Twistors,

Hyper-Kähler Manifolds, &

Complex Moduli

Claude LeBrun
Stony Brook University

Canadian Mathematical Society
Winter Meeting, Toronto, Ontario
December 3, 2022

Key references:

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Twistors, Hyper-Kähler Manifolds,
and Complex Moduli,

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Twistors, Hyper-Kähler Manifolds,
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Special Metrics and Groups Actions in Geometry,
Springer INdAM series, vol. 23, 2017.

And

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Topology versus Chern Numbers
for Complex 3-Folds,

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Pacific Journal of Mathematics **191** (1999) 123–131.

Main Problem:

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$$\mathcal{A} \subset H^1(Y, \mathcal{O}(T^{1,0}_{J_0}Y)).$$

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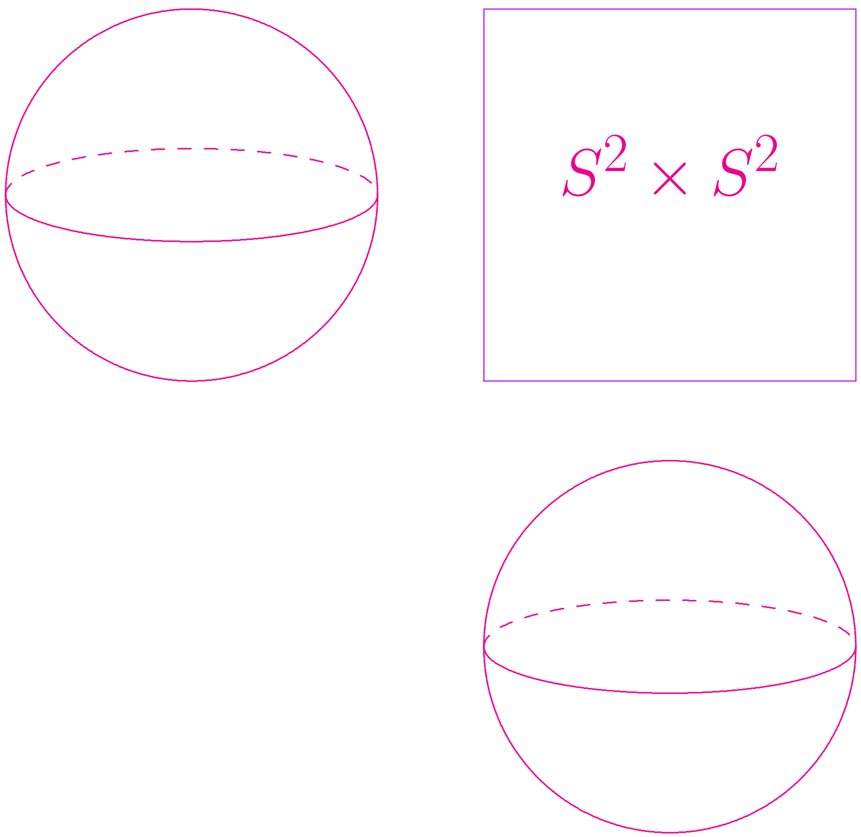

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"even" Hirzebruch surfaces

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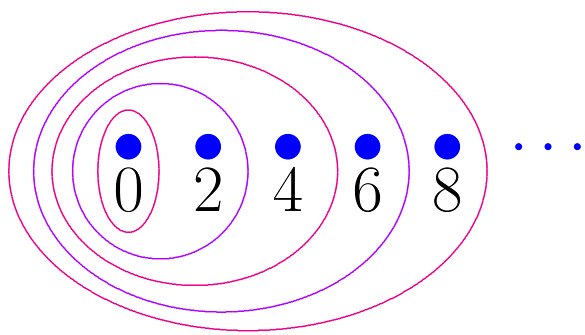

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In general, the answer is No!

Our route to this conclusion:

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Theory of Riemannian Holonomy

Recall. . .

(M^n, g): holonomy

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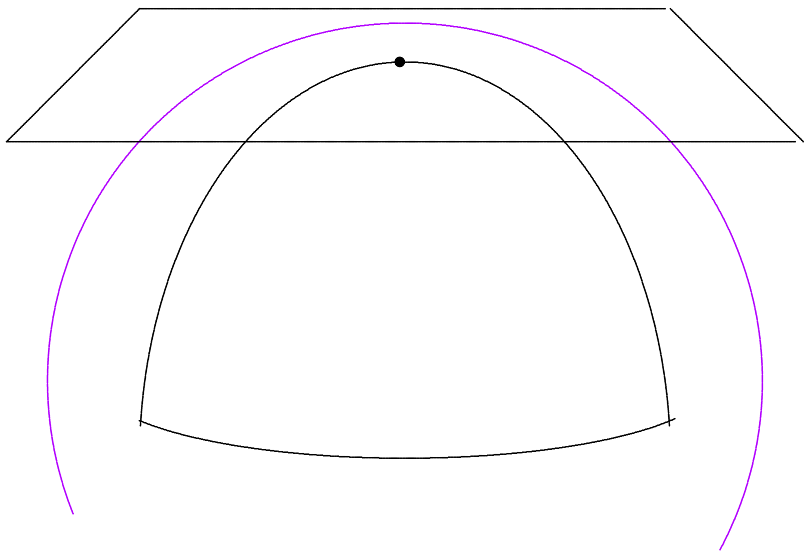

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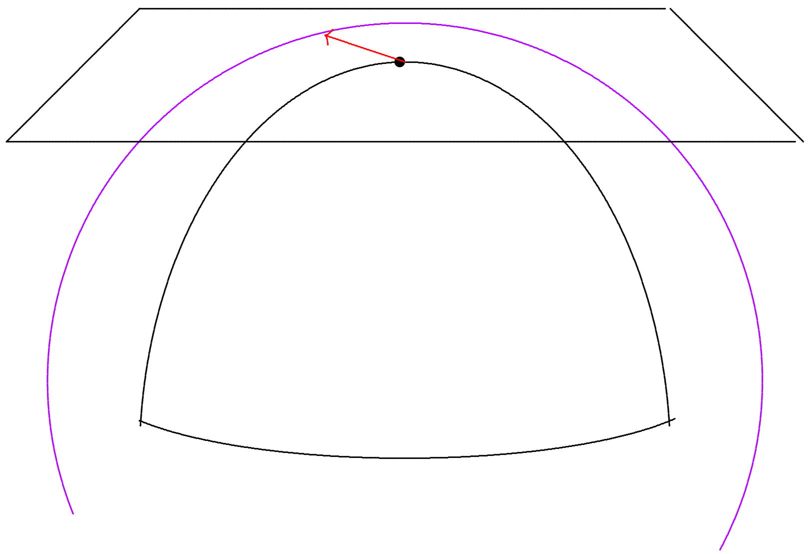

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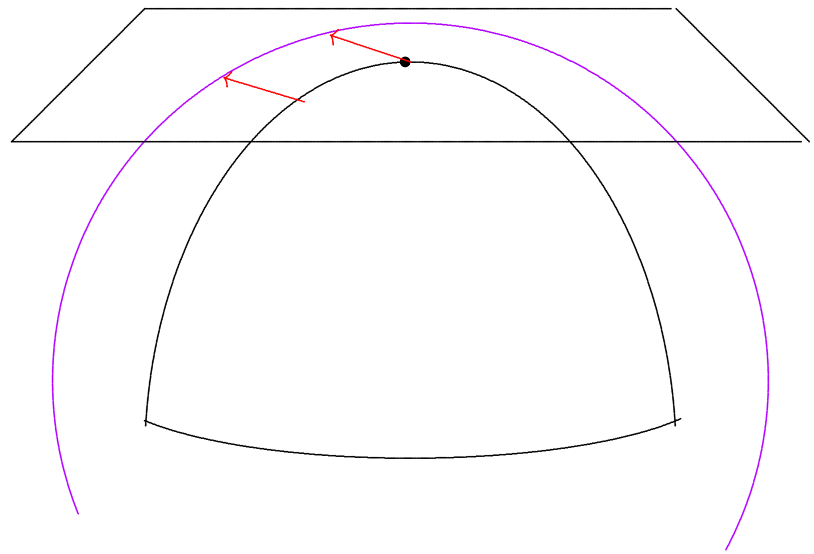

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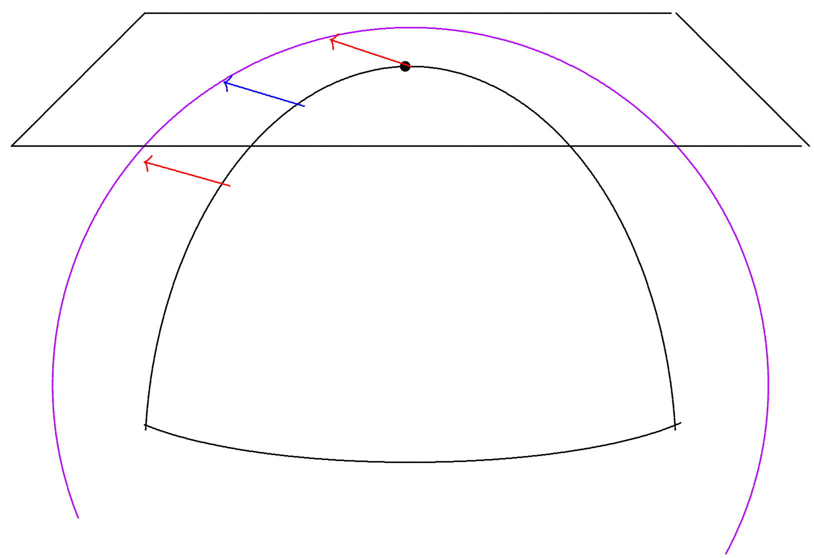

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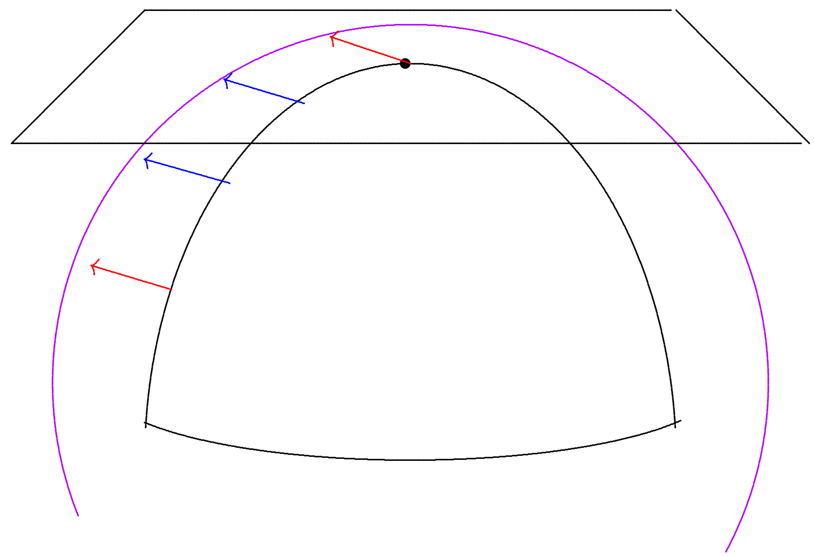

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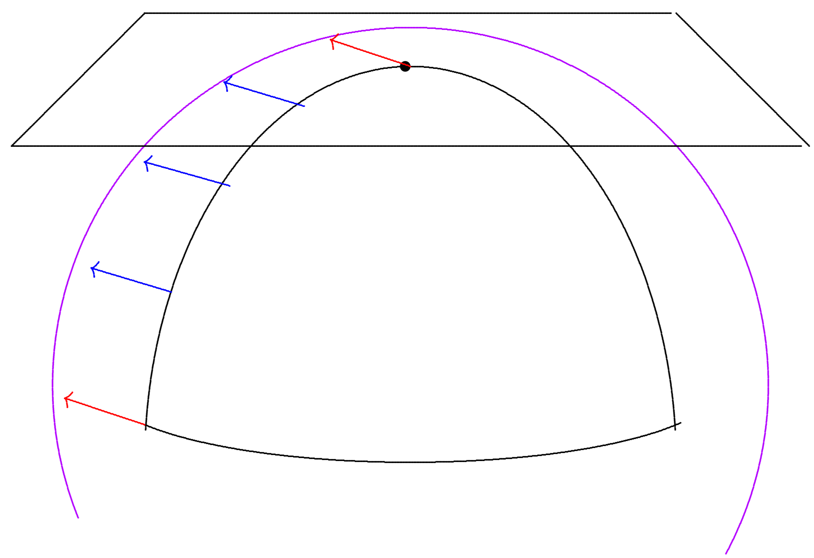

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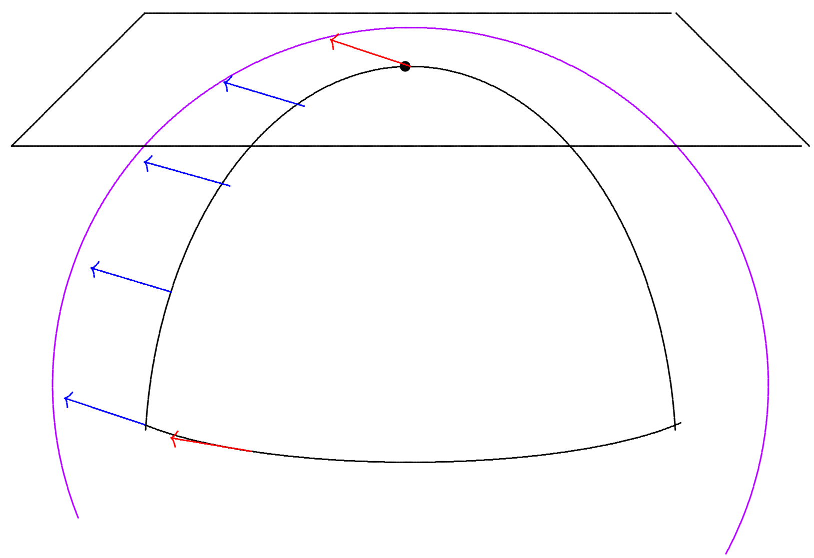

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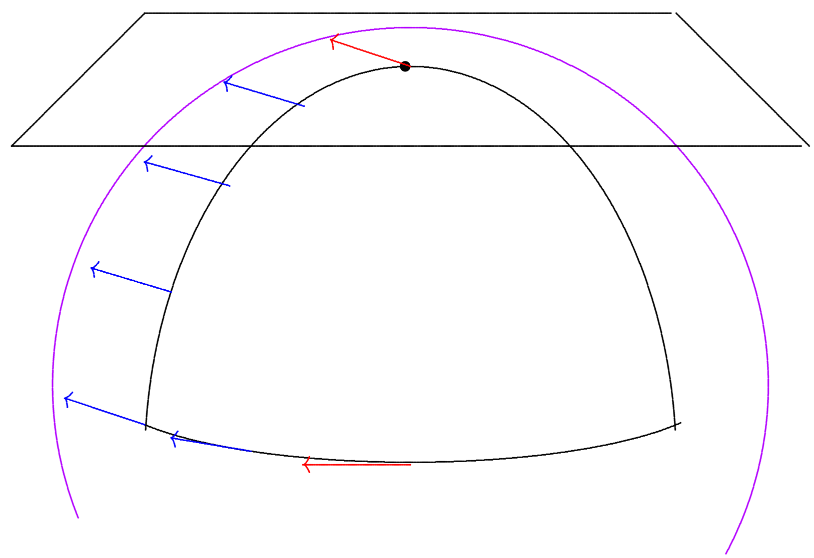

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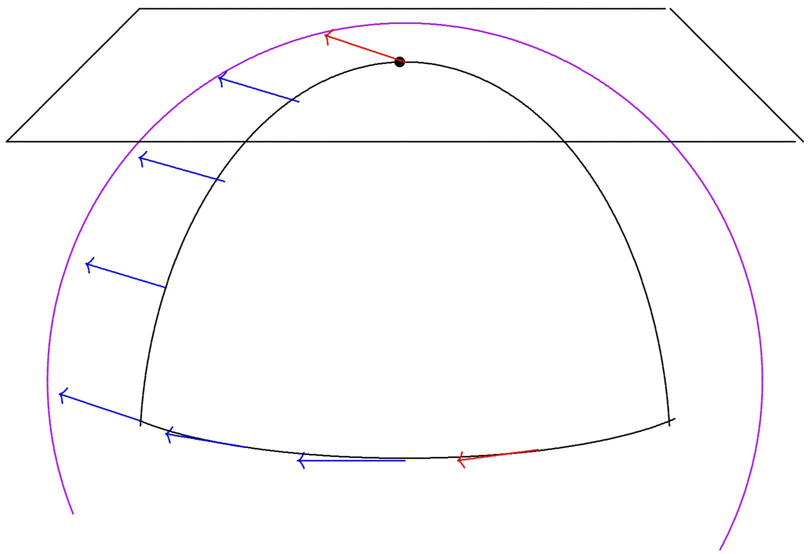

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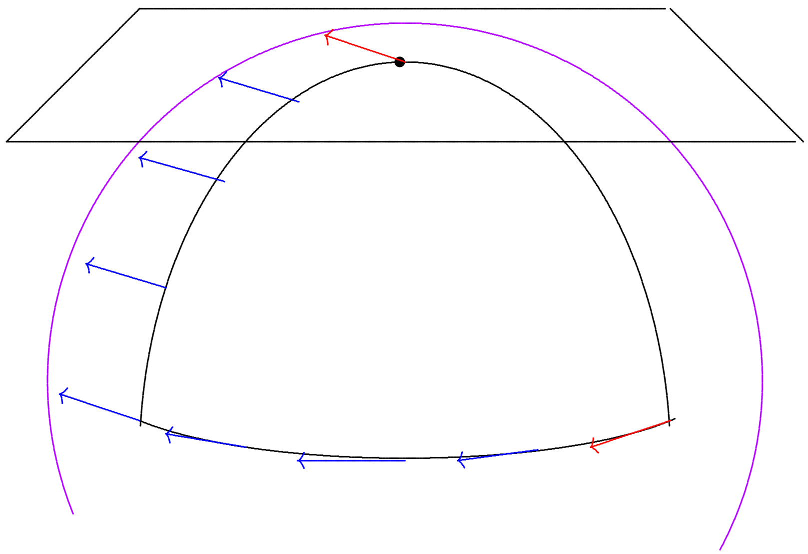

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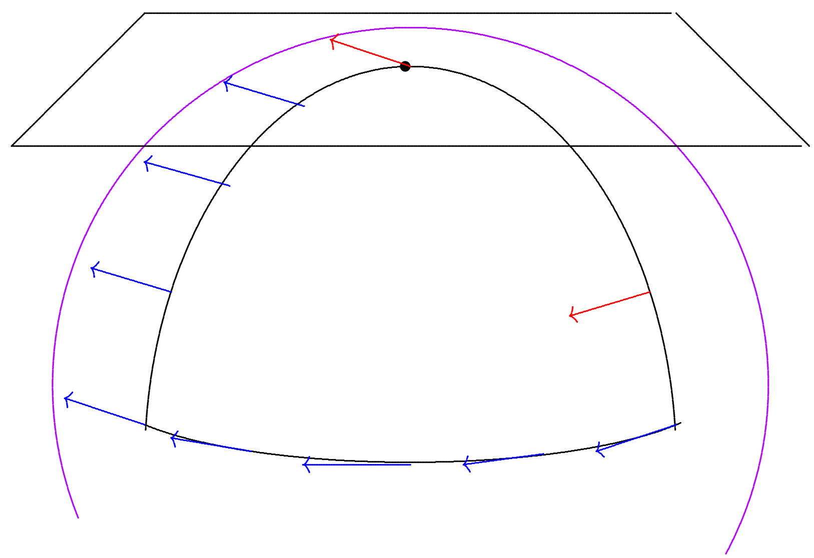

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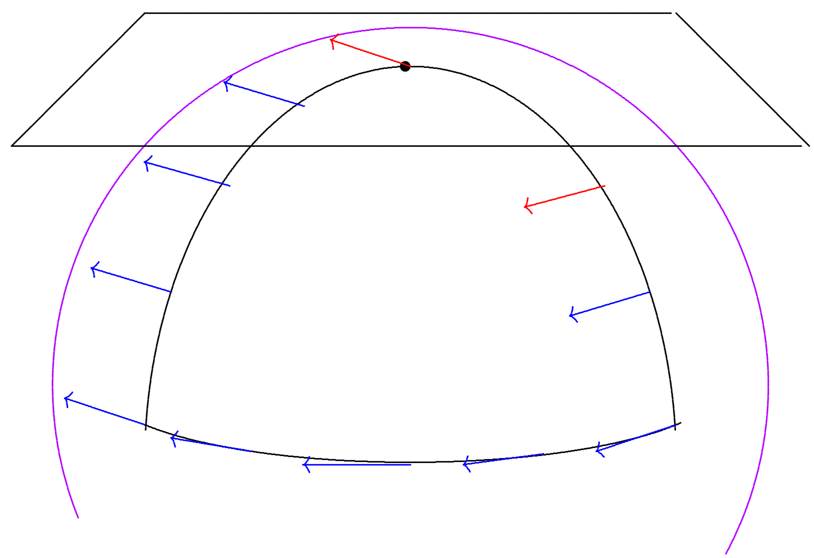

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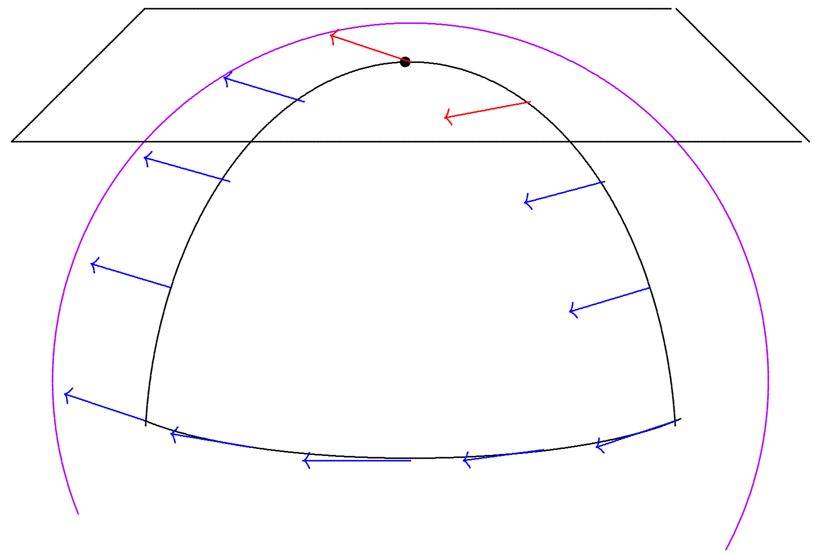

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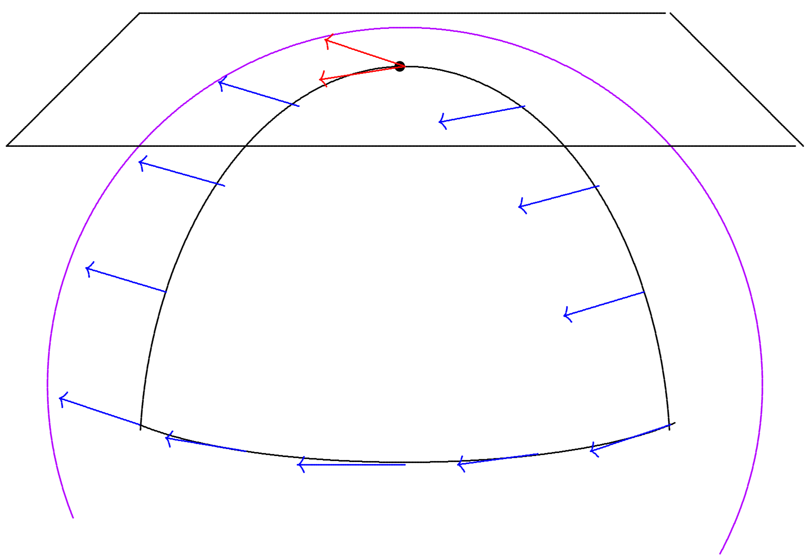

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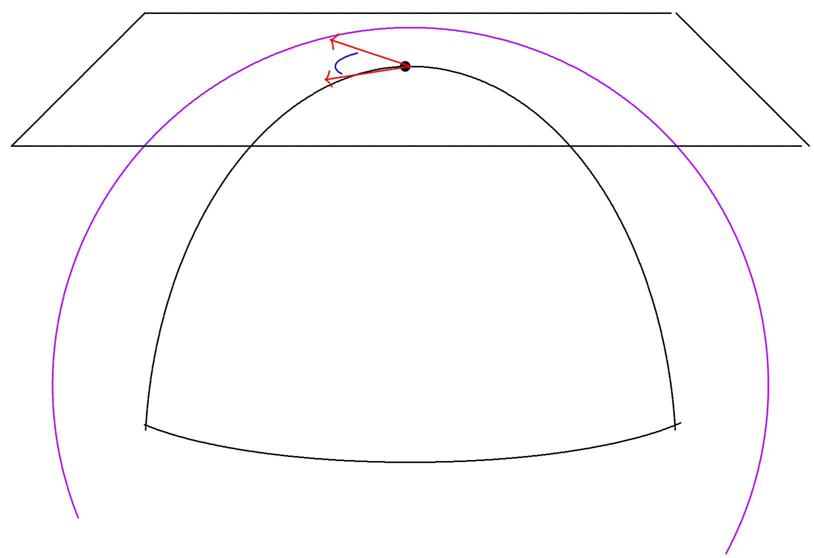
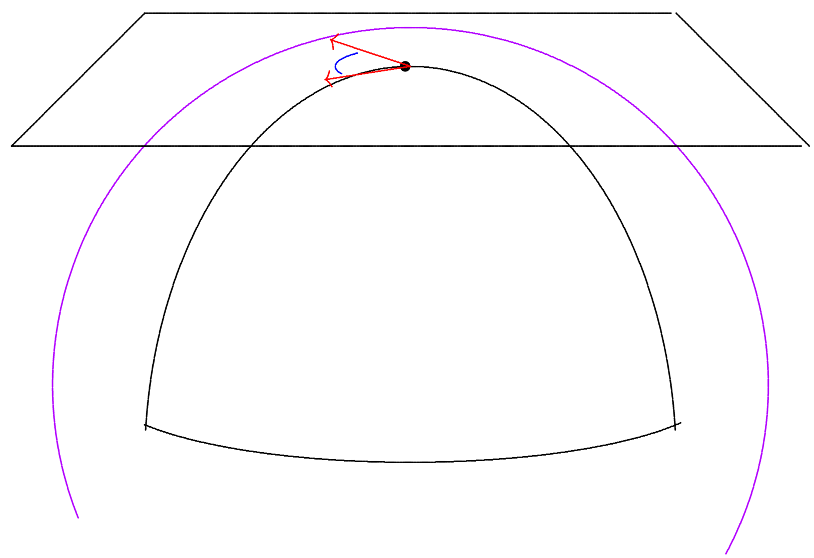


(M^n, g):

holonomy $\subset \mathbf{O}(n)$



Kähler metrics:

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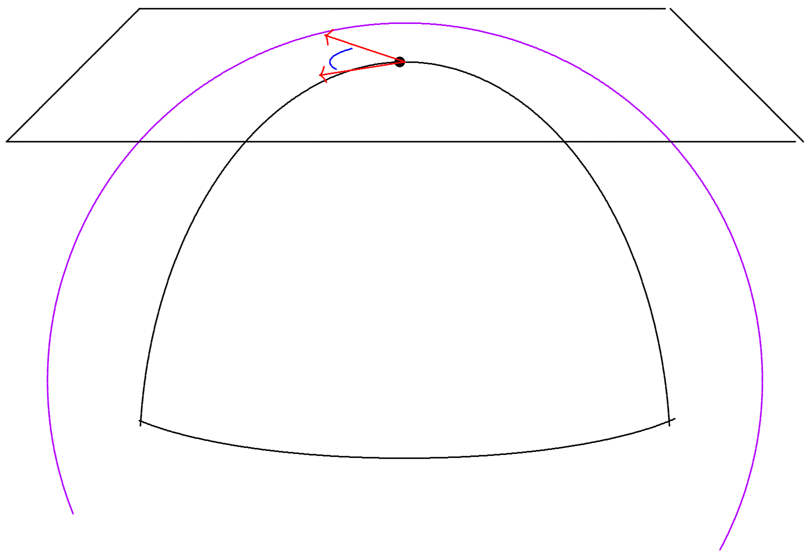
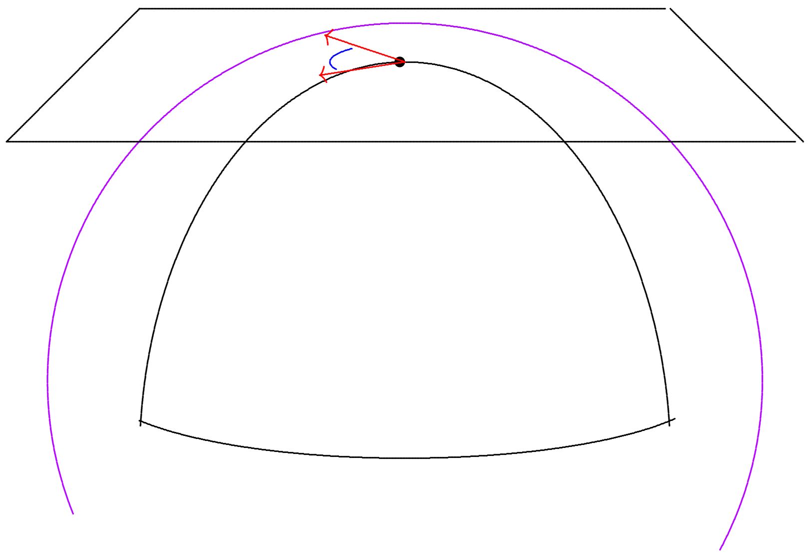


Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$



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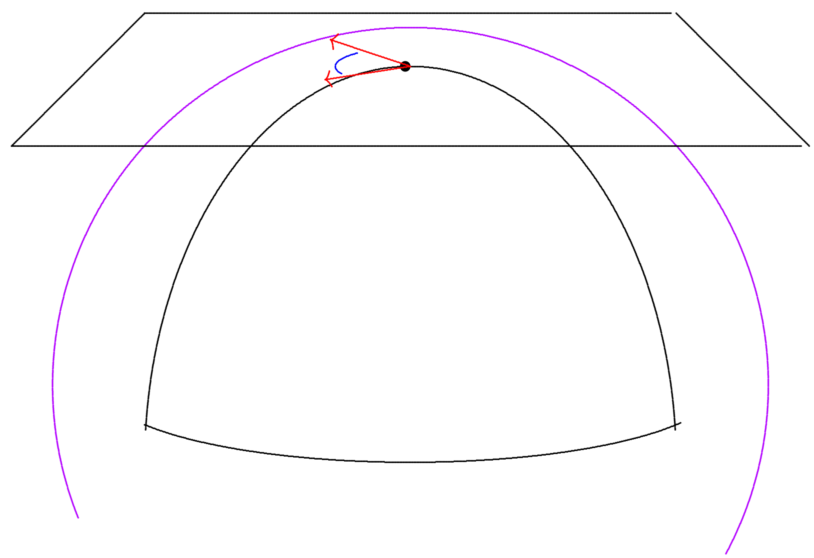
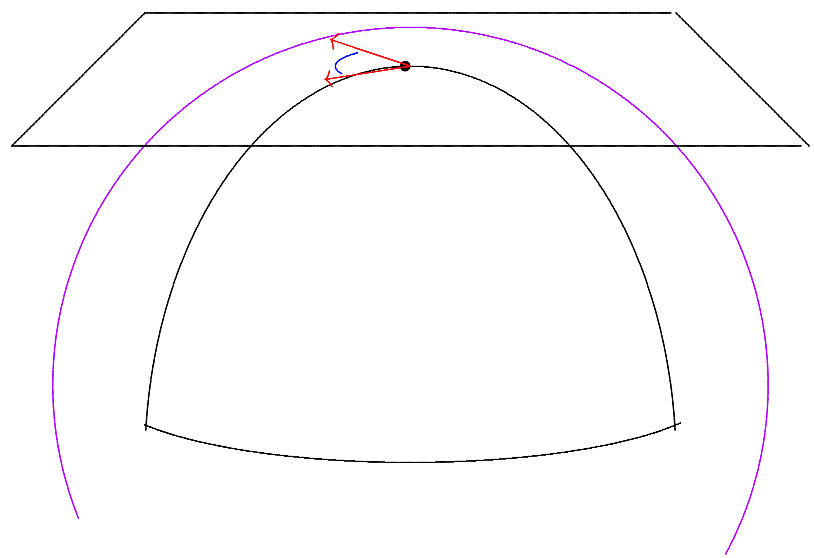


$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

Ricci-flat Kähler metrics:

(M^{2m}, g): holonomy



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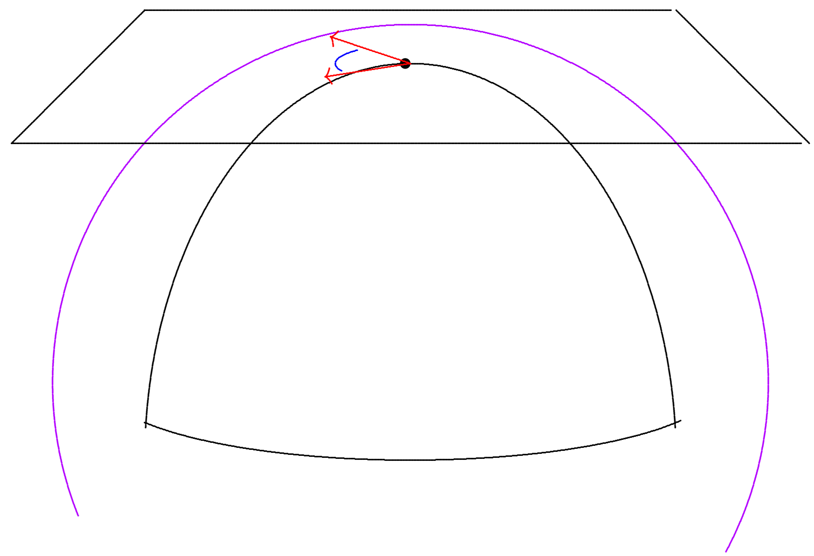
(M^{2m}, g): Ricci-flat Kähler $\Longleftarrow$ holonomy $\subset \mathbf{SU}(m)$



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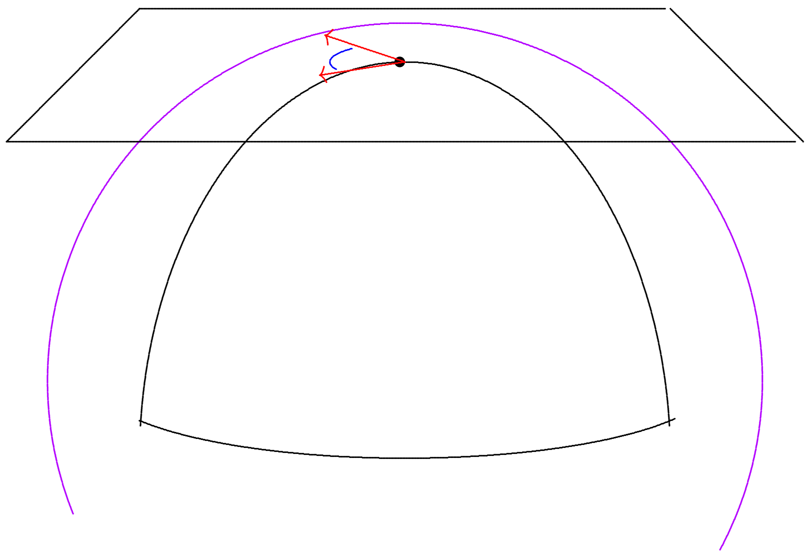

$\mathbf{SU}(m) \subset \mathbf{U}(m)$: $\{A \mid \det A = 1\}$

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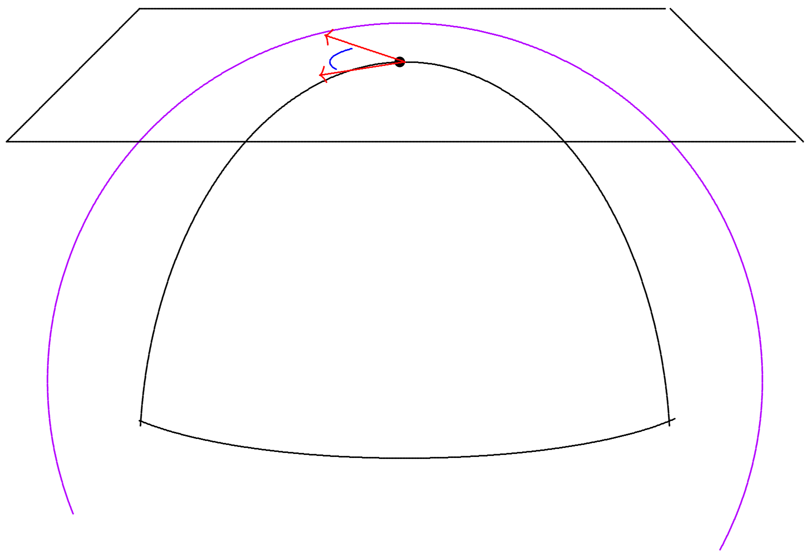

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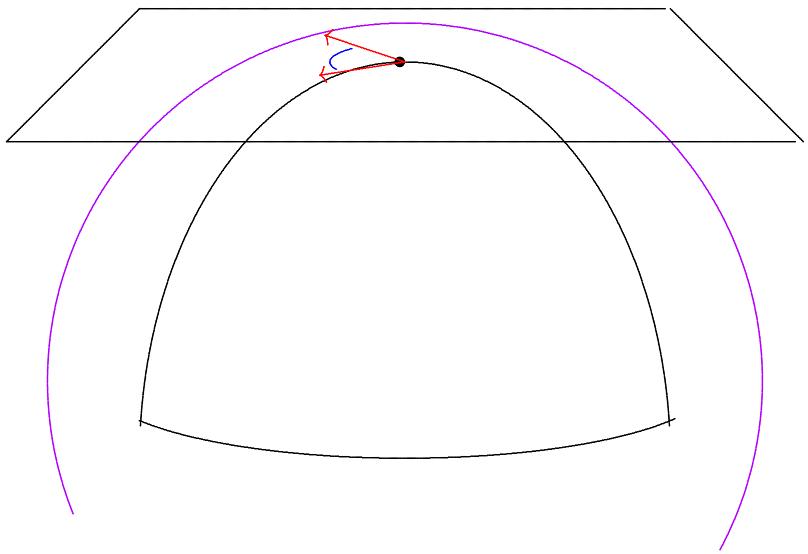
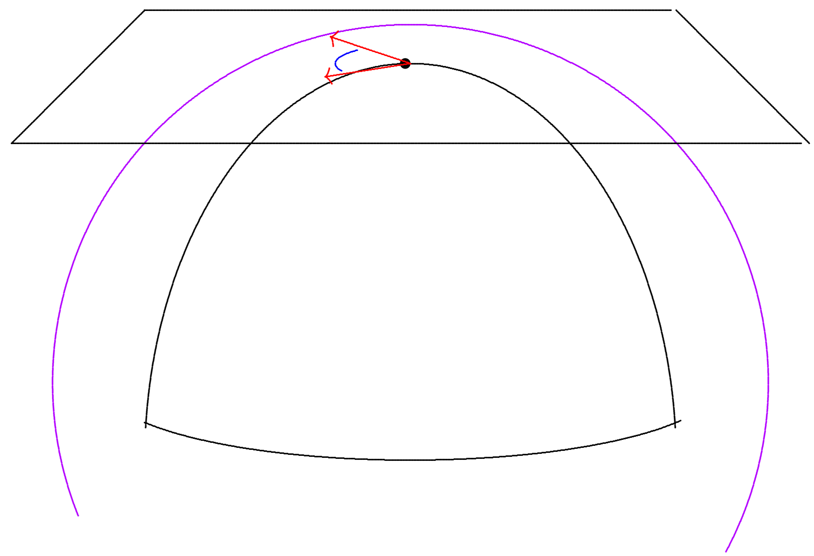


if M is simply connected.

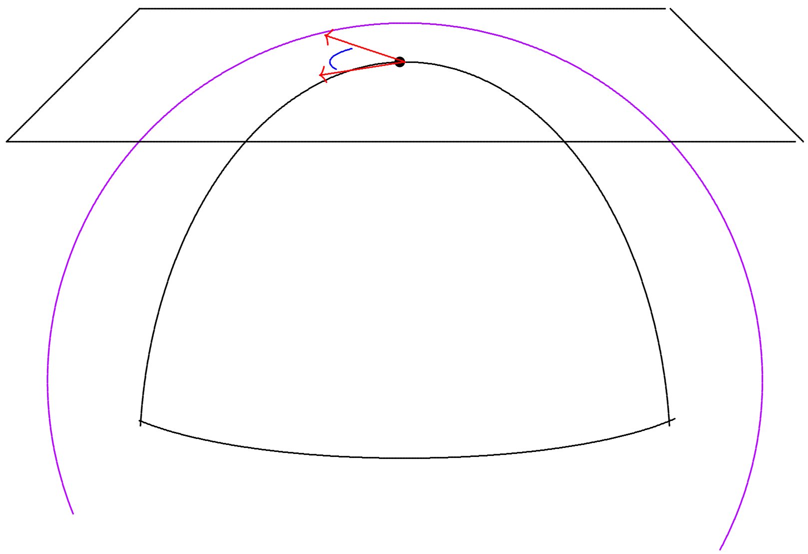
Calabi-Yau metrics:

(M^{2m}, g): Calabi-Yau $\Longleftrightarrow$ holonomy $\subset \mathbf{SU}(m)$



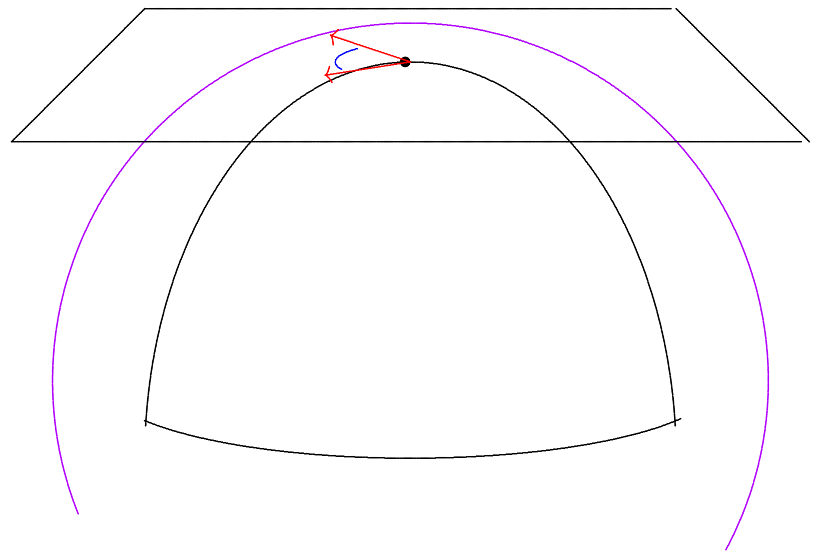
Hyper-Kähler metrics:

(M^{4k}, g) holonomy



Hyper-Kähler metrics:

(M^{4k}, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(k)$



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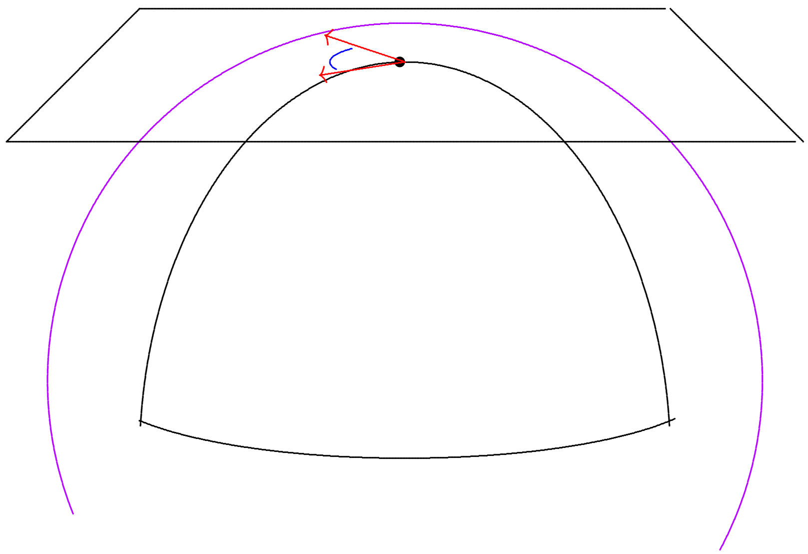

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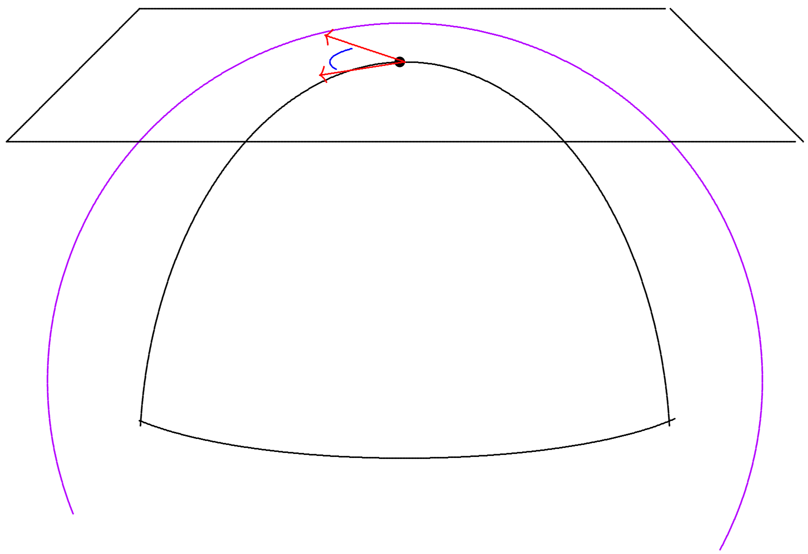

$\mathbf{Sp}(k) \subset \mathbf{SU}(2k)$

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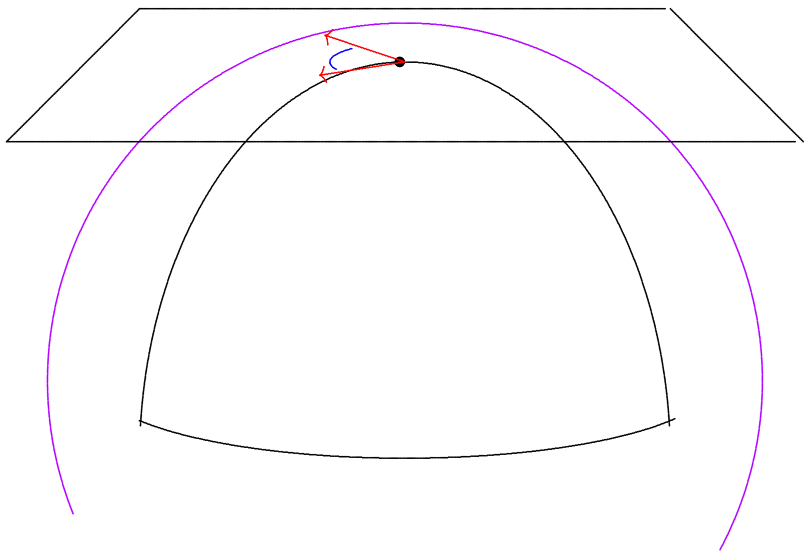

$\mathbf{Sp}(k) \subset \mathbf{SU}(2k)$

in many ways!

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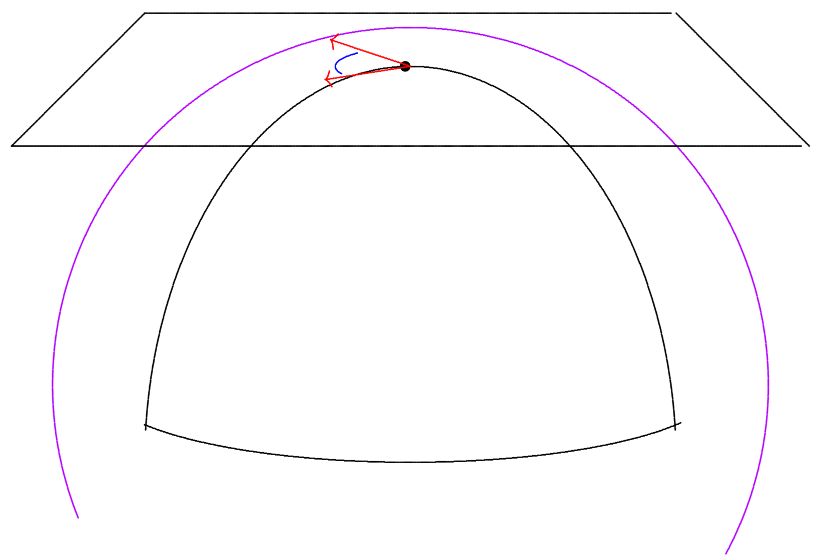

$$\mathbf{Sp}(k) \subset \mathbf{SU}(2k)$$

in many ways! (For example, permute i, j, k. . .)

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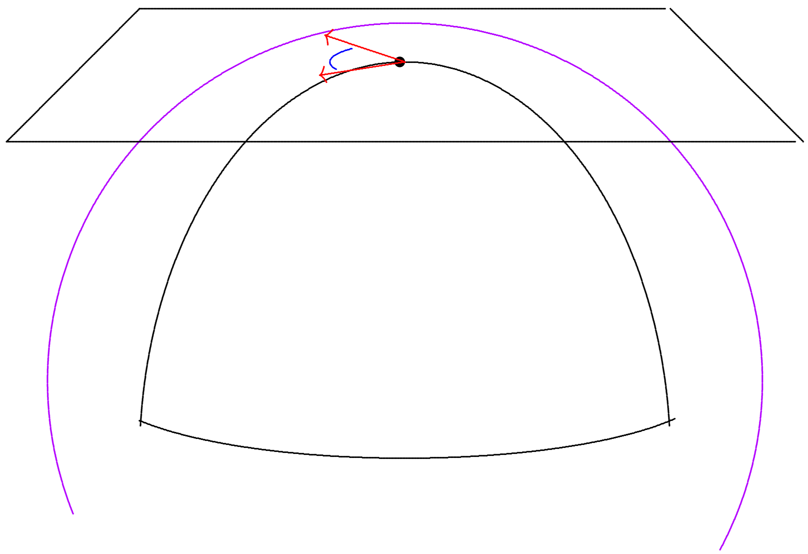

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Ricci-flat and Kähler,

for many different complex structures!

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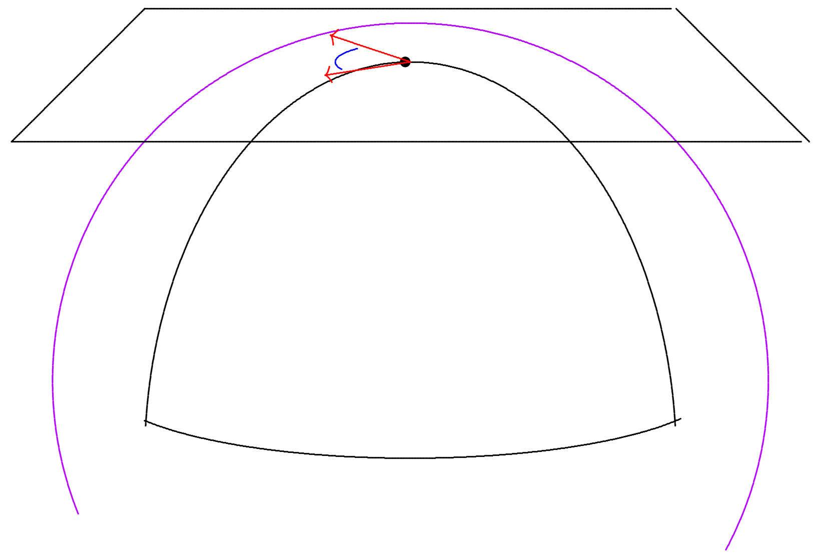
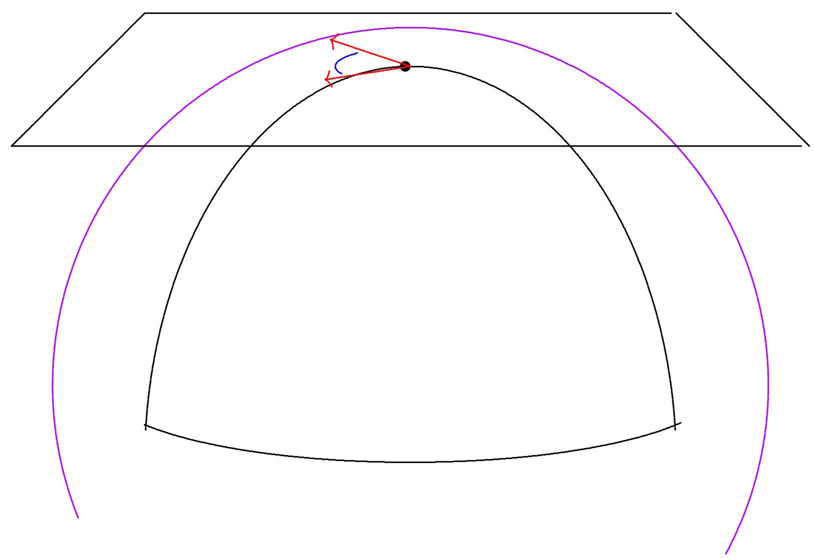


$\mathbf{Sp}(k) \subset \mathbf{SU}(2k)$

Hyper-Kähler metrics:

(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



$$\mathbf{Sp}(1) = \mathbf{SU}(2)$$

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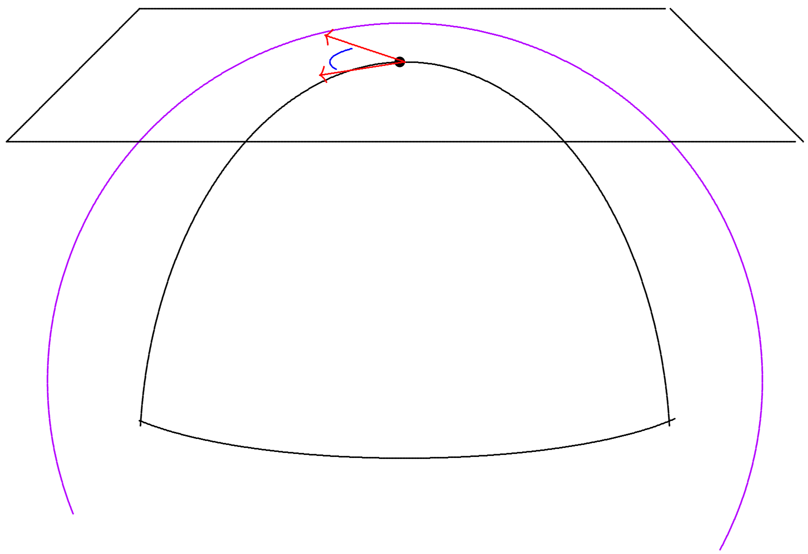

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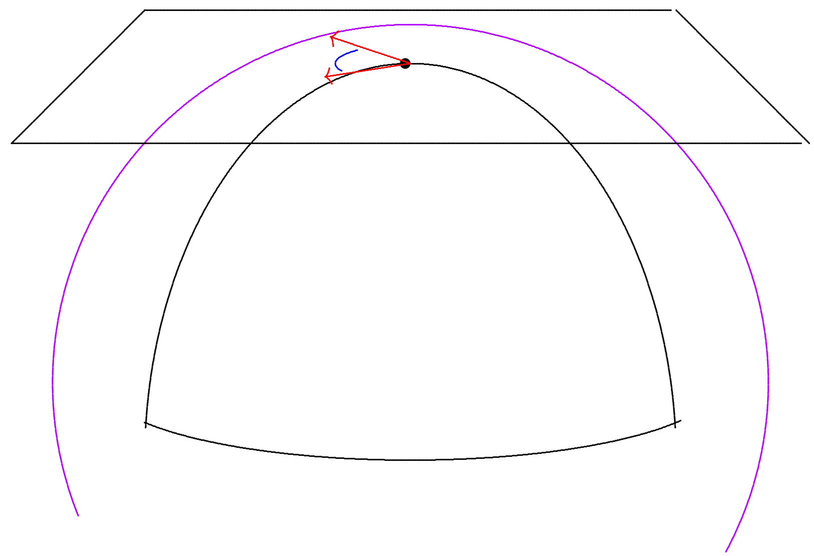
For (M^4, g):

hyper-Kähler $\Longleftrightarrow$ Calabi-Yau.

Hyper-Kähler metrics:

(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



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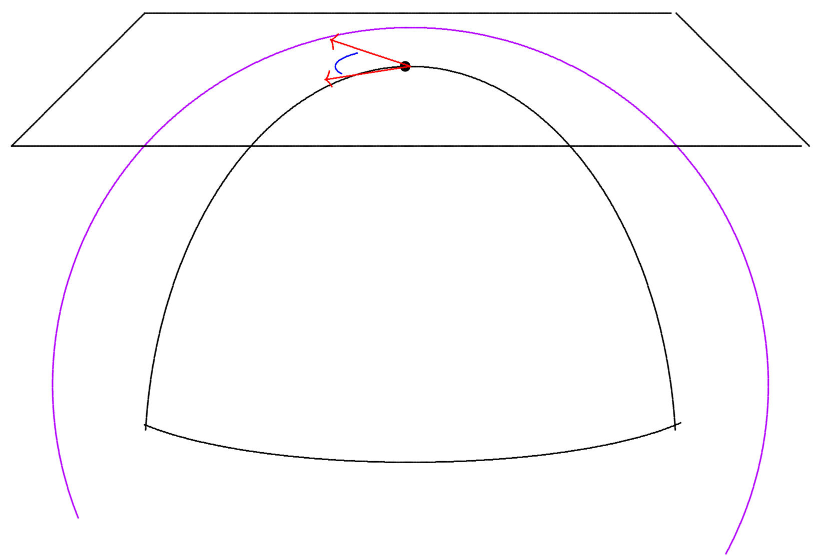

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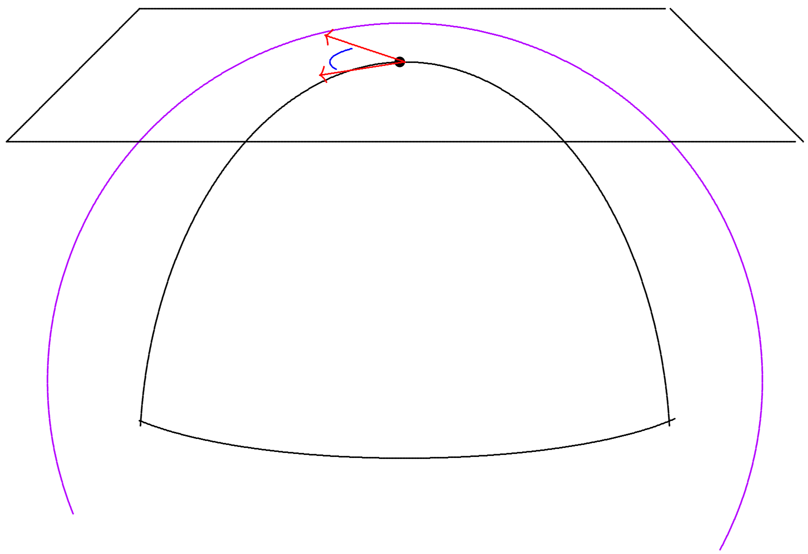

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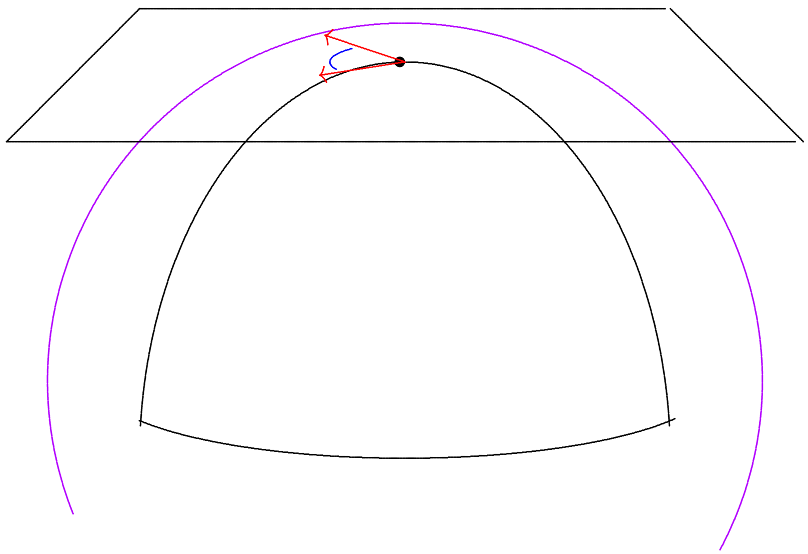

$$\mathbf{Sp}(k) \subset \mathbf{SU}(2k)$$

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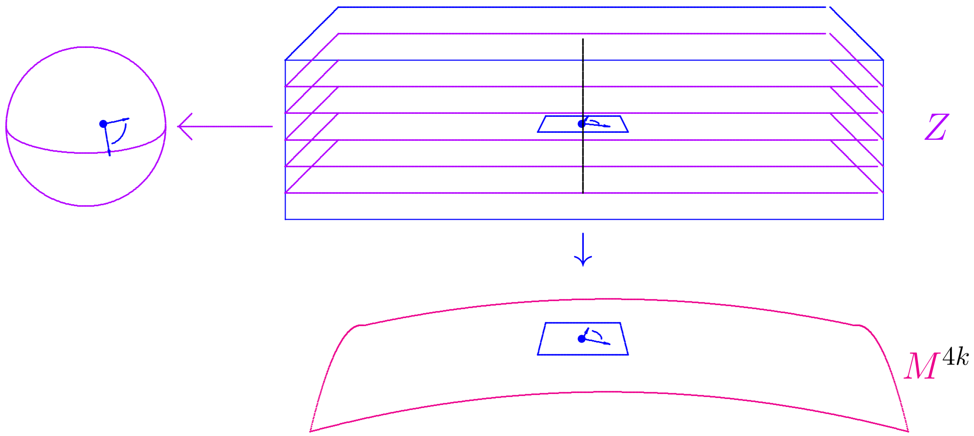
All these complex structures can be repackaged



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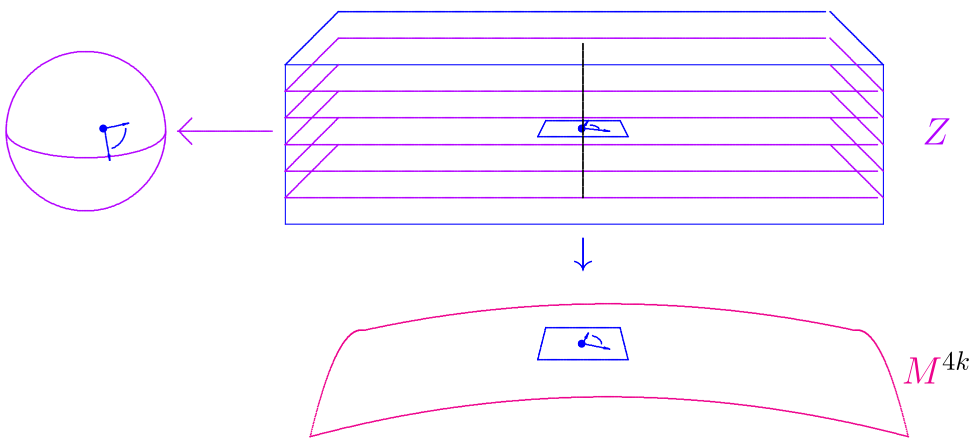

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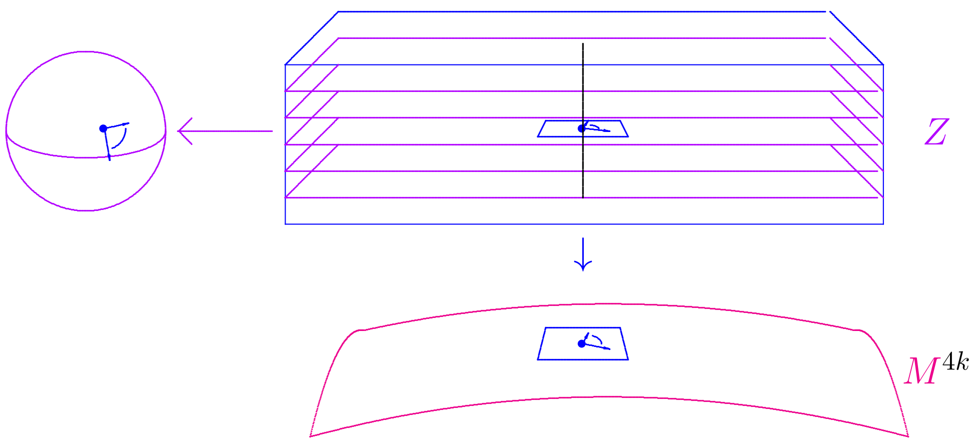

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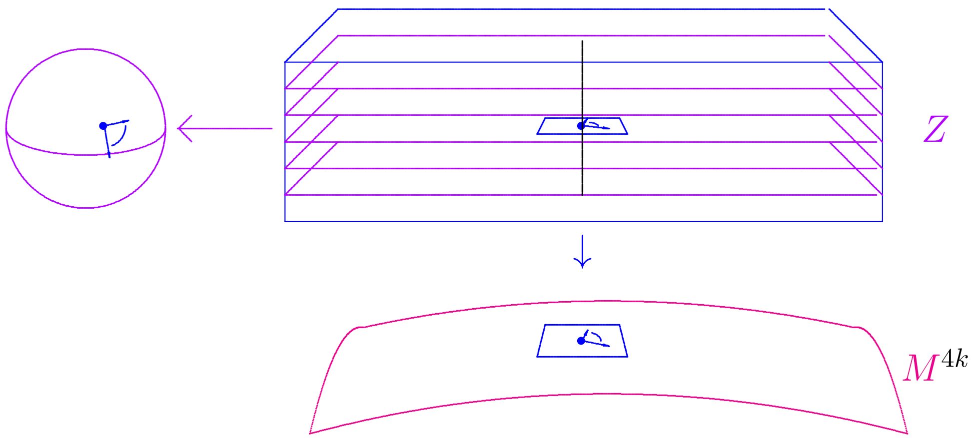

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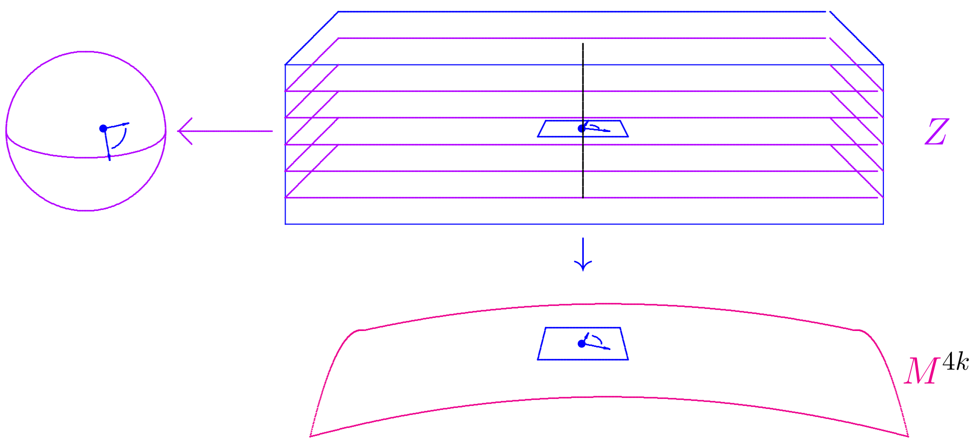

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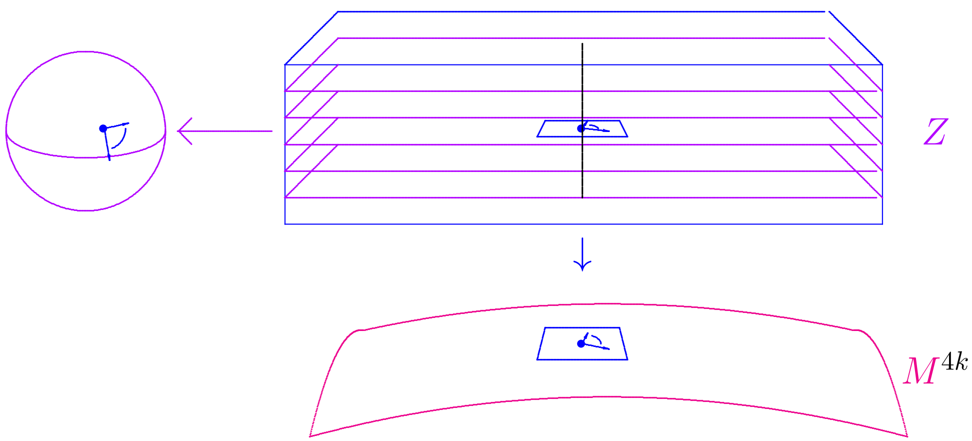

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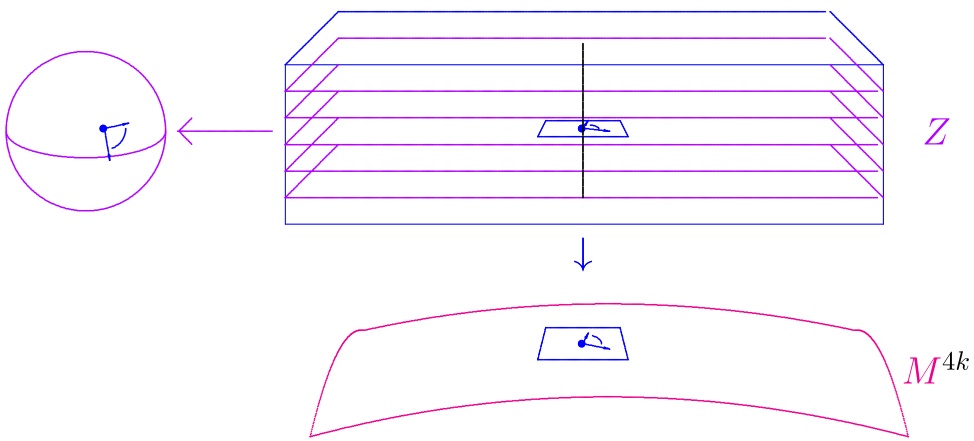

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Multiplicativity of Todd genus + Cheeger-Gromoll.

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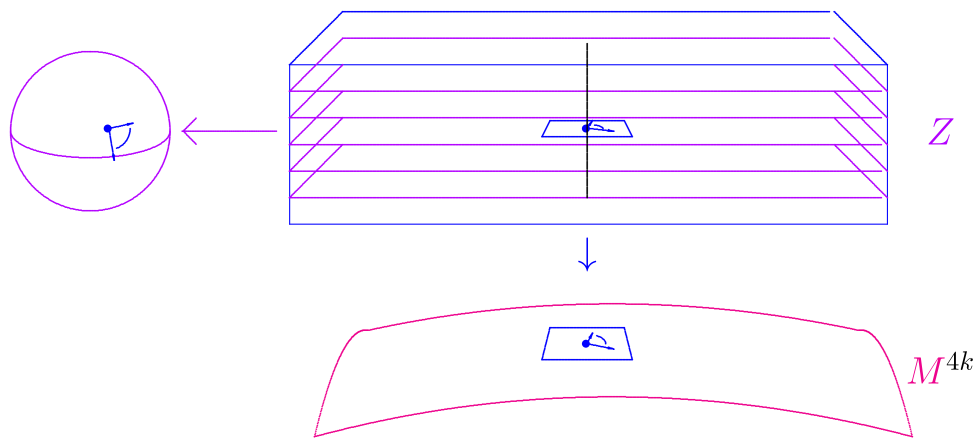

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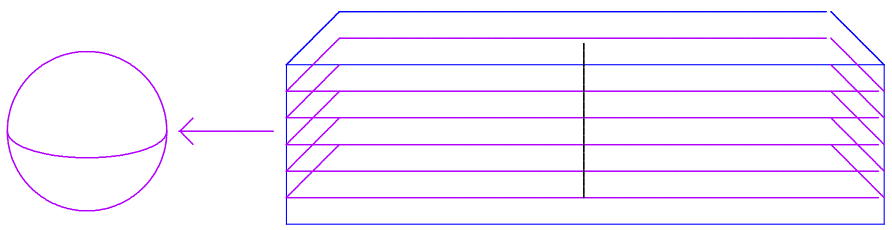

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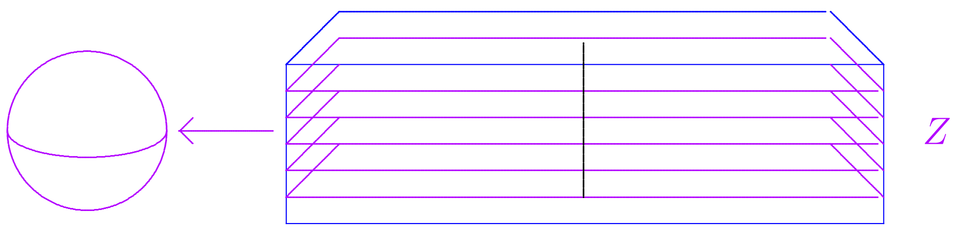

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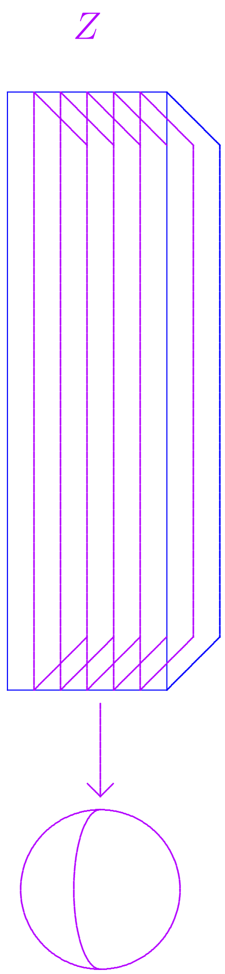

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Set $\ell = \deg(f)$.

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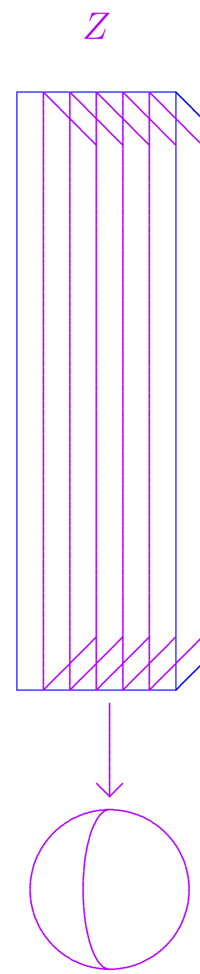

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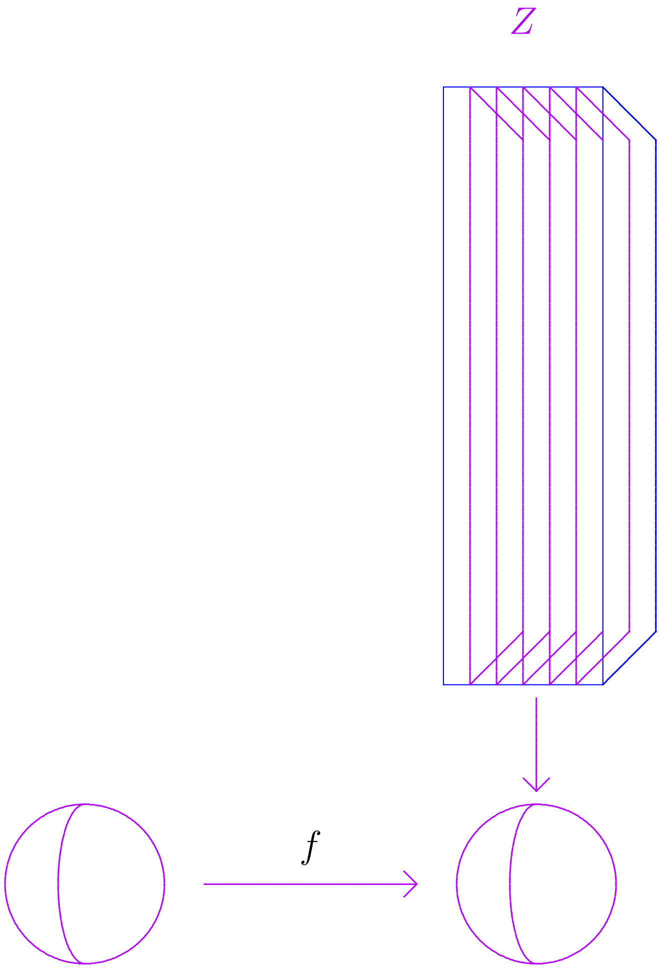

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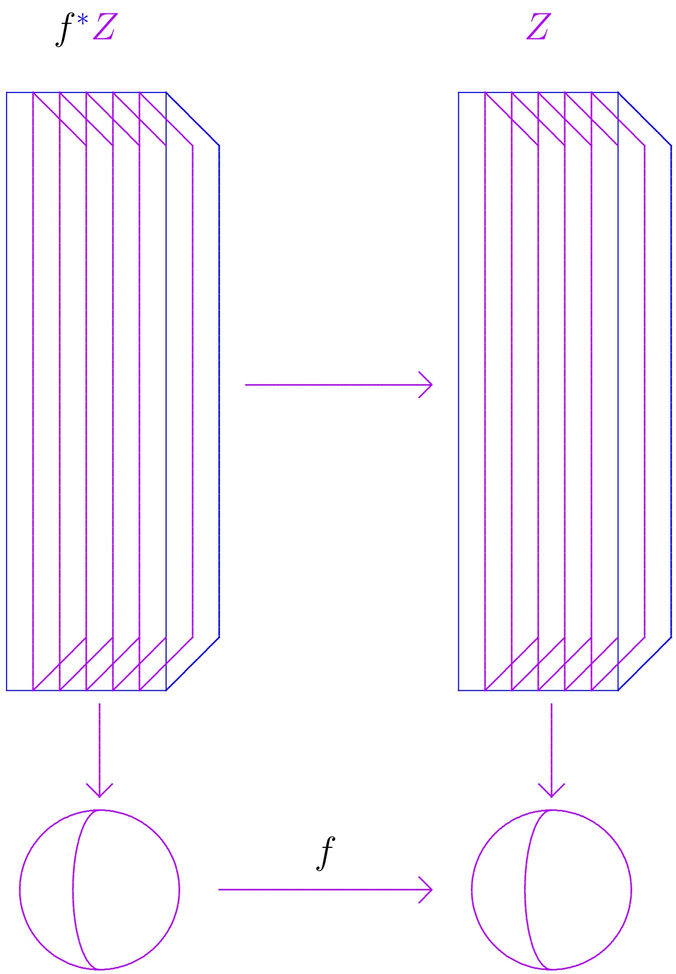

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Pull-back f^*Z is exactly $(\varpi \times \mathrm{id})^{-1}(\mathrm{graph}_f)$.

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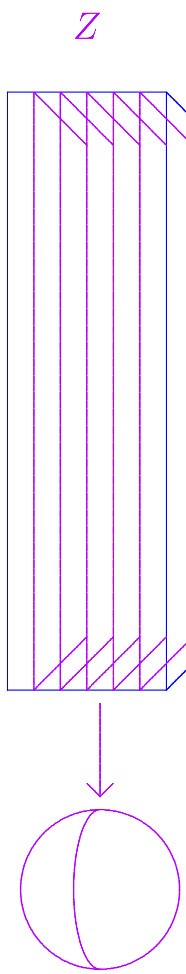

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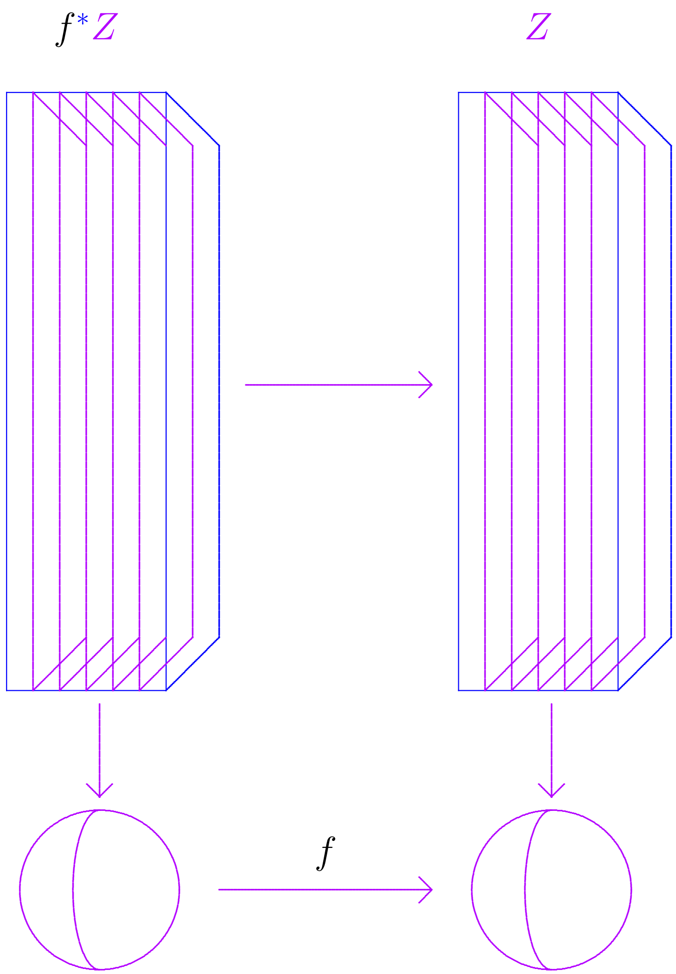

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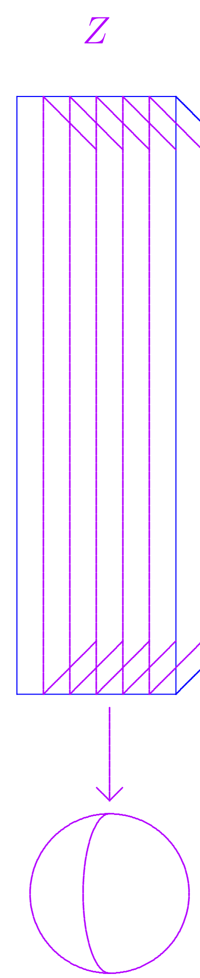

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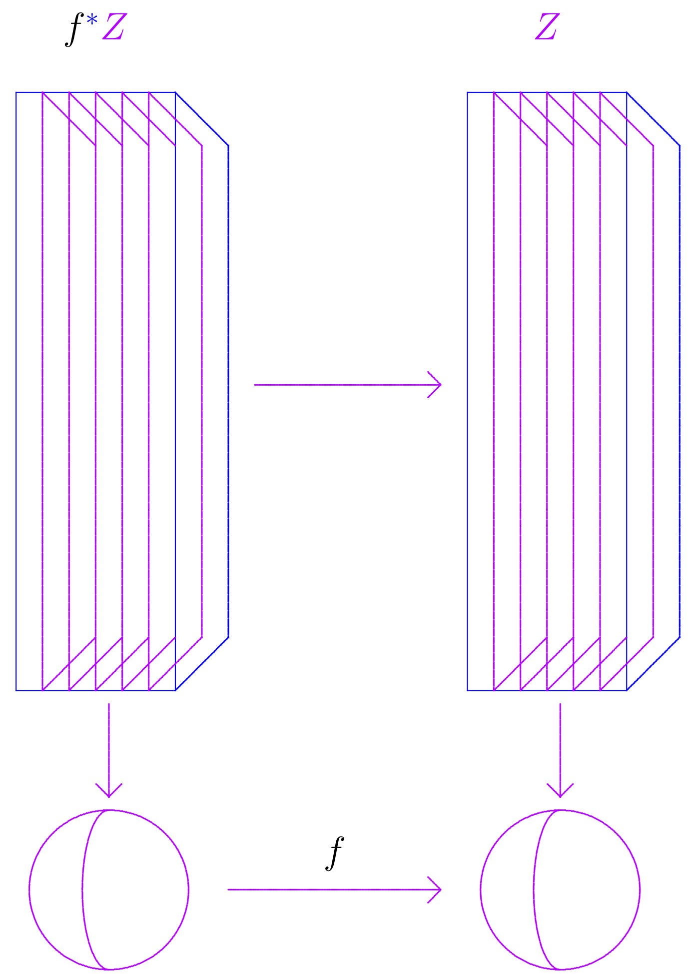

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- Kodaira-Spencer map of the family

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Well, thanks for your attention!

It's a real pleasure being here!

Thanks for the invitation!