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Einstein metrics, complex surfaces, and symplectic 4-manifolds

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Abstract

Which smooth compact 4-manifolds admit an Einstein metric with non-negative Einstein constant? A complete answer is provided in the special case of 4-manifolds that also happen to admit either a complex structure or a symplectic structure.

A Riemannian manifold (M, g) is said to be *Einstein* if it has constant Ricci curvature, in the sense that the function

$$v \longmapsto r(v, v)$$

on the unit tangent bundle $\{v \in TM \mid \|v\|_g = 1\}$ is constant, where r denotes the Ricci tensor of g. This is of course equivalent to demanding that g satisfy the *Einstein equation*

$$r = \lambda g$$

for some real number λ. A fundamental open problem in global Riemannian geometry is to determine precisely which smooth compact n-manifolds admit Einstein metrics. For further background on this problem, see [**4**].

When $n = 4$, the problem is deeply intertwined with geometric and topological phenomena unique to this dimension; and our discussion here will therefore solely focus on this idiosyncratic case. But this paper will focus on even narrower versions of the problem. Let

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us thus first consider the special class of smooth 4-manifolds that arise from compact complex surfaces by forgetting the complex structure. Which of these admit Einstein metrics? If we are willing to also constrain the Einstein constant λ to be non-negative, the following complete answer can now be given:

THEOREM A. *Let M be the underlying smooth 4-manifold of a compact complex surface. Then M admits an Einstein metric with $\lambda \geqslant 0$ if and only if it is diffeomorphic to one of the following: a del Pezzo surface, a $K3$ surface, an Enriques surface, an Abelian surface, or a hyper-elliptic surface.*

Recall that complex surfaces with $c_1 > 0$ are called *del Pezzo surfaces*. The complete list of these [**10**, **22**] consists of $\mathbb{CP}_1 \times \mathbb{CP}_1$ and of $\mathbb{CP}_2$ blown up at k points in general position, where $0 \leqslant k \leqslant 8$. Up to diffeomorphism, the possibilities are thus $S^2 \times S^2$ and $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $0 \leqslant k \leqslant 8$; here # denotes the connected sum, and $\overline{\mathbb{CP}}_2$ denotes $\mathbb{CP}_2$ equipped with its non-standard orientation.

A celebrated result of Tian [**30**] asserts that most del Pezzo surfaces actually admit $\lambda > 0$ *Kähler–Einstein* metrics; for earlier related results, see [**26**, **31**]. However, there are two exceptional cases that are not covered by Tian's existence theorem; namely, no Kähler-Einstein metric can exist on on $\mathbb{CP}_2$ blown up at $k = 1$ or 2 points, because [**23**] both of these complex manifolds have non-reductive automorphism groups. Nonetheless, an explicit $\lambda > 0$ Einstein metric on $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ was constructed Page [**25**]; and while Page's construction seemed to have nothing at all to do with Kähler geometry, Derdziński [**11**] eventually showed that Page's metric is in fact *conformally Kähler*—that is, it is actually a Kähler metric times a smooth positive function. However, it was only quite recently [**9**] that various breakthroughs in the theory of extremal Kähler metrics made it possible to prove the existence of an analogous metric on $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$. The upshot is the following:

THEOREM 1 (Chen–LeBrun–Weber). *Let (M^4, J) be any compact complex surface with $c_1 > 0$. Then there is a $\lambda > 0$ Einstein metric on M which is conformally equivalent to a Kähler metric on (M, J).*

In the $\lambda = 0$ case, the existence problem for Kähler–Einstein metrics was definitively settled by Yau [**33**], whose celebrated solution of the Calabi conjecture implies that any compact complex manifold of Kähler type with $c_1^{\mathbb{R}} = 0$ admits a unique Ricci-flat Kähler metric in each Kähler class. Kodaira's classification scheme files the compact complex surfaces with these properties into four pigeon-holes [**3**, **13**]. First, there are the $K3$ surfaces, defined as the simply connected complex surfaces with $c_1 = 0$; they are all deformation equivalent [**7**, **15**], and are thus all diffeomorphic to any smooth quartic in $\mathbb{CP}_3$. Next, there are the Enriques surfaces, which are $\mathbb{Z}_2$-quotients of suitable $K3$ surfaces; again, there is only one diffeotype. Then there are the Abelian surfaces, which are by definition the complex tori $\mathbb{C}^2/\Lambda$; obviously, these are all diffeomorphic to the 4-torus. And finally, there are the hyper-elliptic surfaces, which are quotients of certain Abelian surfaces by a finite group G of complex affine-linear maps; there are exactly seven possibilities for G, namely $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, and there is exactly one diffeotype for each of these seven possibilities.

The existence results we have described above only produce Einstein metrics which are closely related to complex structures. But what if we merely require that an Einstein metric and a complex structure somehow manage to uneasily coexist on the same manifold, without necessarily being on friendly terms? Contrary to what one might expect, Theorem A asserts that, provided we constrain the Einstein constant to be non-negative, the conformally Kähler possibilities already exhaust the entire list of possible diffeotypes. This generalizes an analogous observation regarding the $\lambda > 0$ case that was first proved in [**9**].

Of course, a conformally Kähler, Einstein metric is related not only to a complex structure, but also to a symplectic form. This makes it very tempting to look for a symplectic analog of Theorem A. What can we say, then, about symplectic 4-manifolds that also admit $\lambda \geqslant 0$ Einstein metrics? Surprisingly enough, the answer turns out to be exactly the same!

THEOREM B. *Let M be a smooth compact* 4*-manifold which admits a symplectic form. Then M also carries a (possibly unrelated) Einstein metric with $\lambda \geqslant 0$ if and only if it is diffeomorphic to a del Pezzo surface, a $K3$ surface, an Enriques surface, an Abelian surface, or a hyper-elliptic surface.*

What we have learned here can thus be summarized as follows:

THEOREM C. *For a smooth compact* 4*-manifold M, the following are equivalent*:
(i) *M admits both a complex structure and an Einstein metric with $\lambda \geqslant 0$;*
(ii) *M admits both a symplectic structure and an Einstein metric with $\lambda \geqslant 0$;*
(iii) *M admits a conformally Kähler, Einstein metric with $\lambda \geqslant 0$.*

One key ingredient in the proof of these statements is the *Hitchin–Thorpe inequality* [**14**, **28**]. Recall that the bundle of 2-forms over an oriented Riemannian 4-manifold has an invariant decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are by definition the (± 1)-eigenspaces of the Hodge star operator. Elements of Λ^+ are called *self-dual* 2*-forms*, and a connection on a vector bundle over (M, g) is said to be *self-dual* if its curvature is a bundle-valued self-dual 2-form. This notion is intimately related to the 4-dimensional case of the Einstein equations, because [**1**] an oriented 4-dimensional Riemannian manifold is Einstein iff the induced connection on $\Lambda^+ \to M$ is self-dual. The positivity of instanton numbers for solutions of the self-dual Yang–Mills equations therefore implies the following:

LEMMA 1 (Hitchin–Thorpe Inequality). *Any compact oriented Einstein* 4*-manifold (M, g) satisfies $p_1(\Lambda^+) \geqslant 0$, with equality iff the induced connection on $\Lambda^+ \to M$ is flat. Moreover, the latter occurs iff (M, g) is finitely covered by a Calabi–Yau $K3$ surface or by a flat* 4*-torus.*

Here, the delicate equality case was first cracked by Hitchin [**14**]. An oriented Riemannian 4-manifold (M, g) induces a flat connection on $\Lambda^+ \to M$ iff the curvature tensor $\mathcal{R}$ of g belongs to $\Lambda^- \otimes \Lambda^-$. Metrics with this property are said to be *locally hyper-Kähler*. Such metrics are in particular Ricci-flat, so the Cheeger–Gromoll splitting theorem [**4**, **8**] implies that a compact locally hyper-Kähler 4-manifold either has finite fundamental group, or else is flat. In the latter case, Bieberbach's theorem [**5**, **29**] then implies that the manifold has a finite regular cover which is a flat 4-torus; in the former case, it is a finite quotient of a simply connected compact manifold with holonomy $Sp(1)$, and the choice of a parallel complex structure then allows one to view this covering space as a $K3$ surface equipped with a Ricci-flat Kähler metric.

Note that, while $\Lambda^+ \hookrightarrow \Lambda^2$ depends on g, its bundle-isomophism type is metric-independent. In fact, $p_1(\Lambda^+)$ actually equals the oriented homotopy invariant

$(2\chi + 3\tau)(M)$, where χ and τ respectively denote the Euler characteristic and signature. For us, however, it is more important to notice that if M admits an orientation-compatible complex structure J, then

$$p_1(\Lambda^+) = c_1^2(M, J),$$

since Λ^+ is bundle-isomorphic to $\mathbb{R} \oplus K$, where $K = \Lambda^{2,0}$ is the canonical line bundle of (M, J). As a matter of convention, almost-complex structures will henceforth always be assumed to be orientation-compatible. In particular, complex surfaces (M, J) will always be given the complex orientation, and symplectic 4-manifolds (M, ω) will always be oriented so that $\omega \wedge \omega$ is a volume 4-form. Thus, the Hitchin–Thorpe inequality becomes

$$c_1^2(M, J) \geqslant 0$$

whenever M carries an almost-complex structure J.

To complete the proofs of Theorems A, B and C, let us use (i), (ii) and (iii) to refer to the corresponding numbered statements in Theorem C, and let us also introduce the final numbered statement:

(iv) M is diffeomorphic to a del Pezzo surface, a $K3$ surface, an Enriques surface, an Abelian surface, or a hyper-elliptic surface.

We have already seen that (iv) $\Rightarrow$ (iii) $\Rightarrow$ [(i) and (ii)]. In light of Lemma 1 and its reformulation as the inequality $c_1^2 \geqslant 0$, it thus suffices to show that:

- if $c_1^2 > 0$, then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv); and
- if $c_1^2 = 0$, then (ii) $\Rightarrow$ (i) $\Rightarrow$ (iv).

We now begin by observing that (i) $\Rightarrow$ (ii) when $c_1^2 > 0$:

LEMMA 2. *Let (M, J) be a compact complex surface. If $c_1^2(M, J) > 0$, then (M, J) is of Kähler type. In particular, M admits a symplectic form ω.*

Proof. Let $K = \Lambda^{2,0}$ denote the canonical line bundle of (M, J). Since we have assumed that $c_1^2(M) > 0$, the Riemann–Roch theorem and Serre duality predict that either $h^0(M, \mathcal{O}(K^\ell))$ or $h^0(M, \mathcal{O}(K^{-\ell}))$ must grow quadratically as $\ell \to +\infty$. It follows that (M, J) is algebraic, and therefore [**3**, **15**] projective. Hence (M, J) is of Kähler type, as claimed.

Next, we show that (ii) $\Rightarrow$ (iv) when $c_1^2 > 0$, using a slight generalization of a result proved in [**24**]:

LEMMA 3. *Let (M, ω) be a symplectic 4-manifold with $c_1^2 > 0$. If M admits a metric of non-negative scalar curvature, then M is diffeomorphic to a del Pezzo surface.*

Proof. We equip M with the spin^c structure induced by any almost-complex structure adapted to the symplectic form ω. Then, even if $b_+(M) = 1$, the hypothesis that $c_1^2 > 0$ guarantees [**12**, **18**] that this spin^c structure has a well-defined Seiberg–Witten invariant, counting the gauge-equivalence classes of solutions of the *unperturbed* Seiberg–Witten equations

$$D_A\Phi = 0, \quad F_A^+ = -\frac{1}{2}\Phi \odot \bar{\Phi}$$

with multiplicities, for an arbitrary Riemannian metric on M. But since any solution of these equations must satisfy both the Weitzenböck formula

$$0 = 2\Delta|\Phi|^2 + 4|\nabla\Phi|^2 + s|\Phi|^2 + |\Phi|^4$$

and the integral identity

$$\int_M |\Phi|^4 d\mu = 8\int_M |F_A^+|^2 d\mu \geqslant 32\pi^2 c_1^2(M) > 0,$$

these equations have no solution at all if the chosen metric g has scalar curvature $s \geqslant 0$. Our hypotheses therefore imply that the Seiberg–Witten invariant must vanish for the relevant spin^c structure. However, a fundamental result of Taubes [**27**] asserts that this invariant must be non-zero for a symplectic 4-manifold with either $b_+(M) \geqslant 2$, or with $b_+(M) = 1$ and $c_1 \cdot [\omega] < 0$. Our symplectic manifold therefore has $b_+(M) = 1$ and $c_1 \cdot [\omega] \geqslant 0$. But since $b_+(M) = 1$, the intersection form is negative-definite on the orthogonal complement of $[\omega]$; our assumption that $c_1^2 > 0$ thus implies that $c_1 \cdot [\omega] \neq 0$, and our symplectic manifold therefore has $b_+(M) = 1$ and $c_1 \cdot [\omega] > 0$. A result of Liu [**21**, theorem B] therefore tells us that (M, ω) must be a symplectic blow-up of $\mathbb{CP}_2$ or a ruled surface. Since we also have $c_1^2 > 0$, it follows that M is diffeomorphic to $S^2 \times S^2$ or to $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$ for some k with $0 \leqslant k \leqslant 8$. Hence M is diffeomorphic to a del Pezzo surface, as claimed.

We now turn to the $c_1^2 = 0$ case. Recall that $b_+(M) \neq 0$ for any symplectic 4-manifold (M, ω), since the symplectic class $[\omega] \in H^2(M, \mathbb{R})$ has positive self-intersection. The following observation therefore implies, in particular, that (ii) ⇒ (i) when $c_1^2 = 0$.

LEMMA 4. *Let M be a smooth compact* 4*-manifold with $p_1(\Lambda^+) = 0$ and $b_+ \neq 0$. If M admits an Einstein metric g, then g is Ricci-flat and Kähler with respect to some orientation-compatible complex structure J on M.*

Proof. By Lemma 1, the Einstein metric g is locally hyper-Kähler. It follows that g has vanishing scalar curvature s and self-dual Weyl curvature W_+, since these are the trace and trace-free parts of the $\Lambda^+ \otimes \Lambda^+$ component of the Riemann tensor $\mathcal{R}$ of g. On the other hand, since $b_+(M) \neq 0$, there must be a self-dual harmonic 2-form $\psi \not\equiv 0$ on (M, g). However, the Weitzenböck formula for self-dual 2-forms [**6**, **16**] reads

$$(d + d^*)^2\psi = \nabla^*\nabla\psi - 2W_+(\psi, \cdot) + \frac{s}{3}\psi,$$

so our harmonic form ψ must satisfy

$$0 = \int \langle\psi, \nabla^*\nabla\psi\rangle d\mu = \int |\nabla\psi|^2 d\mu$$

and we therefore have $\nabla\psi = 0$. In particular, the point-wise norm of ψ is a non-zero constant, and by replacing ψ with a constant multiple, we may assume that $\|\psi\|_g \equiv \sqrt{2}$. The endomorphism $J : TM \to TM$ given by $v \mapsto (v \lrcorner \psi)^\sharp$ is then parallel, and satisfies $J^2 = -1$. Thus J is a complex structure on M, and the Ricci-flat metric g now becomes a Kähler metric on the complex surface (M, J).

Finally, we show that (i) ⇒ (iv) when $c_1^2 = 0$.

LEMMA 5. *If a smooth compact* 4*-manifold M admits both an Einstein metric and a complex structure with $c_1^2 = 0$, then M is diffeomorphic to a $K3$ surface, an Enriques surface, an Abelian surface, or a hyper-elliptic surface.*

Proof. By Lemma 1, M has a finite cover N with b_1 even. Let J_0 be any given complex structure on M, let $\varpi : N \to M$ denote the covering map, and let $\hat{J}_0$ denote the pull-back of J_0 to M, so that ϖ becomes a holomorphic map from $(N, \hat{J}_0)$ to (M, J_0). Since the Fröhlicher spectral sequence of any compact complex surface degenerates at the E_1 level, the fact that $b_1(N)$ is even implies [**3**, theorem IV·2·6] that that the real-linear injection $H^0(N, \Omega^1) \to H^1(N, \mathbb{R})$ defined by $\alpha \mapsto [\mathrm{R}e\ \alpha]$ is an isomorphism. Thus, if φ is any closed 1-form on M, $[\varpi^*\varphi] = [\mathrm{R}e\ \alpha] \in H^1(N, \mathbb{R})$ for some $\alpha \in H^0(N, \Omega^1)$. But it then follows that $[\varphi] = [\varpi_*\varpi^*(\varphi/n)] = [\mathrm{R}e\ \varpi_*(\alpha/n)]$, where n is the degree of ϖ, and where the push-down ϖ_* is the fiber sum of the local push-forwards via the local diffeomorphism ϖ. Hence $H^0(M, \Omega^1) \to H^1(M, \mathbb{R})$ is also surjective, and hence an isomorphism. Thus $b_1(M)$ is even. Since $b_1(M) \equiv b_+(M) + 1 \bmod 2$ by the integrality of the Todd genus, it therefore follows that $b_+(M)$ is odd, and so, in particular, non-zero.

Lemma 4 therefore shows that M admits a complex structure J for which the given Einstein metric g is Ricci-flat and Kähler. Pulling J back to the finite cover $\varpi : N \to M$ of Lemma 1 thus realizes (M, J) as the quotient of either a Calabi-Yau $K3$ surface or a flat Abelian surface by a finite group of *holomorphic* isometries. If the covering ϖ is non-trivial, it therefore follows [**13**] that (M, J) is either an Enriques surface or a hyper-elliptic surface, and the claim therefore follows.

By contrast, it seems much harder to determine precisely which complex surfaces admit a general Riemannian Einstein metric if we also allow for the $\lambda < 0$ case. Certainly, the Hitchin–Thorpe inequality tells us rather immediately that the underlying smoth 4-manifold of a properly elliptic complex surface (that is, a surface of Kodaira dimension 1) can never admit an Einstein metric. But, by contrast, there are plenty of surfaces of general type (Kodaira dimension 2) which *do* admit Einstein metrics. Indeed, the Aubin/Yau existence theorem [**2**, **32**] tells us that there is a Kähler–Einstein metric with $\lambda < 0$ on any compact complex surface with $c_1 < 0$. These are precisely those *minimal* complex surfaces of general type which contain no (-2)-curves. Now, for surfaces of general type, minimality turns out to have a differentiable meaning, and not just a holomorphic one: it means that the relevant 4-manifold cannot be smoothly decomposed as a connected sum $X\#\overline{\mathbb{CP}}_2$. Unfortunately, however, this is not at present known to be a necessary condition for the existence of an Einstein metric. However, one can at least prove some weaker results in this direction. For example [**20**], if X is a minimal complex surface of general type, its k-point blow-up $X\#k\overline{\mathbb{CP}}_2$ cannot carry an Einstein metric if $k \geqslant c_1^2(X)/3$. (By contrast, the Hitchin–Thorpe inequality only gives an obstruction if $k \geqslant c_1^2(X)$; for an intermediate result, see [**19**].) That is, we can at least say the following:

THEOREM 2. *Let M be the underlying* 4*-manifold of a compact complex surface* (M, J)*. If M admits an Einstein metric g, then either M is as in Theorem* A*, or else* (M, J) *is a surface of general type which is "not too non-minimal," in the sense that it is obtained from its minimal model X by blowing up $k < c_1^2(X)/3$ points.*

In the latter case, we of course have $c_1^2(M) > 2/3c_1^2(X)$. But any minimal surface of general type satisfies [**3**, **13**] the Noether inequality $c_1^2(X) \geqslant b_+(X) - 5$. Putting these together, and remembering that b_+ is unchanged by blowing up, we therefore obtain a non-trivial geographical inequality which, for trivial reasons, also happens to hold for the manifolds of Theorem A:

COROLLARY 3. *Let M be the underlying 4-manifold of a compact complex surface (M, J). If M admits an Einstein metric g, then M is of Kähler type, and satisfies*

$$c_1^2(M) > \frac{2}{3}(b_+(M) - 5).$$

In particular, these 4-manifolds M all admit symplectic structures. On the other hand, there is no known result that obviously promises such a Noether-type inequality for symplectic 4-manifolds that admit Einstein metrics. It would be very interesting to prove anything in this direction!

Perhaps the most fascinating open problem in the subject is to determine whether there exist Einstein metrics on compact complex surfaces that are not conformally Kähler (with respect to any complex structure). For surfaces with $c_1^2 = 0$, Hitchin's results on the boundary case of the Hitchin–Thorpe inequality allow us to see that no such metrics can exist. But the only other complex surfaces for which such a result has been proved are the ball quotients, which saturate the Miyaoka–Yau inequality [**17**]. In a related vein, one might instead hope to improve the "not too non-minimal" statement in Proposition 2. Is it really ever possible to find an Einstein metric on the underlying 4-manifold of a non-minimal complex surface of general type? If so, such a metric would certainly have to be qualitatively different from a Kähler–Einstein metric, in many different respects!

Dedication. This article is dedicated to Prof. Akira Fujiki, and a preliminary version was included in the informal lecture-note volume **Complex Geometry in Osaka: in honour of Akira Fujiki's 60th birthday**, S. Goto *et al.* editors, Osaka University 2008.

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