

*Mass, Kähler Manifolds, &*

*Symplectic Geometry*

Claude LeBrun

Stony Brook University

Incontro INdAM:

“Current Developments in Complex Geometry,”  
al Palazzone di Cortona, il 5 giugno, 2019

Mass, Kähler Manifolds,  
and Symplectic Geometry

arXiv: 1810.11417 [math.DG]

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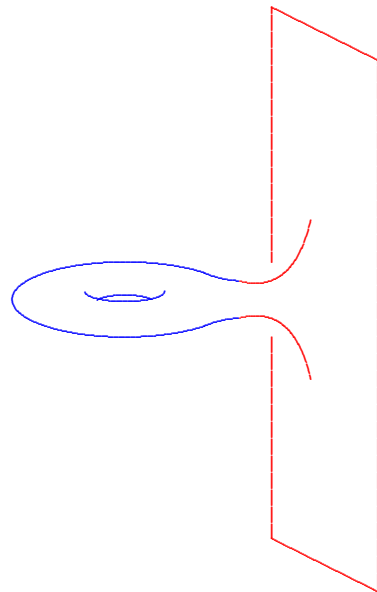
Builds on previous paper

Mass in Kähler Geometry

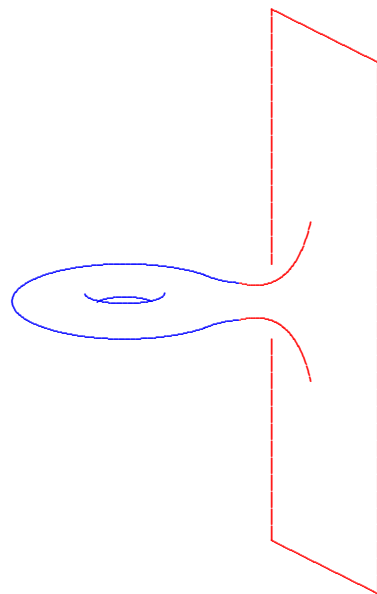
Comm. Math. Phys. 347 (2016) 621–653.

(Joint with Hans-Joachim Hein)

**Definition.** A complete, non-compact Riemannian  $n$ -manifold  $(M^n, g)$

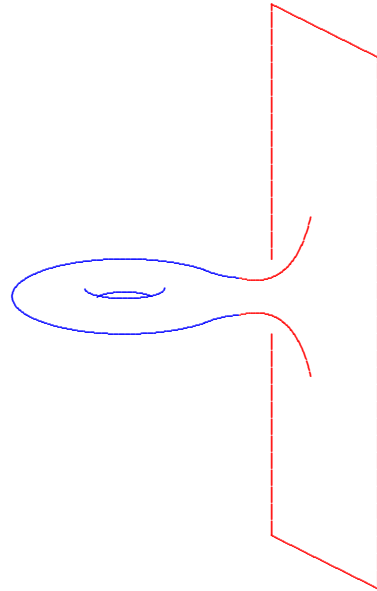


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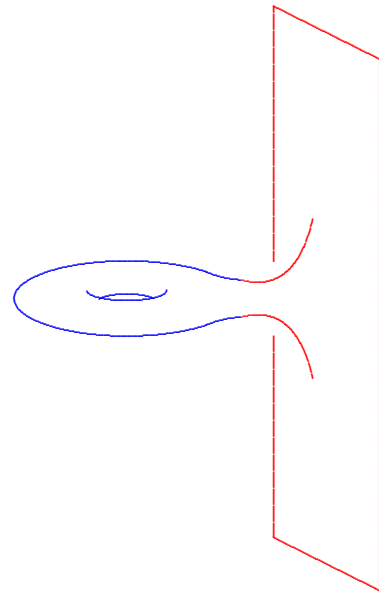
$$n \geq 3$$

**Definition.** A complete, non-compact Riemannian  $n$ -manifold  $(M^n, g)$  is called asymptotically Euclidean



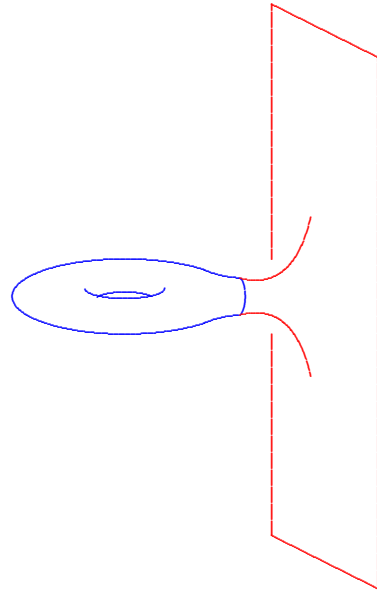
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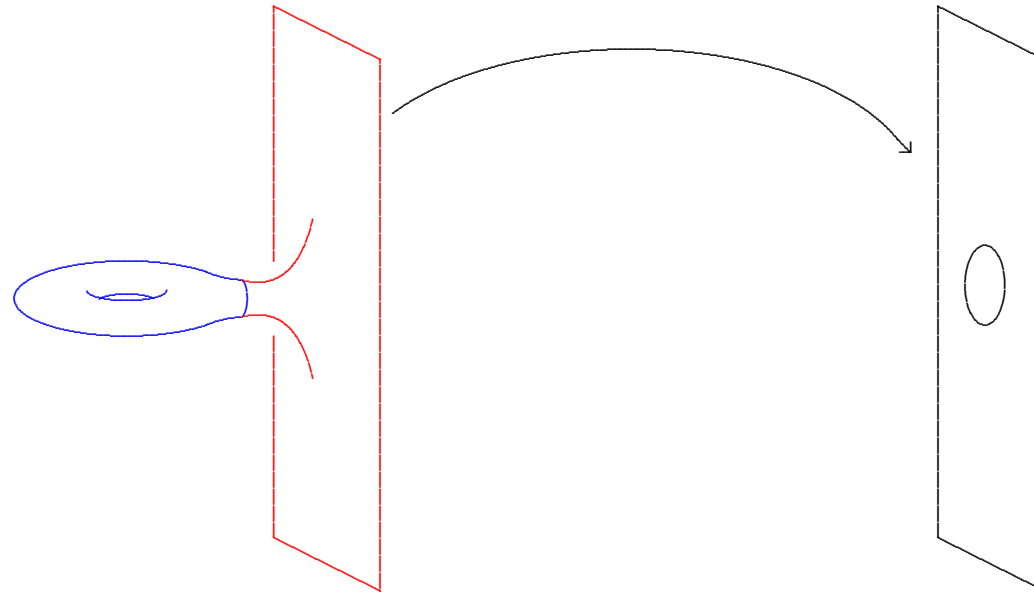
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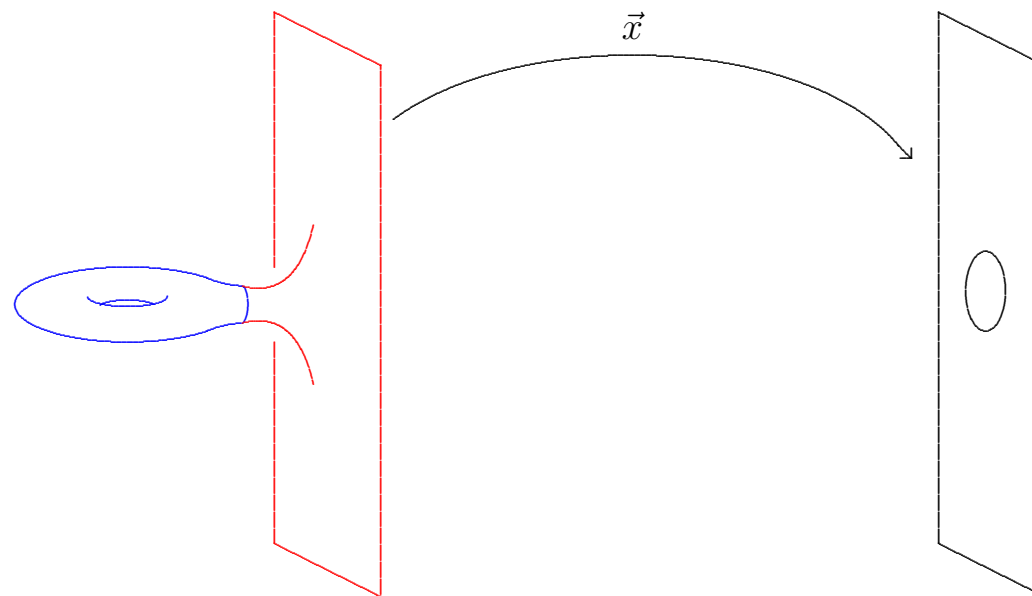




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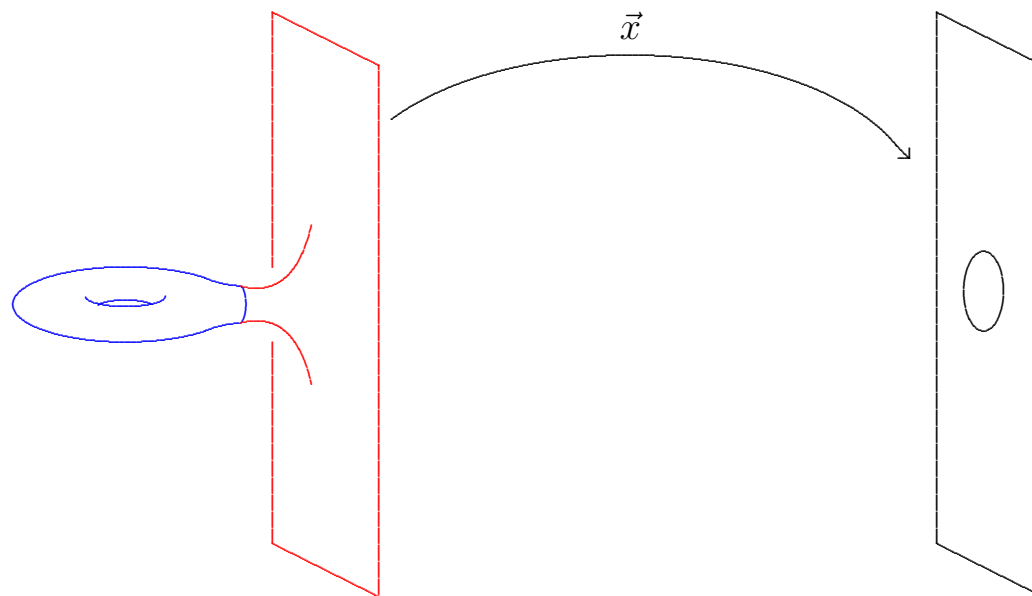


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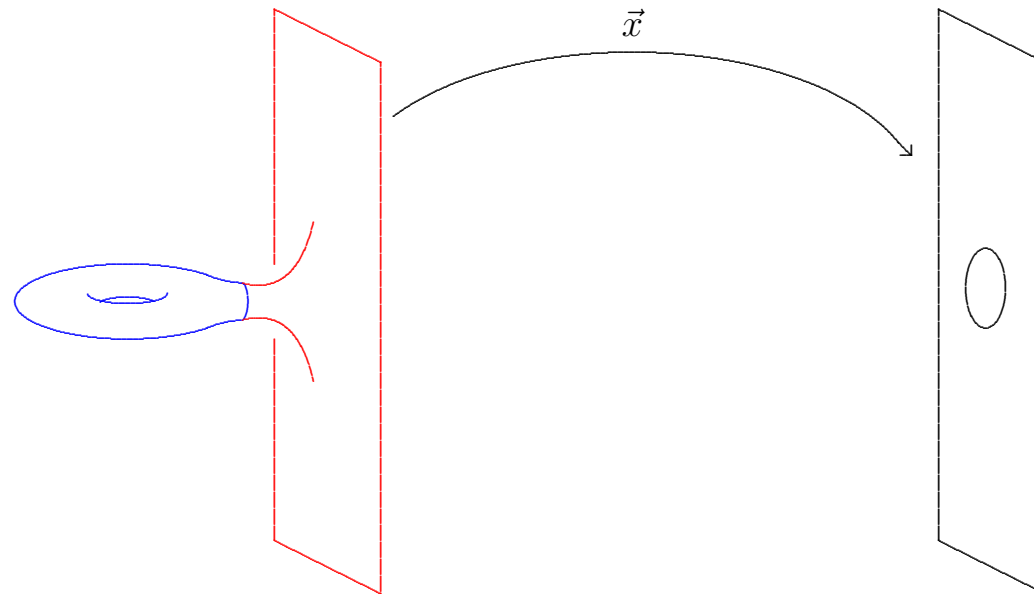
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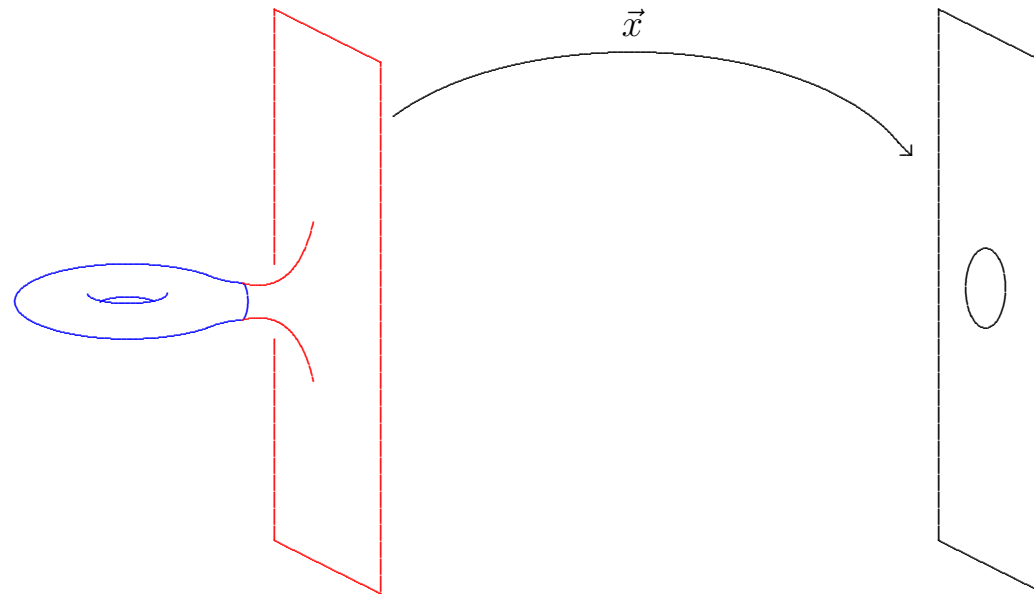
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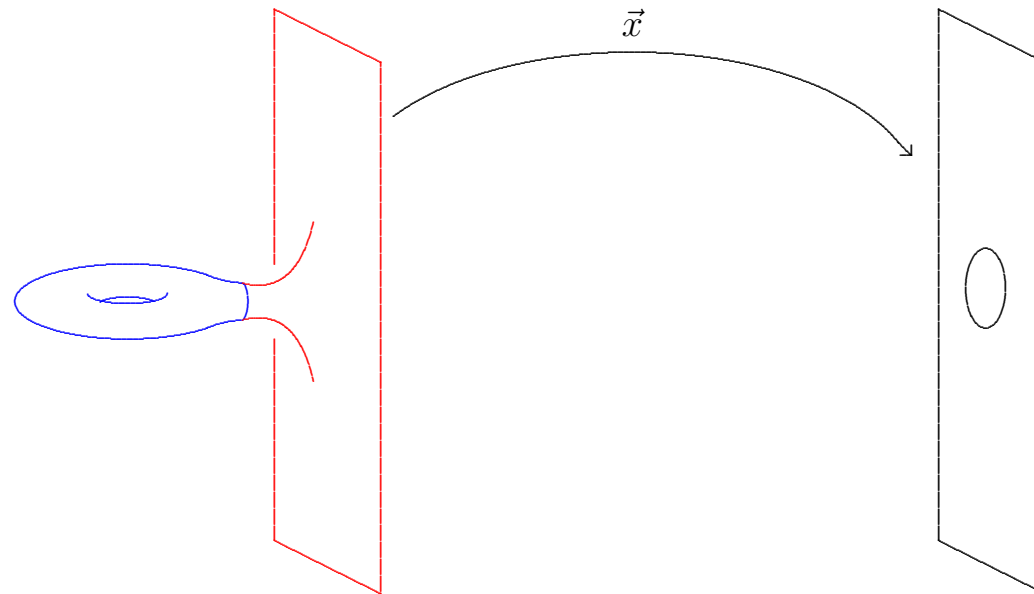
**Chruściel-type fall-off:**

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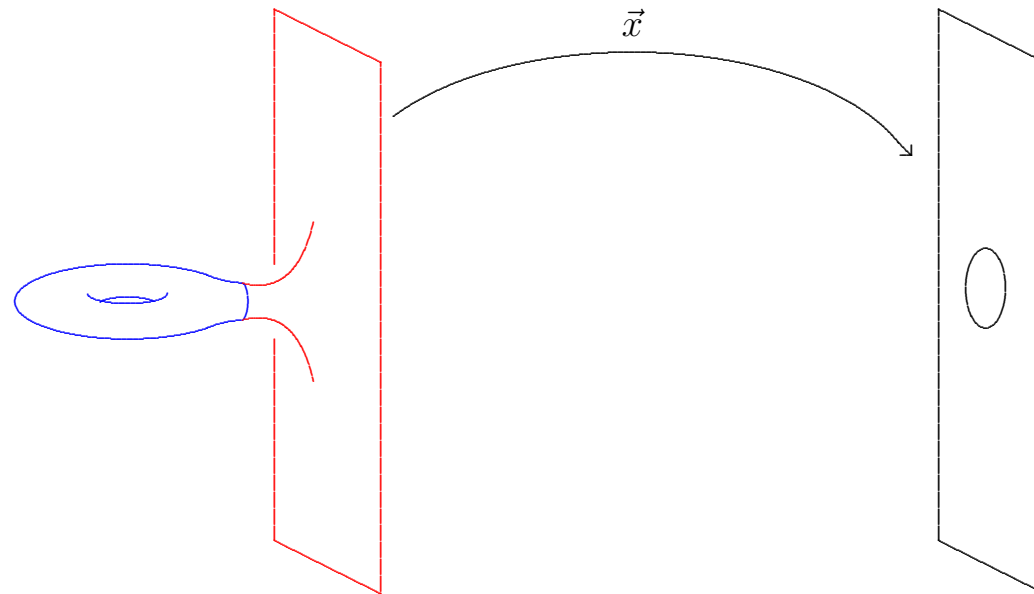
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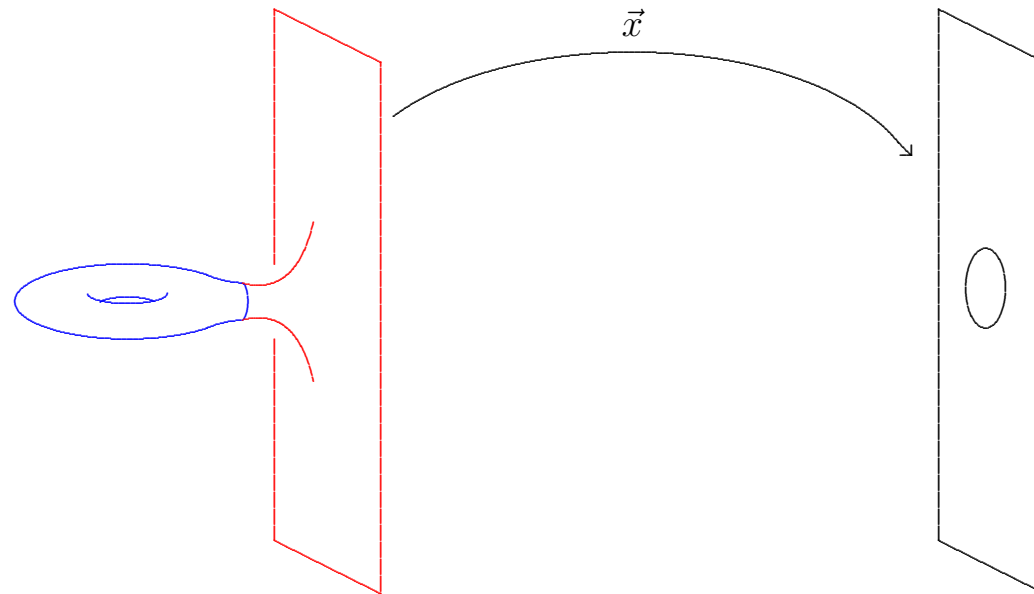
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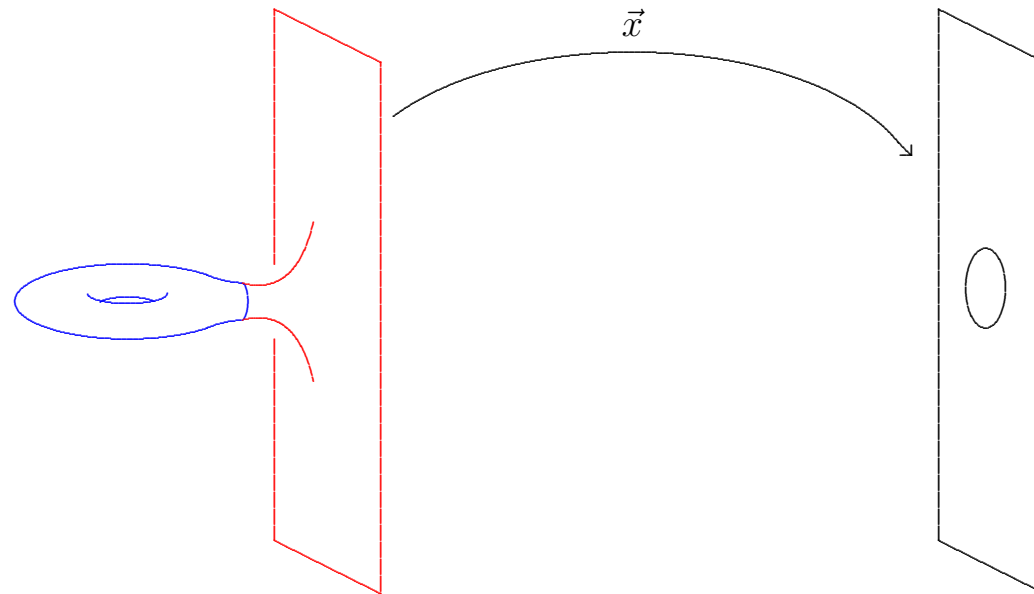


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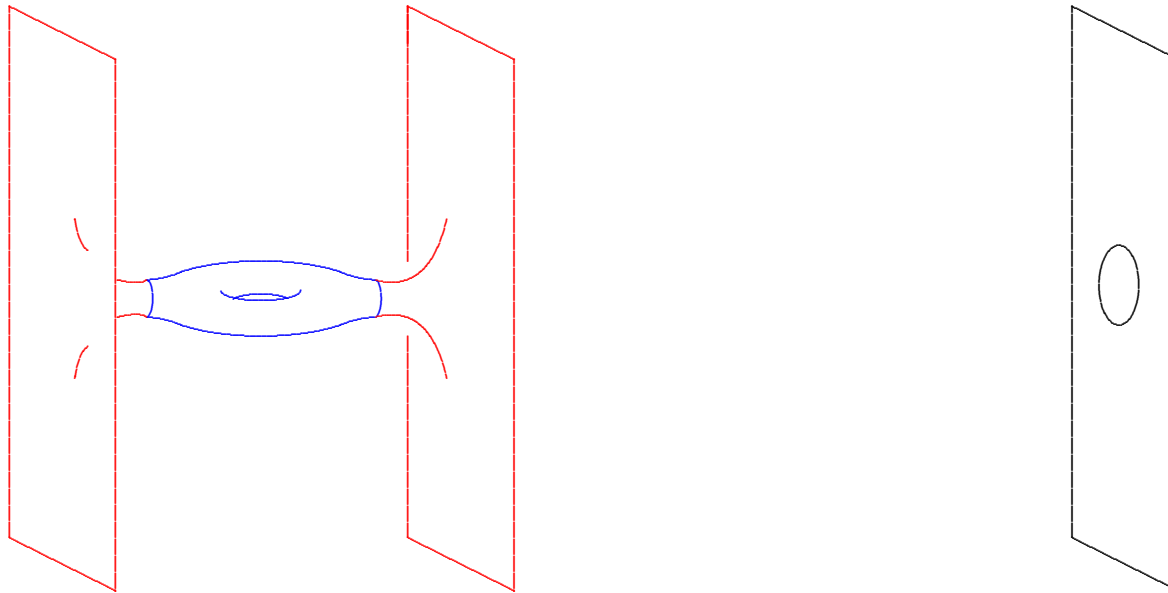
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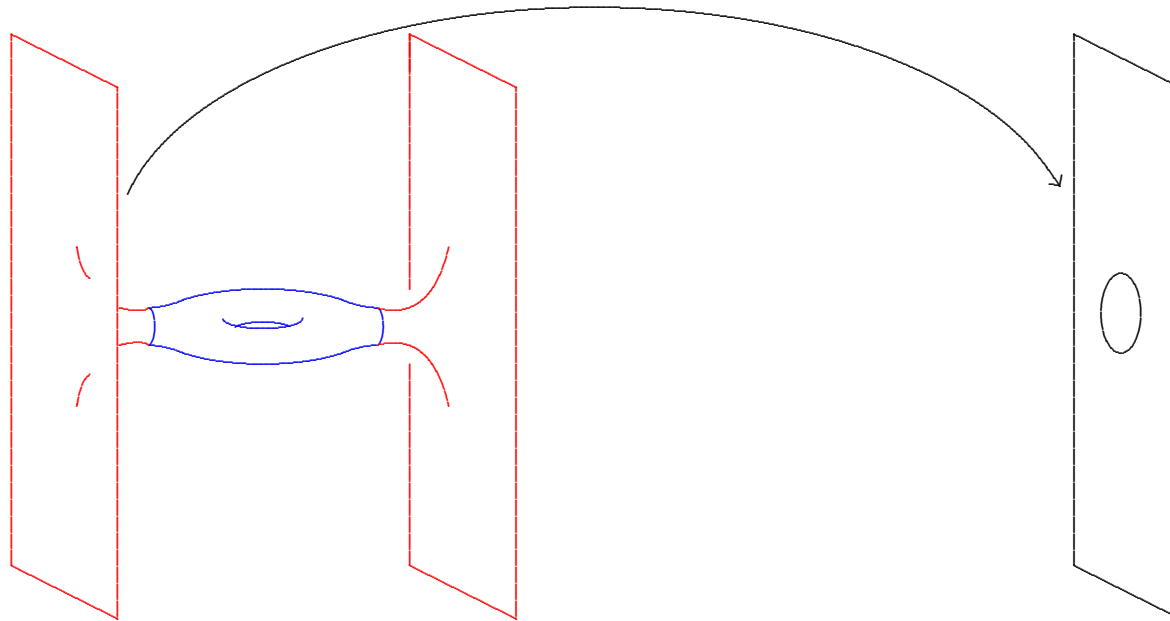
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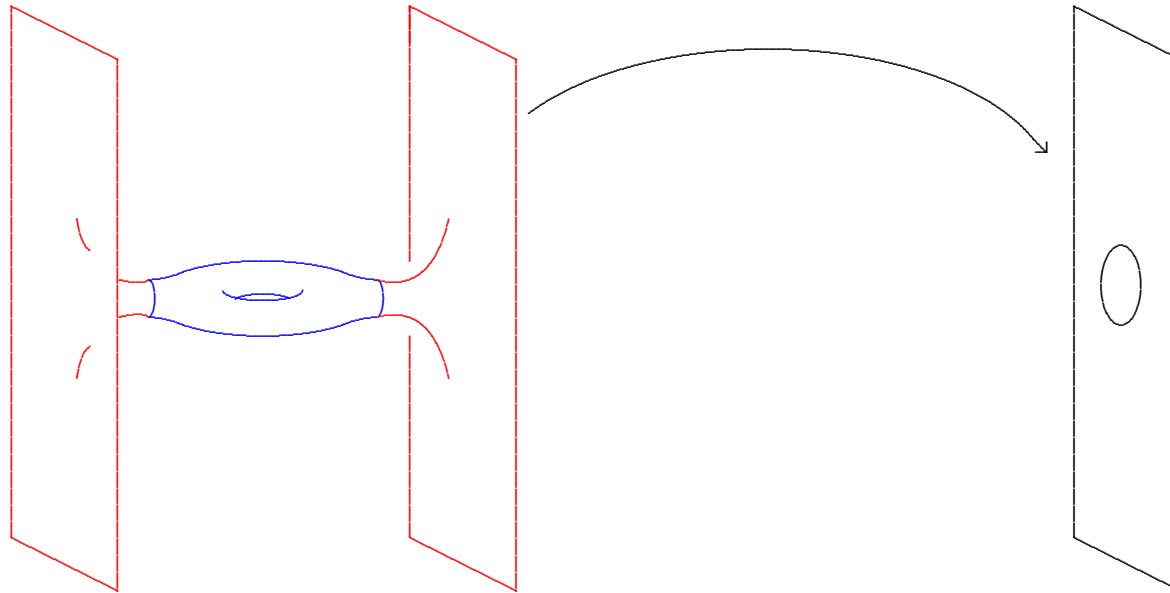
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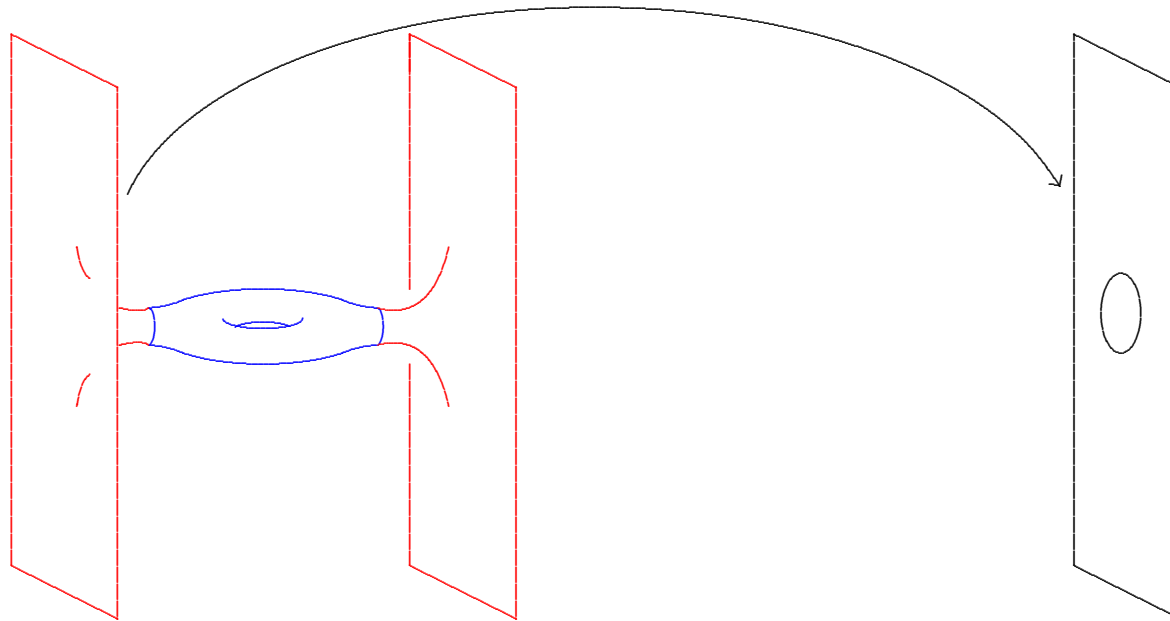
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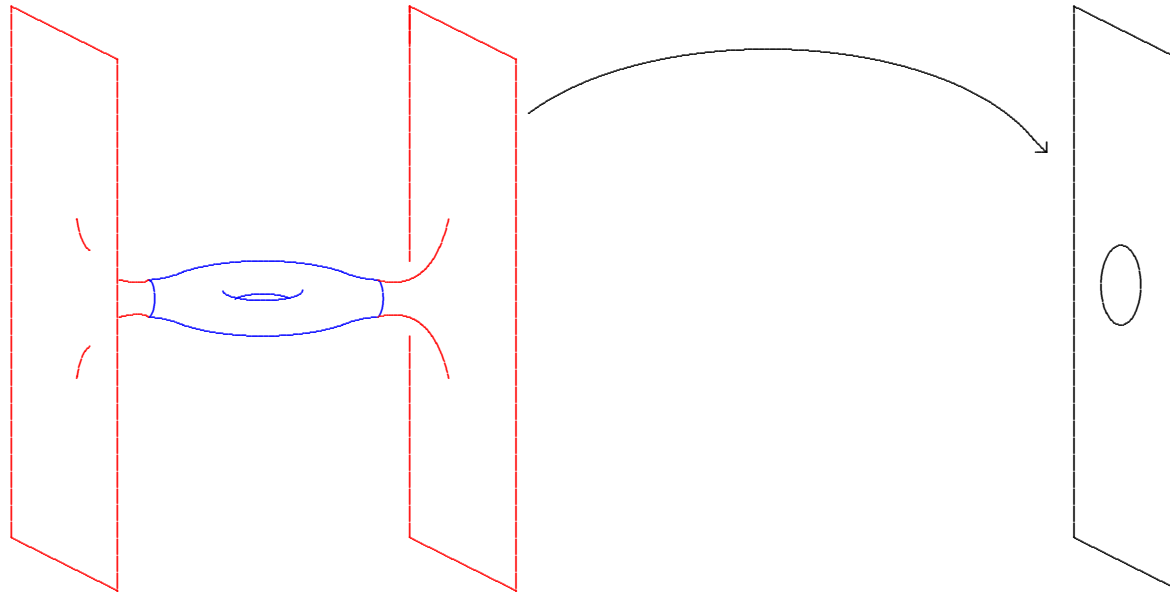
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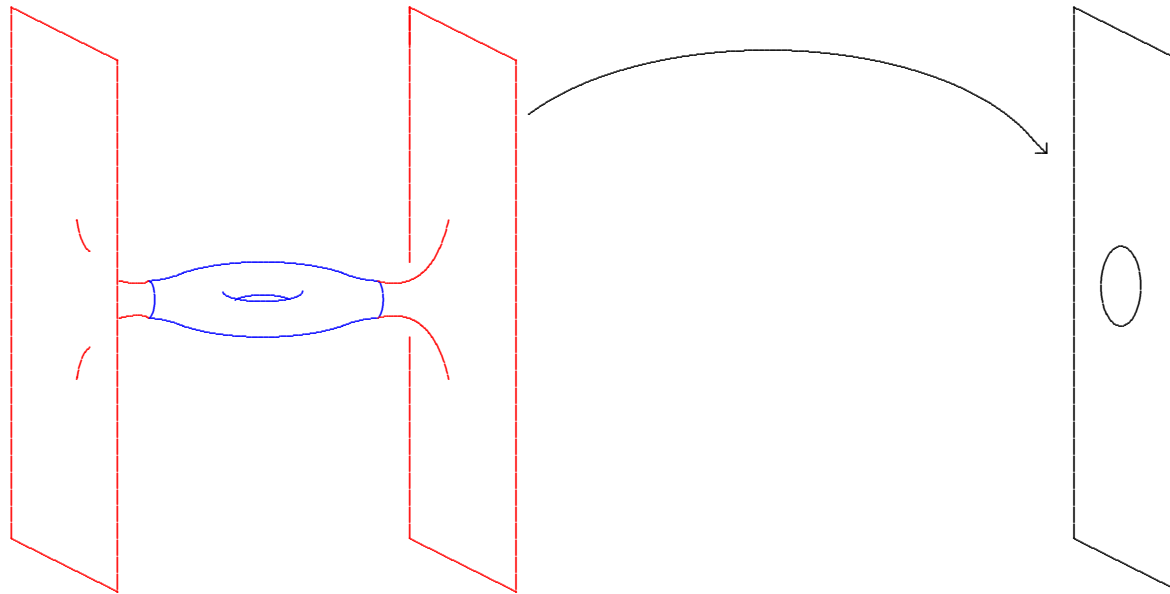
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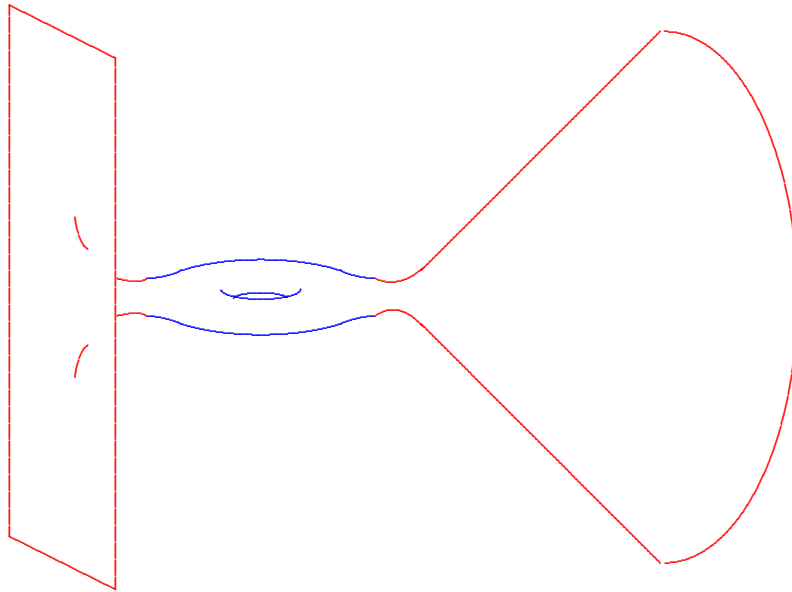
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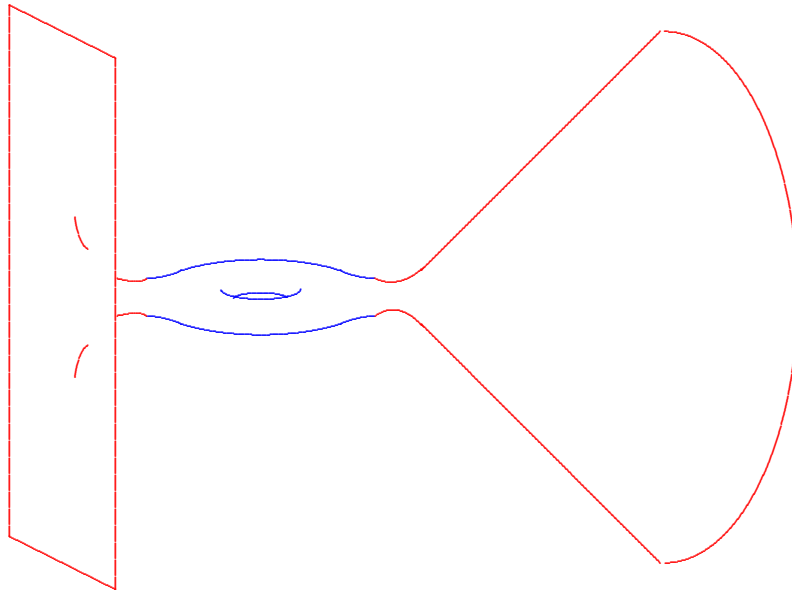
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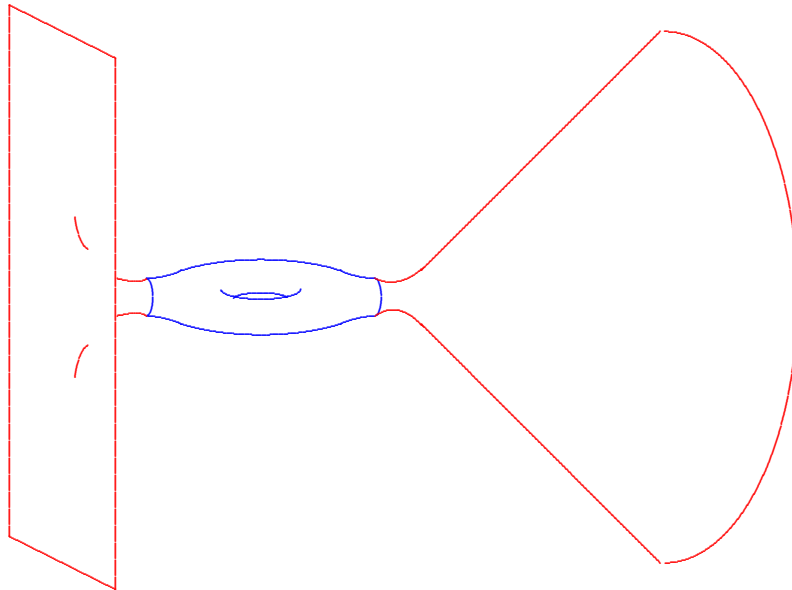




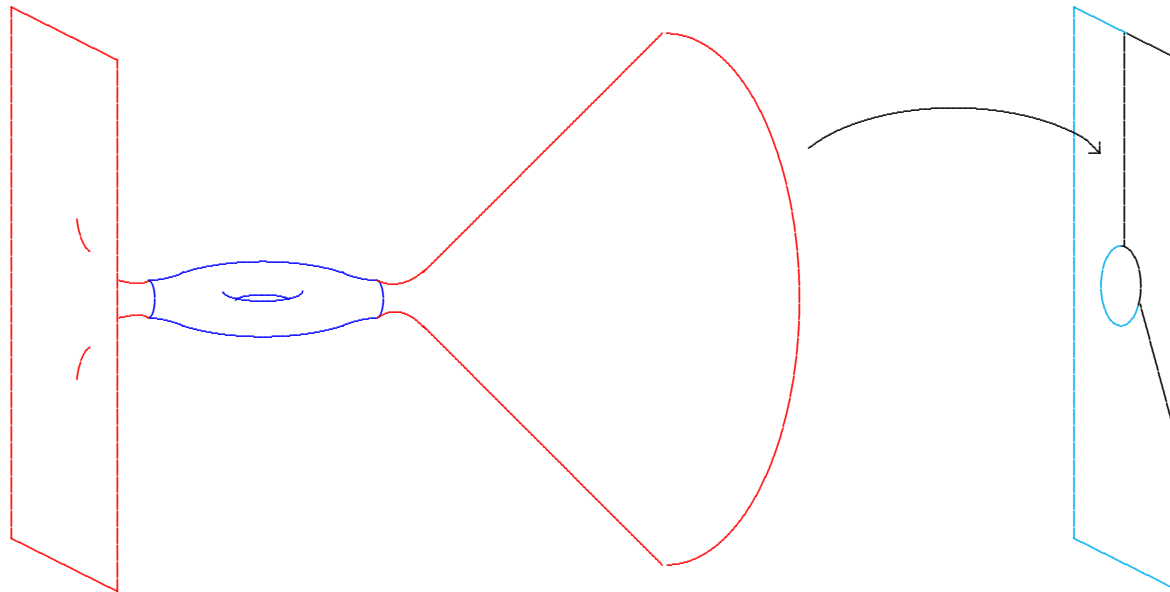
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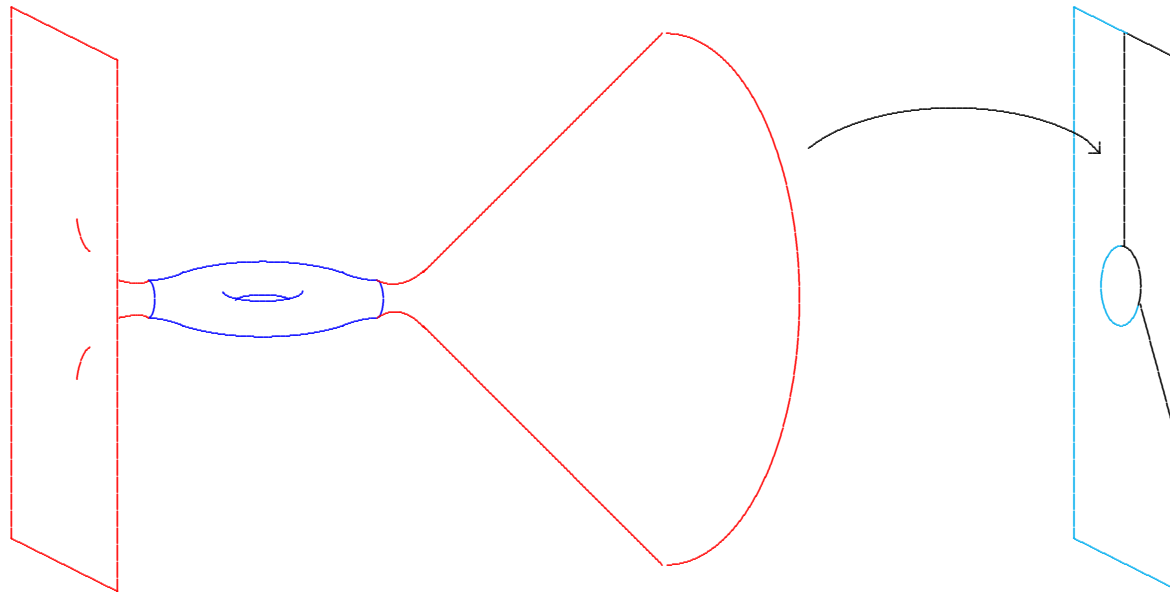
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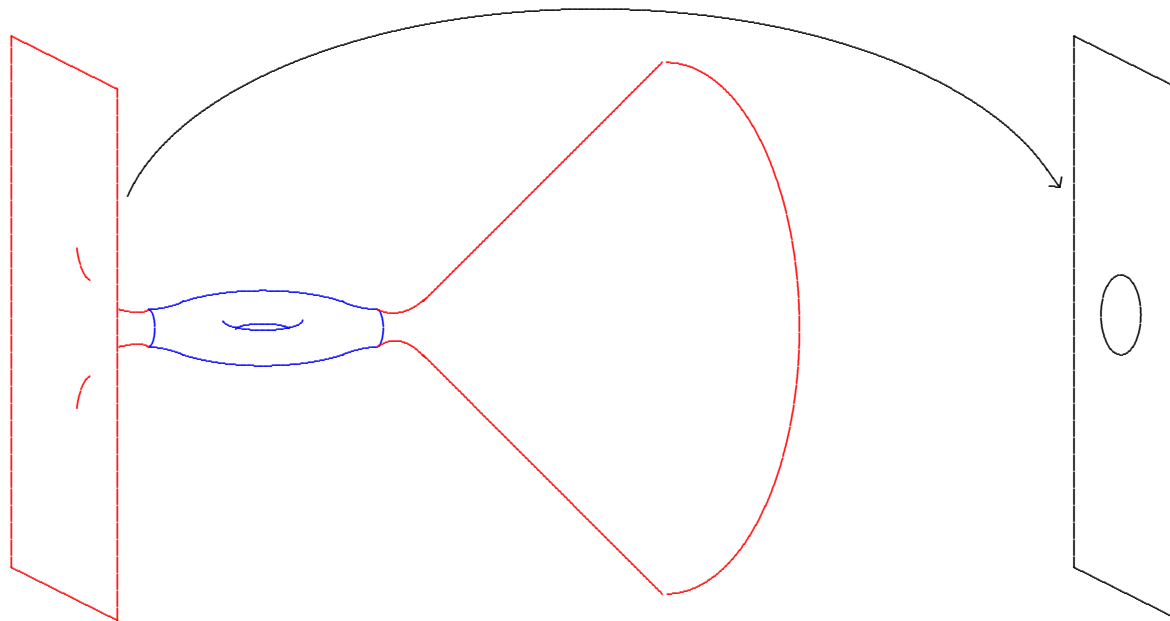
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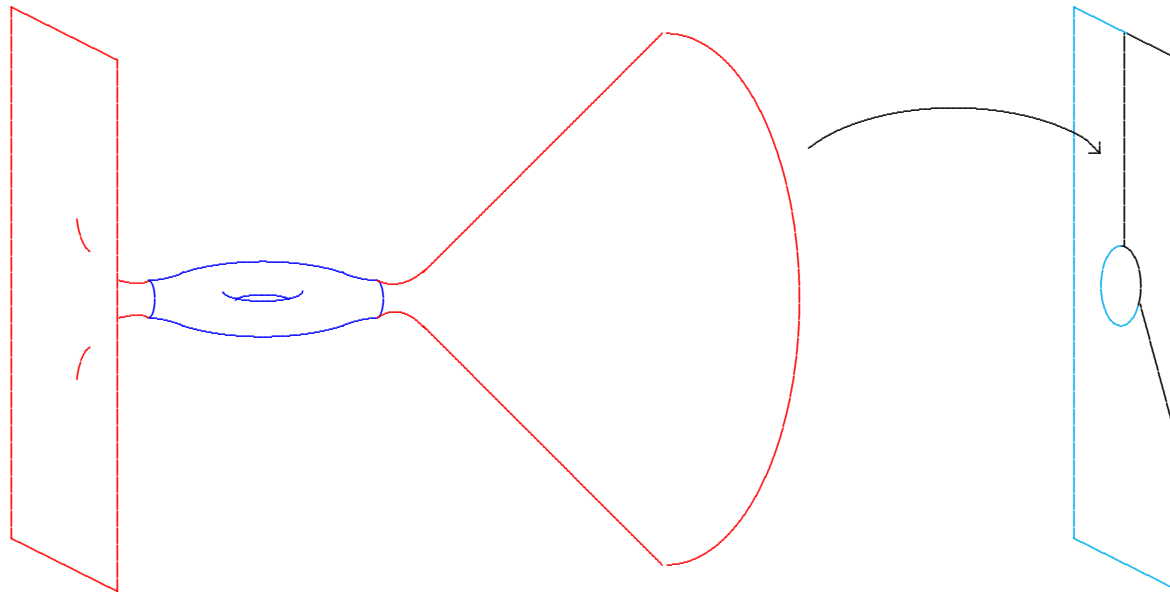
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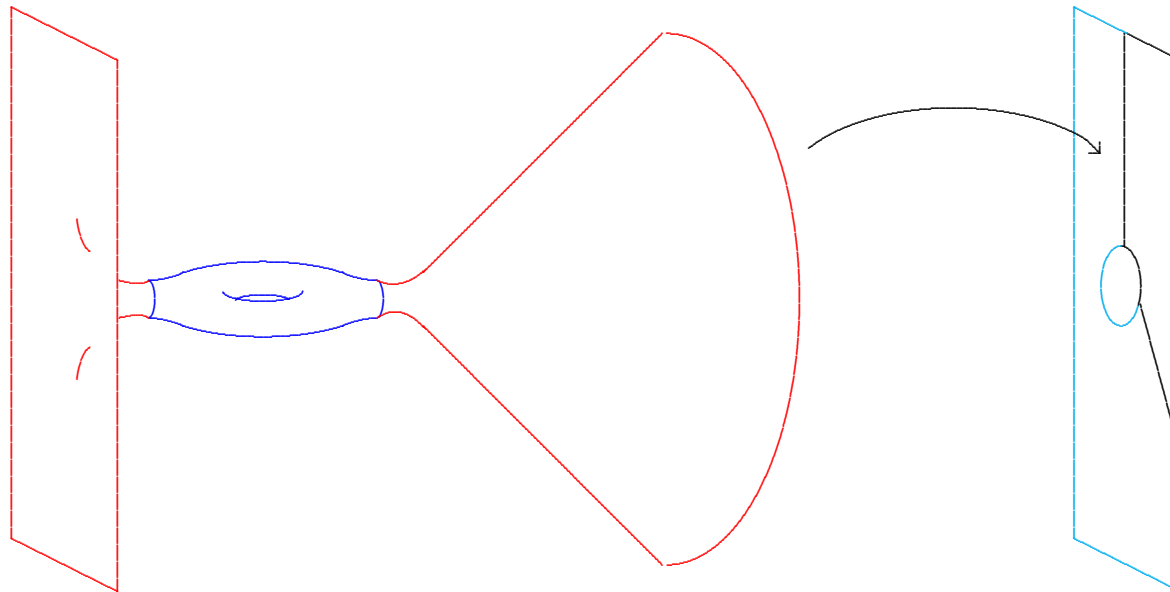
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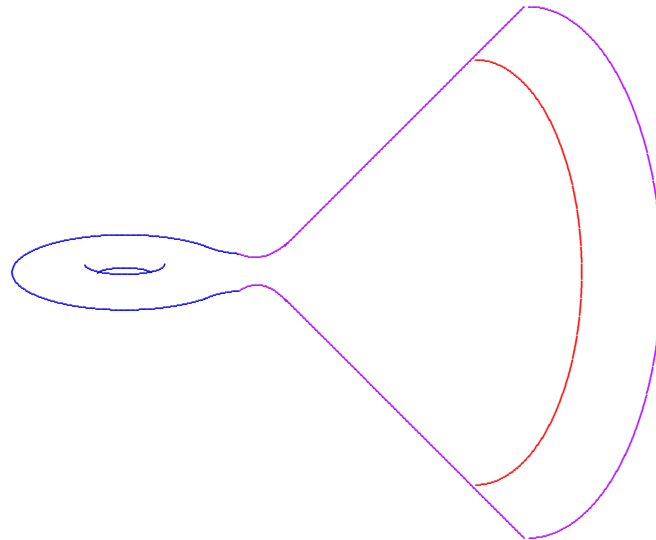
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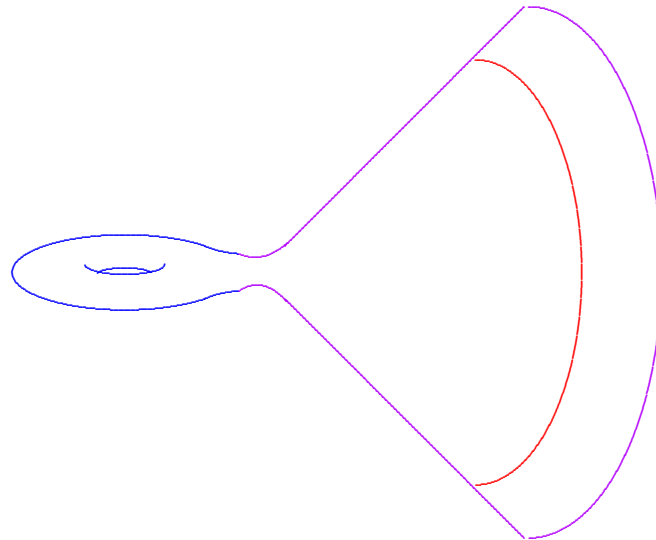


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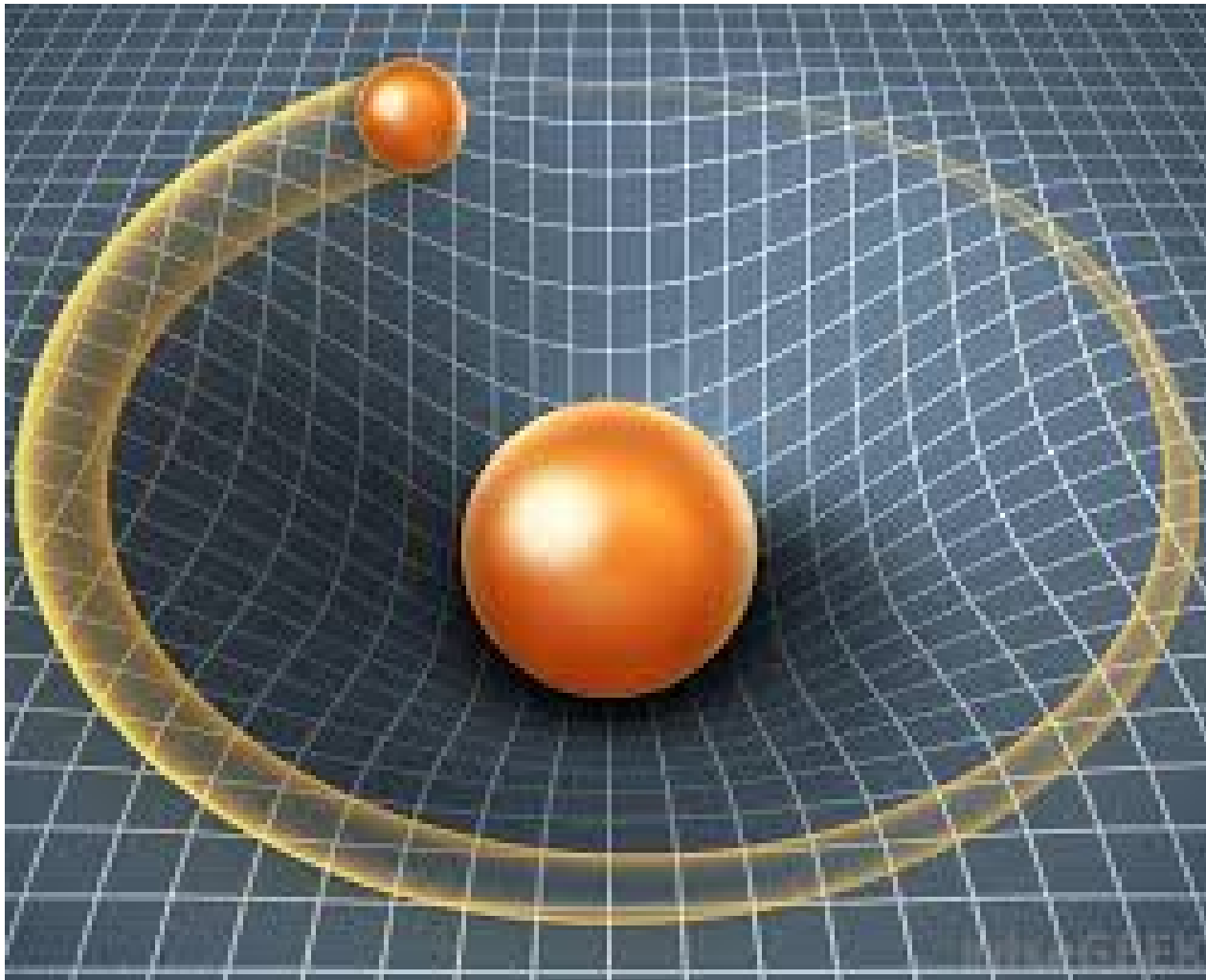
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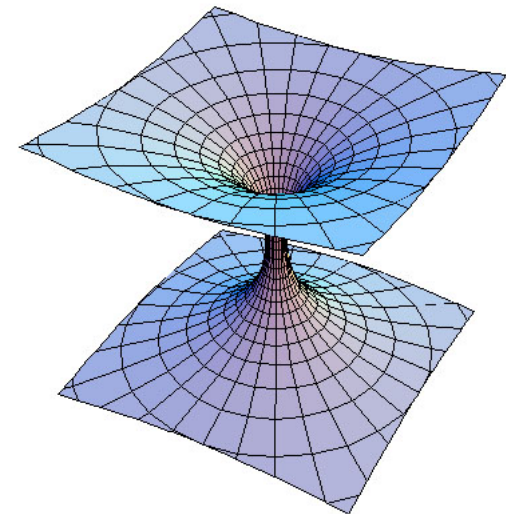
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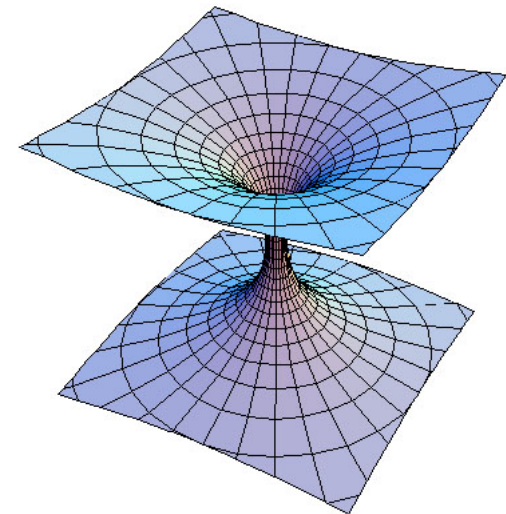
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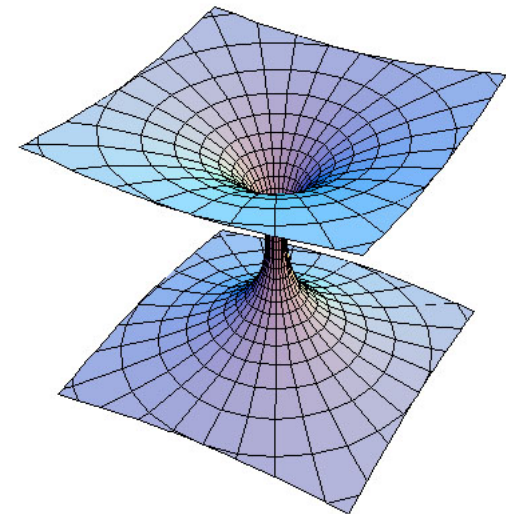
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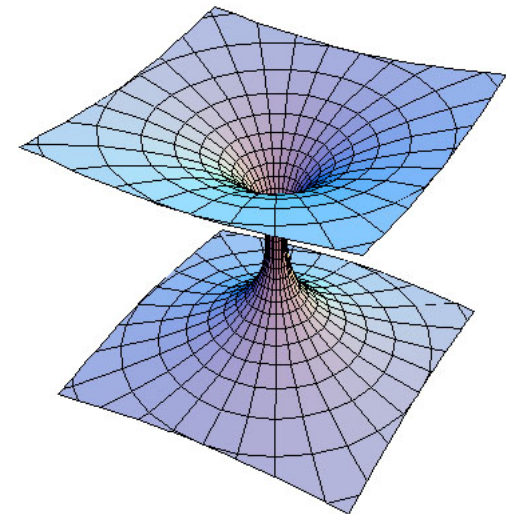
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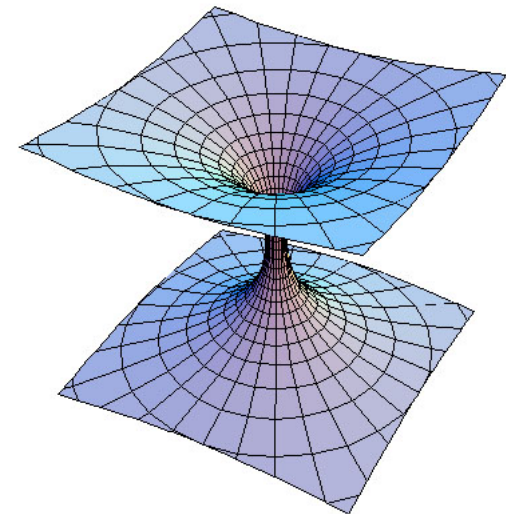
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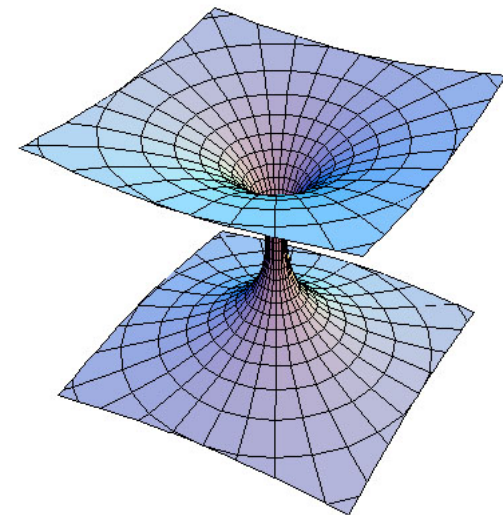
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But notice that this crude model for the mass in particular assumes faster metric fall-off!

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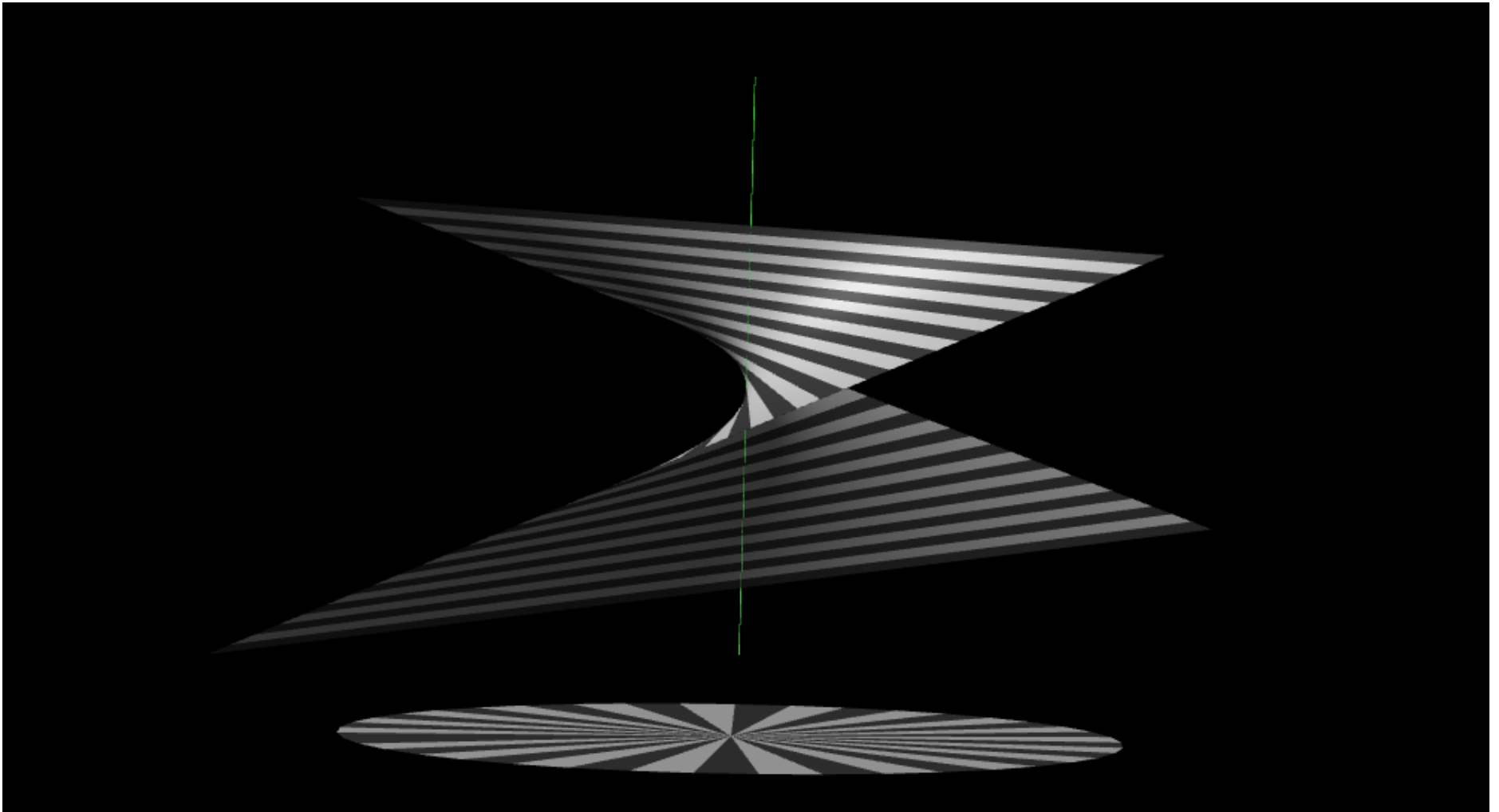


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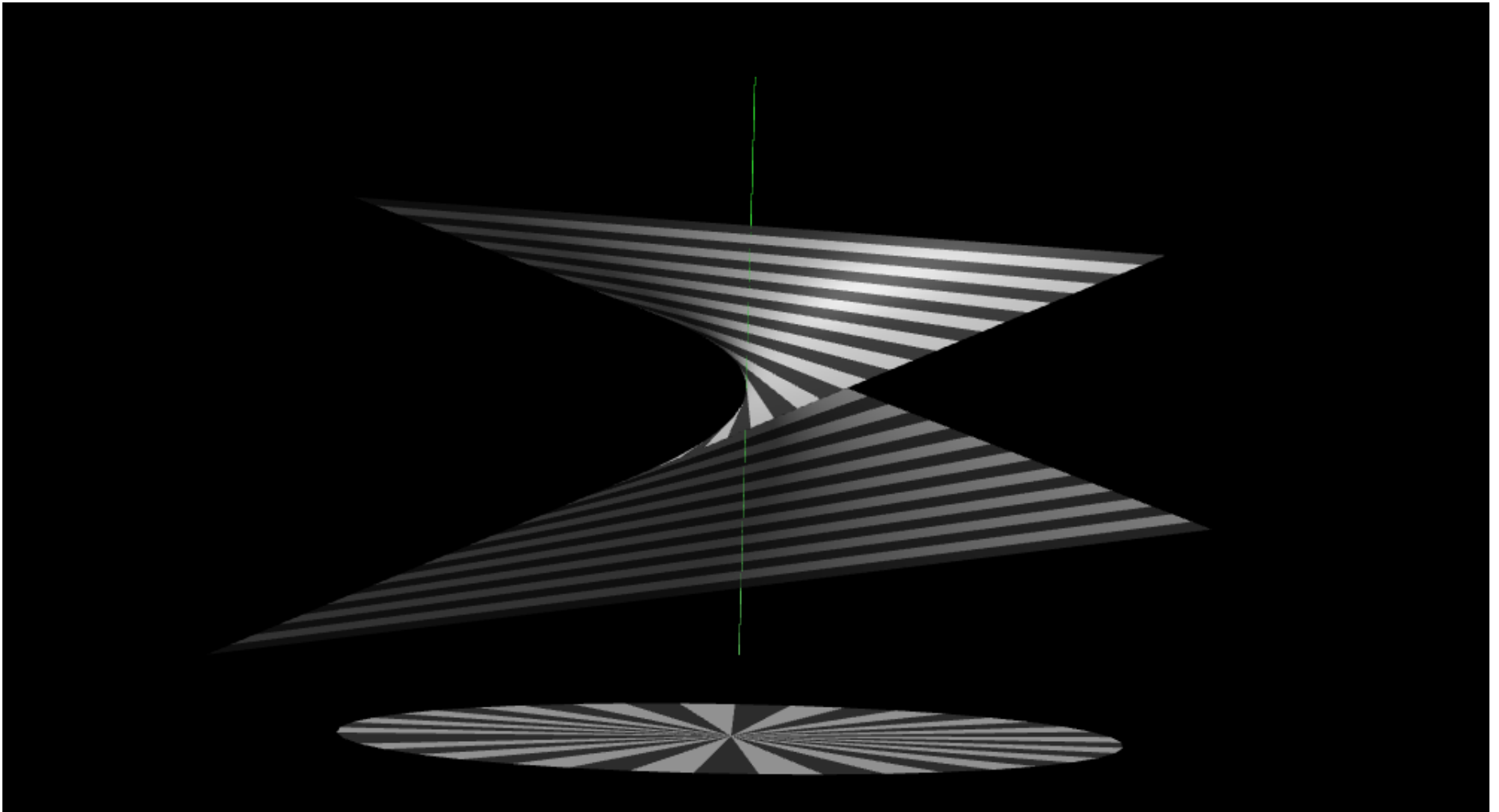
Restrict Euclidean  $\times$  round metric to  $\begin{vmatrix} z_1 & z_2 \\ \zeta_1 & \zeta_2 \end{vmatrix} = 0$ .

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Comm. Math. Phys. 347 (2016) 621–653.

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unless  $\varepsilon > \frac{1}{2}$ , when Chruściel fall-off sufficed.

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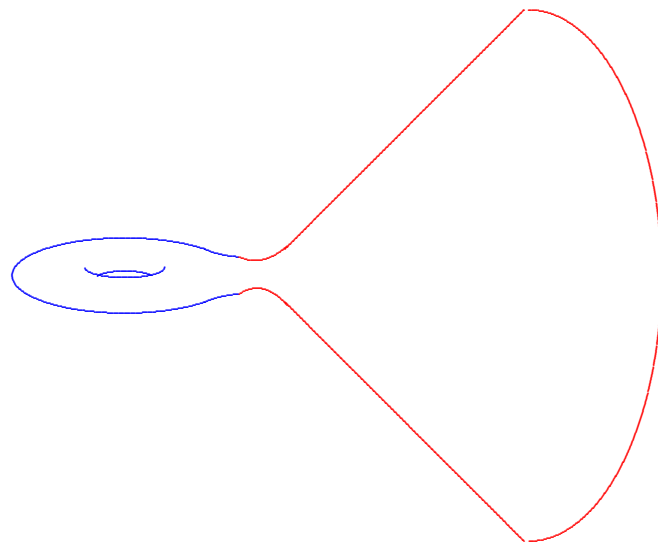
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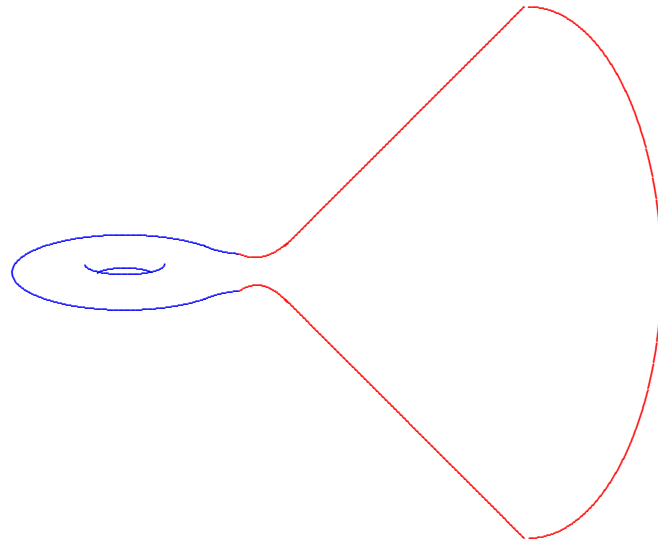
In particular, Chruściel fall-off suffices to imply all the following results:

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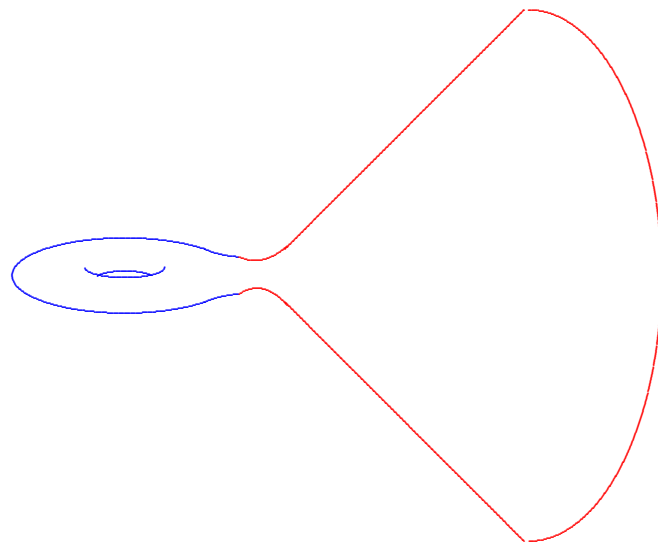
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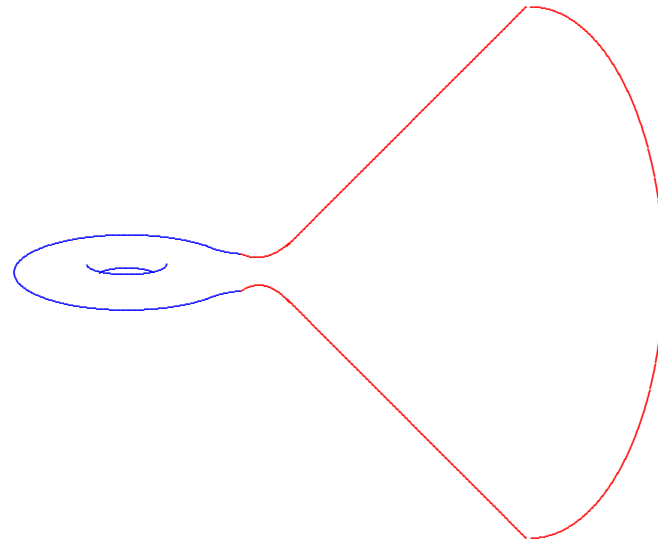
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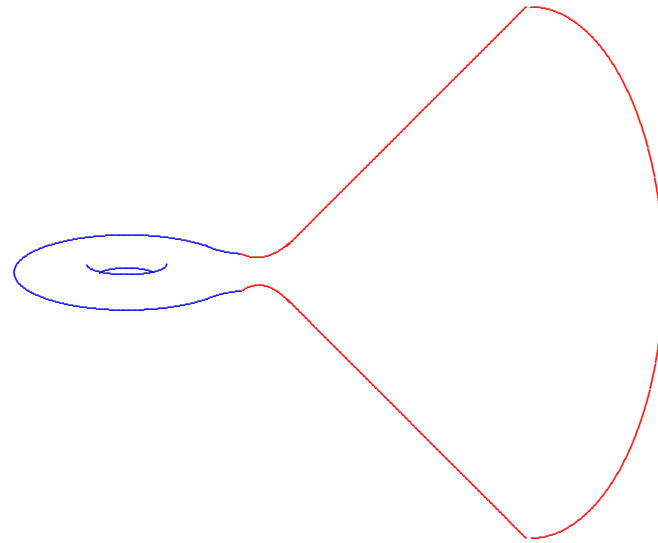


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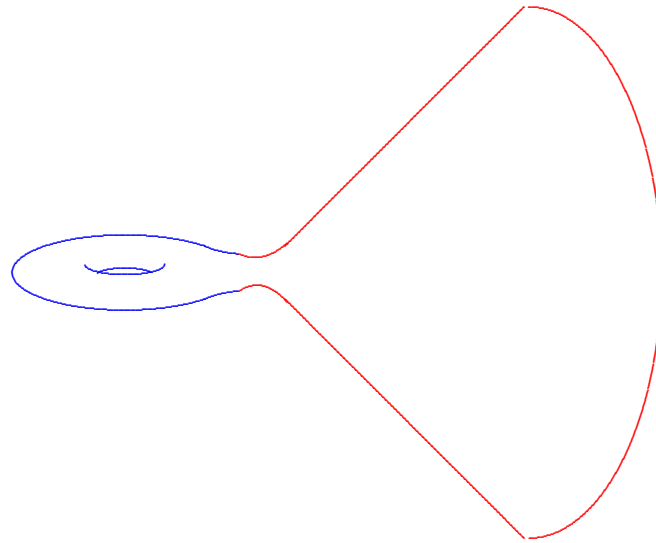
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Mass of an ALE Kähler manifold is unambiguous.

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Mass of an ALE Kähler manifold is unambiguous.

Does not depend on the choice of an end!

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$$m(M, g) = - \langle \clubsuit(c_1), [\omega]^{m-1} \rangle + \frac{(m-1)!}{4(2m-1)\pi^m} \int_M s_g d\mu_g$$

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- $s$  = scalar curvature;
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So the mass is a “boundary correction” to the topological formula for the total scalar curvature.

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New proof shows this follows from Chruściel fall-off.

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**Theorems A & B** are corollaries concerning scalar-flat Kähler metrics.



Another key consequence. . .



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This has an interesting corollary...



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Now use Bishop-Gromov inequality.



Some applications ...



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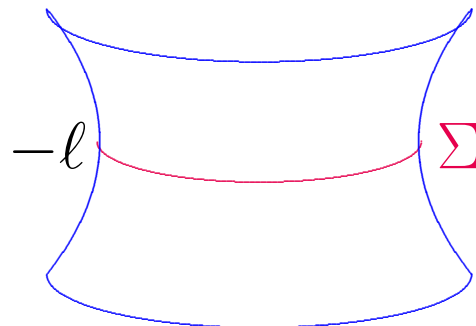
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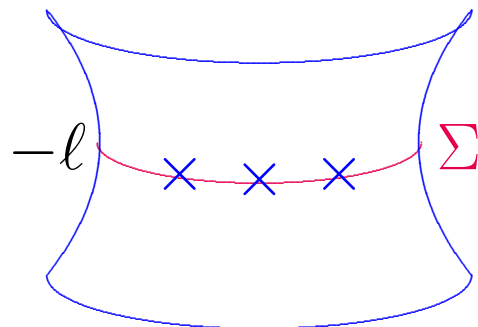




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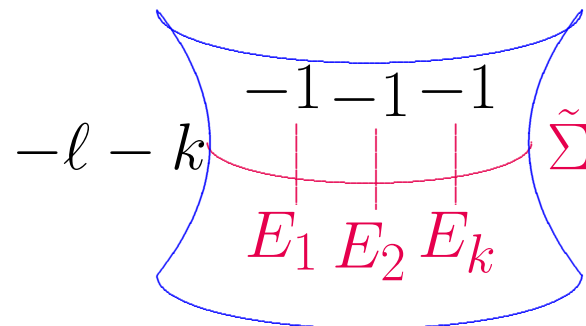
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**L, Calderbank-Singer, Lock-Viaclovsky...**

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**Theorems A & B** are corollaries concerning scalar-flat Kähler metrics.

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How does one prove main results?



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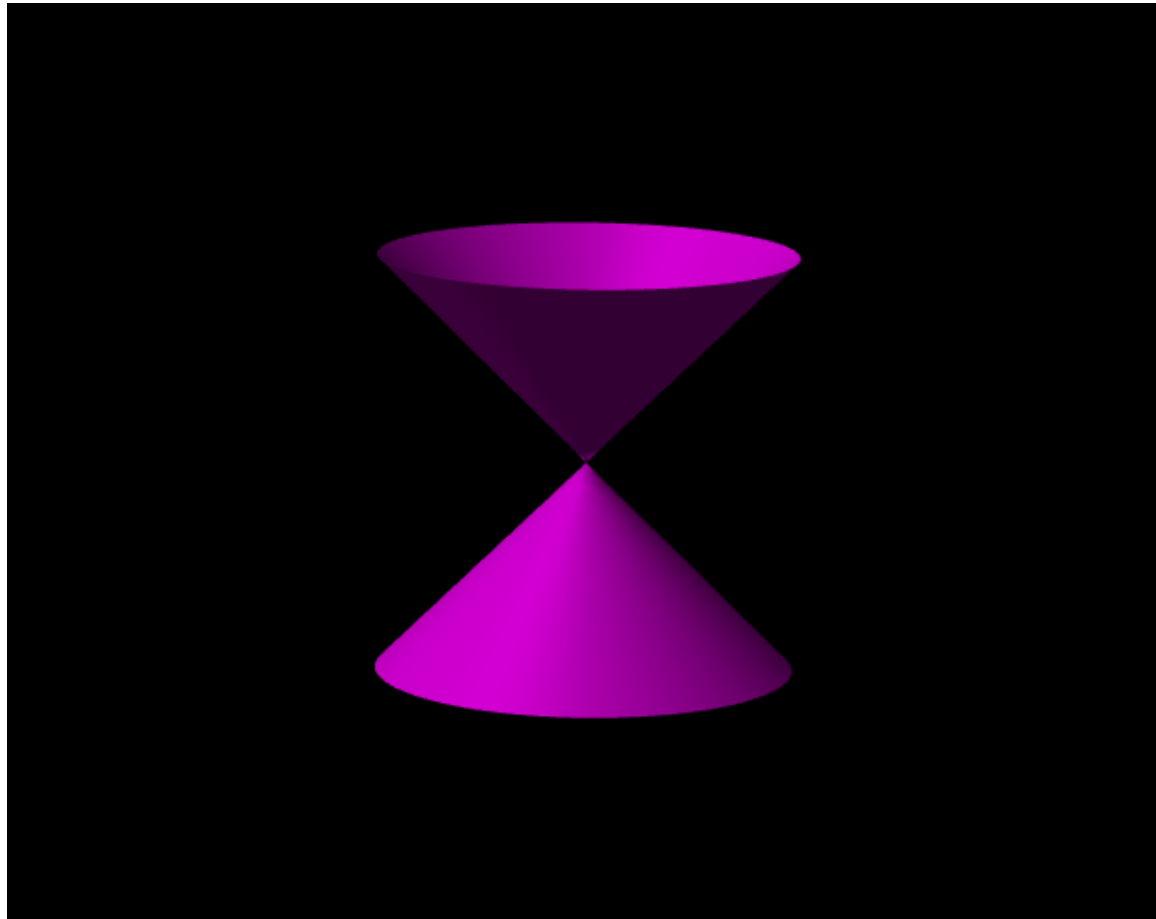
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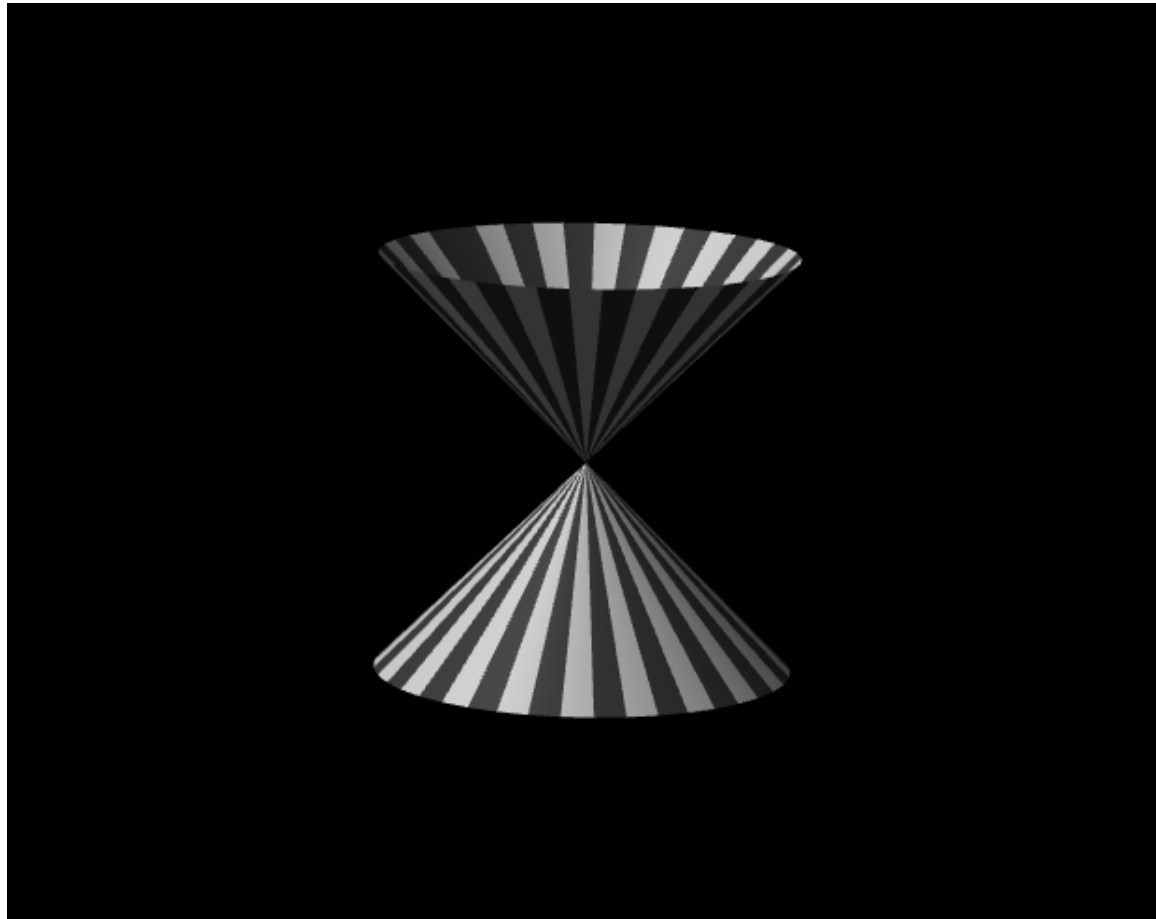




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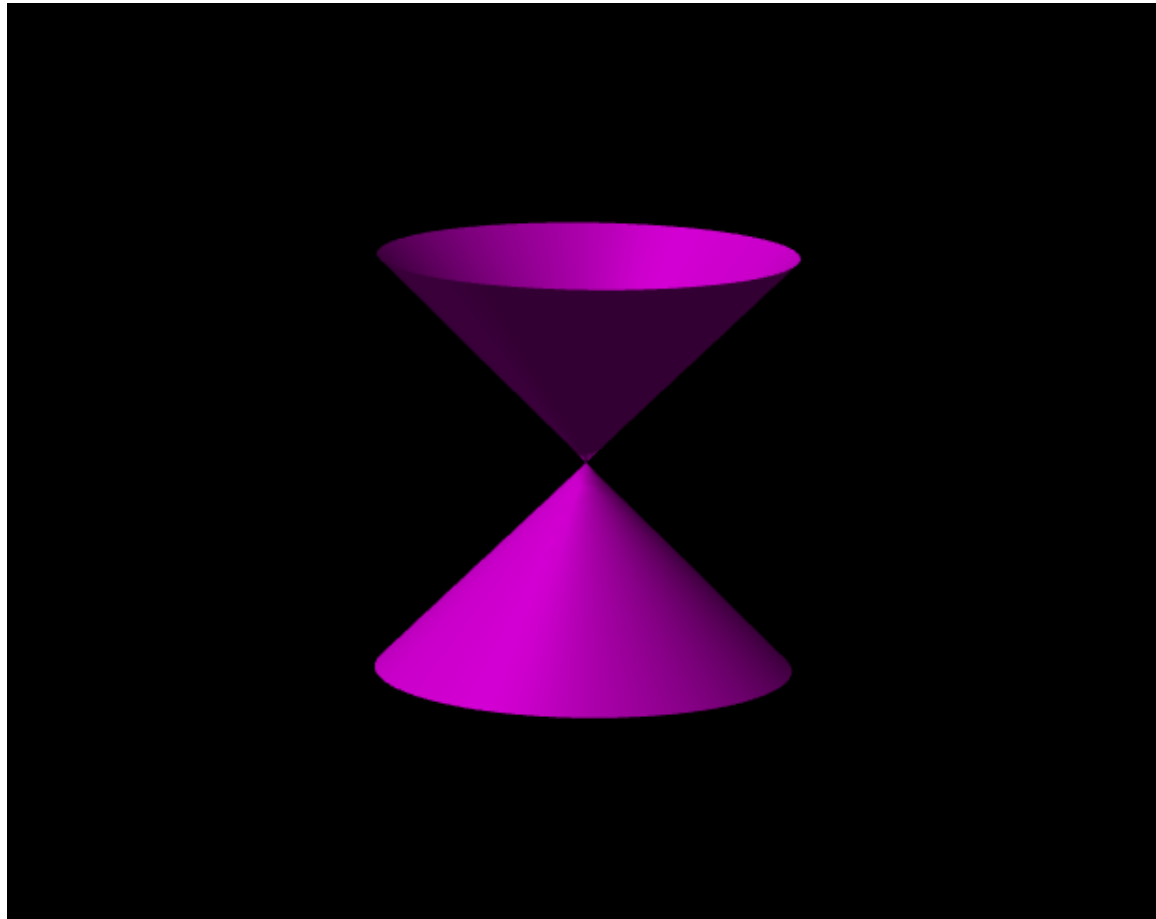
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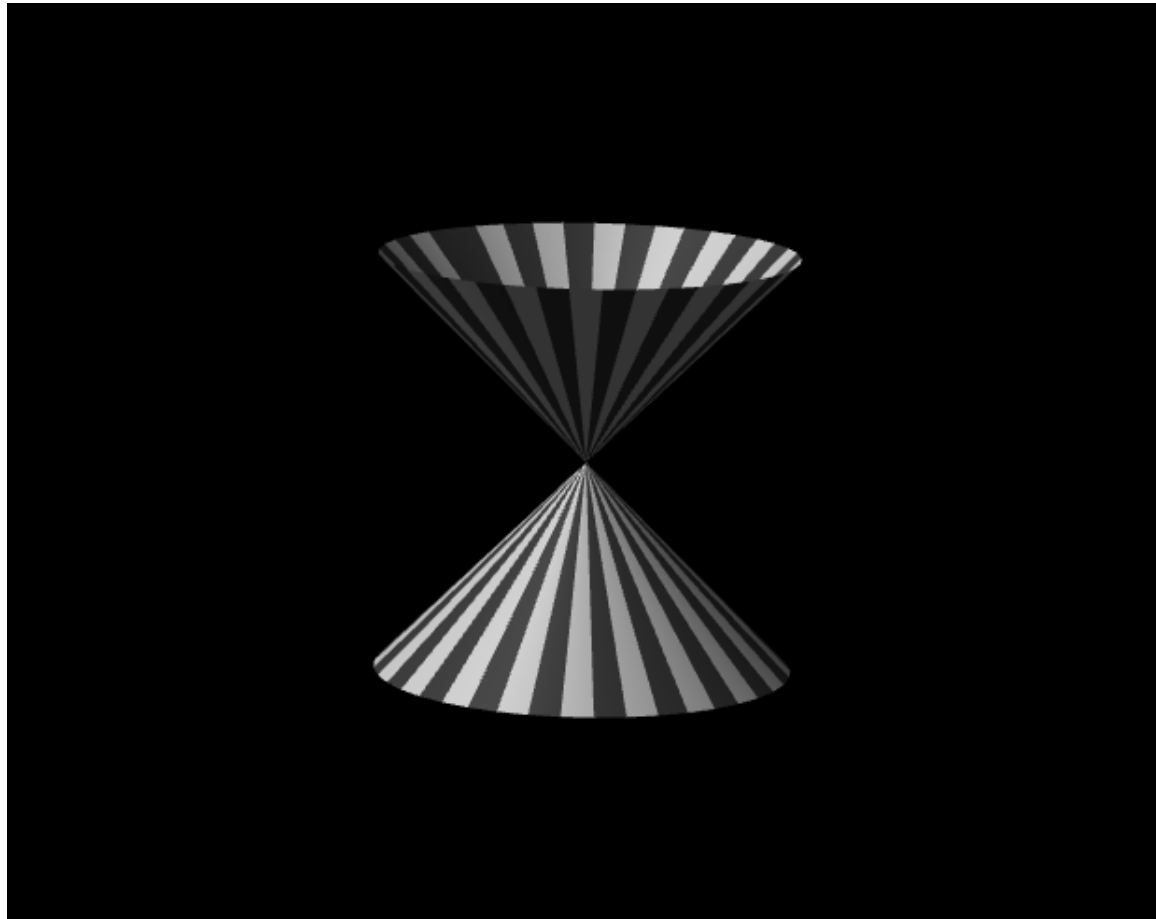
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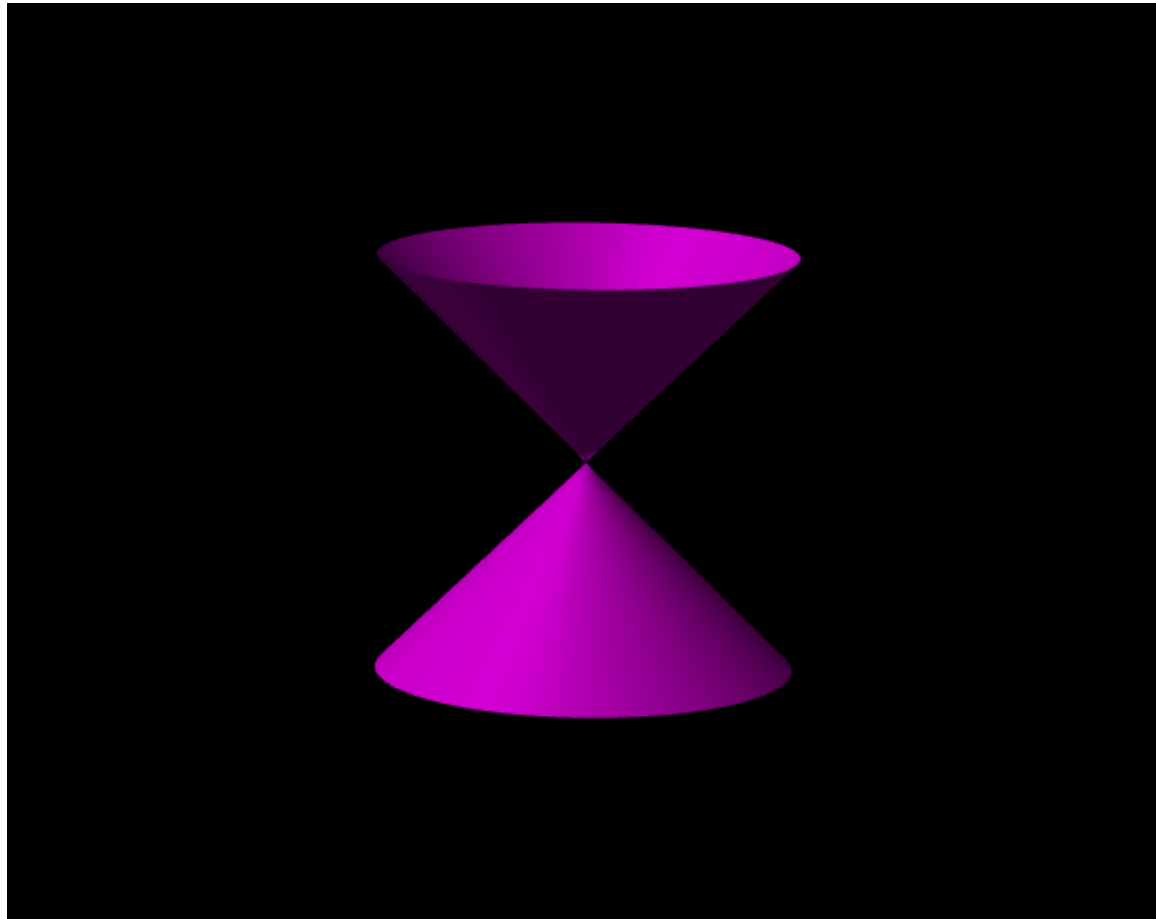
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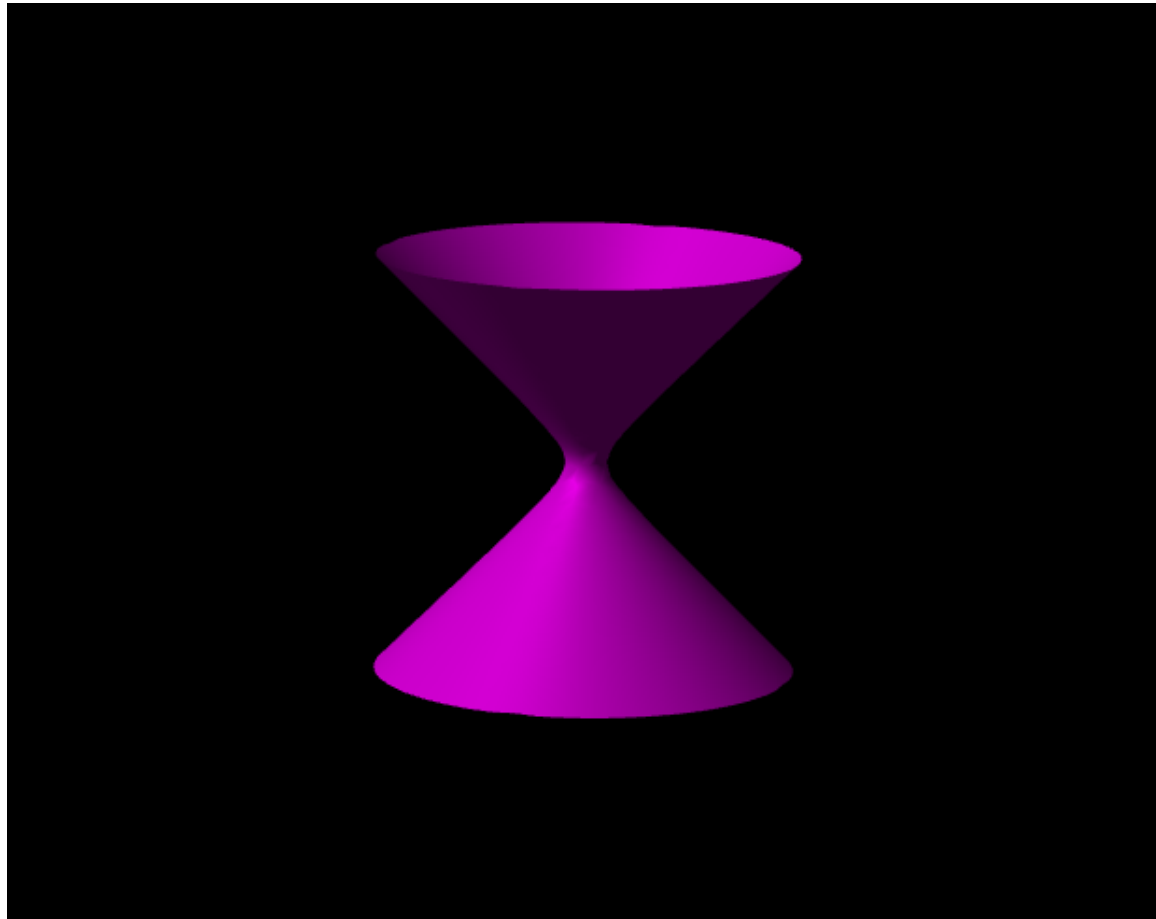
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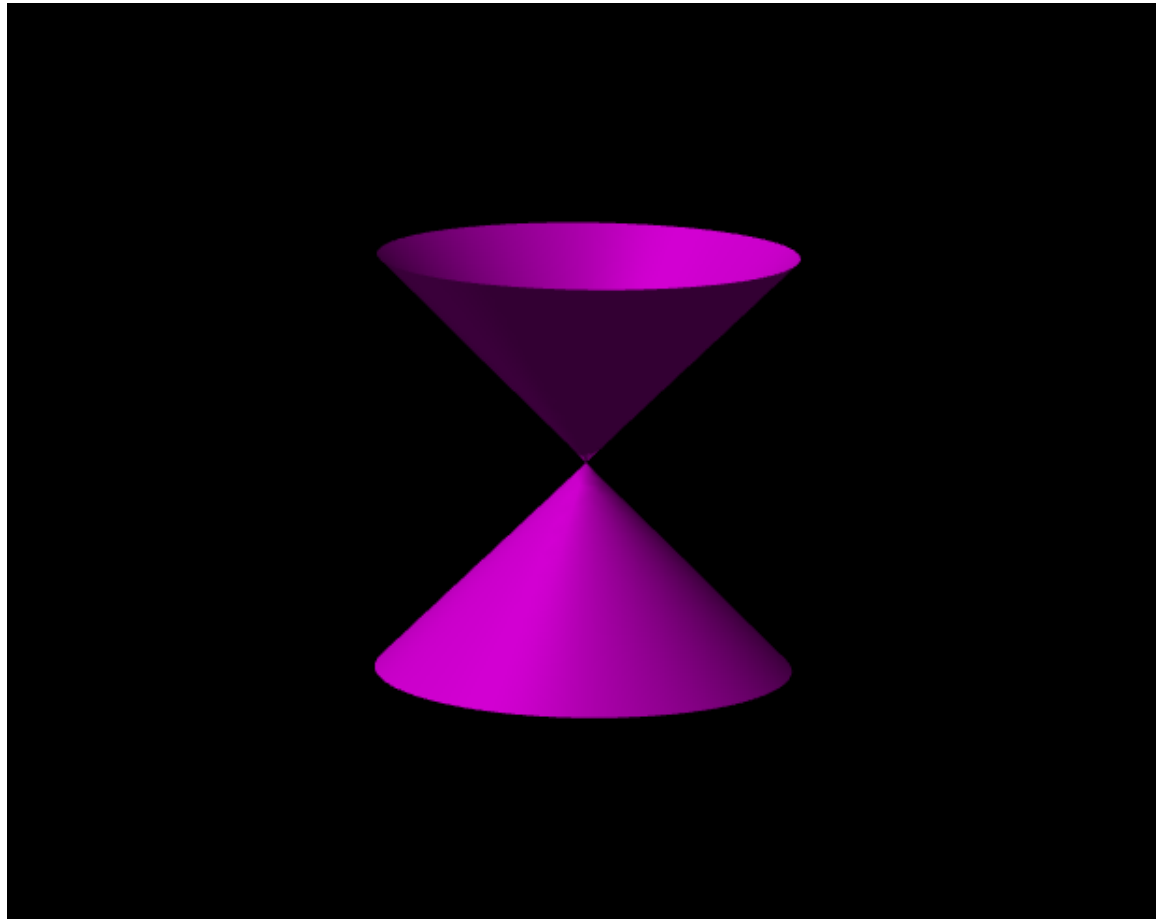
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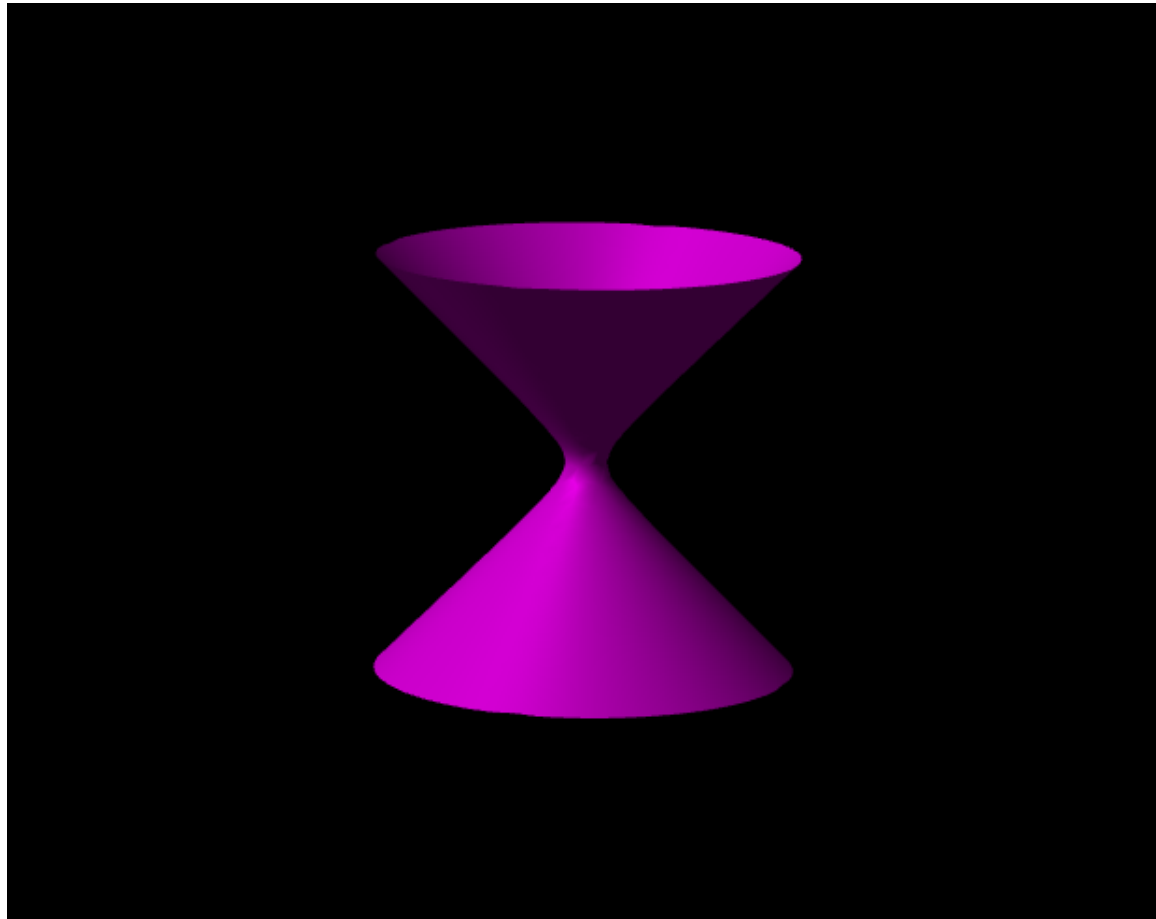
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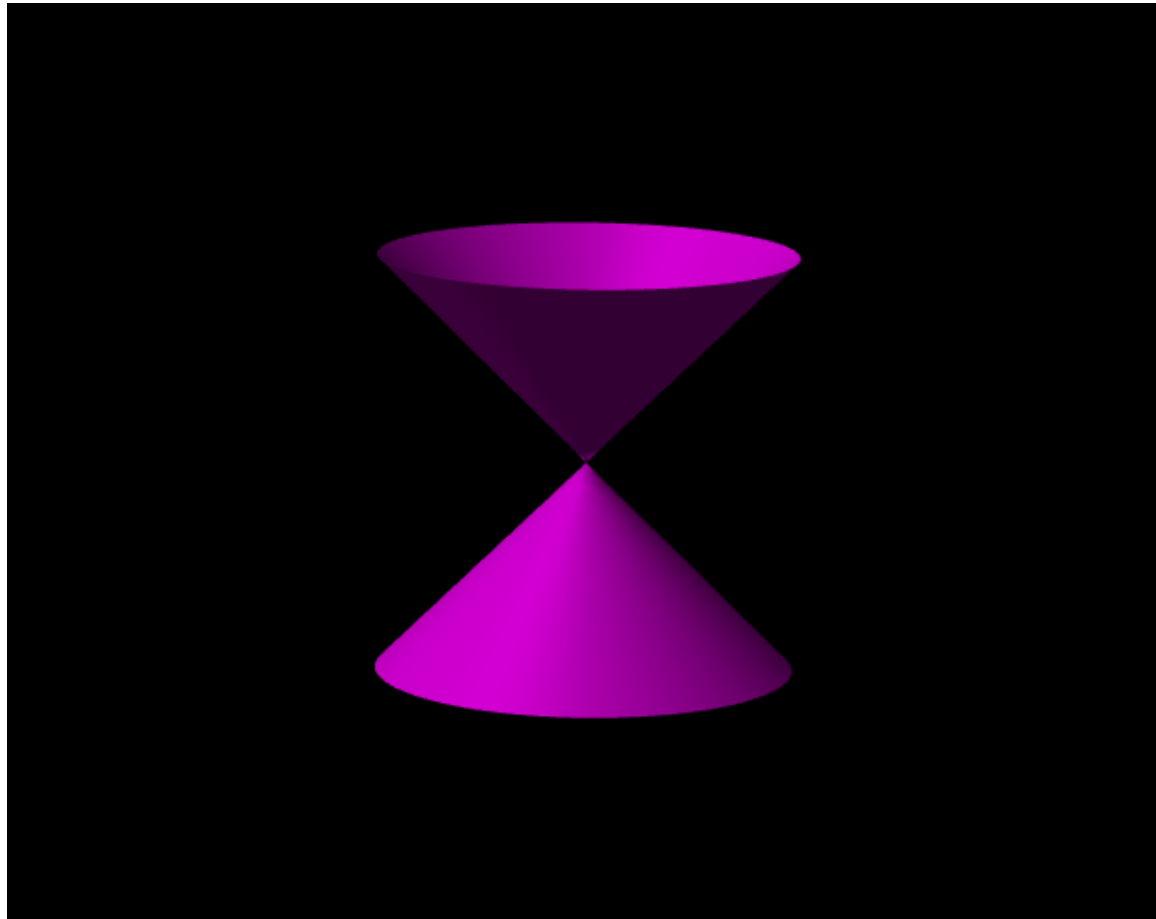
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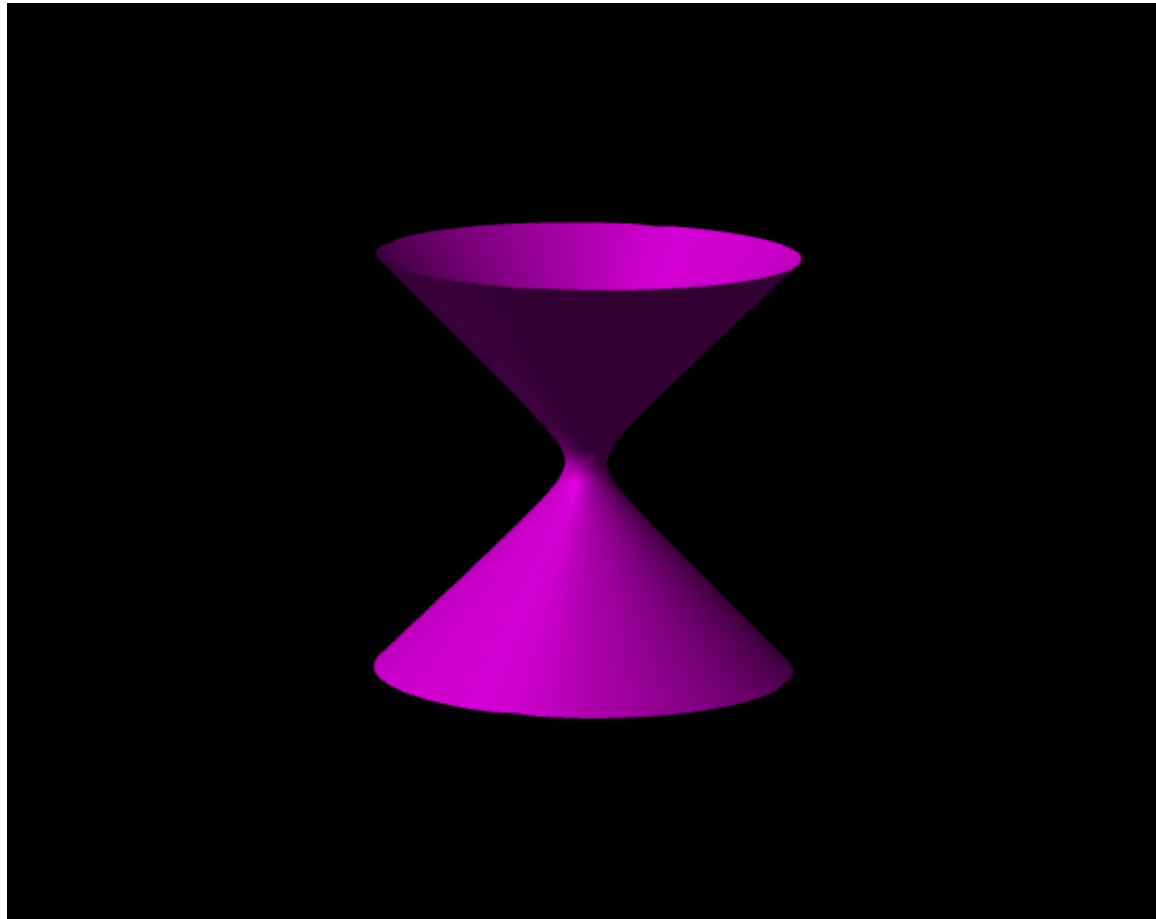




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Fortunately, however, the symplectic structure is always standard at infinity!

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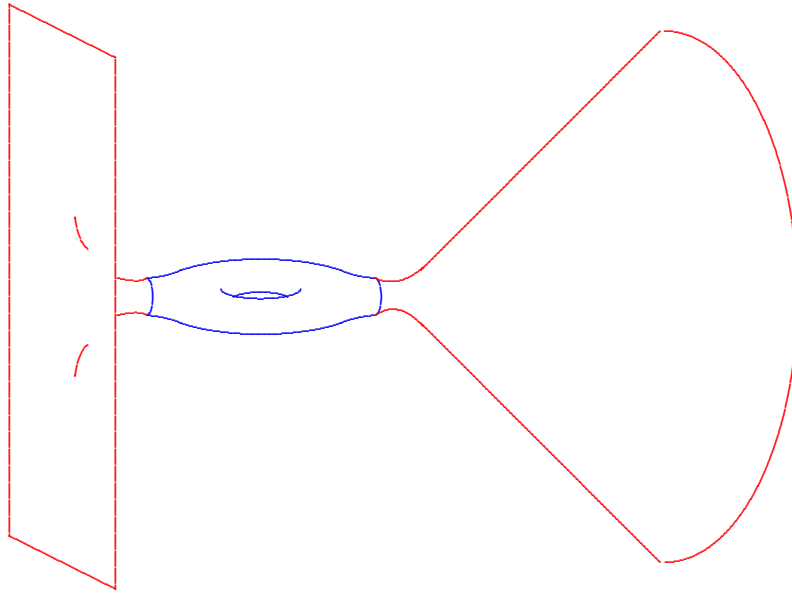
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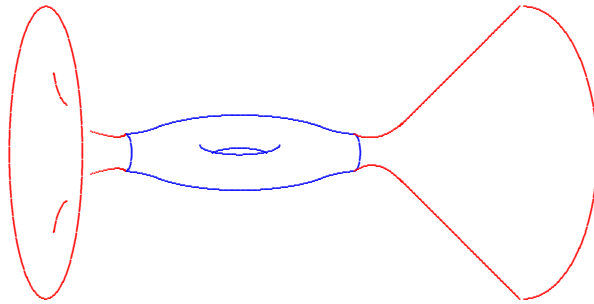
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Quantitative version of Moser stability argument...

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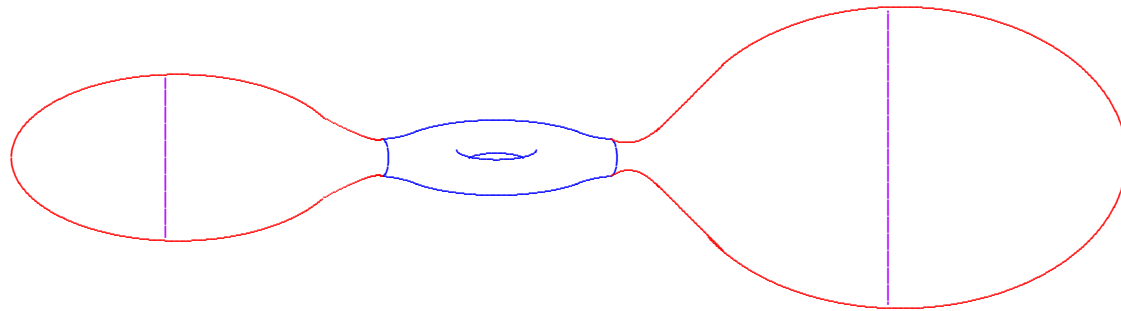


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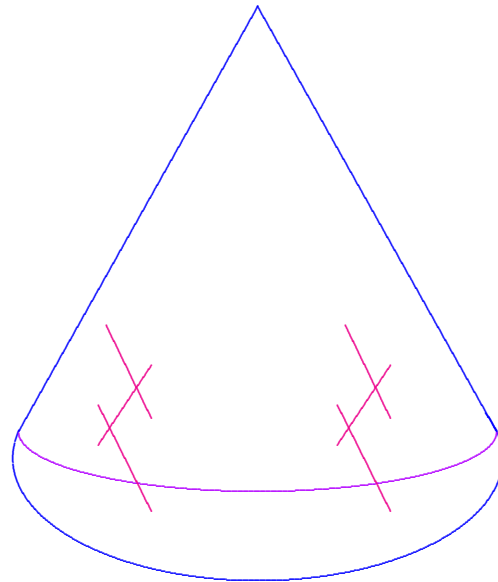
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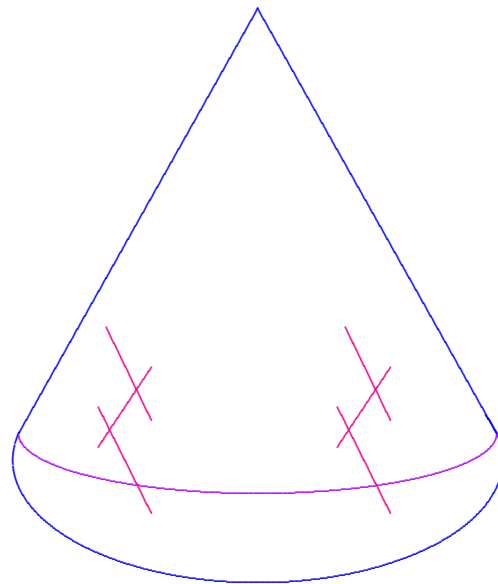




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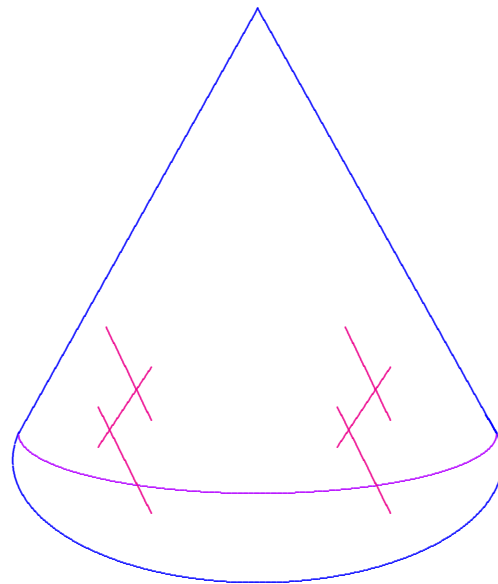


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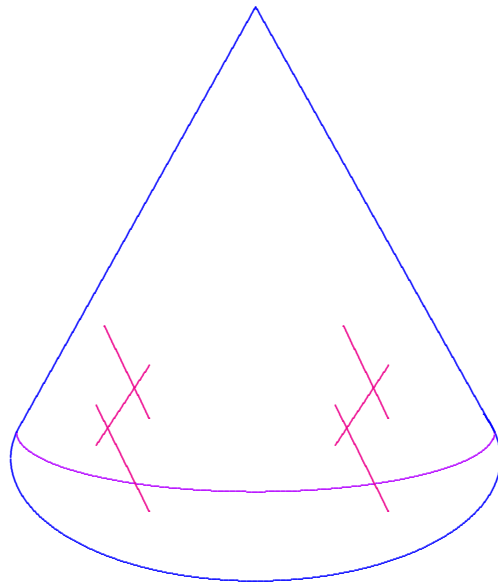
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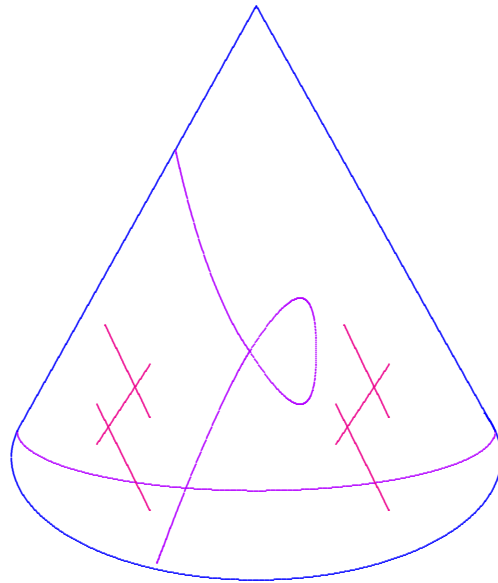
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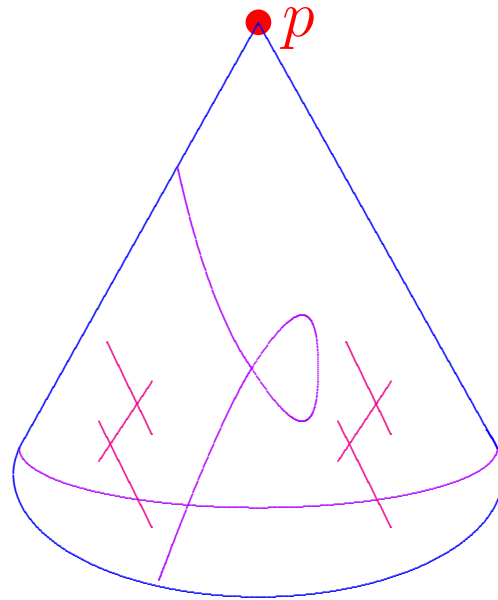
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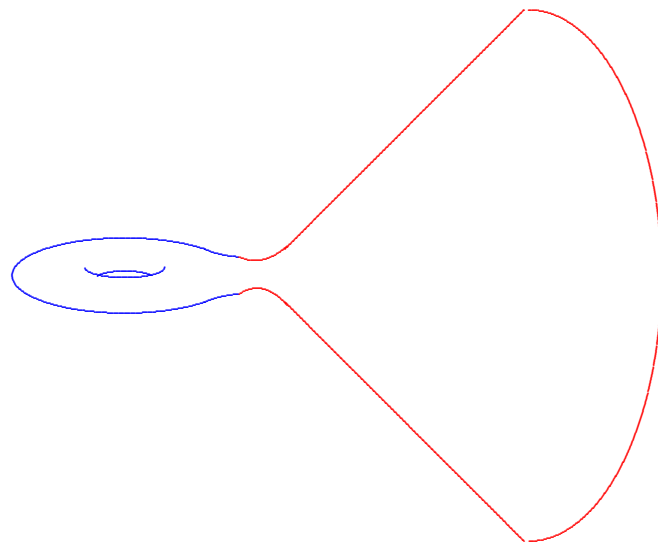
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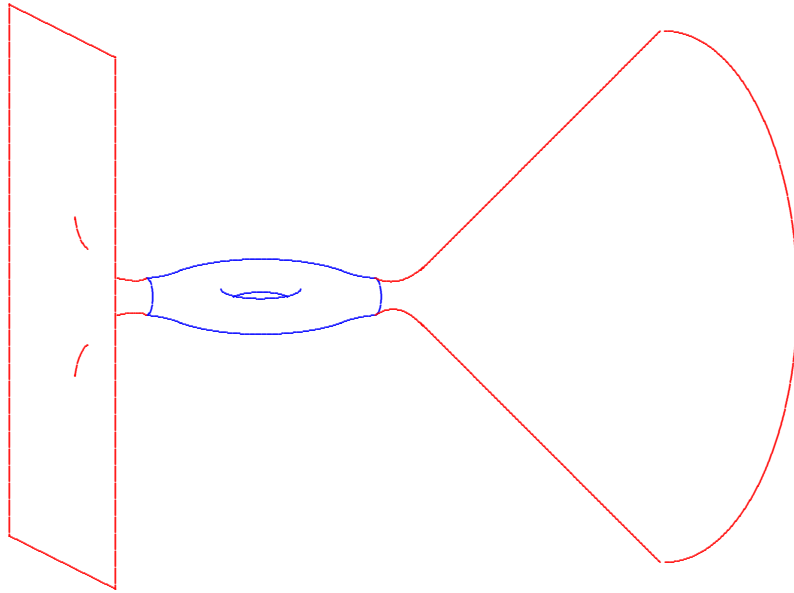
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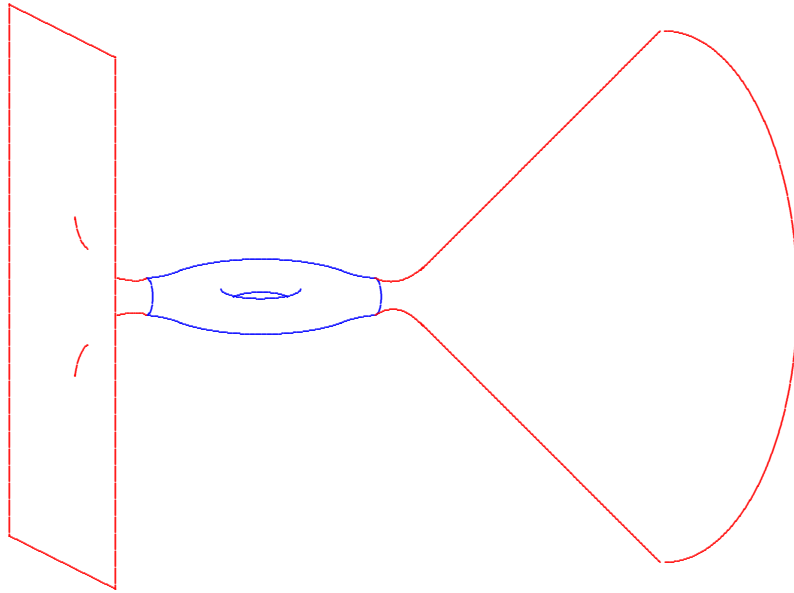


**Lemma.** *Any ALE Kähler manifold has only one end.*

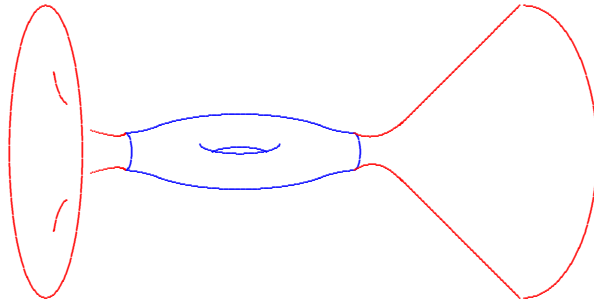




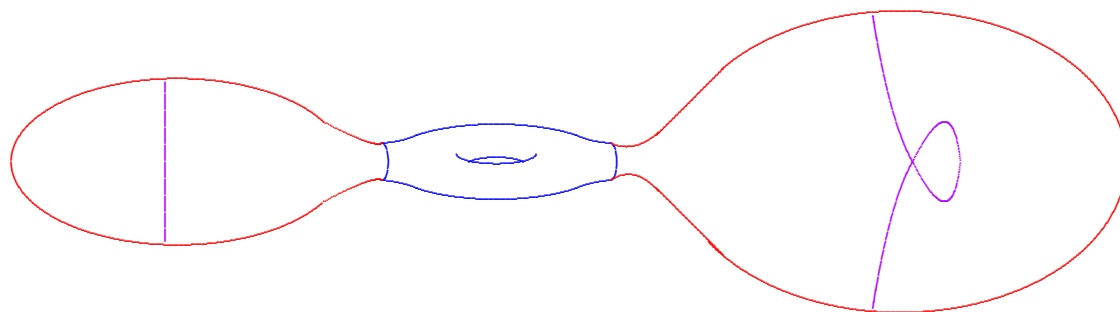
What if  $M^4$  has more than one end?



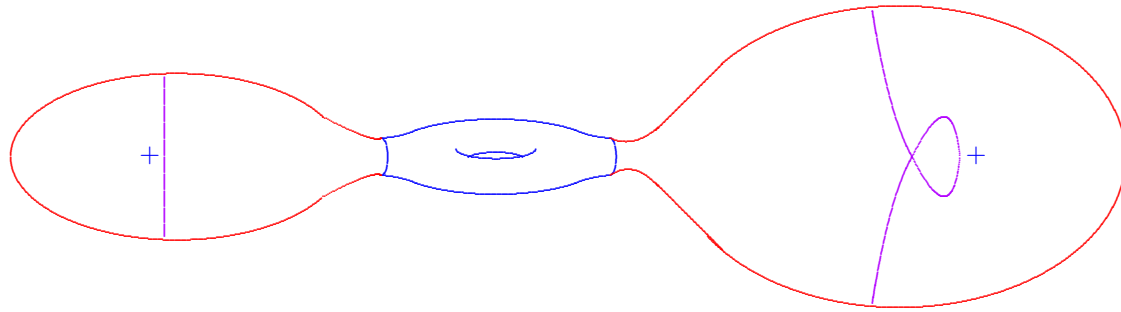
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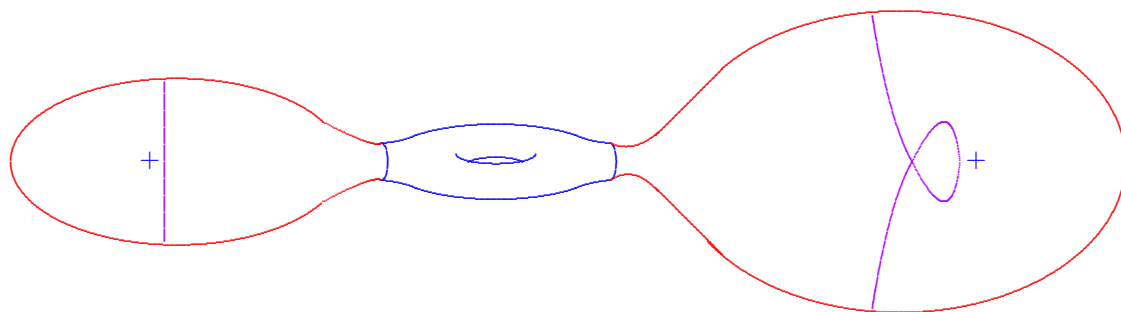


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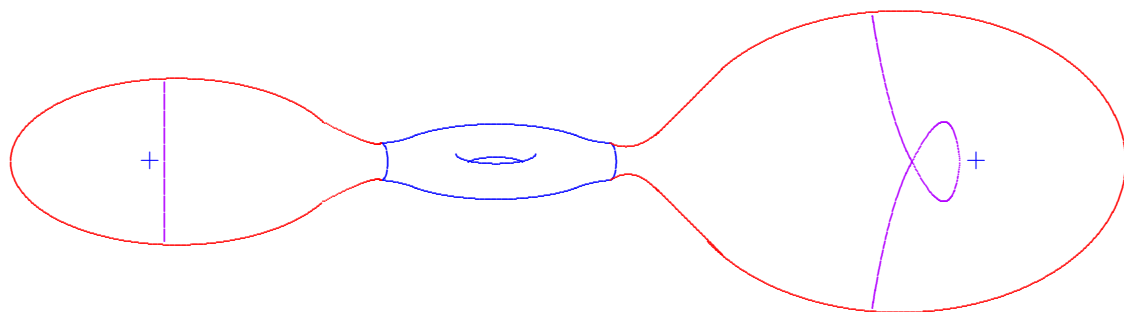
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McDuff  $\implies \widehat{M} \approx$  rational complex surface.

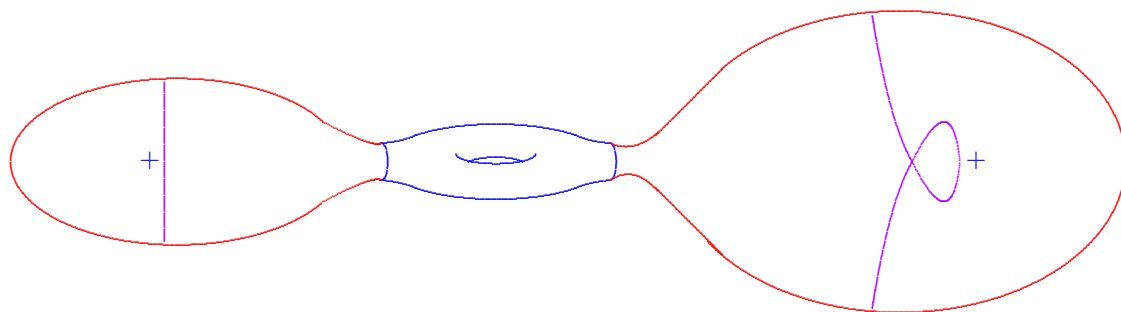


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McDuff  $\implies$  intersection form  $(+- \cdots -)$ .

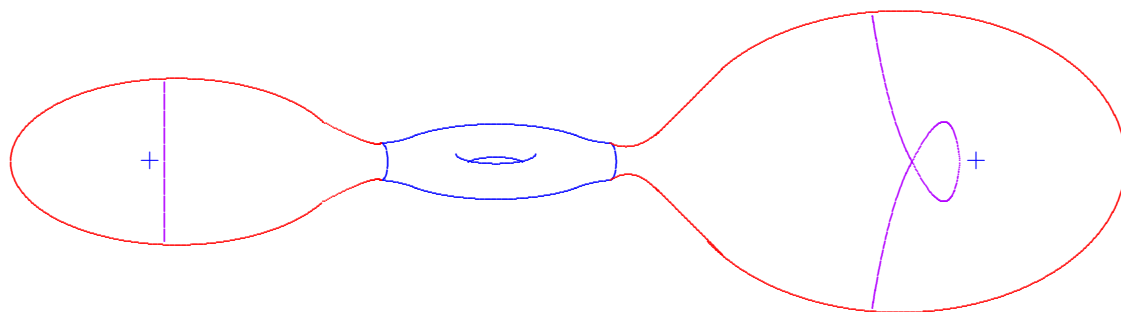




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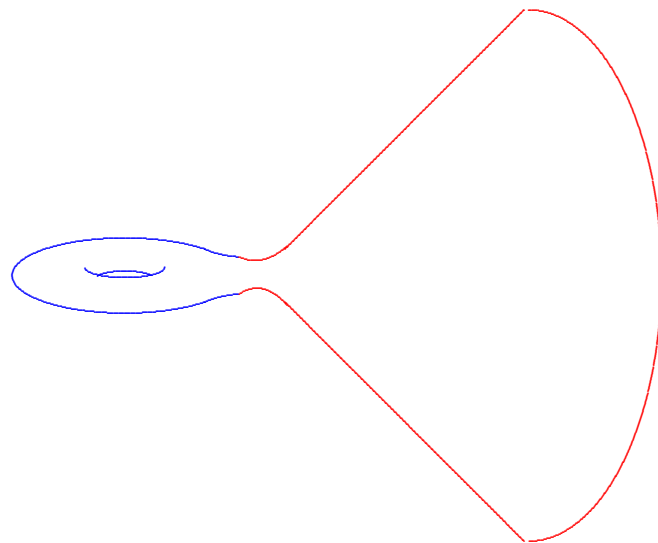
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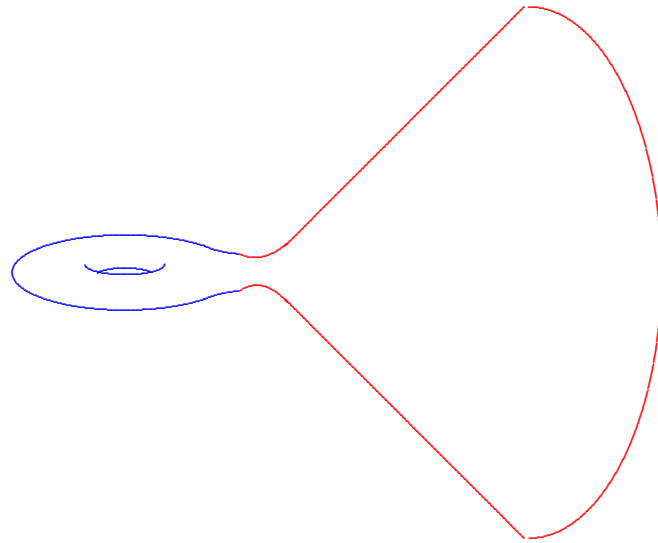
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[In higher dimensions, one similarly shows that  $(M, J)$  can be compactified as Kähler orbifold. The Hodge theorem on intersection form instead tells one that form on  $H^{1,1}(\widehat{M}, \mathbb{R})$  is of type  $(+ - \cdots -)$ .]

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The following result provides the key...

**Proposition.** *Let  $g$  be a  $C^2$  Kähler metric on  $(\mathbb{R}^{2m} - \mathbf{D}^{2m})/\Gamma$ ,  $m \geq 2$ , where  $\Gamma \subset \mathbf{SO}(2m)$  is some finite group that acts without fixed-points on  $S^{2m-1}$ .*

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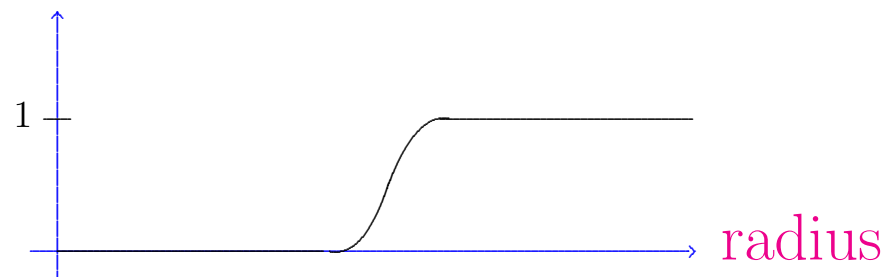
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We'll now deduce the mass formula. . .

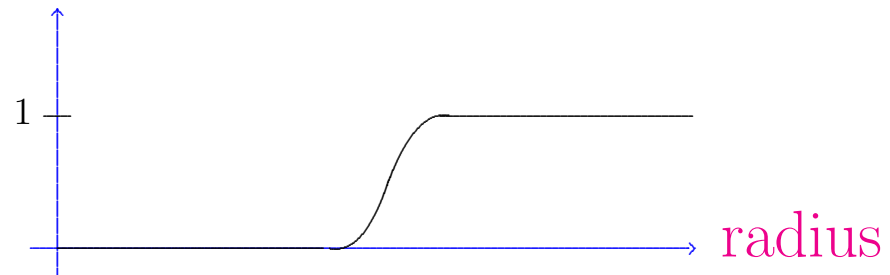
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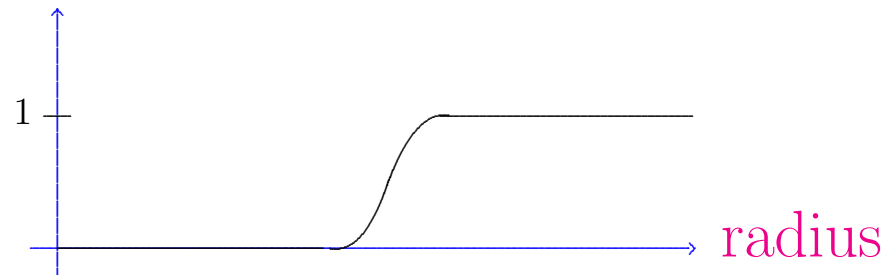




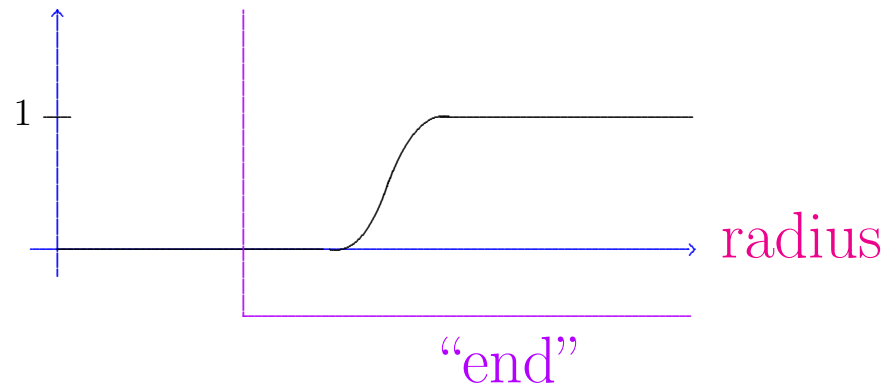
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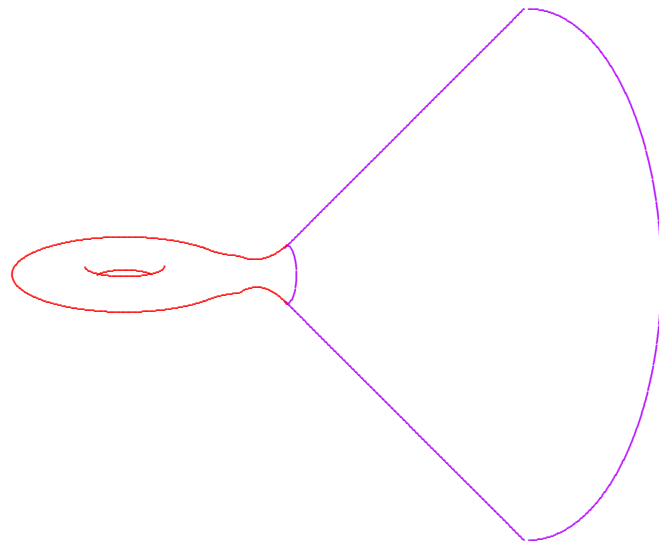
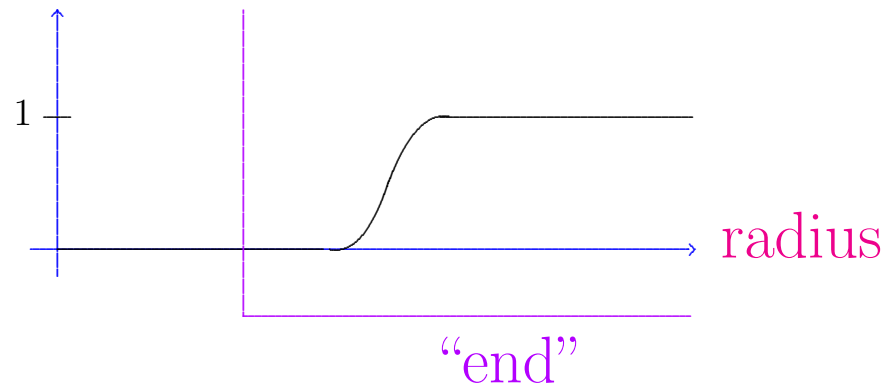
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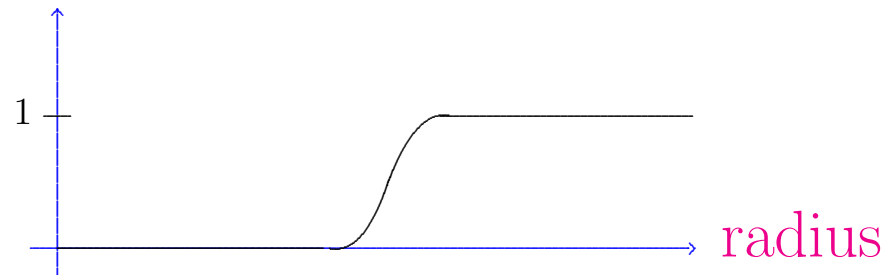
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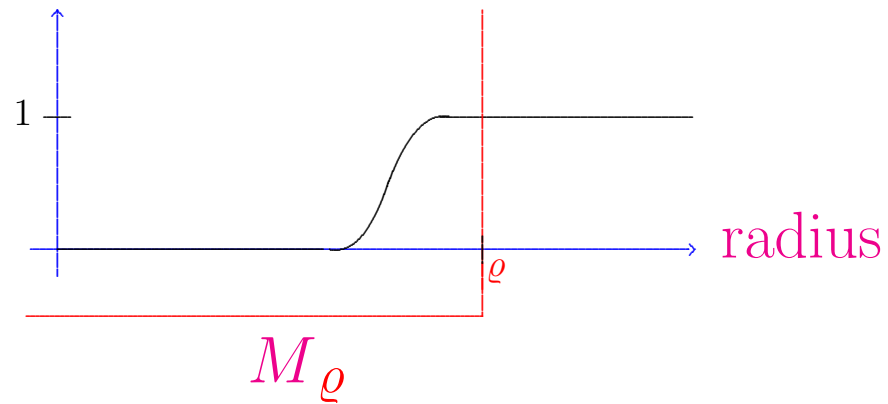
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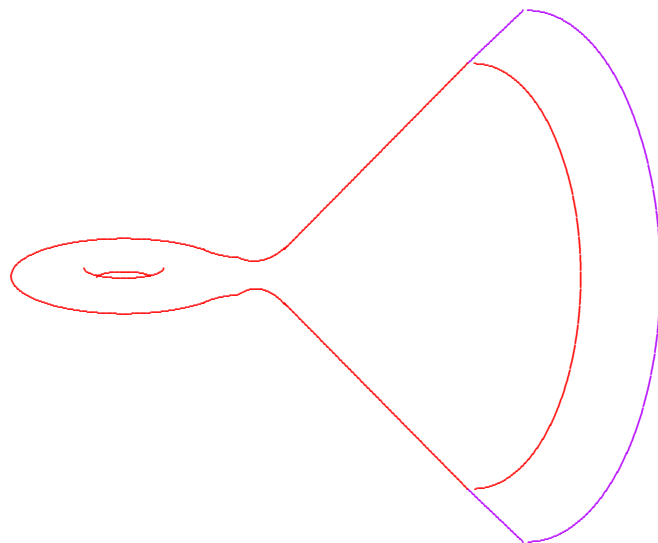
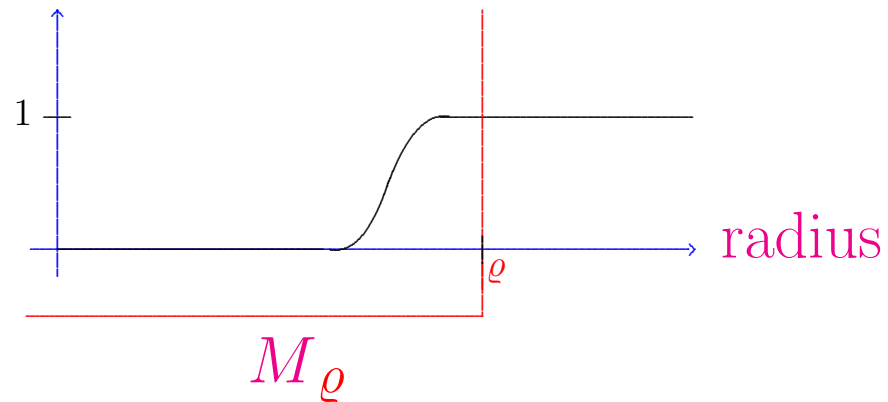
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$$\langle 2\pi\clubsuit(c_1), [\omega]^{m-1} \rangle = \frac{(m-1)!}{2} \int_{M_\varrho} s \, d\mu - \int_{\partial M_\varrho} \theta \wedge \omega^{m-1}$$

by Stokes' theorem.

Let  $f : M \rightarrow \mathbb{R}$  be smooth cut-off function:  
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because there is only one end!

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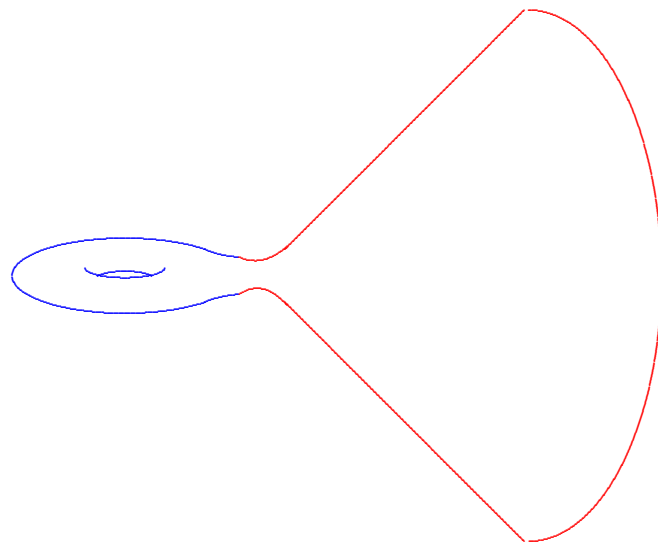
Limit as  $\varrho \rightarrow \infty$  now yields the mass formula.

$$\frac{4\pi^m(2m-1)}{(m-1)!} \mathfrak{m}(M, g) = -\frac{4\pi}{(m-1)!} \langle \clubsuit(c_1), [\omega]^{m-1} \rangle + \int_M s_g d\mu_g$$



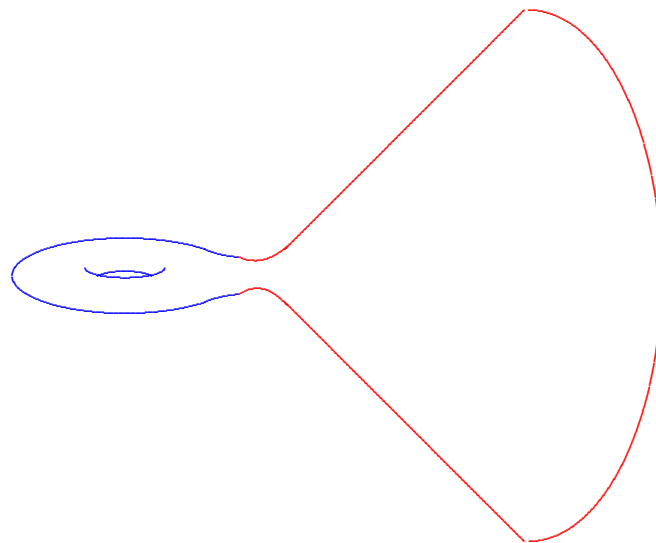
$$m(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1} \rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m} \int_M s_g d\mu_g$$

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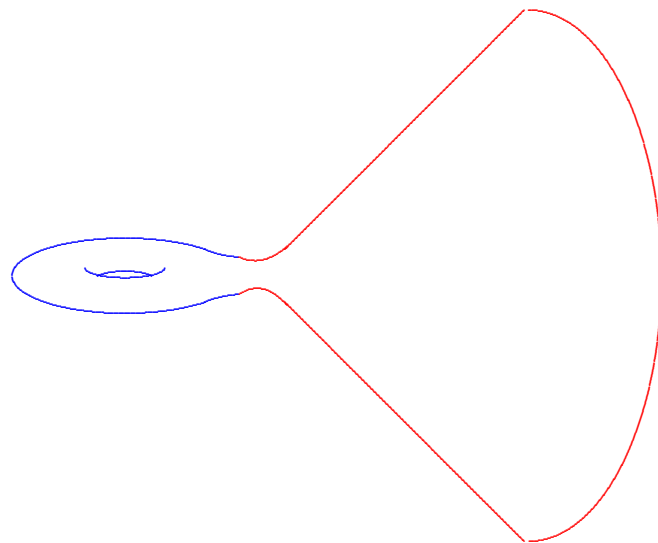
But the Penrose-type inequality is more subtle.

**Theorem E** (Penrose Inequality). *Let  $(M^{2m}, g, J)$  be an **AE** Kähler manifold with scalar curvature  $s \geq 0$ . Then  $(M, J)$  carries a canonical divisor  $D$  that is expressed as a sum  $\sum_j \mathbf{n}_j D_j$  of compact complex hypersurfaces with positive integer coefficients, with the property that  $\bigcup_j D_j \neq \emptyset$  whenever  $M \not\cong \mathbb{R}^{2m}$ . In terms of this divisor, we then have*

$$m(M, g) \geq \frac{(m-1)!}{(2m-1)\pi^{m-1}} \sum \mathbf{n}_j \text{Vol}(D_j)$$

*with  $= \iff (M, g, J)$  is scalar-flat Kähler.*

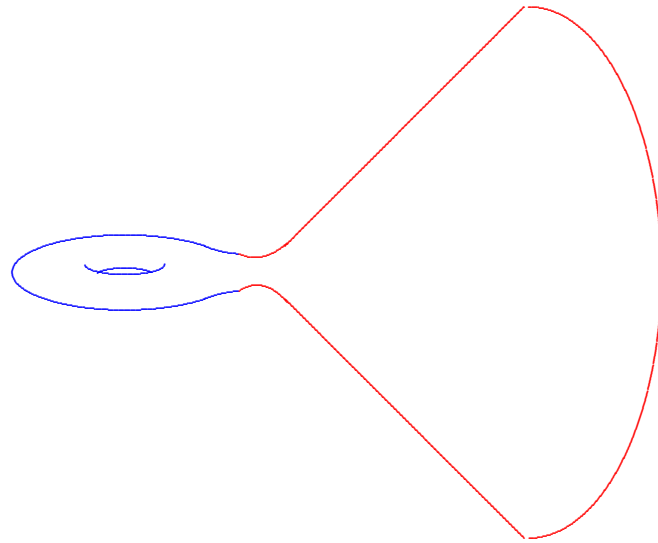
$$m(M, g) = \frac{\langle \clubsuit(-c_1), [\omega]^{m-1} \rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m} \int_M s_g d\mu_g$$





$m = 2 :$

$$m(M, g) = \frac{\langle \clubsuit(-c_1), [\omega] \rangle}{3\pi} + \frac{1}{12\pi} \int_M s_g d\mu_g$$



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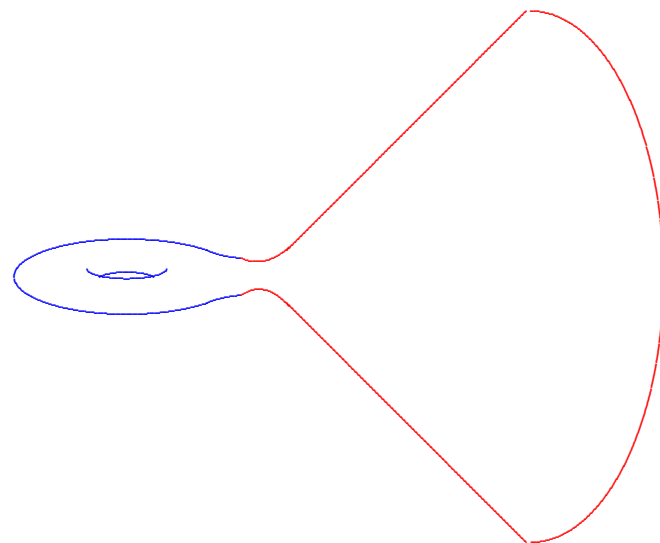
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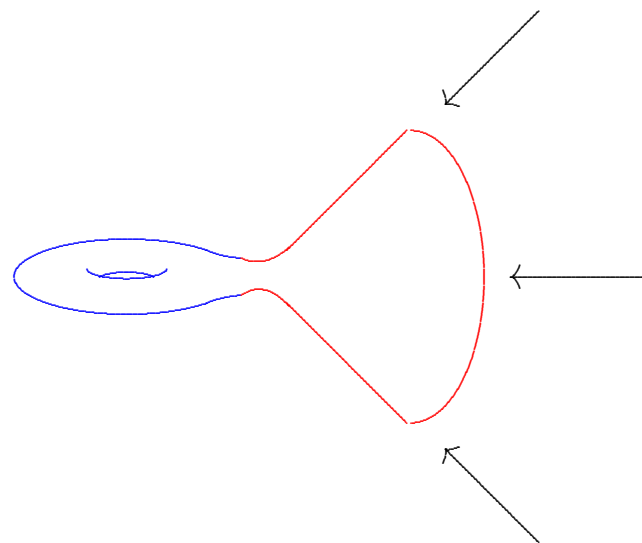
**Technical challenge:** Loss of control of derivatives!

Distance-decreasing map:

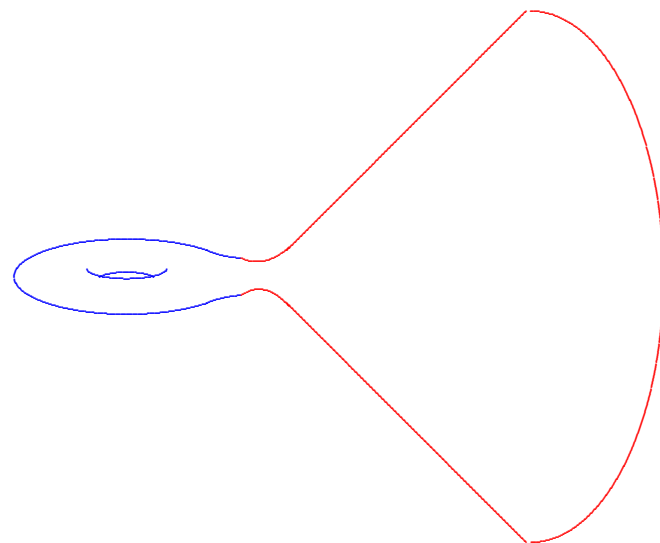




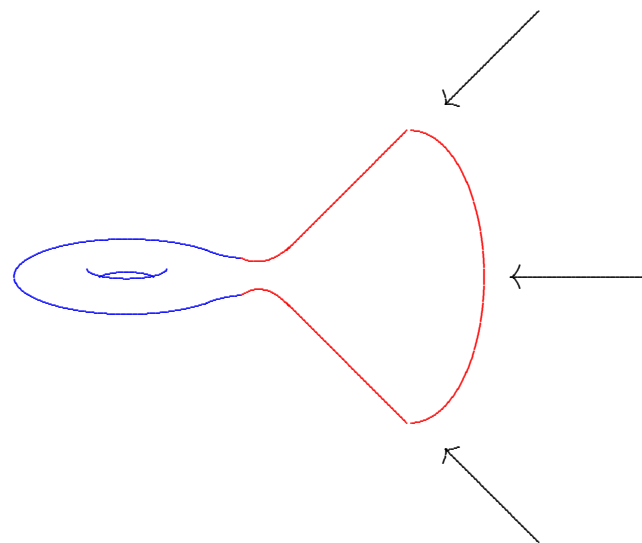
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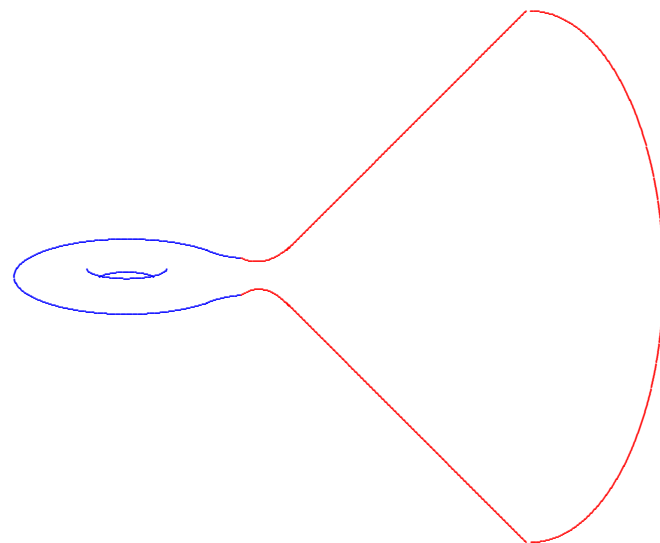
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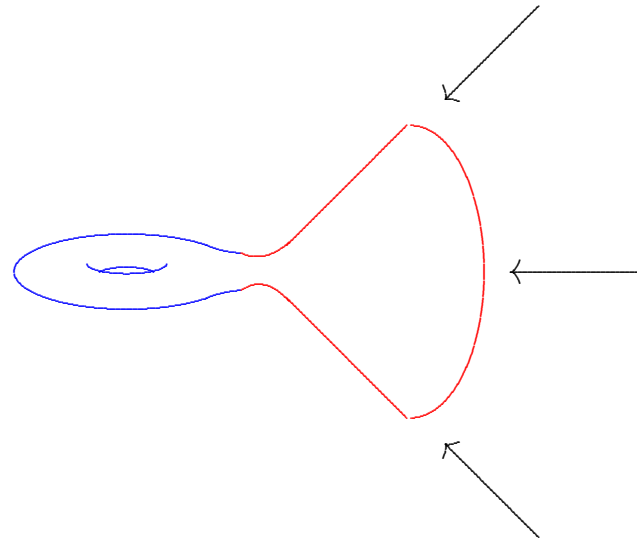
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Robust under distortion of metric in outer region.

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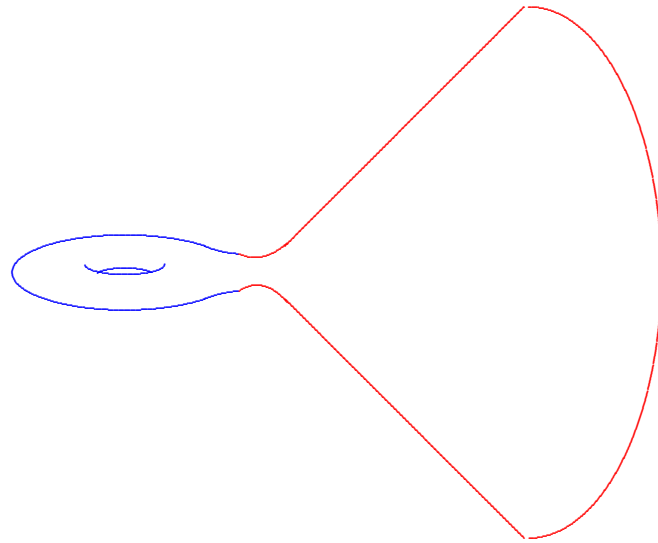
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In  $(M, J)$ , this gives desired Poincaré dual of  $-c_1$ .

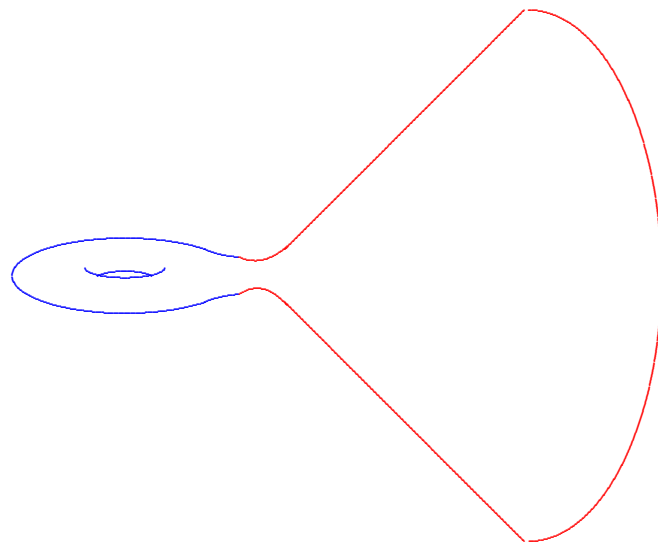


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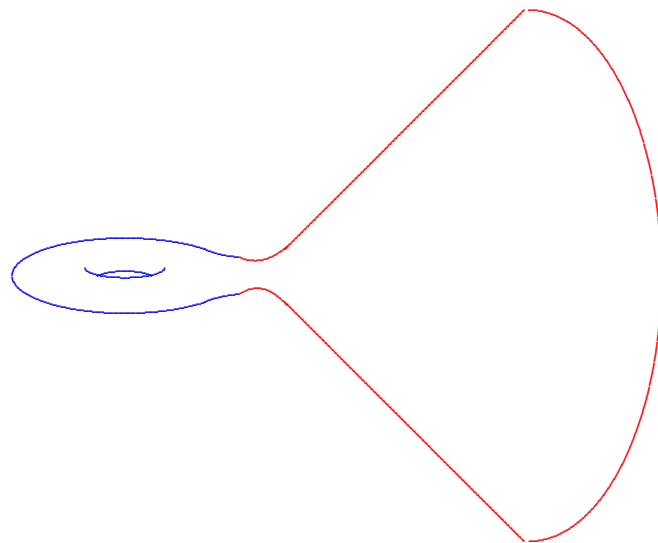


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**In fine,**

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organizzatori,

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非常感谢你邀请我!

Fēicháng gǎnxiè nǐ yāoqǐng wǒ!



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