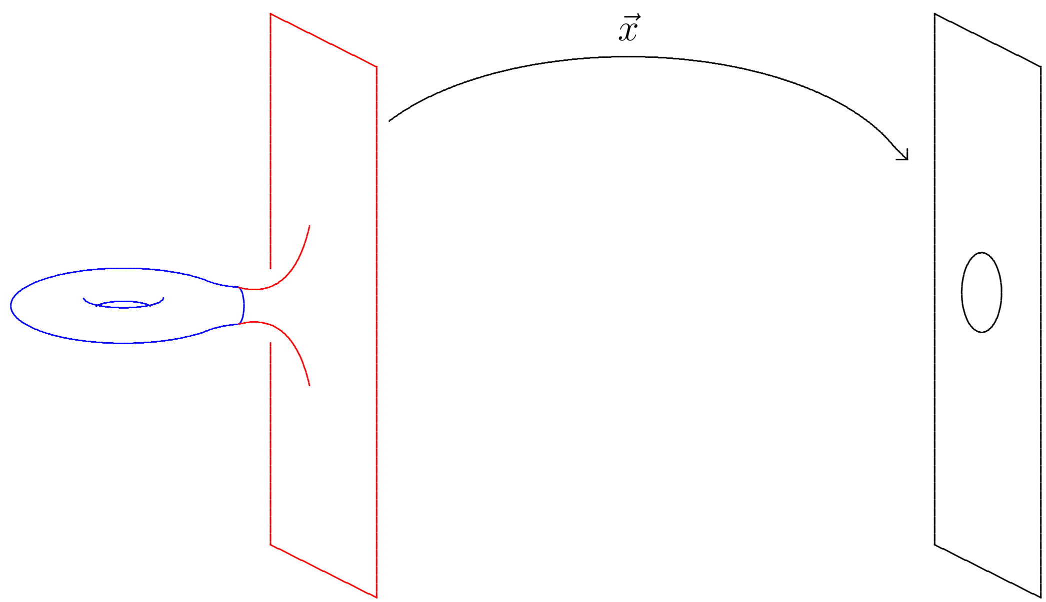

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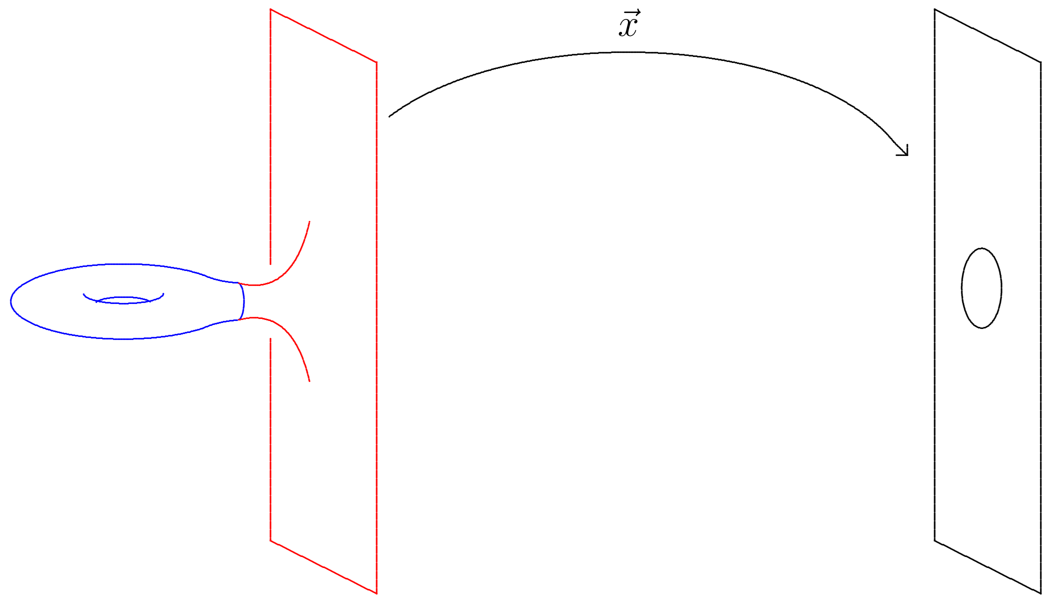

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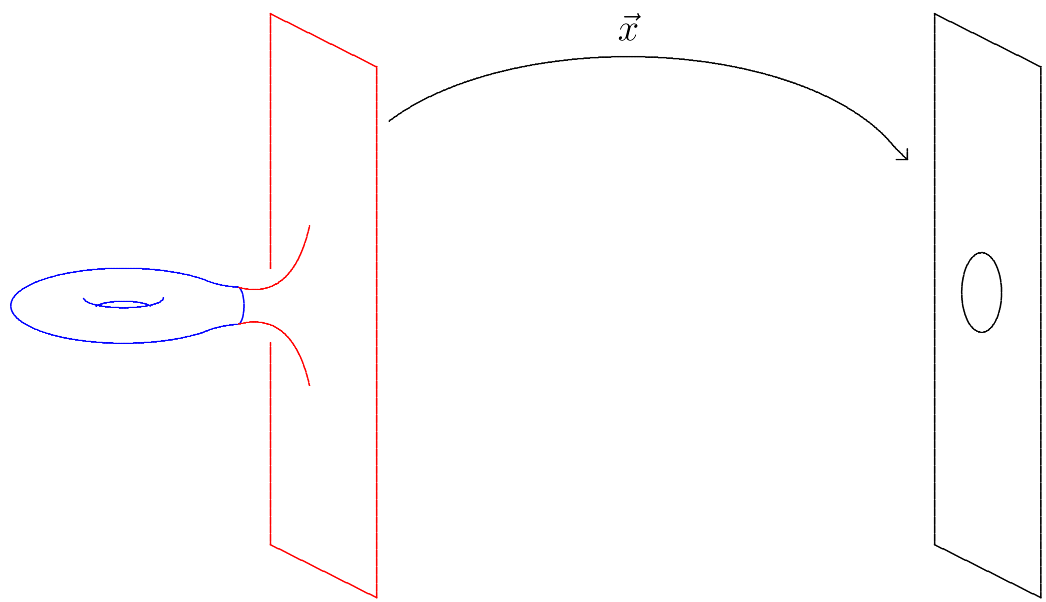

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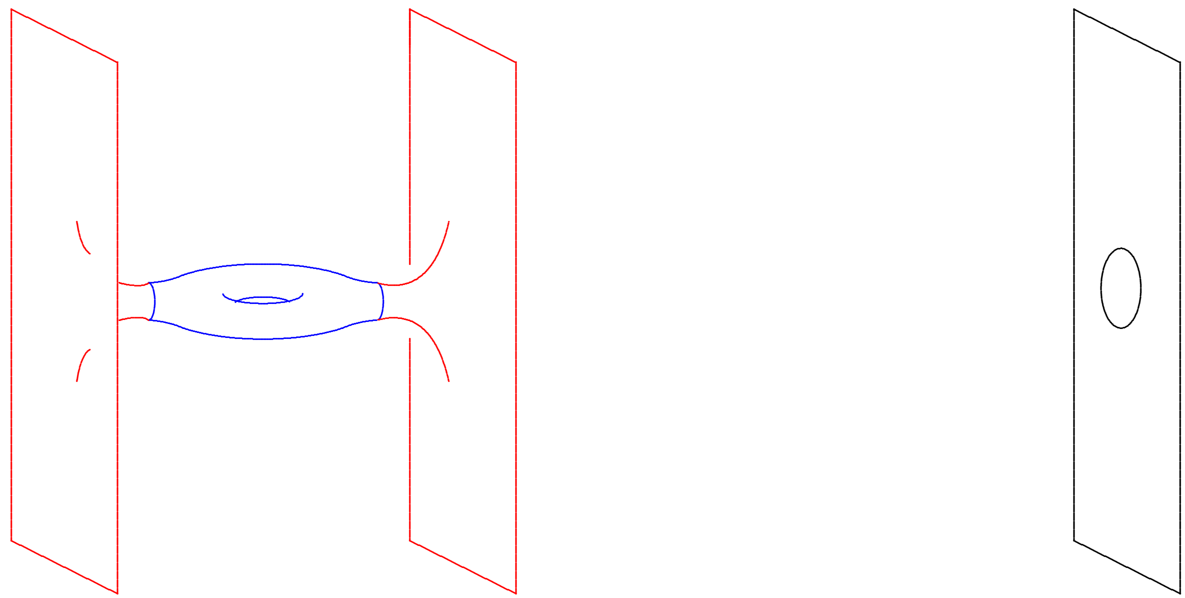

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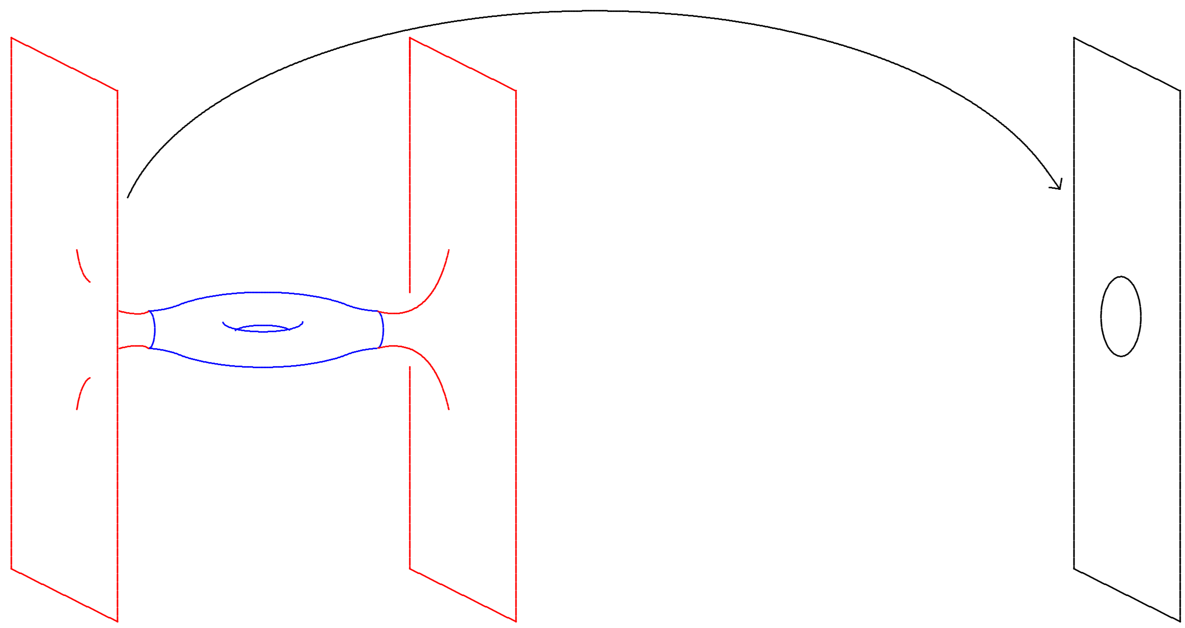

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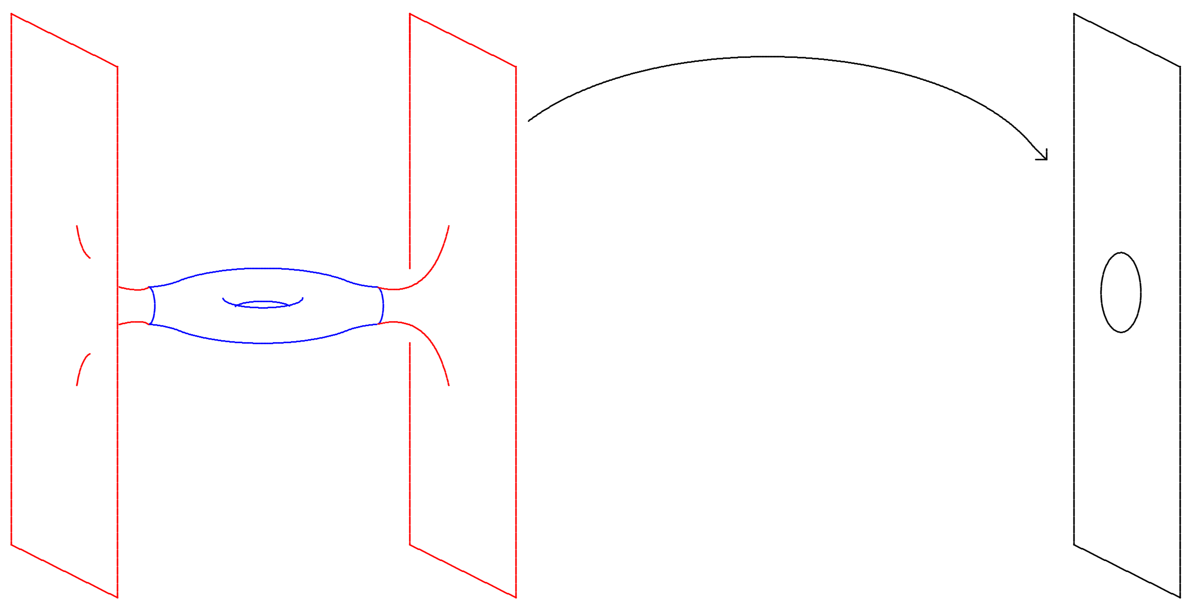

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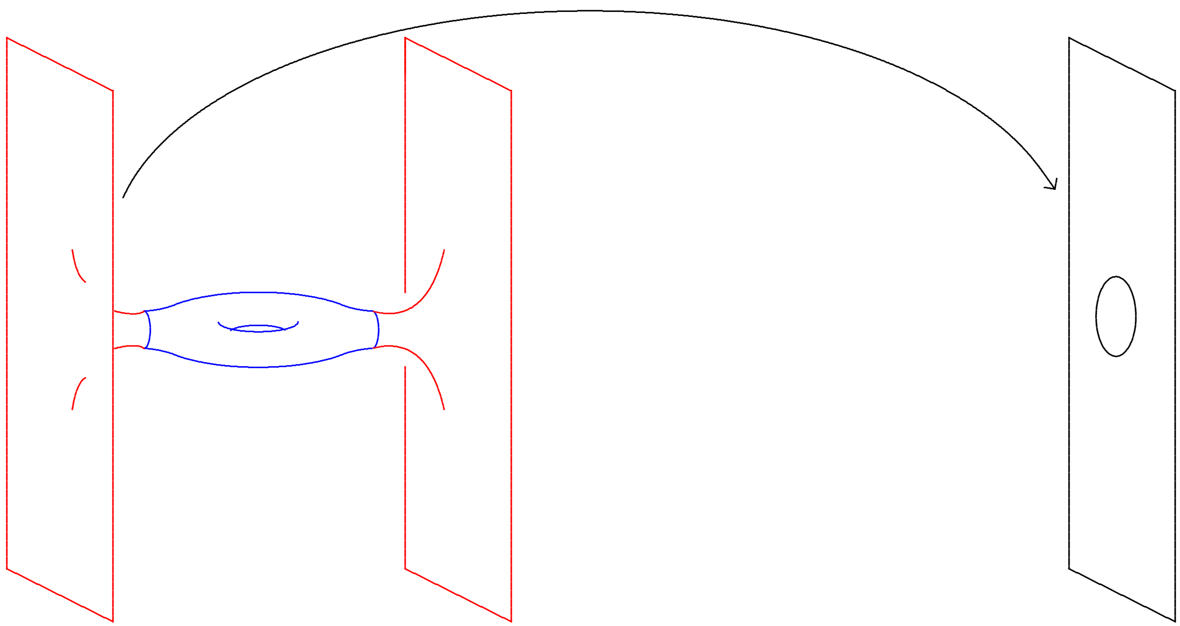

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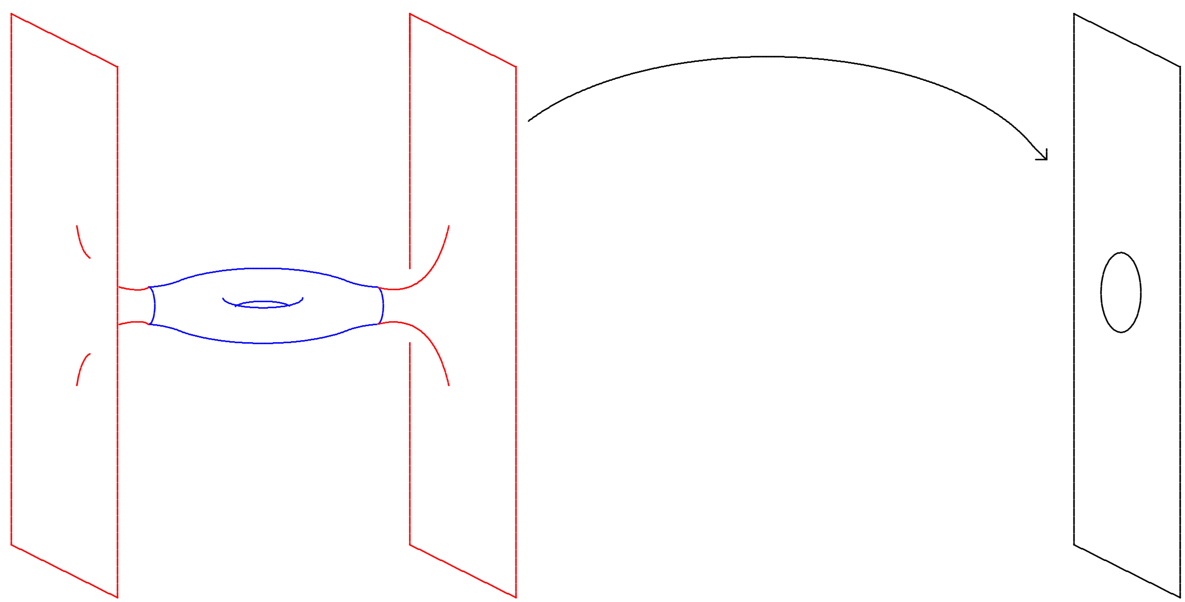

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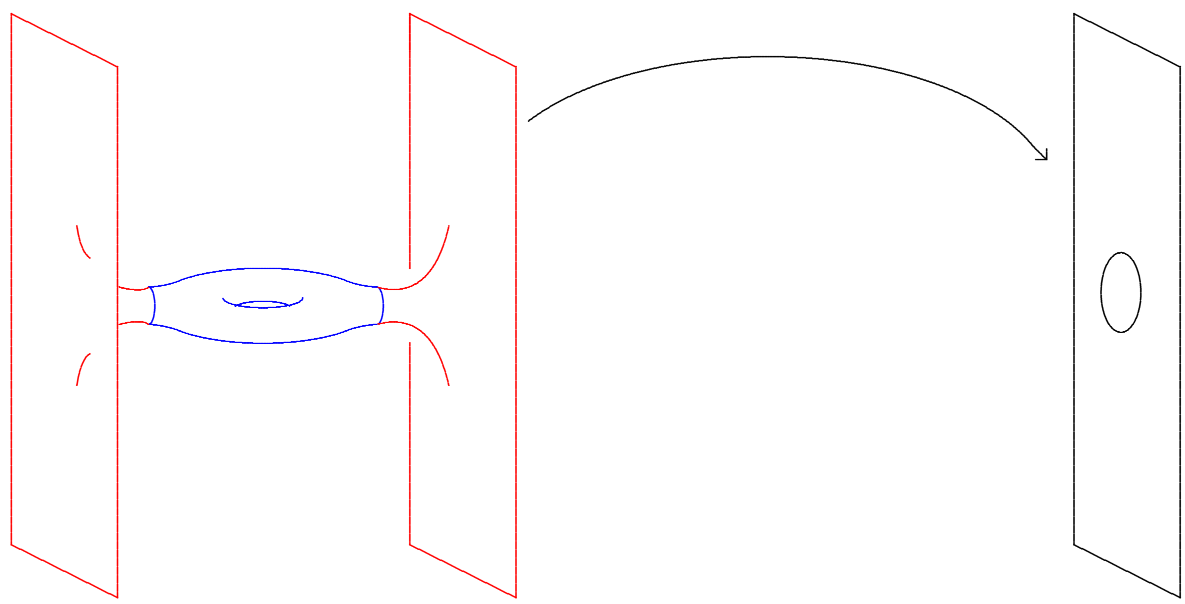

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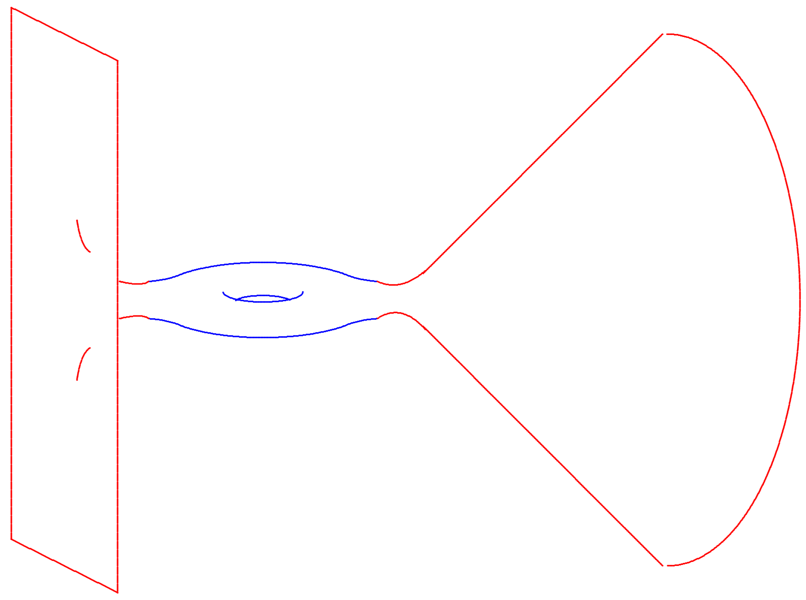

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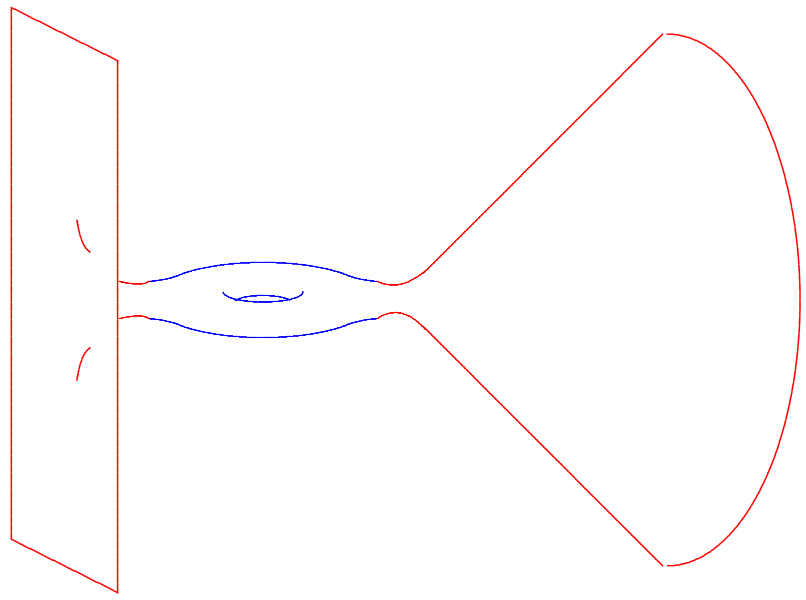

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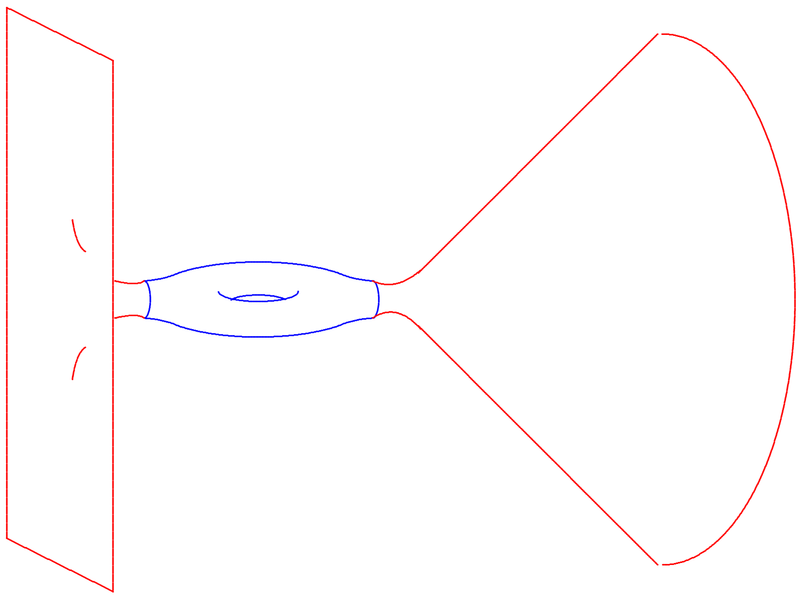

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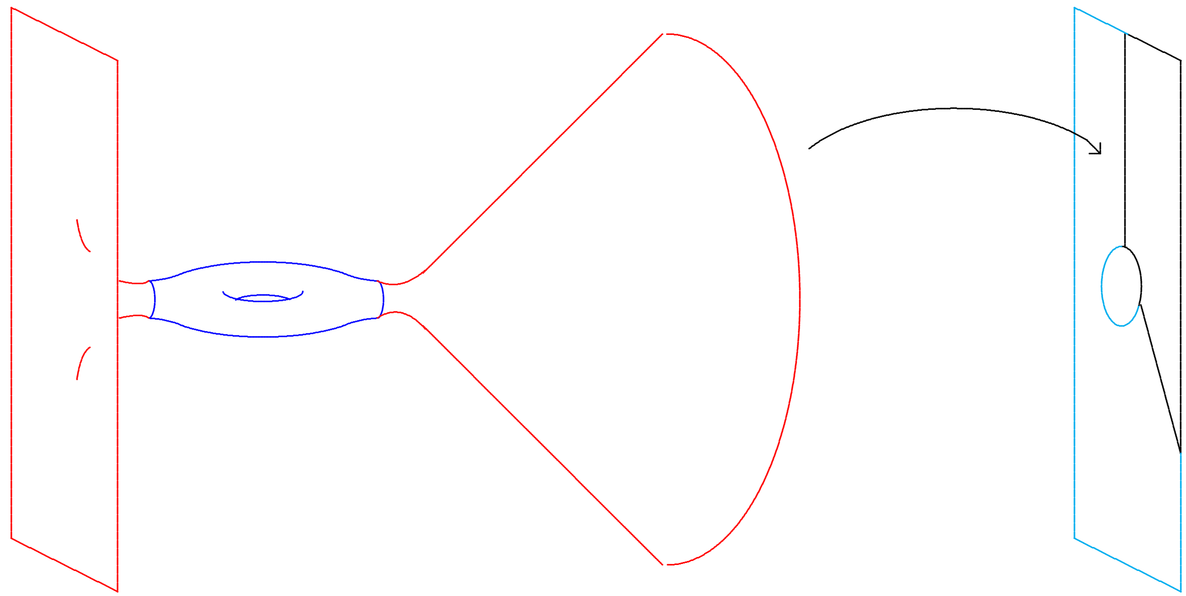

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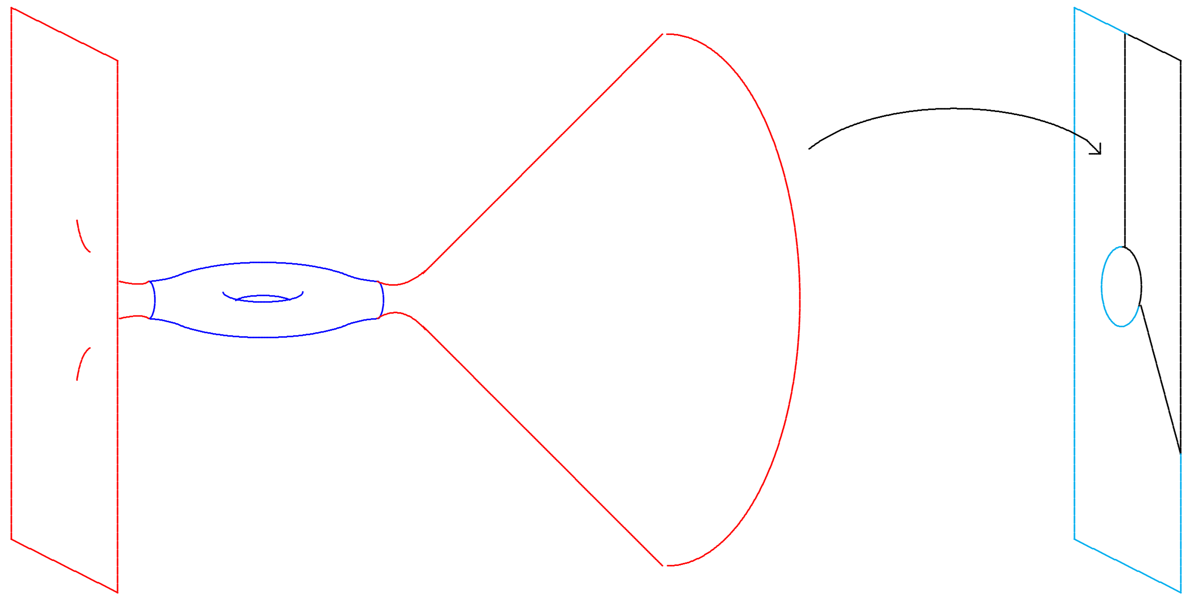

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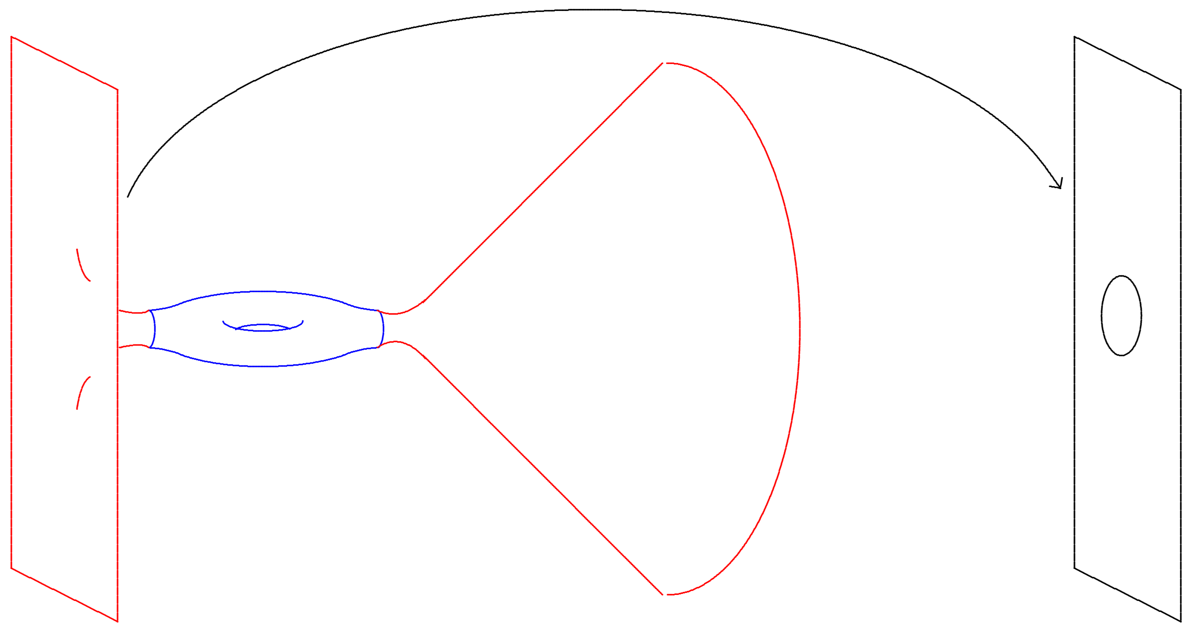

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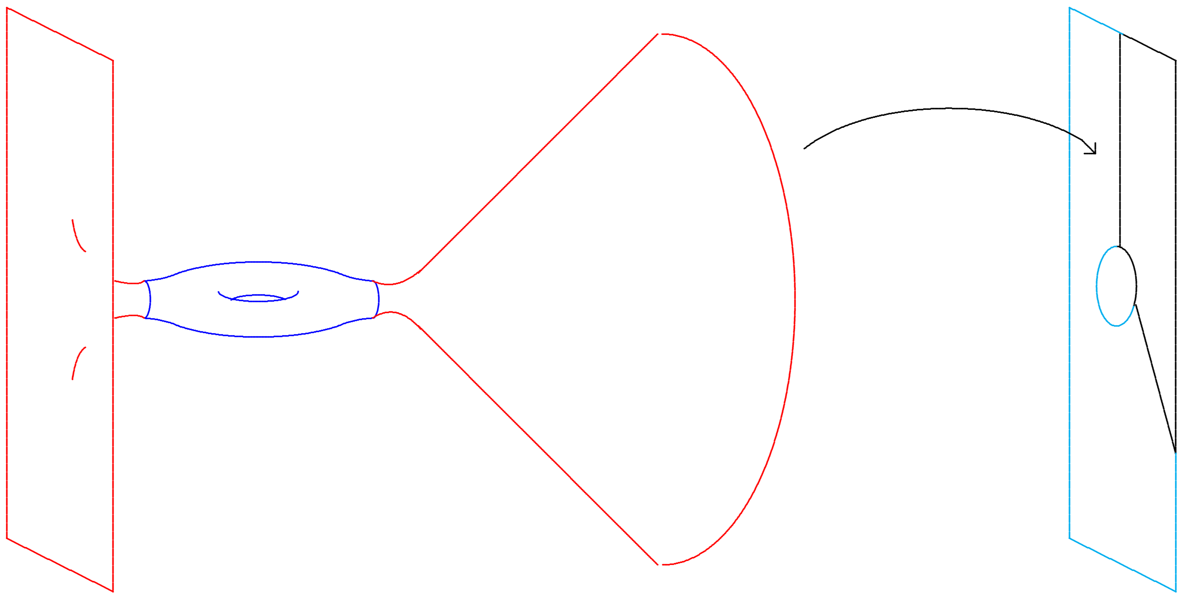

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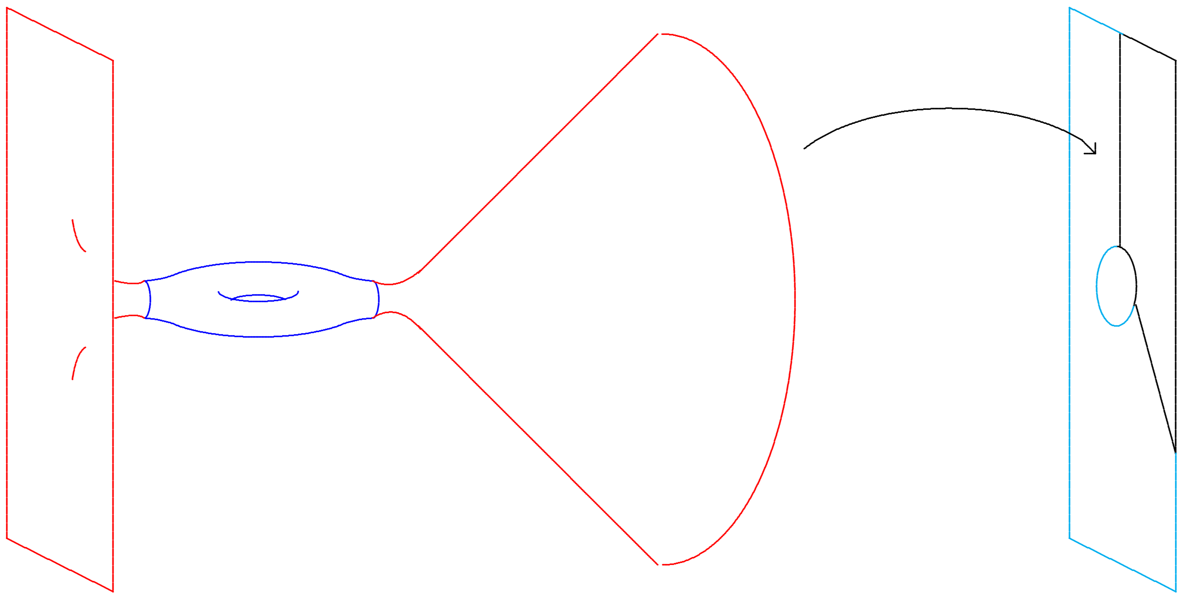

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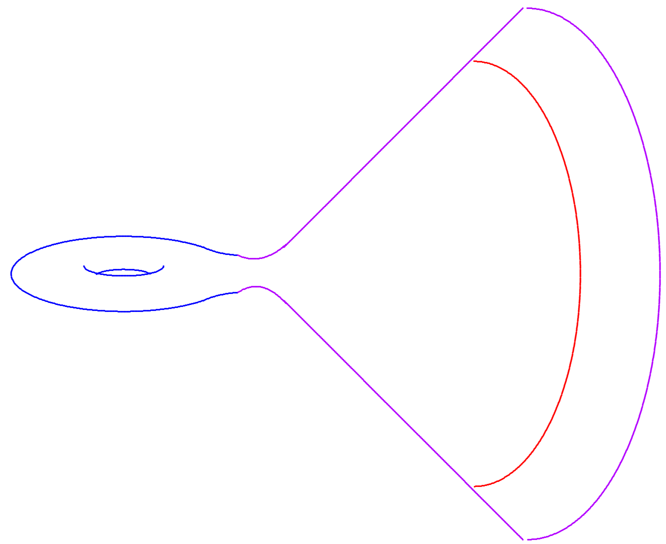

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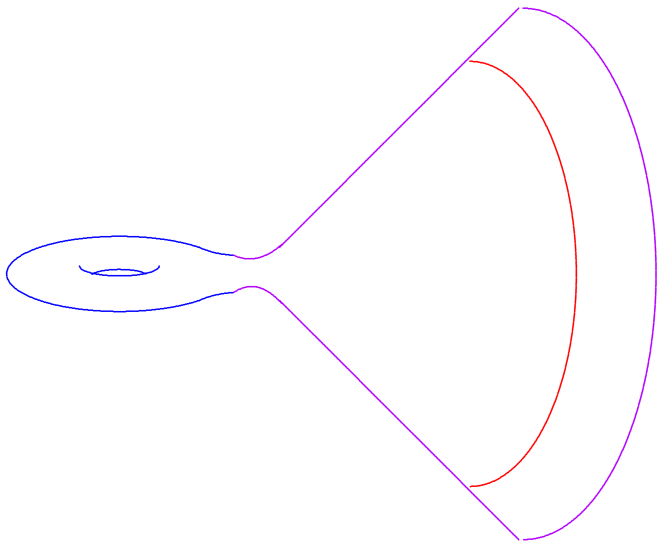

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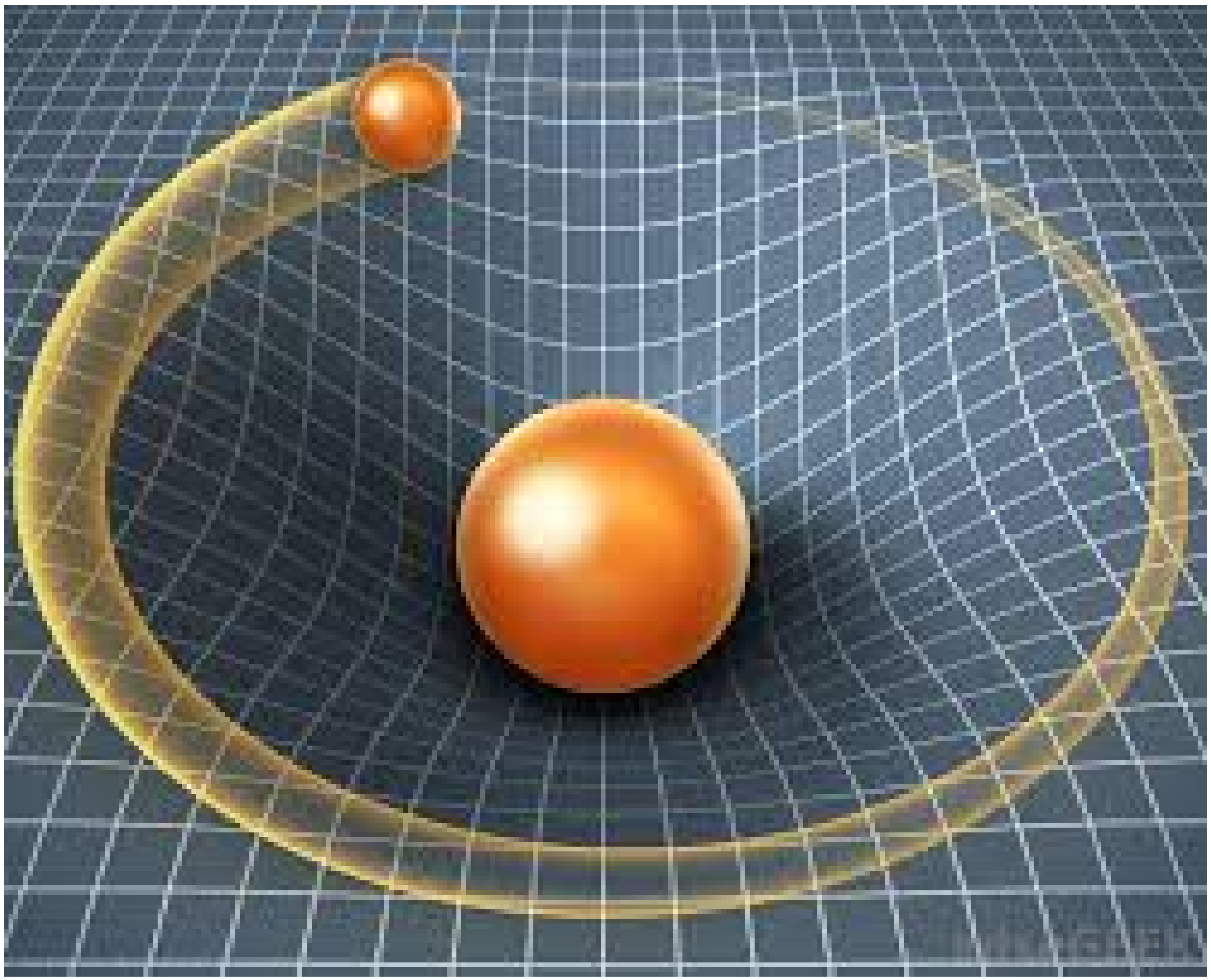

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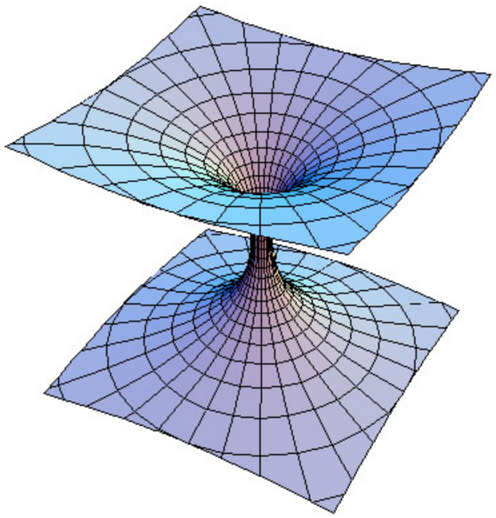

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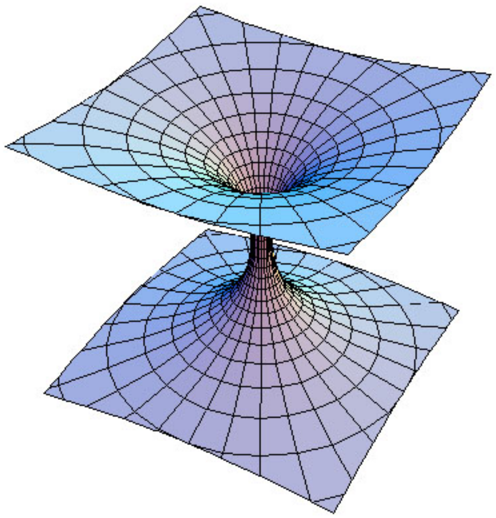

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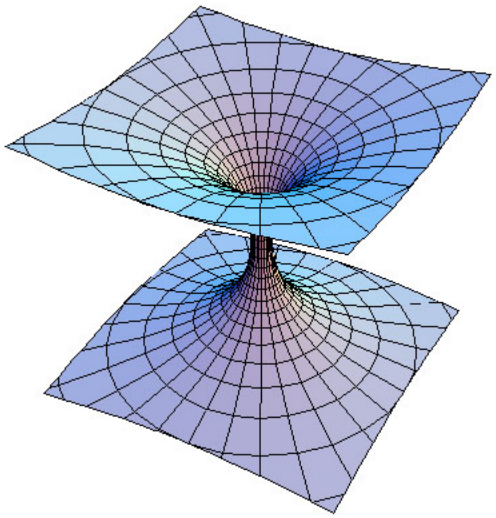

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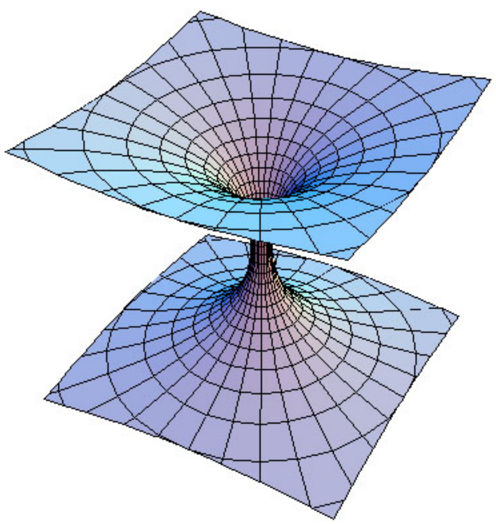

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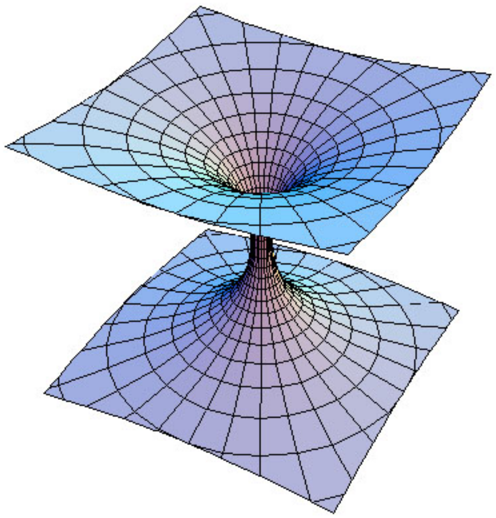

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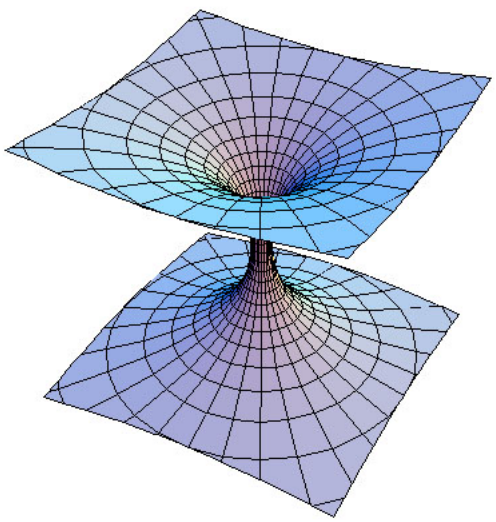

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But notice that this crude model for the mass in particular assumes faster metric fall-off!

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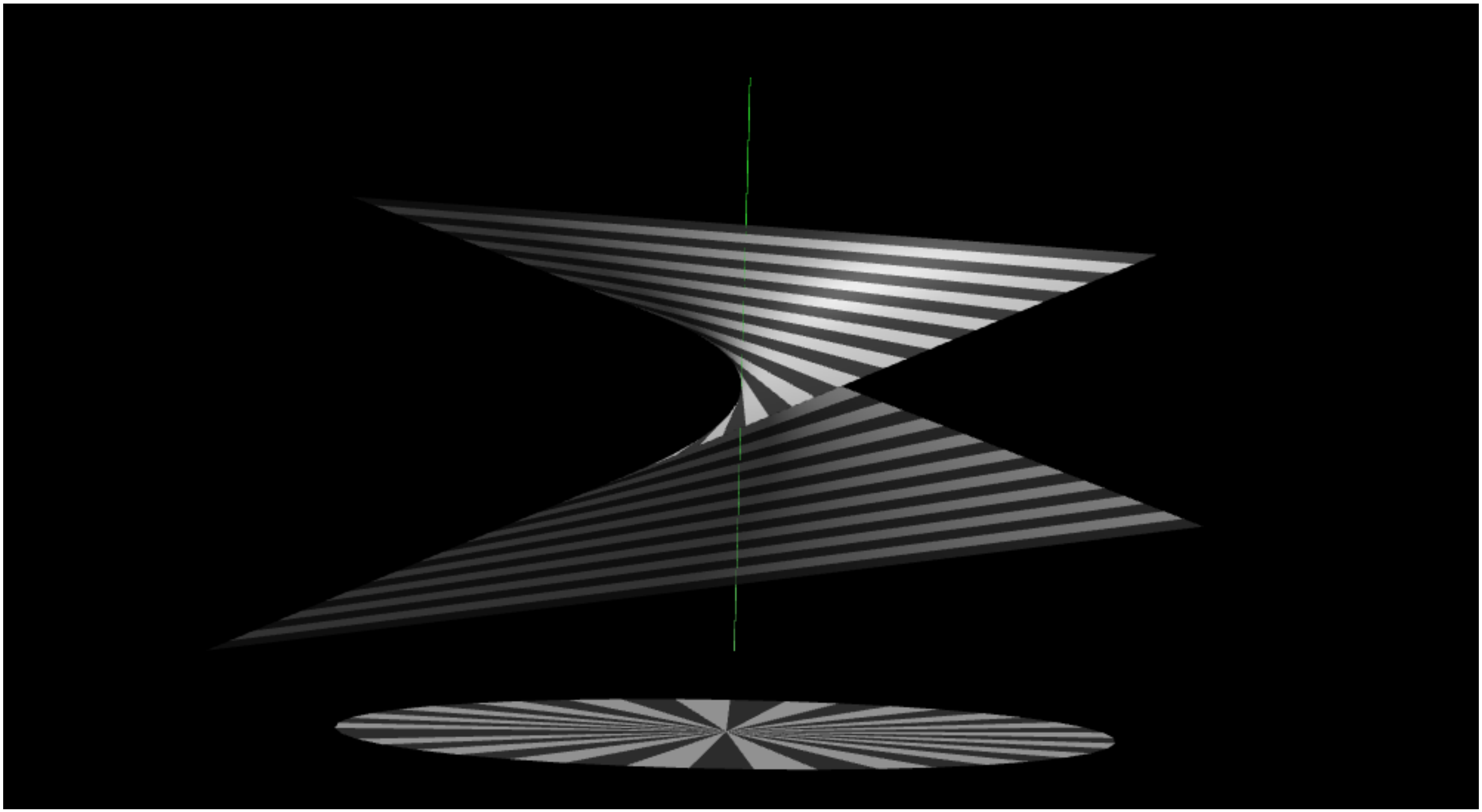

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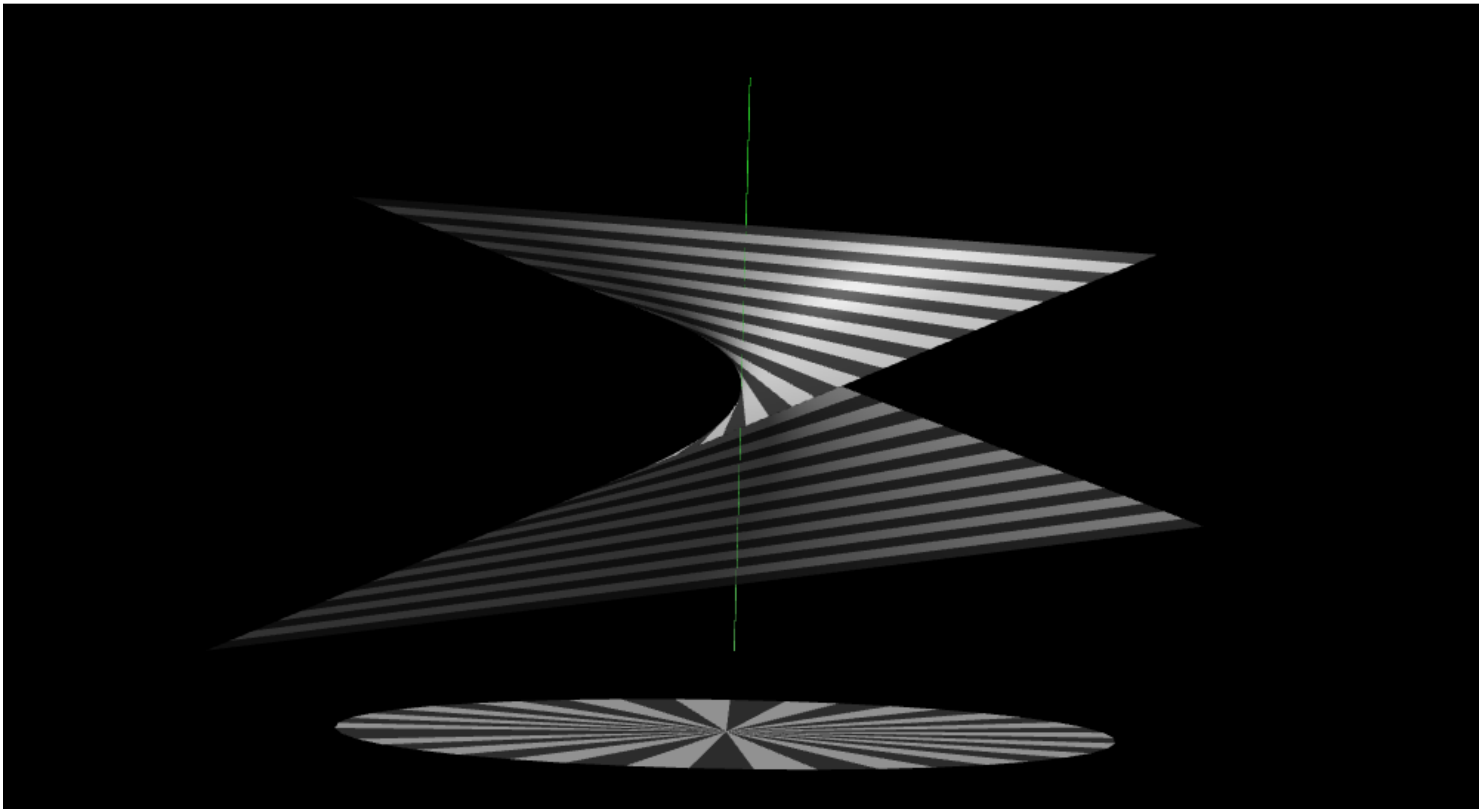

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Comm. Math. Phys. 347 (2016) 621–653.

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unless $\varepsilon > \frac{1}{2}$, when Chruściel fall-off sufficed.

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In particular, Chruściel fall-off suffices to imply all the following results:

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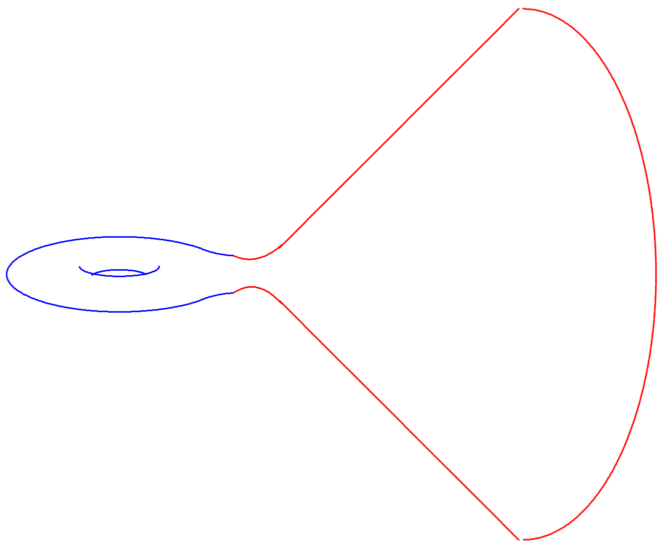

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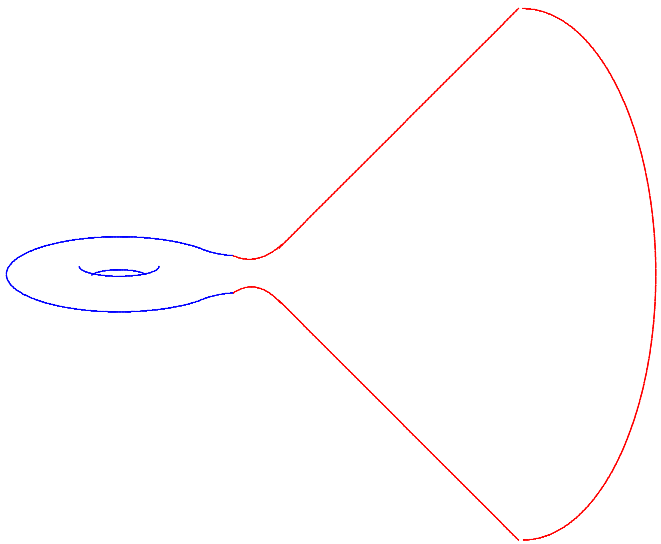

$$n = 2m \geq 4$$

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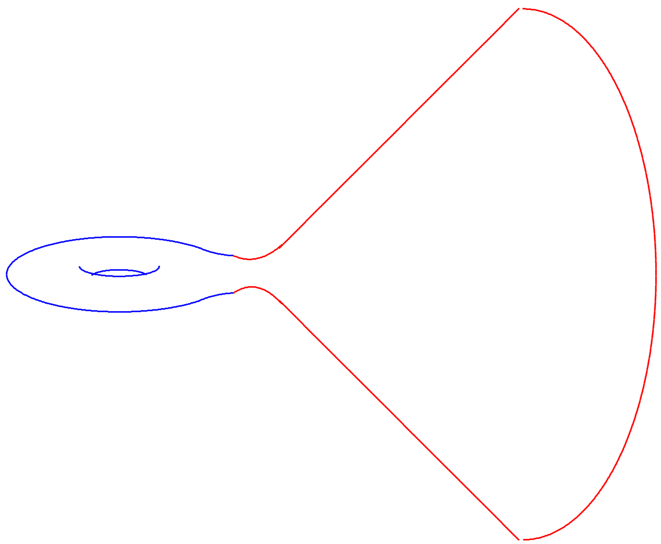

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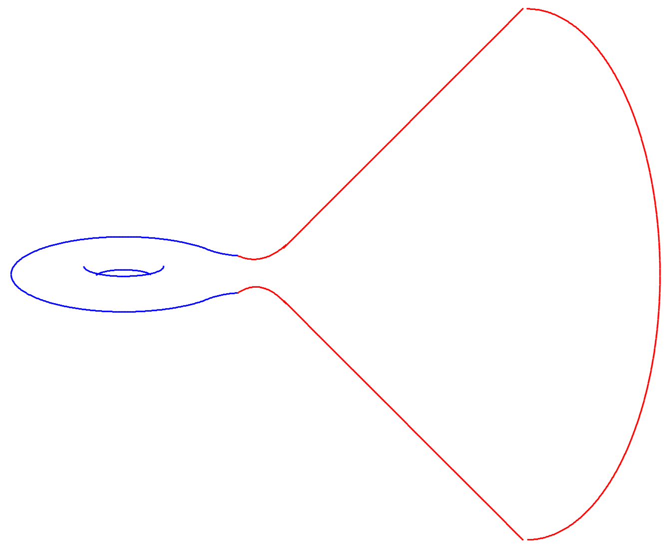

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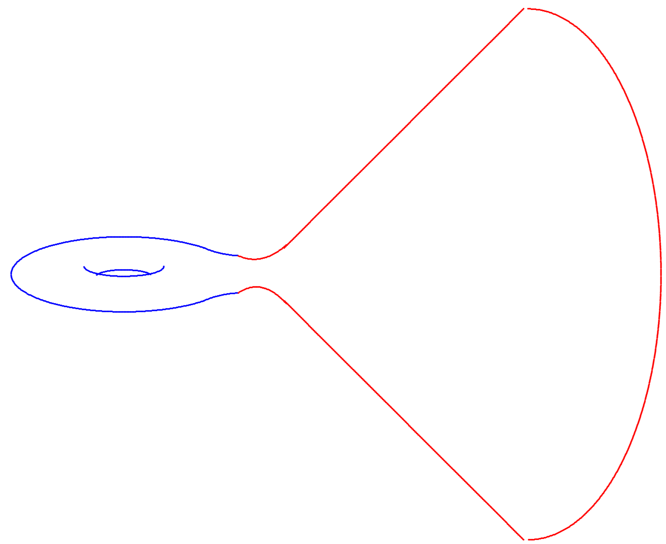

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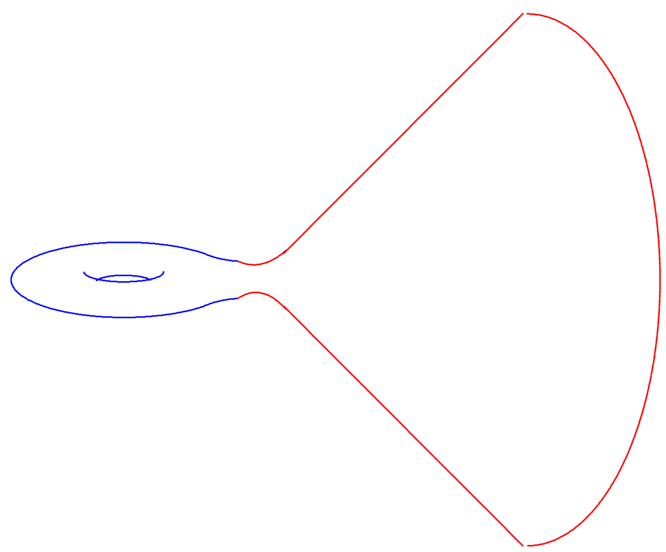

Upshot:

Mass of an ALE Kähler manifold is unambiguous.

Does not depend on the choice of an end!

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Theorems A & B are corollaries concerning scalar-flat Kähler metrics.

Another key consequence…

Theorem E (Penrose Inequality).

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This has an interesting corollary...

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Now use Bishop-Gromov inequality.

Some applications . . .

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Corollary. *Any ALE scalar-flat Kähler manifold (M, g, J) of complex dimension m has mass given by*

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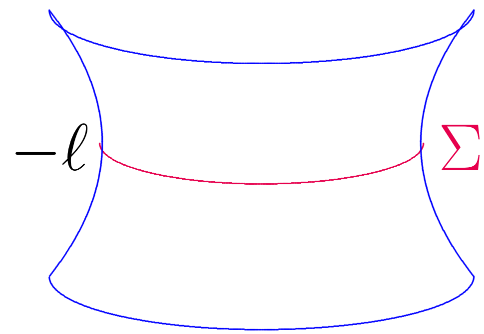
Example. Blow up Chern class $-\ell$ line bundle over $\mathbb{CP}_1$ at k points on zero section Σ.



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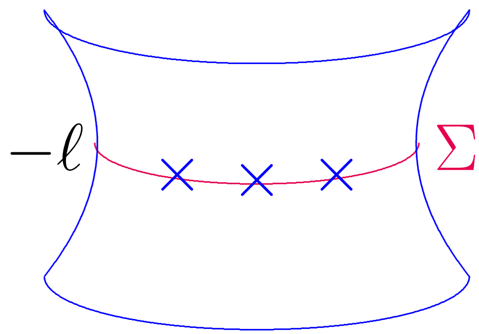

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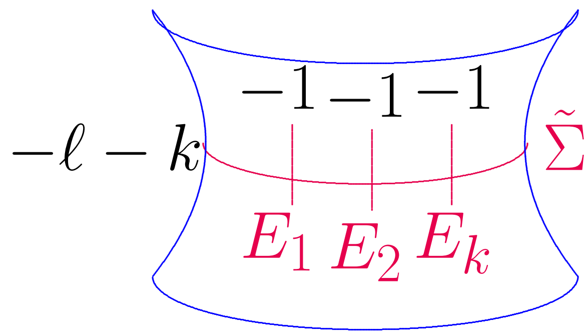

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That is, $\mathfrak{m}(M, g, J)$ *is completely determined by*

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In particular, this shows mass is coordinate-invariant, without ever invoking Bartnik-Chruściel!

Theorems A & B are corollaries concerning scalar-flat Kähler metrics.

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Theorems A & B are corollaries concerning scalar-flat Kähler metrics. But for such metrics, faster fall-off is guaranteed, so new proof is not actually needed!

How does one prove main results?

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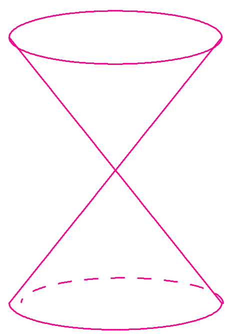

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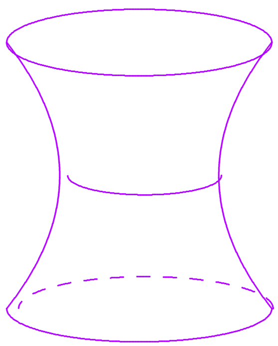

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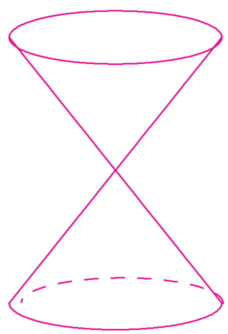

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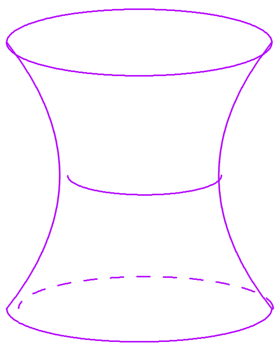

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Fortunately, however, the symplectic structure is always is standard at infinity!

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Quantitative version of Moser stability argument. . .

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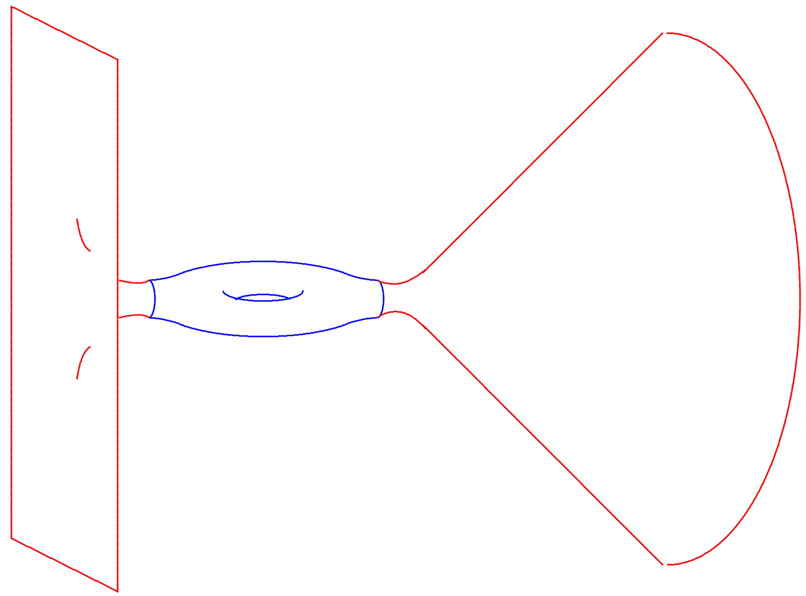

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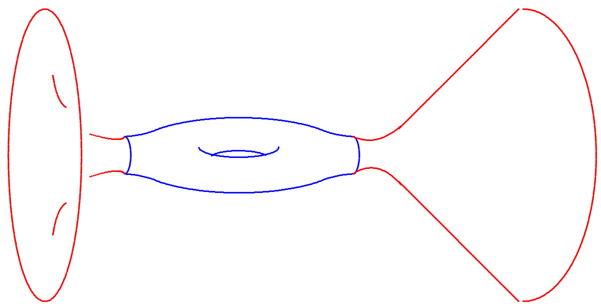

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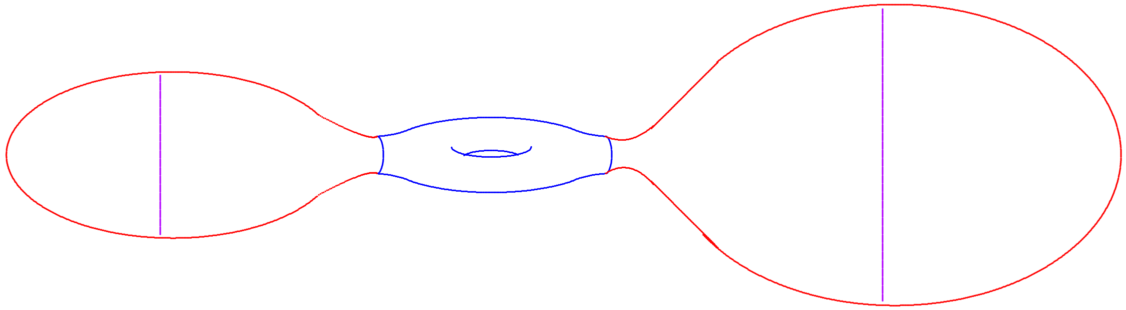

- Truncate at large radius;

- Glue in standard plugs.

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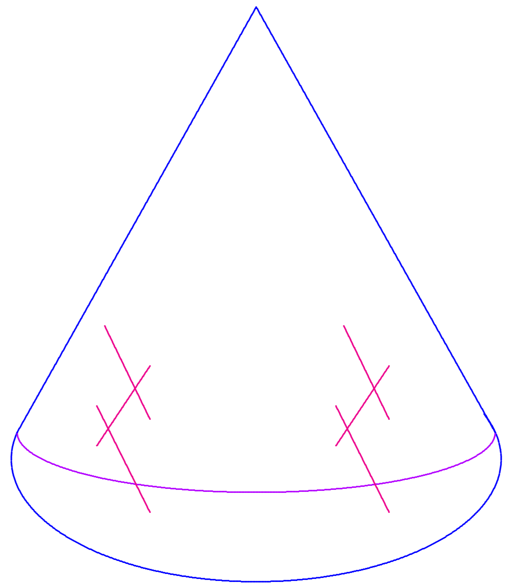

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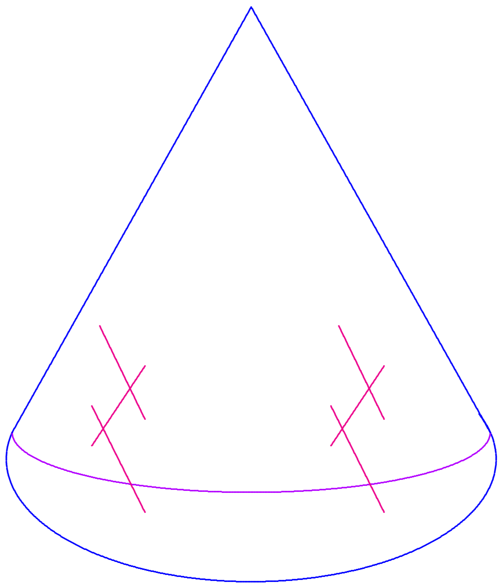

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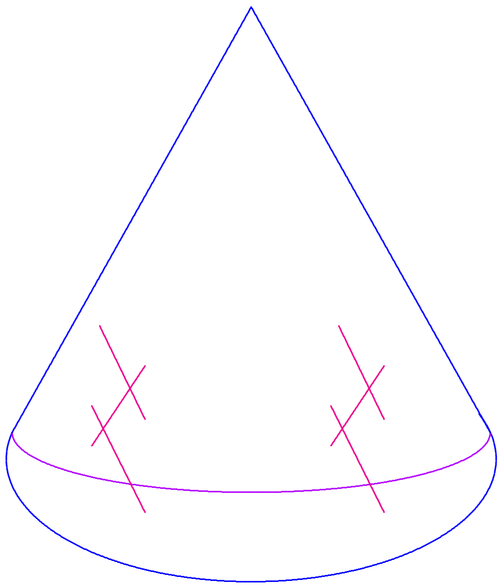

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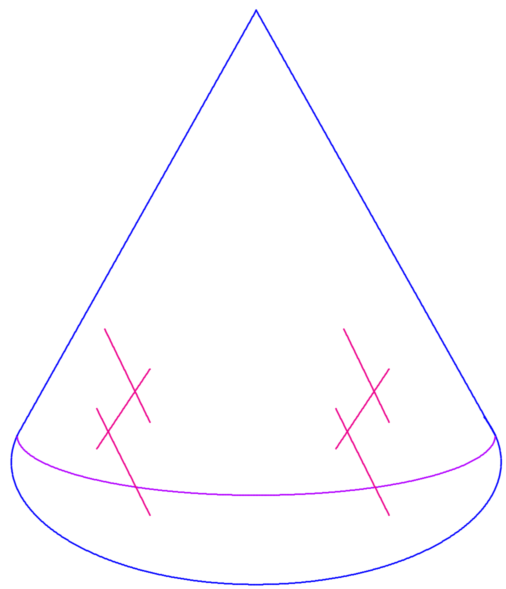

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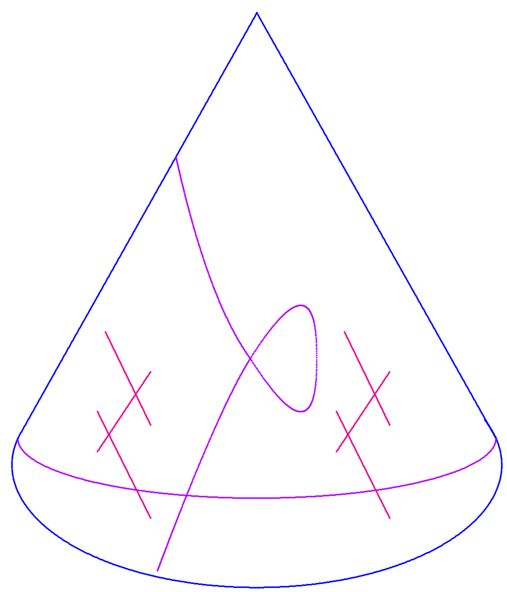

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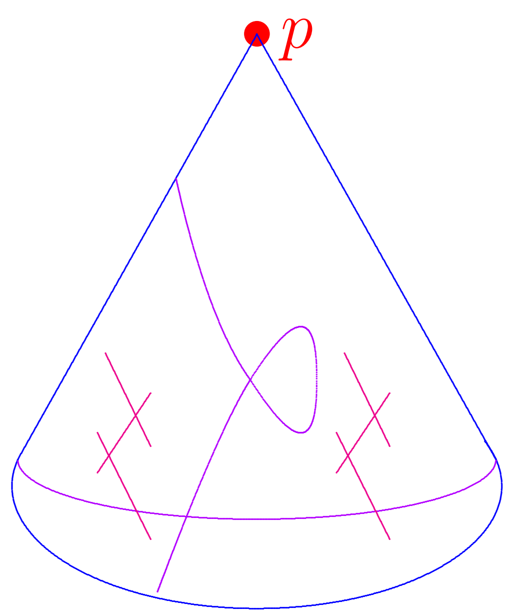

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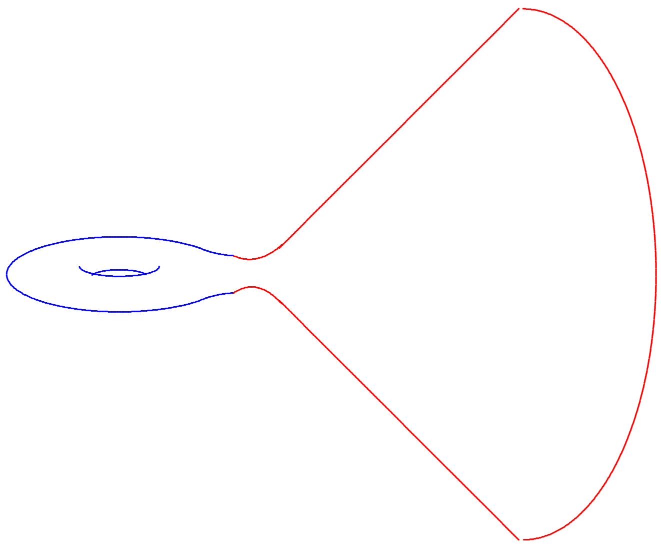
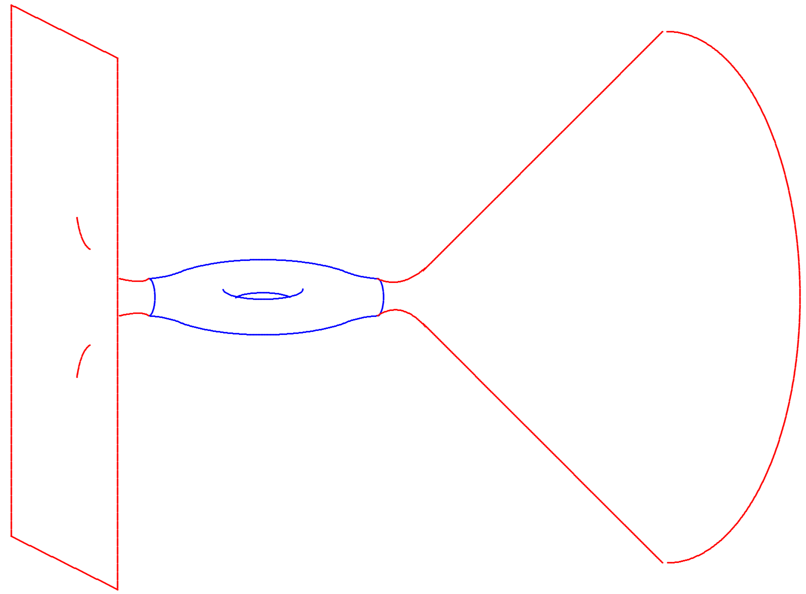
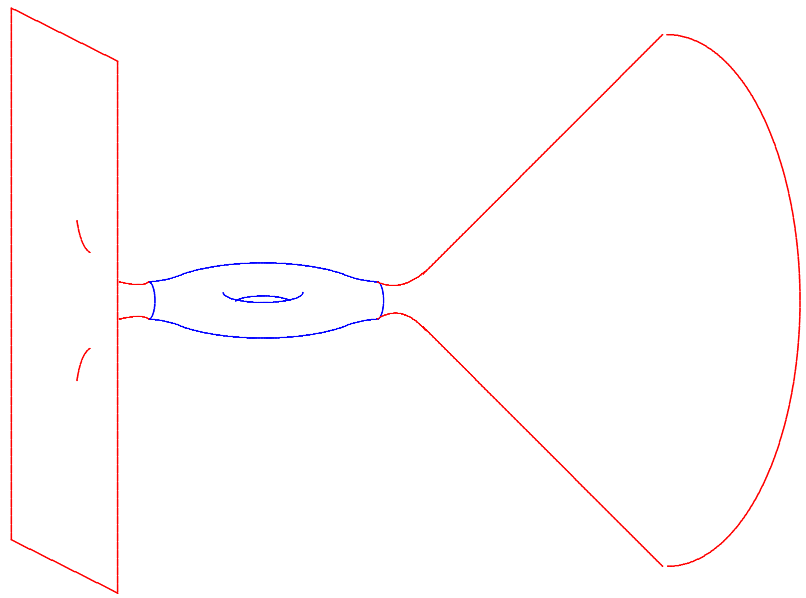
Lemma. *Any ALE Kähler manifold has only one end.*

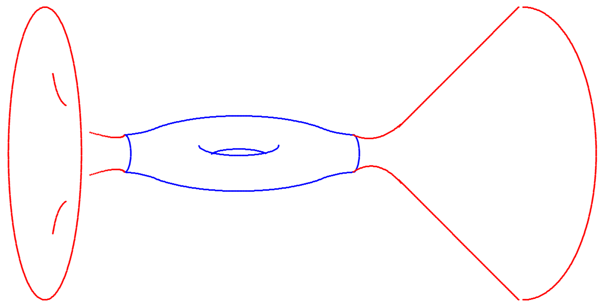




What if M^4 has more than one end?



Compactify M as symplectic 4-manifold $\widehat{M}$.



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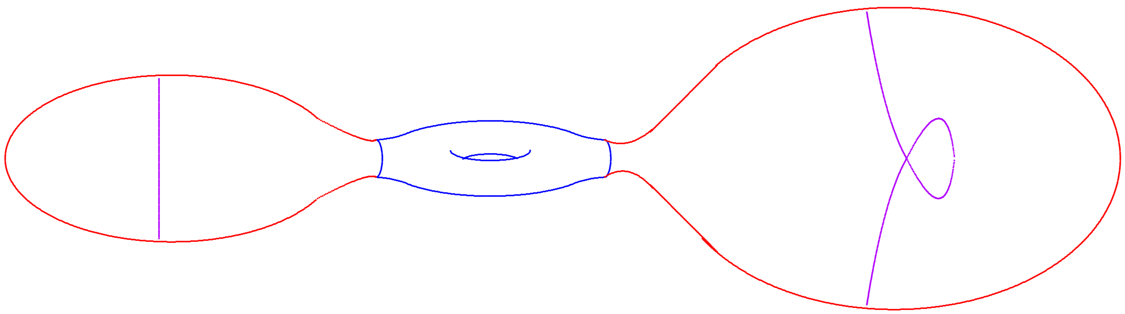

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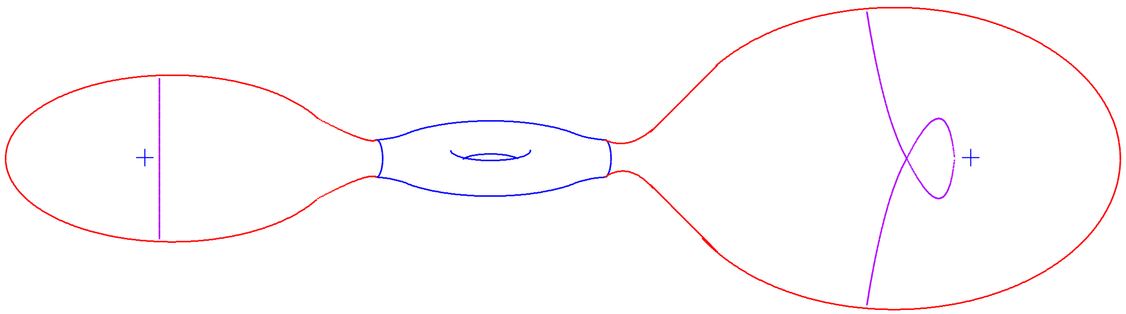

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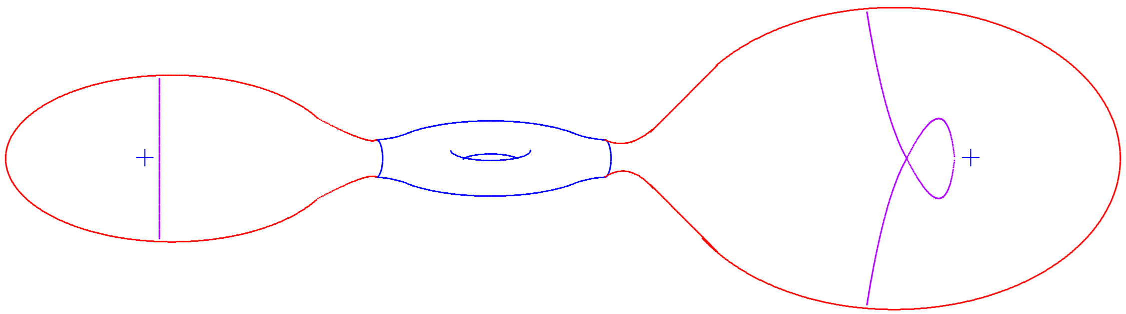

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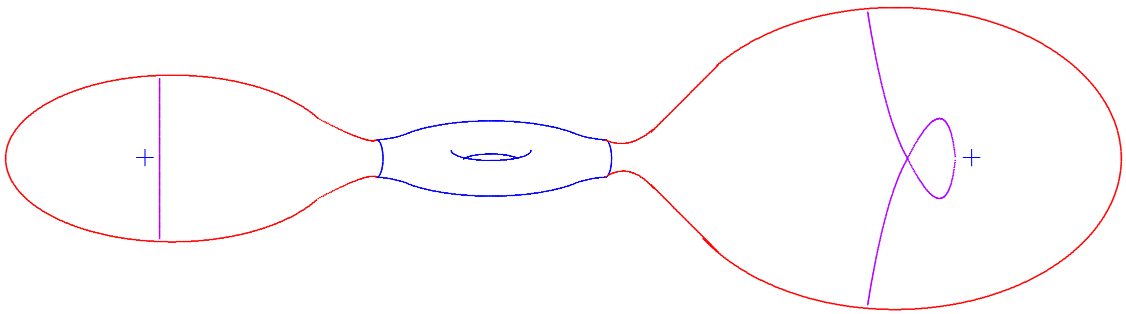
McDuff $\Longrightarrow$ $\widehat{M} \approx$ rational complex surface.



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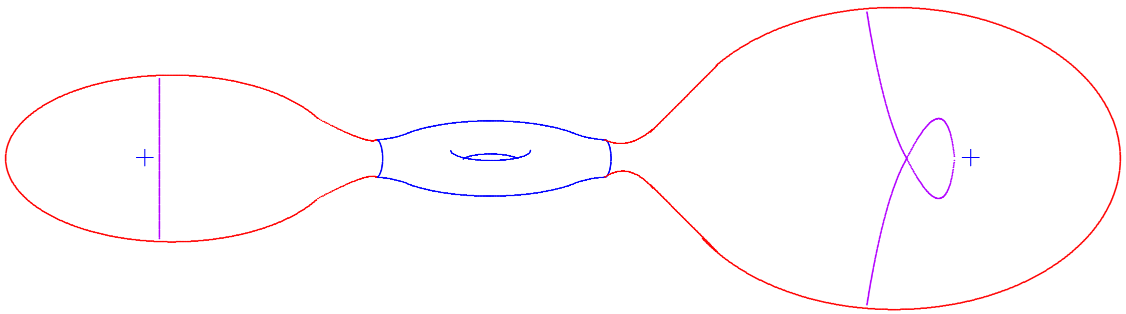
McDuff $\Longrightarrow$ intersection form $(+-\cdots-)$.



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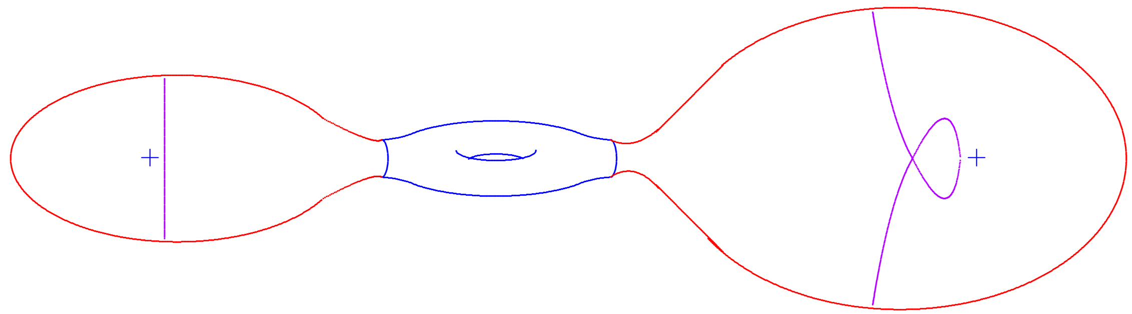
McDuff $\Longrightarrow b_+(M) = 1$.



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Since each end contributes positive direction. . .

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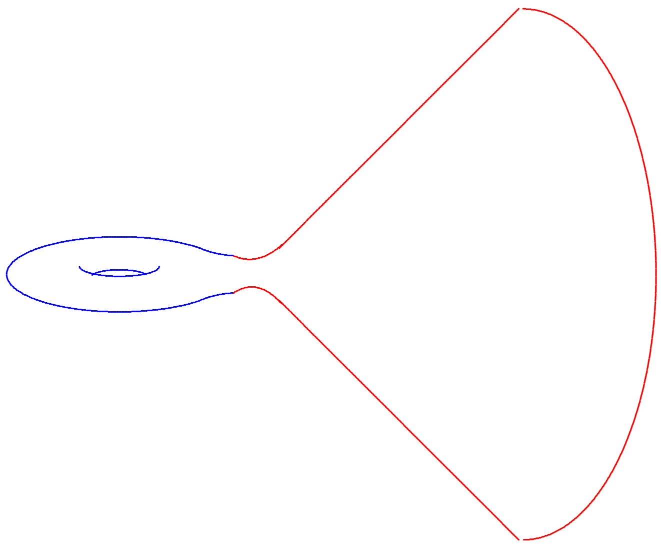

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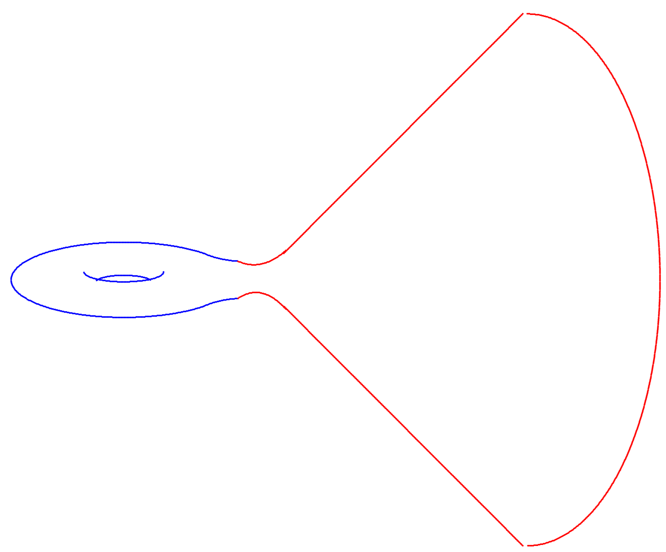

[In higher dimensions, one similarly shows that (M, J) can be compactified as Kähler orbifold. The Hodge theorem on intersection form instead tells one that form on $H^{1,1}(\widehat{M}, \mathbb{R})$ is of type $(+- \cdots -)$.]

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Would be happy to explain this during Q & A time.

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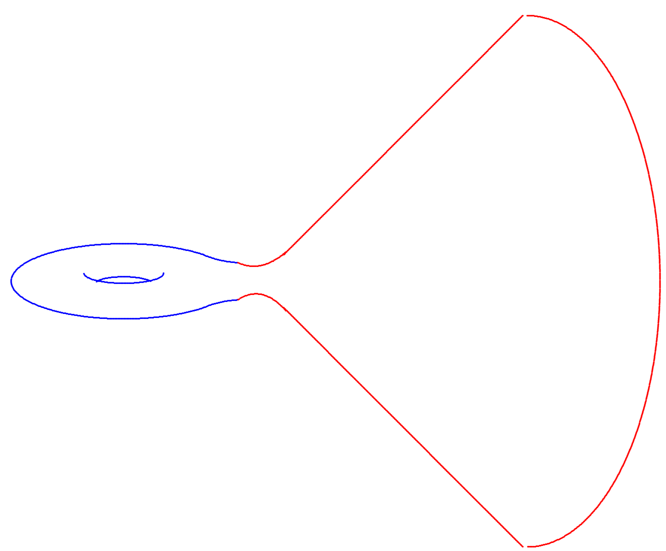
But the Penrose-type inequality is more subtle.

Theorem E (Penrose Inequality). *Let* (M^{2m}, g, J) *be an AE Kähler manifold with scalar curvature* $s \geq 0$. *Then* (M, J) *carries a canonical divisor* D *that is expressed as a sum* $\sum_j \mathbf{n}_j D_j$ *of compact complex hypersurfaces with positive integer coefficients, with the property that* $\bigcup_j D_j \neq \varnothing$ *whenever* $M \not\approx \mathbb{R}^{2m}$. *In terms of this divisor, we then have*

$$\mathfrak{m}(M, g) \geq \frac{(m-1)!}{(2m-1)\pi^{m-1}} \sum \mathbf{n}_j \, Vol(D_j)$$

with $= \iff (M, g, J)$ *is scalar-flat Kähler.*

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1} \rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m} \int_M s_g d\mu_g$$



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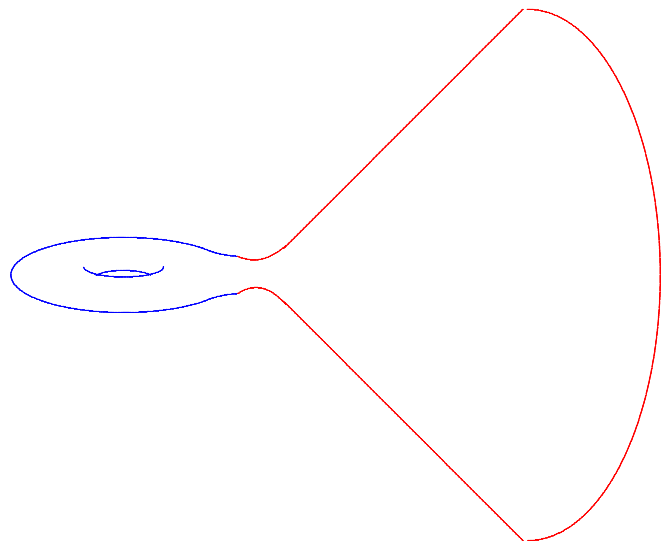
Technical challenge: Loss of control of derivatives!

Distance-decreasing map:



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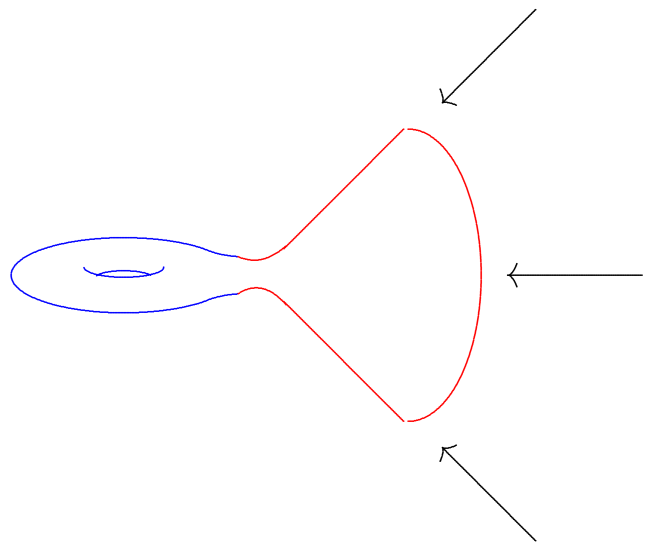

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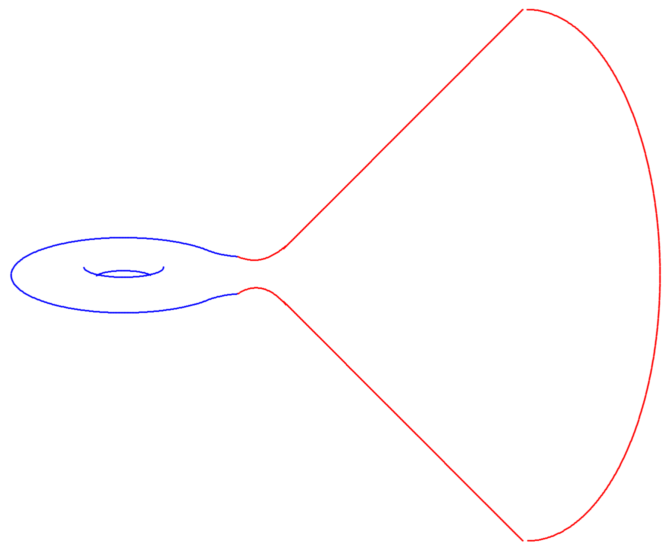

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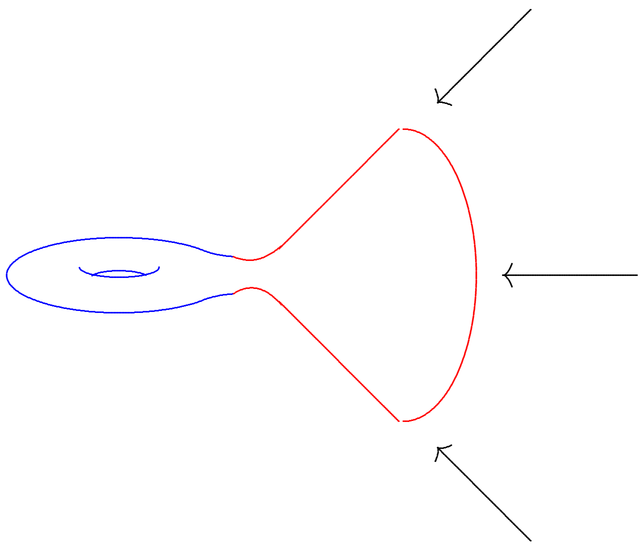

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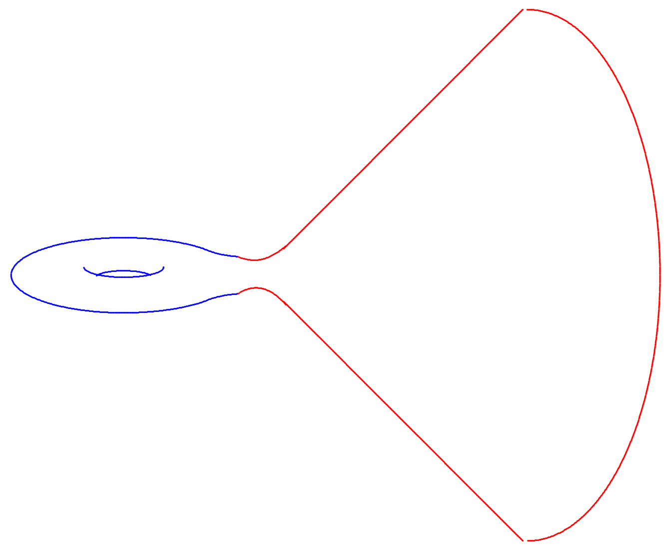

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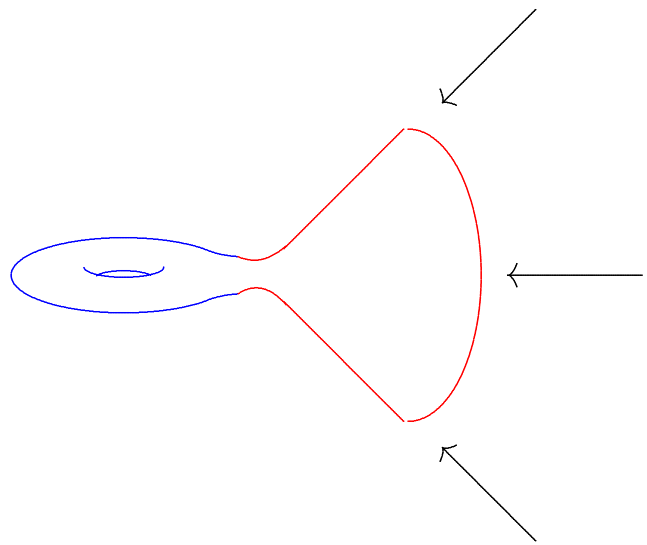

Robust under distortion of metric in outer region.

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In (M, J), this gives desired Poincaré dual of $-c_1$.

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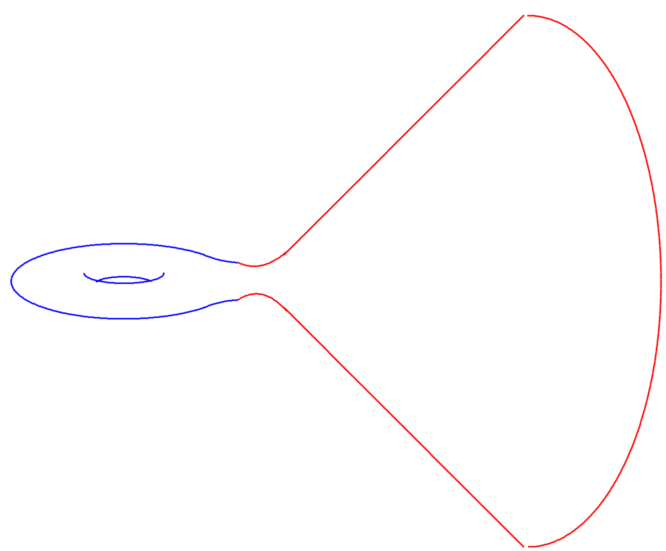

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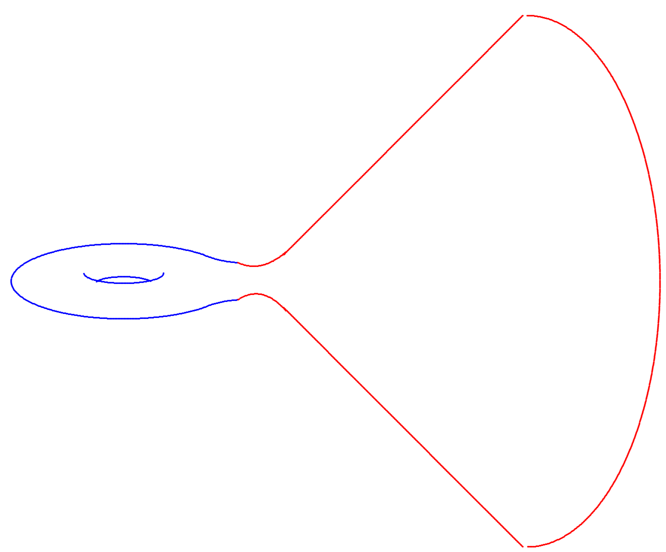

Ta for the mateship!

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How does one prove mass formula?

Proposition. *Let* g *be a* C^2 *Kähler metric on* $(\mathbb{R}^{2m} - \mathbf{D}^{2m})/\Gamma$, $m \geq 2$, *where* $\Gamma \subset \mathbf{SO}(2m)$ *is some finite group that acts without fixed-points on* S^{2m-1}.

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Integrating on S_ϱ/Γ therefore yields:

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This now implies the mass formula. . .

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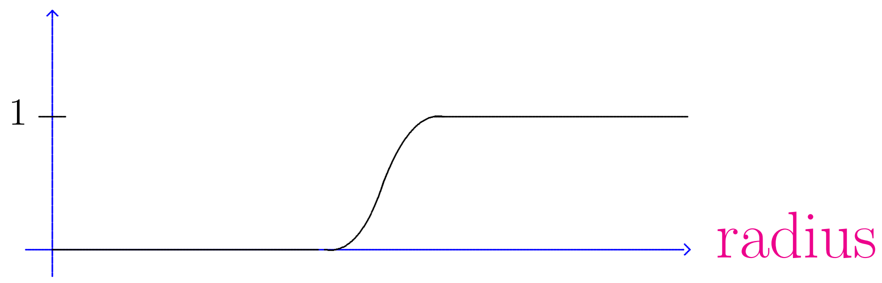

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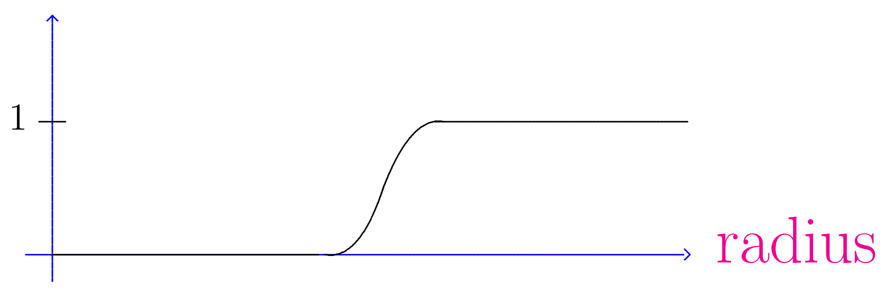

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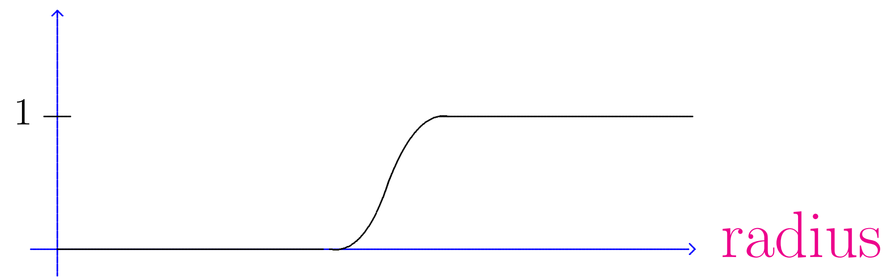

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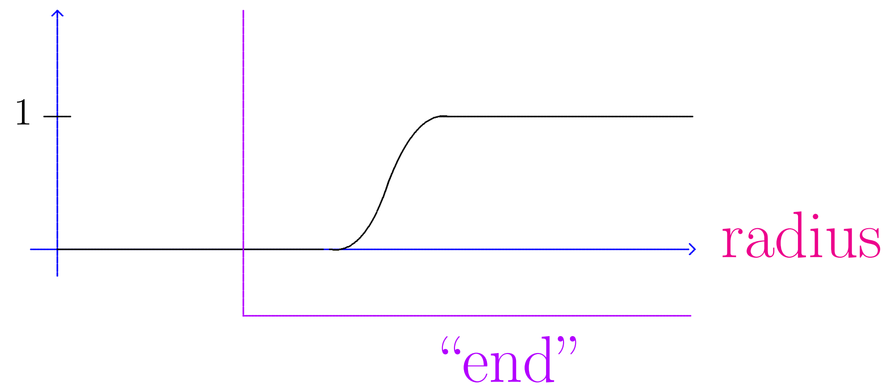

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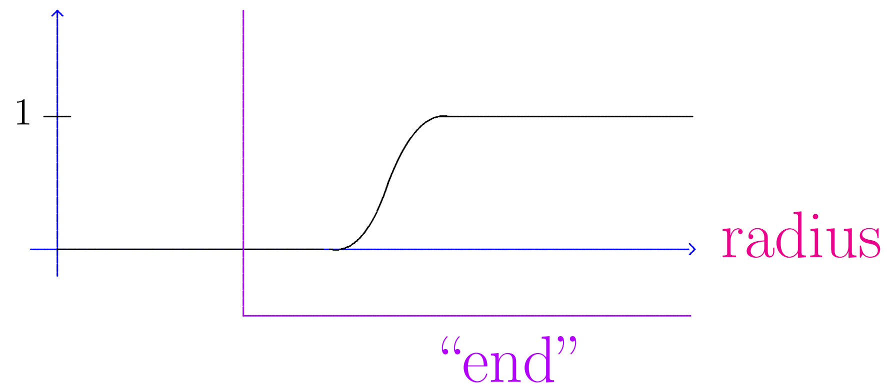

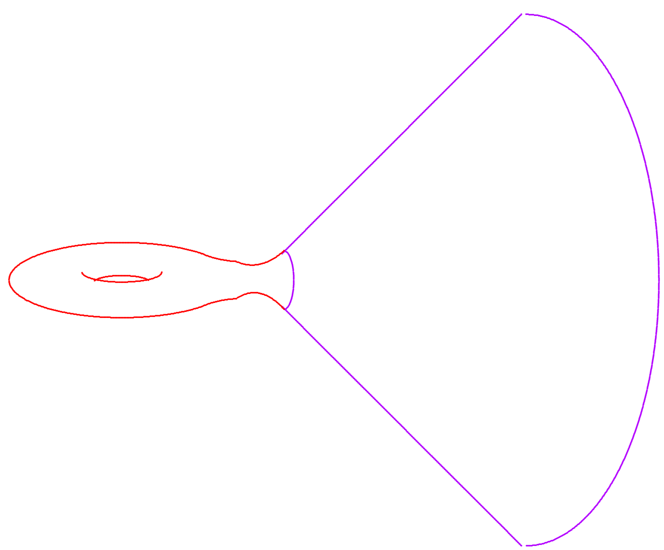

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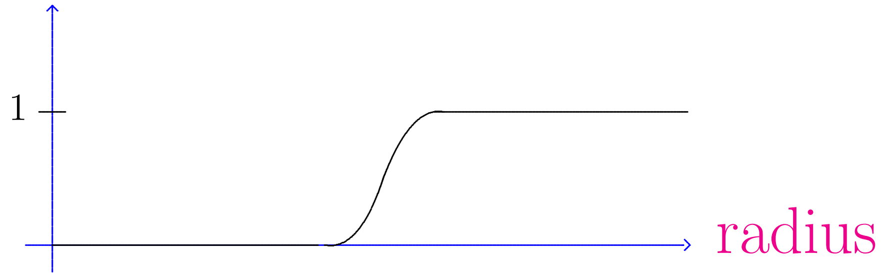

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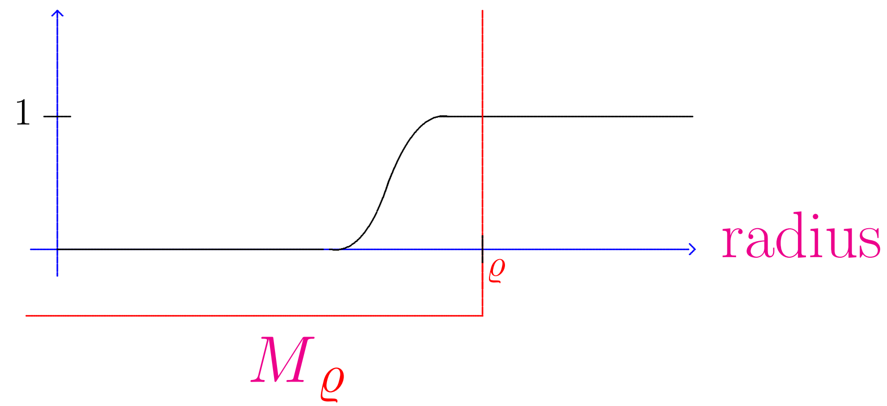

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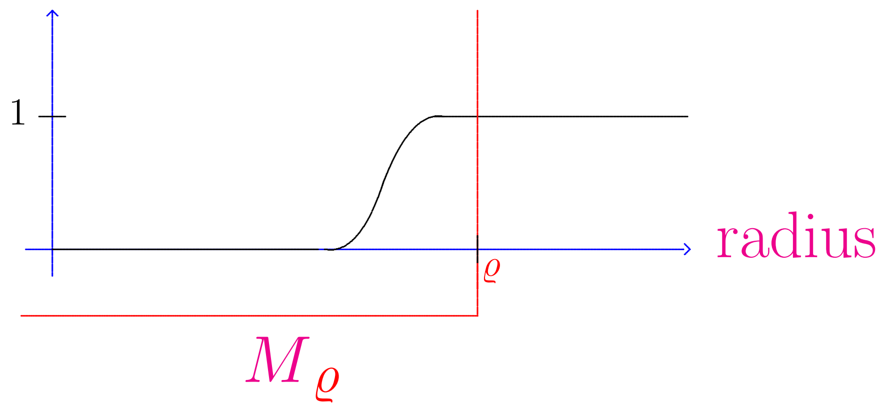

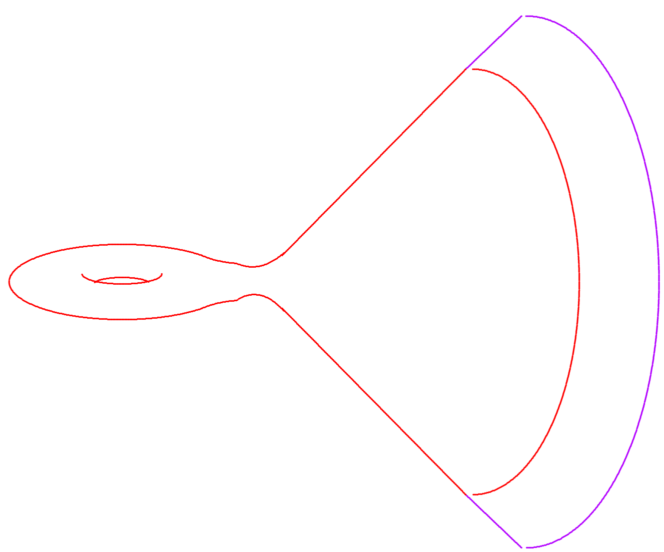

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Compactly supported, because $d\theta = \rho$ near infinity.

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$$\langle 2\pi\clubsuit(c_1), [\omega]^{m-1}\rangle = \frac{(m-1)!}{2}\int_{M_\varrho} s\, d\mu - \int_{M_\varrho} d(f\theta\wedge\omega^{m-1})$$

Let $f : M \to \mathbb{R}$ be smooth cut-off function:

$\equiv 0$ away from end,

$\equiv 1$ near infinity.

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by Stokes' theorem.

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because there is only one end!

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Set

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$$[\psi] = \clubsuit([\rho]) = 2\pi\clubsuit(c_1) \in H^2_c(M)$$

$$\frac{2}{(m-1)!}\int_{S_\varrho/\Gamma} \theta\wedge\omega^{m-1} = -\frac{4\pi\langle\clubsuit(c_1), [\omega]^{m-1}\rangle}{(m-1)!} + \int_{M_\varrho} s\, d\mu$$

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$$\int_{S_\varrho/\Gamma} \left[g_{kj,k} - g_{kk,j}\right] \mathbf{n}^j d\mathfrak{a}_E =$$

$$-\frac{4\pi\langle\clubsuit(c_1), [\omega]^{m-1}\rangle}{(m-1)!} + \int_{M_\varrho} s \, d\mu \quad + O(\varrho^{-2\varepsilon})$$

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Limit as $\varrho \to \infty$ now yields the mass formula.

$$\frac{4\pi^m(2m-1)}{(m-1)!}\mathfrak{m}(M,g) = -\frac{4\pi}{(m-1)!}\langle \clubsuit(c_1), [\omega]^{m-1}\rangle + \int_M s_g d\mu_g$$

$$\mathfrak{m}(M,g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g \, d\mu_g$$

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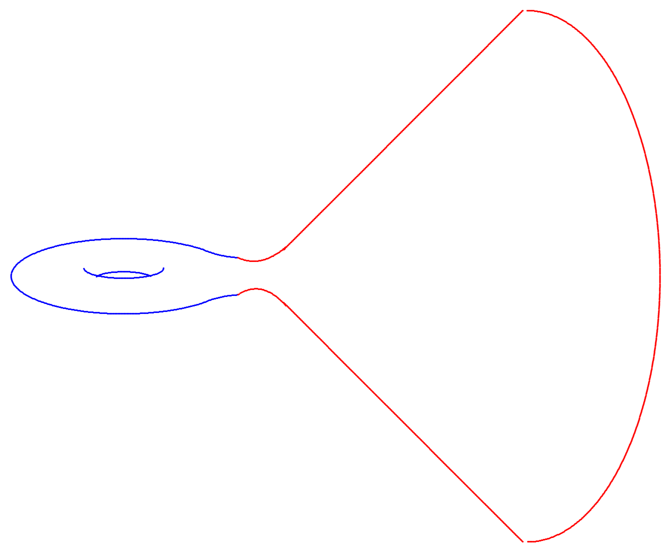

$$\mathscr{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1} \rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m} \int_M s_g \, d\mu_g$$

$$\mathfrak{m}(M,g) = -\frac{\langle ♣(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g d\mu_g$$