On Four-Dimensional

Einstein Manifolds

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Mathematics Colloquium,
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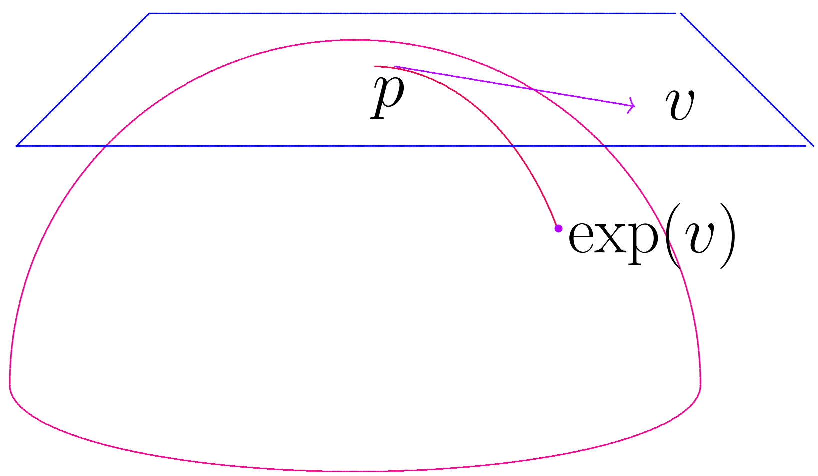

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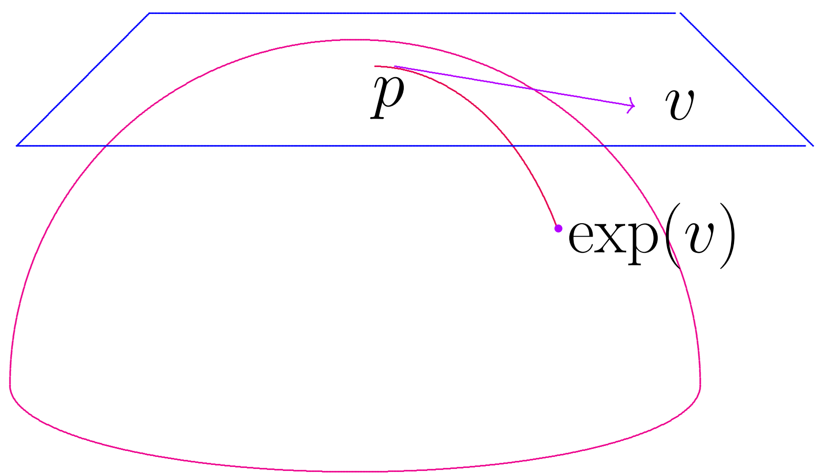

Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

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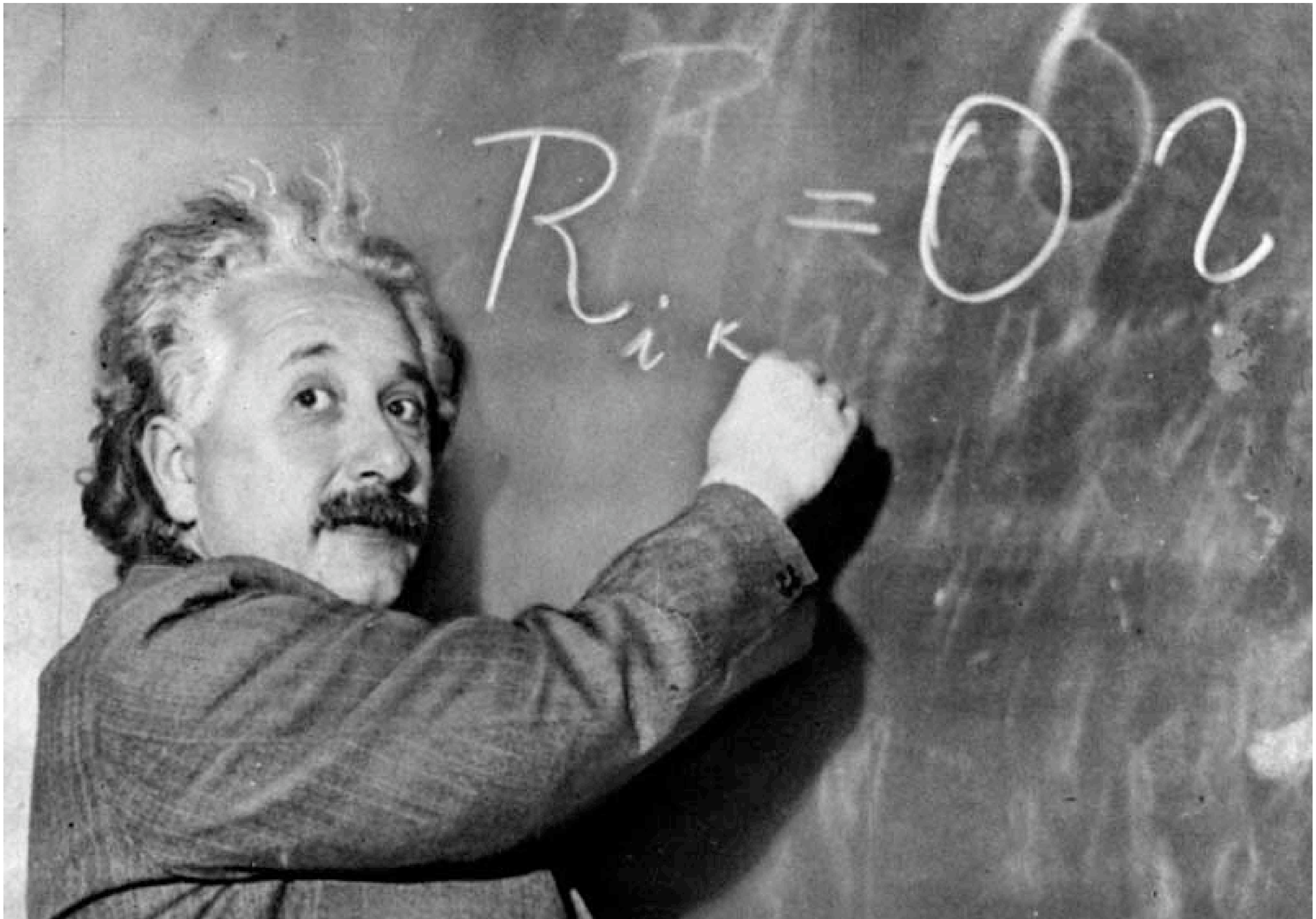

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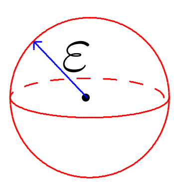

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[3 of the 8 Thurston geometries.]

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When $n = 4$, situation is more encouraging...

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By contrast, high-dimensional Einstein metrics too common; have little to do with geometrization.

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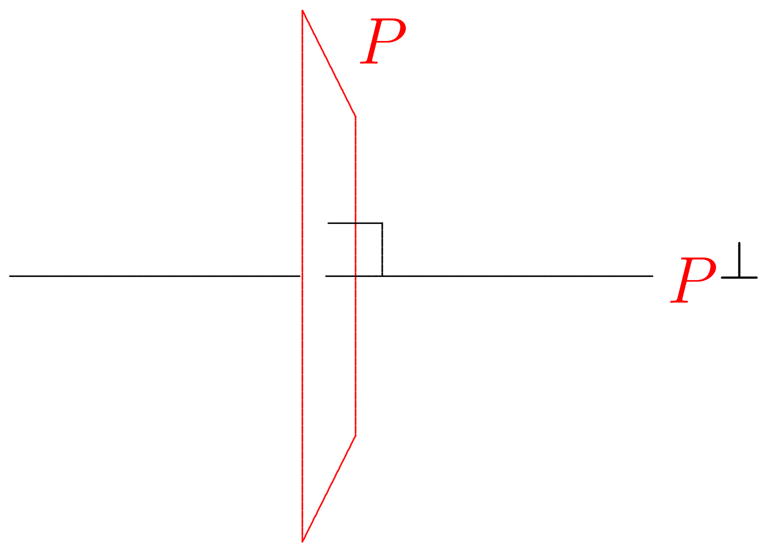

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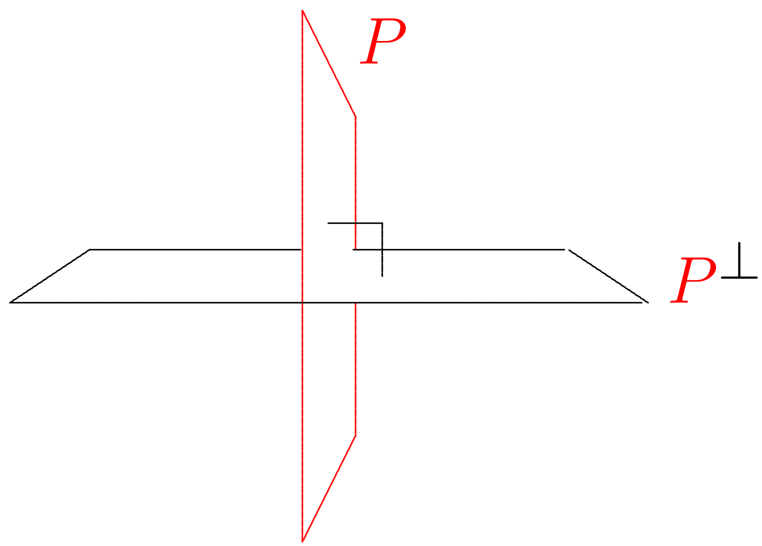

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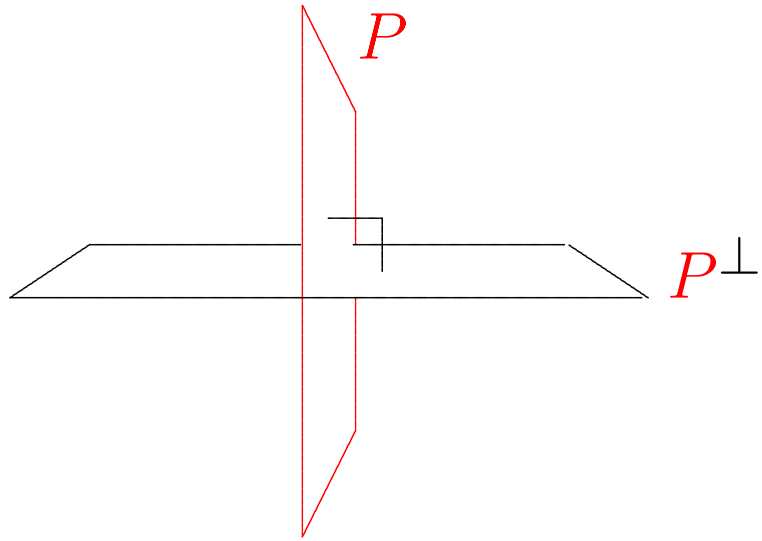

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Associated Kähler form:

$$\omega = i \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \; dz^j \wedge d\bar{z}^k$$

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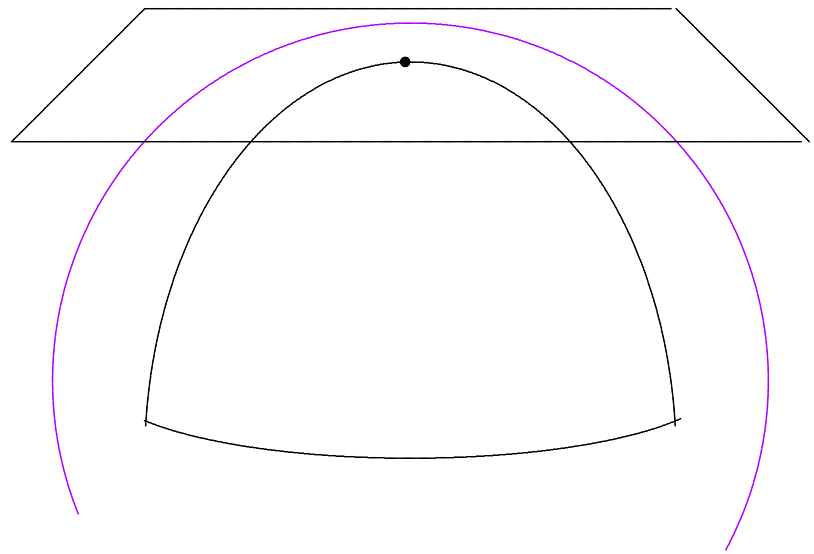

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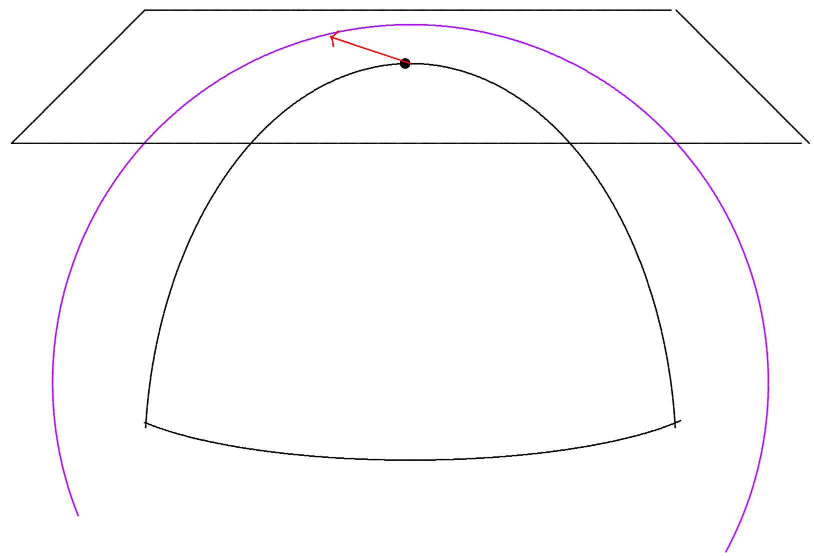

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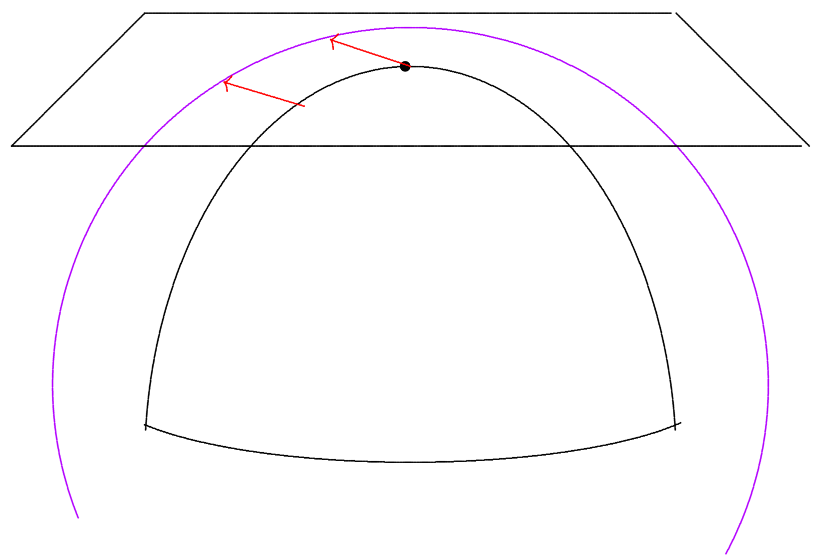

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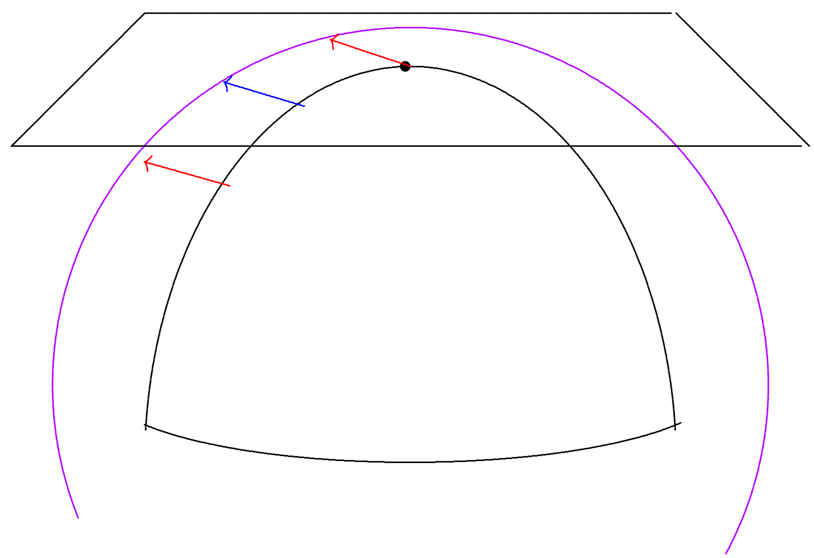

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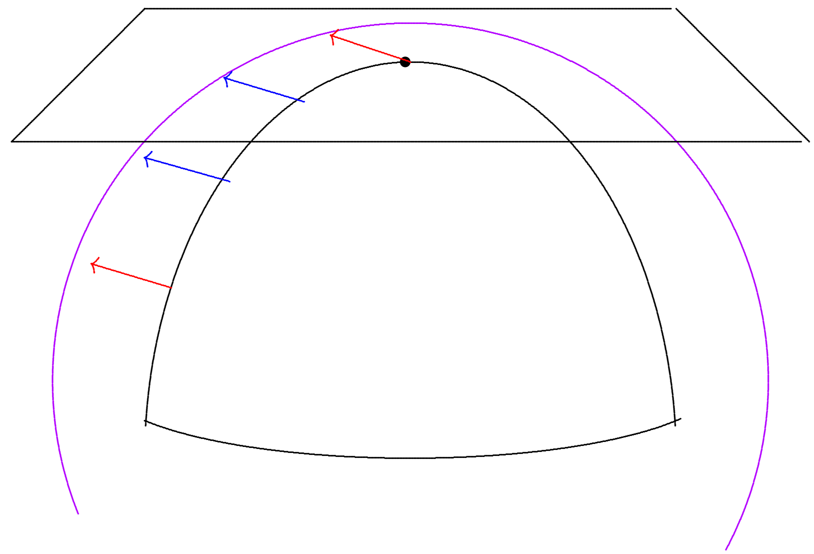

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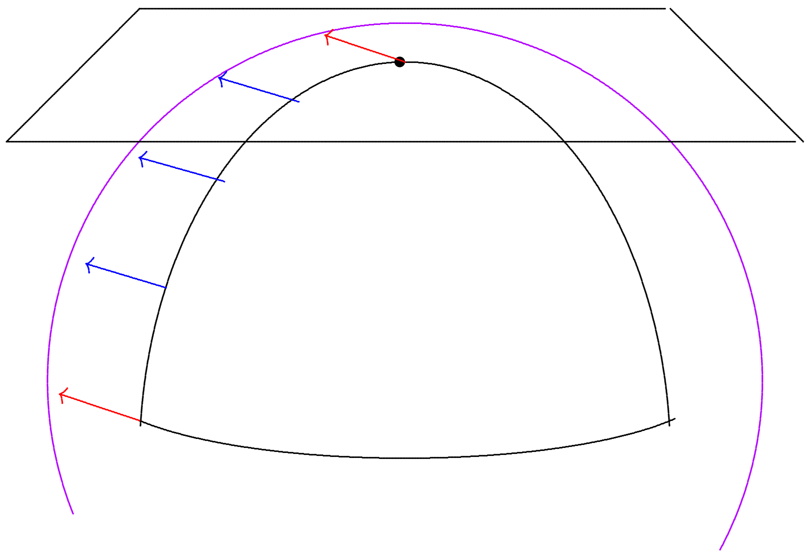

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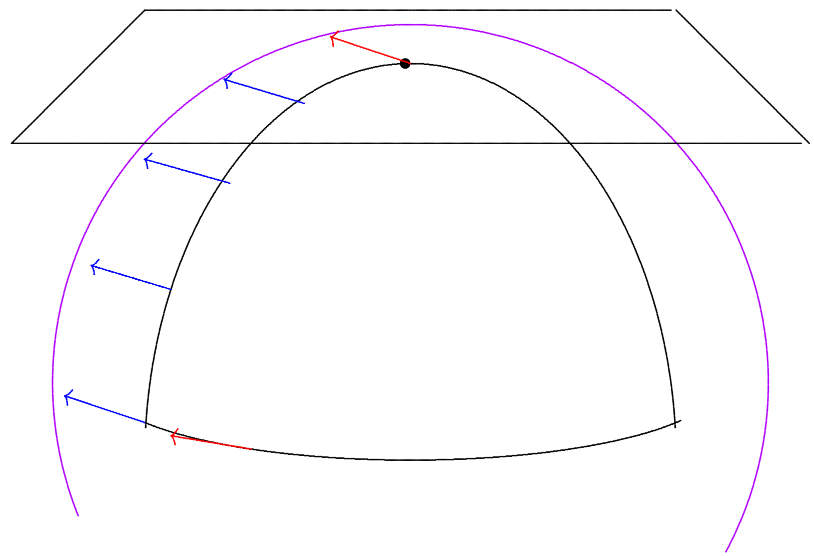

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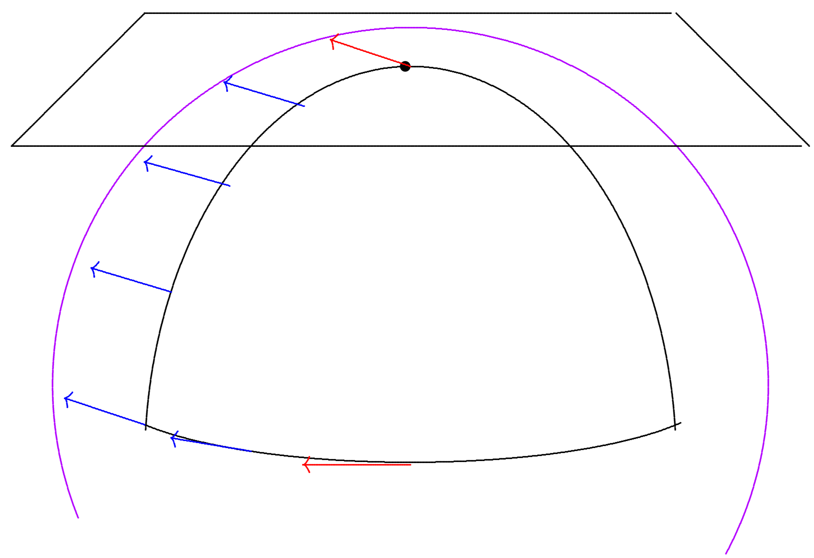

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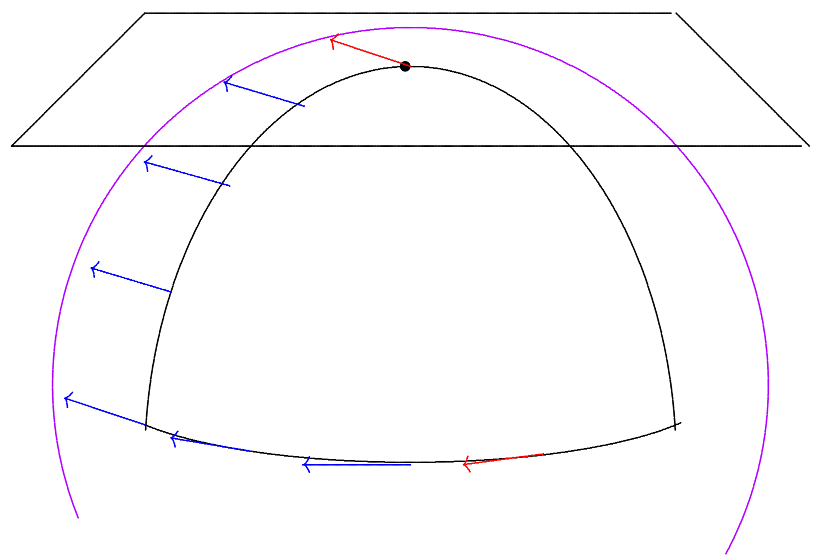

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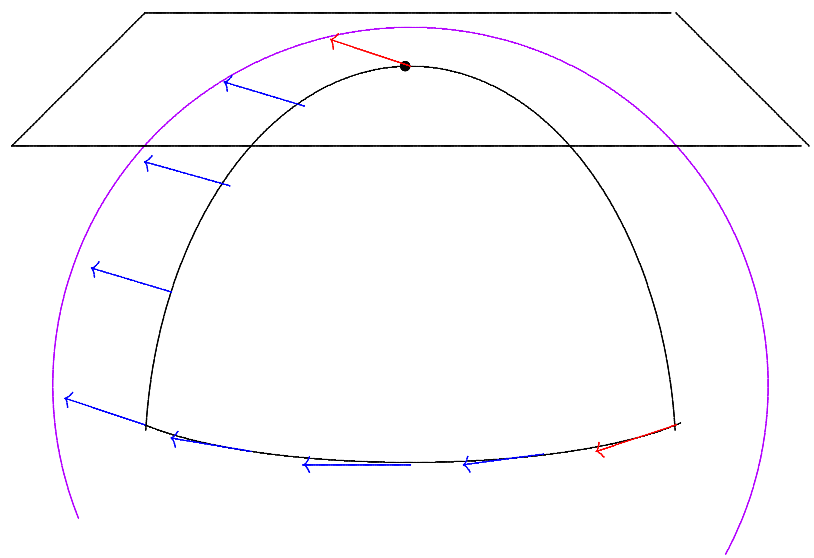

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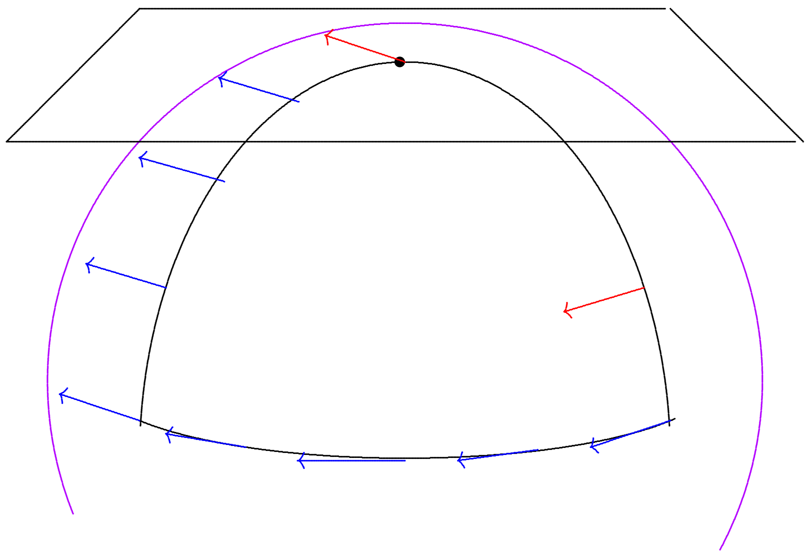

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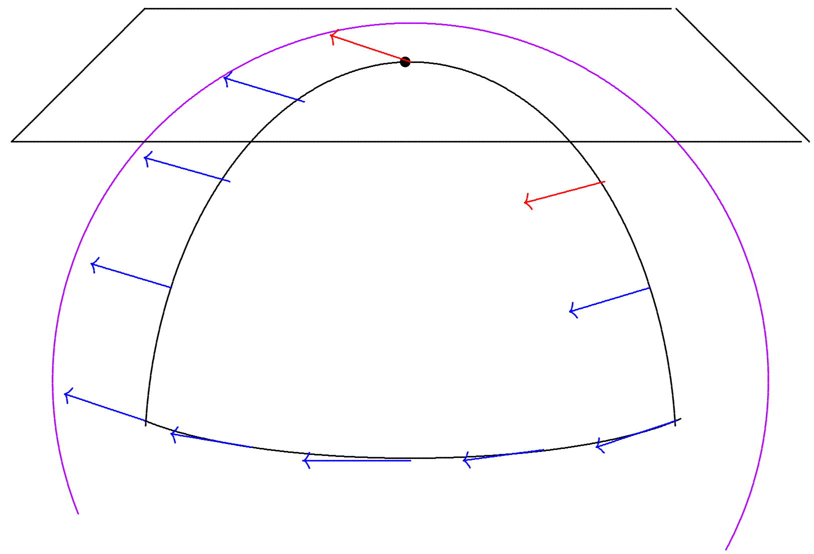

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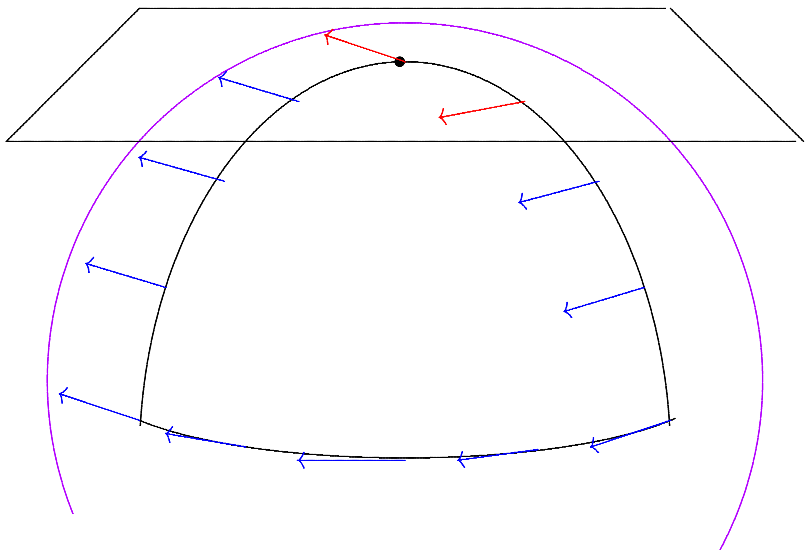

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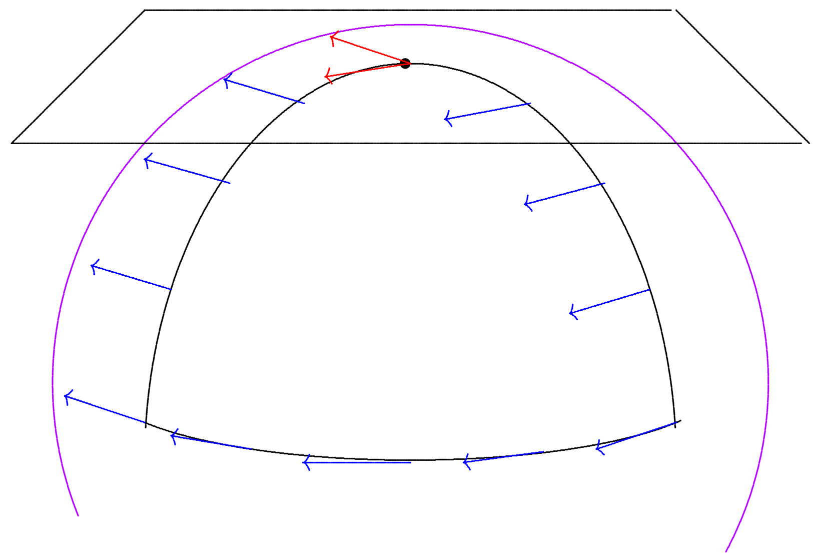

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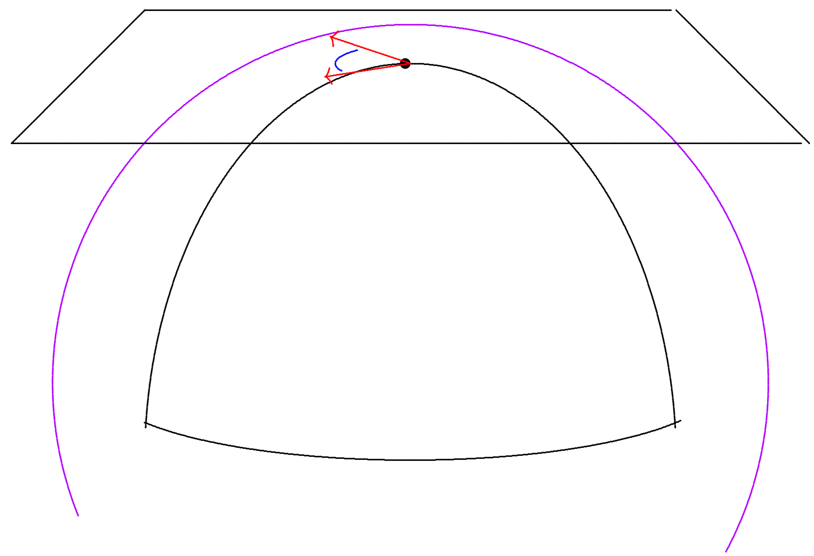
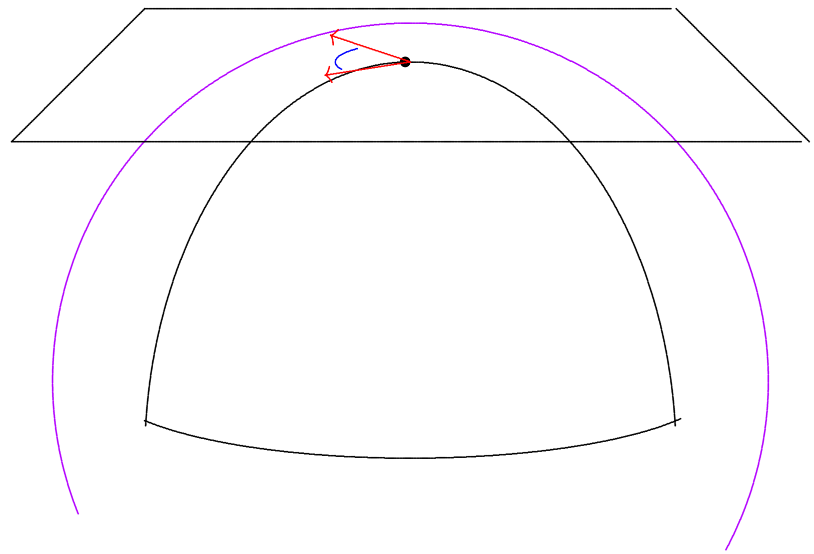
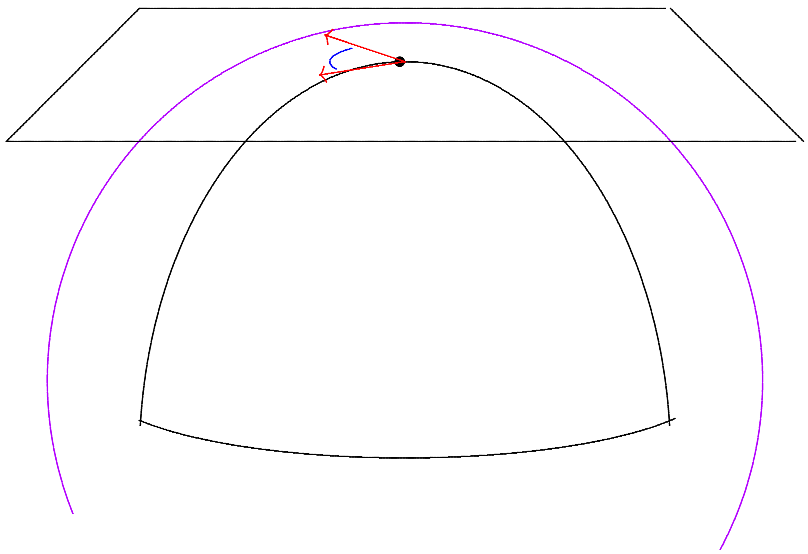


(M^n, g): holonomy $\subset \mathbf{O}(n)$



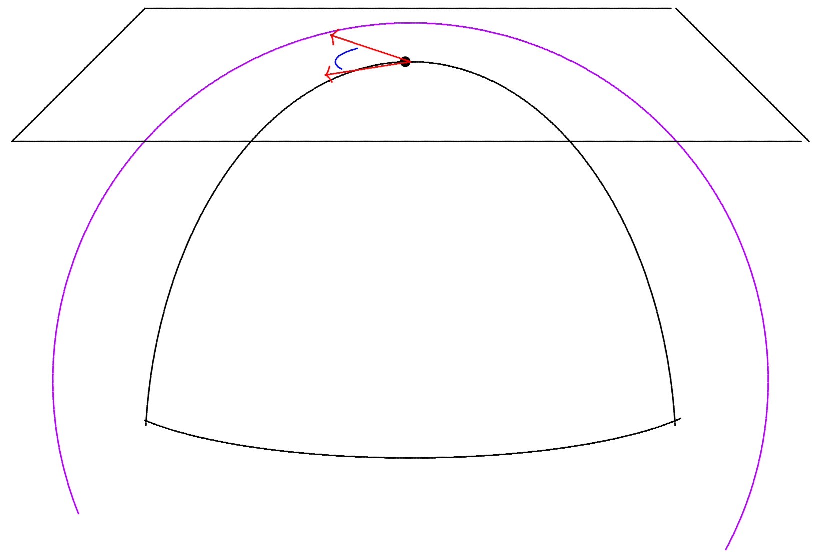
Kähler metrics:

(M^{2m}, g): holonomy



Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$



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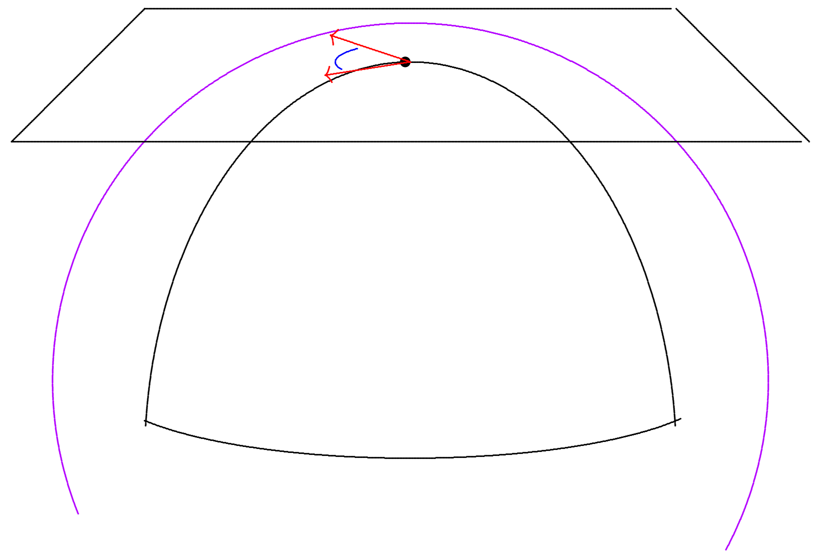

$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

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If M admits a Kähler metric, it of course admits a symplectic form ω.

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$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

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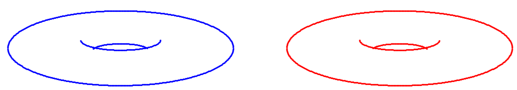
Connected sum #:



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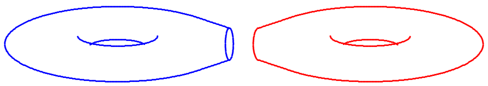

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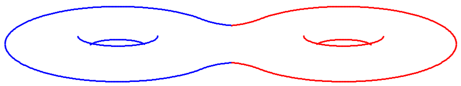

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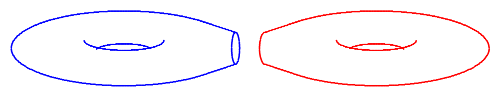

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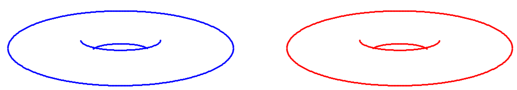

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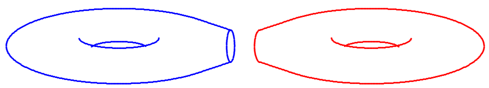

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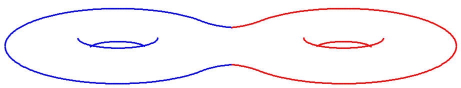

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—André Weil, 1958

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Typical model: Smooth quartic in $\mathbb{CP}_3$.

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Kummer: $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$.

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T^4 = Picard torus of curve of genus 2.

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Remove singularities by deforming equation.

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Generic quartic is a $K3$ surface.

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Calabi/Yau: Admits Ricci-flat Kähler metrics.

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Del Pezzo surfaces,
K3 surface, Enriques surface,
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Existence: Yau, Tian, Page, Chen-L-Weber, et al.

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$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \textit{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

Existence: Yau, Tian, Page, Chen-L-Weber, et al.

No others: Hitchin-Thorpe, Seiberg-Witten, . . .

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Extensive results in $\lambda < 0$ case, too.

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Extensive results in $\lambda < 0$ case, too.

But that would be a topic for a different lecture!

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

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$$
\begin{aligned}
&\mathbb{CP}_2 \# k\overline{\mathbb{CP}_2}, \quad 0 \le k \le 8,\\
&S^2 \times S^2,\\
&K3,\\
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&T^4,\\
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&T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4).
\end{aligned}
$$

Definitive list . . .

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$
$$K3,$$
$$K3/\mathbb{Z}_2,$$
$$T^4,$$
$$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$$
$$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4).$$

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But we understand some cases better than others!

$$\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8,$$
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$$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$$
$$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4).$$

Below the line:

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$$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$$
$$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4).$$

Below the line:

Every Einstein metric is Ricci-flat Kähler.

But we understand some cases better than others!

$$
\begin{array}{l}
\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\
S^2 \times S^2, \\
\hline
K3, \\
K3/\mathbb{Z}_2, \\
T^4, \\
T^4/\mathbb{Z}_2,\, T^4/\mathbb{Z}_3,\, T^4/\mathbb{Z}_4,\, T^4/\mathbb{Z}_6, \\
T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2),\, T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4).
\end{array}
$$

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$

But we understand some cases better than others!

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M) = \{\text{Einstein } h\}/(\text{Diffeos} \times \mathbb{R}^+)$

But we understand some cases better than others!

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8,$$
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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ completely understood.

But we understand some cases better than others!

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, \quad 0 \leq k \leq 8,$$
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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8,$$

$$S^2 \times S^2,$$

$$K3,$$

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$$T^4,$$

$$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$$

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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

Above the line:

$$\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

Above the line:

Know an Einstein metric on each manifold.

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

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$$K3/\mathbb{Z}_2,$$
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$$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$$
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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

Above the line:

Moduli space $\mathscr{E}(M) \neq \varnothing$.

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

Above the line:

Moduli space $\mathscr{E}(M) \neq \varnothing$. But is it connected?

$$\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

Del Pezzo surfaces:

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(M^4, J) for which c_1 is a Kähler class $[\omega]$.

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Blow-up of $\mathbb{CP}_2$ at k distinct points,
in general position,

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Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Del Pezzo surfaces:

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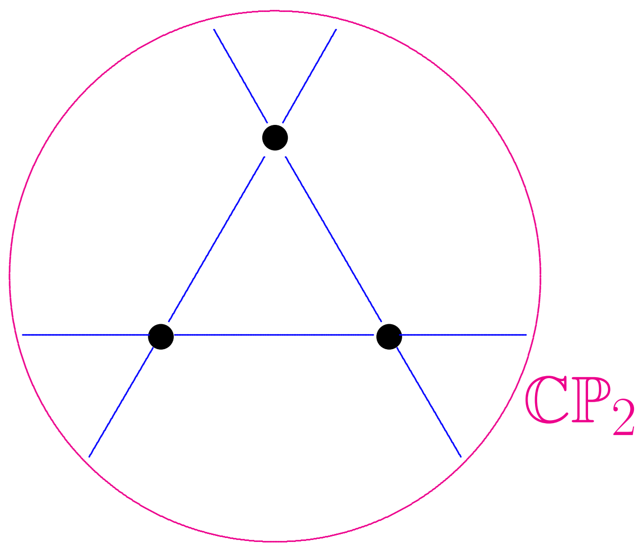
Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



Blowing up:

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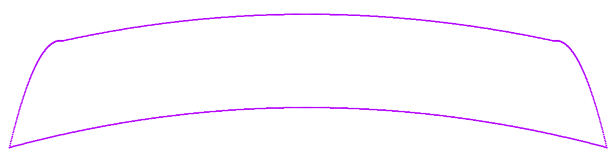
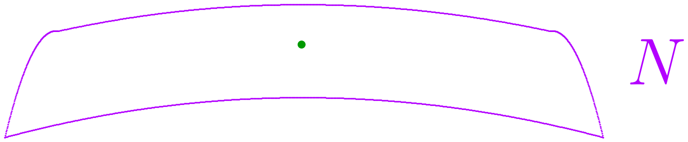
If N is a complex surface,



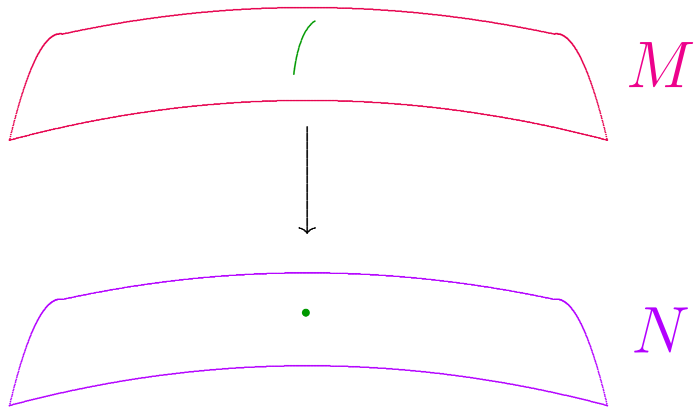
Blowing up:

If N is a complex surface, may replace $p \in N$



Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$

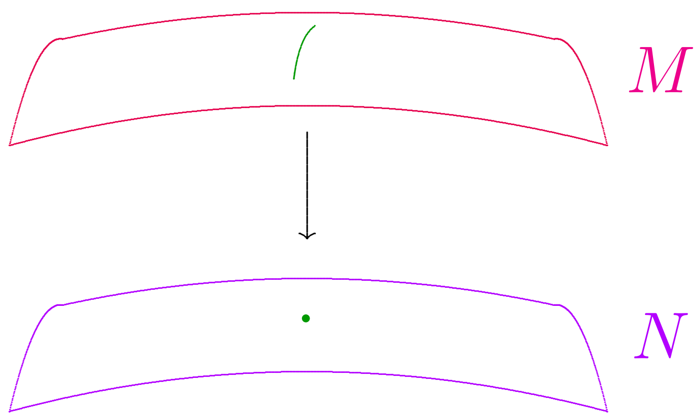


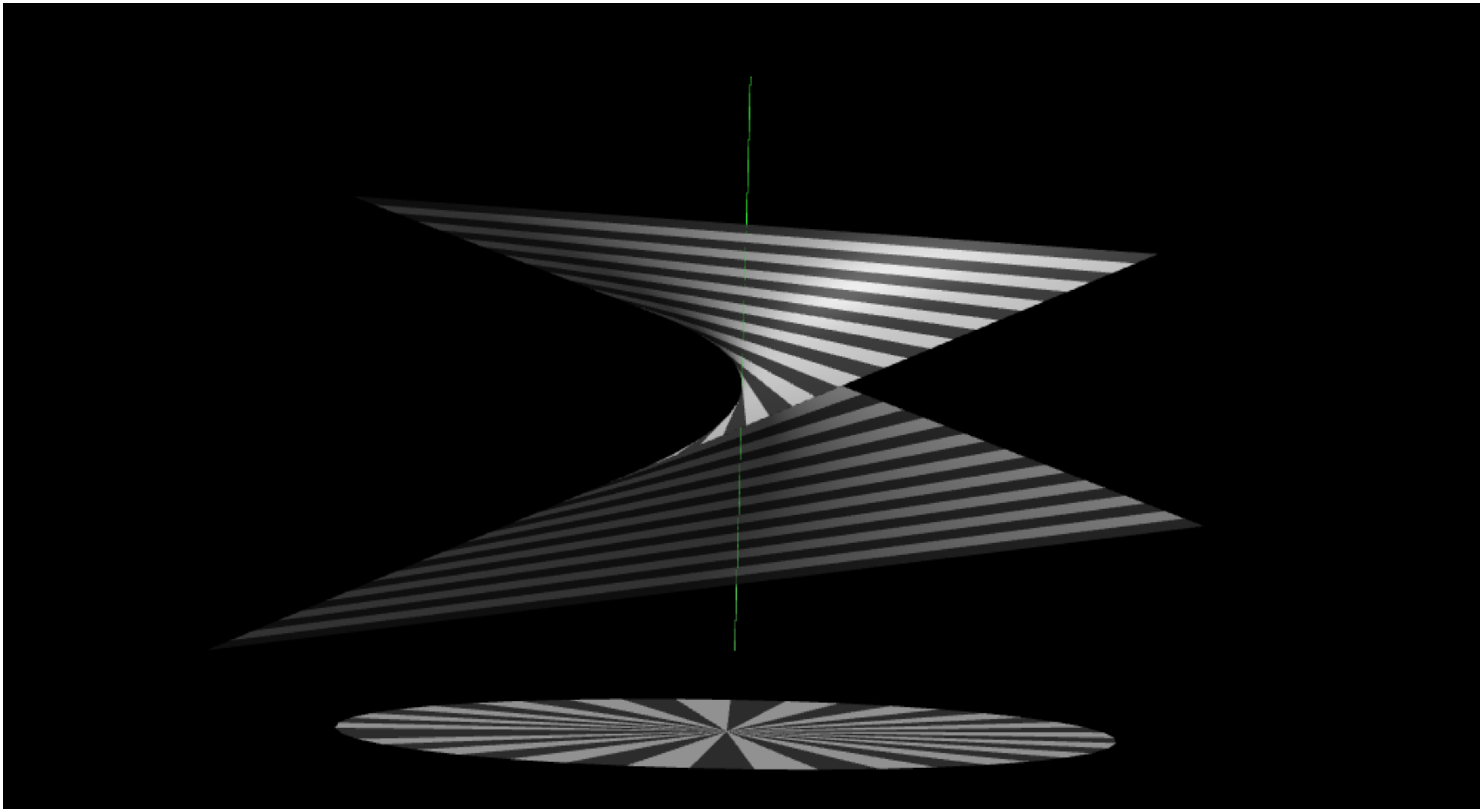
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}_2}$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.





Blowing up:

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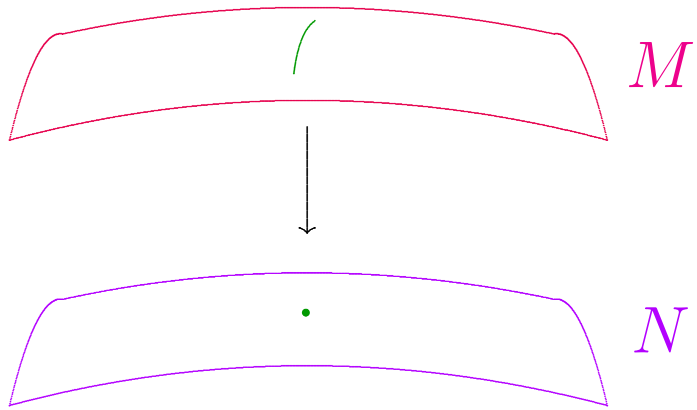

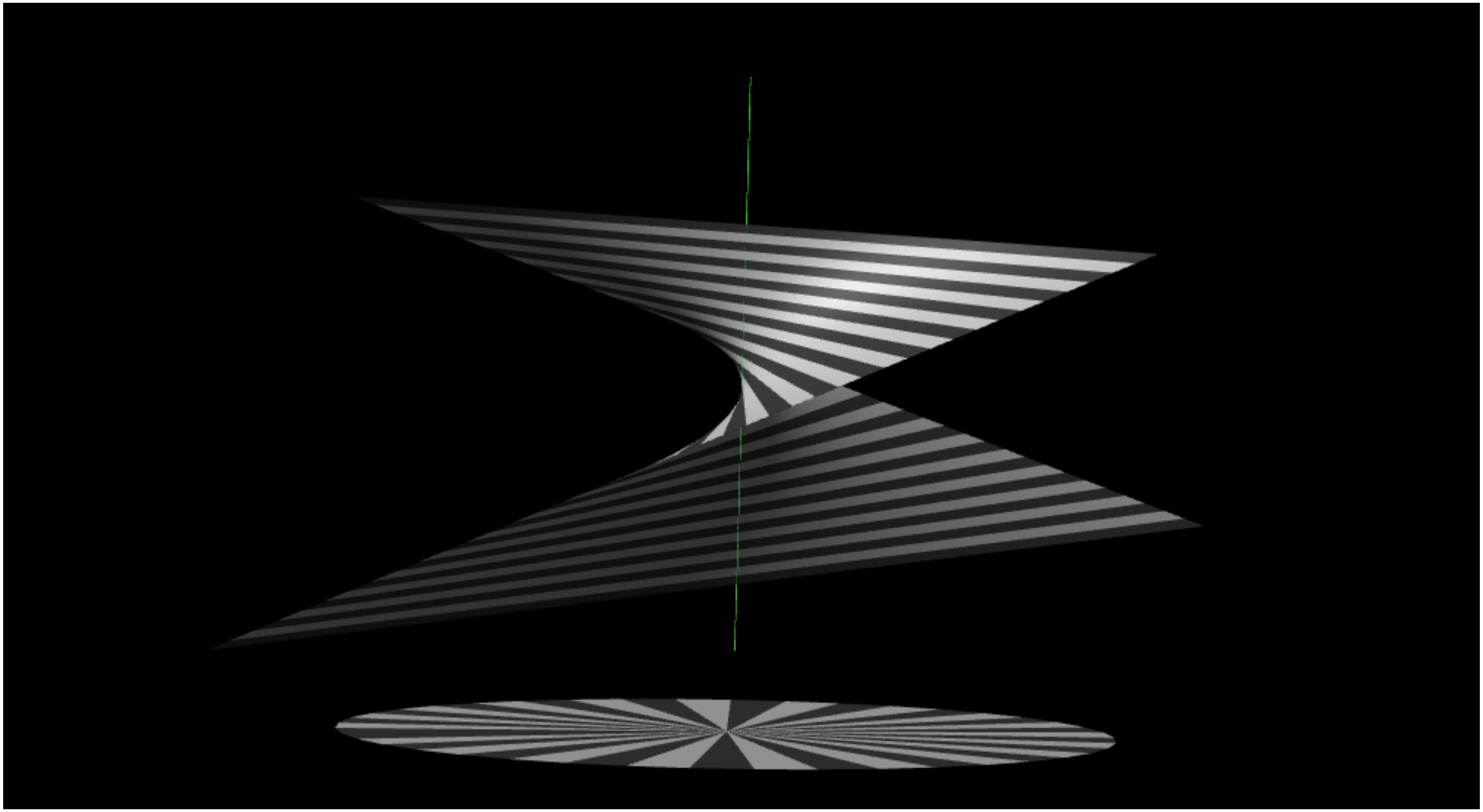

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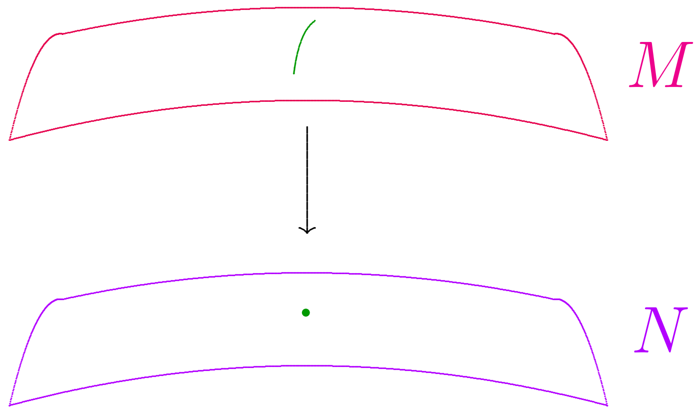

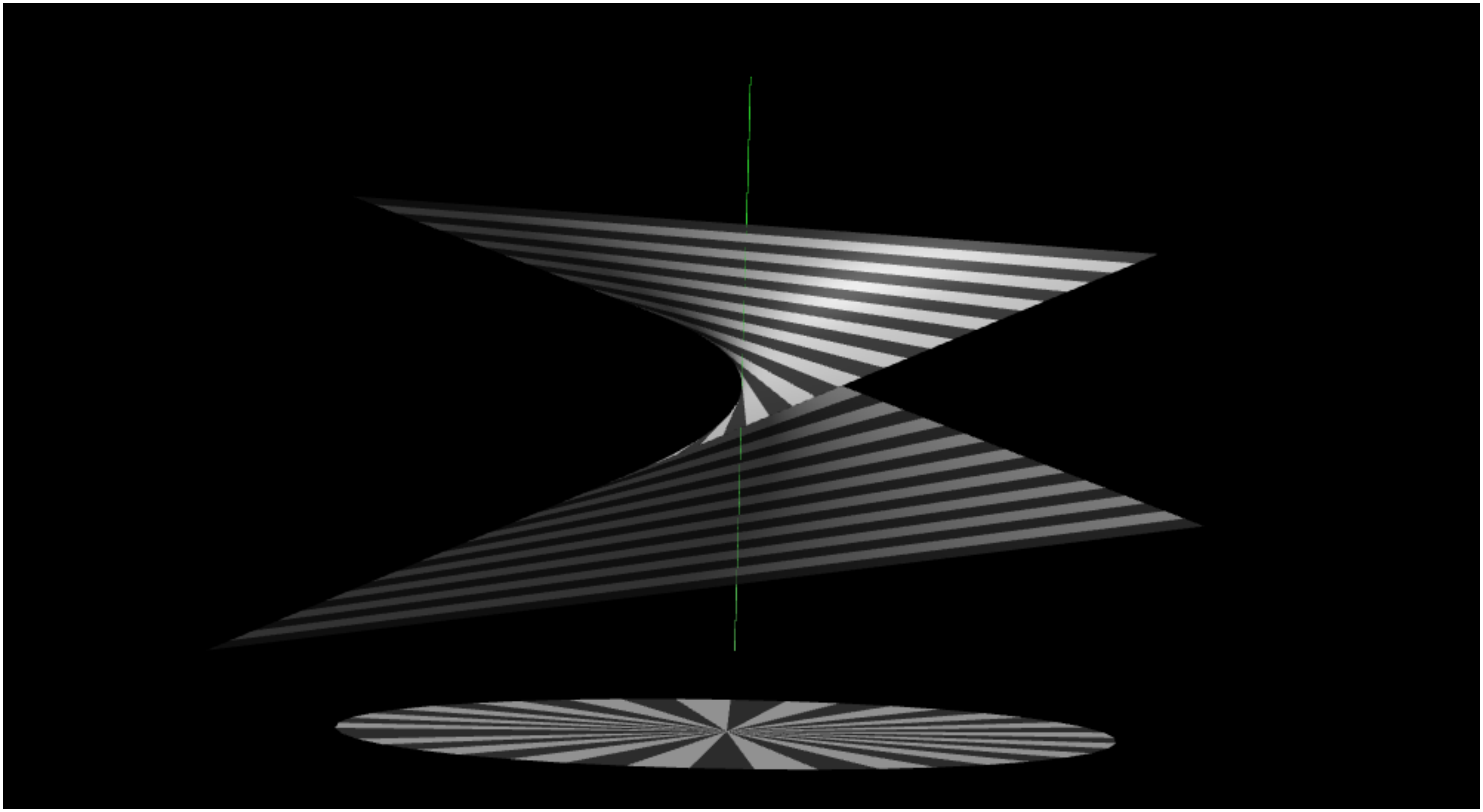

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If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

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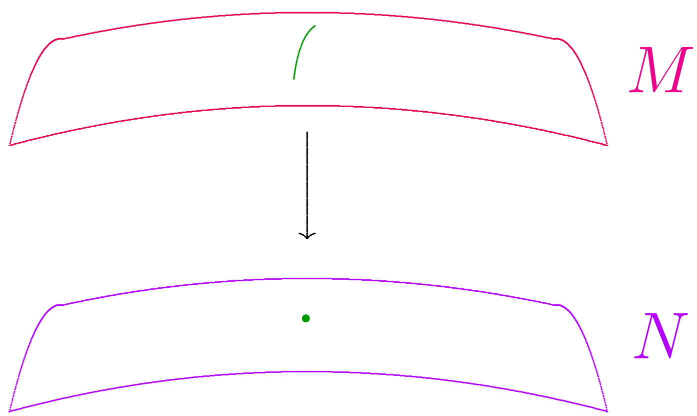
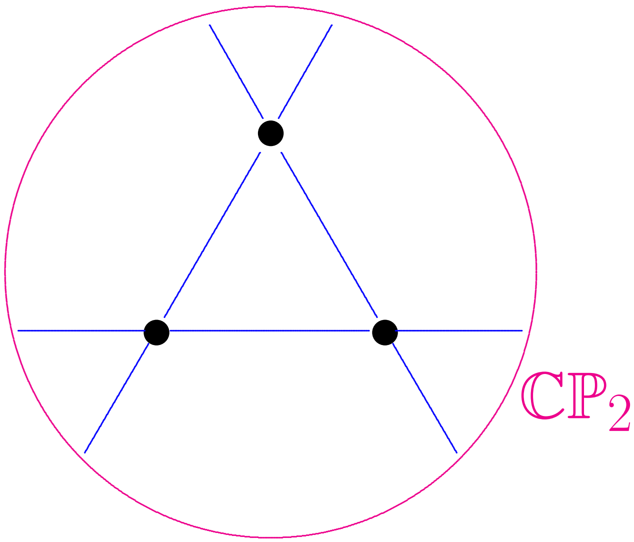
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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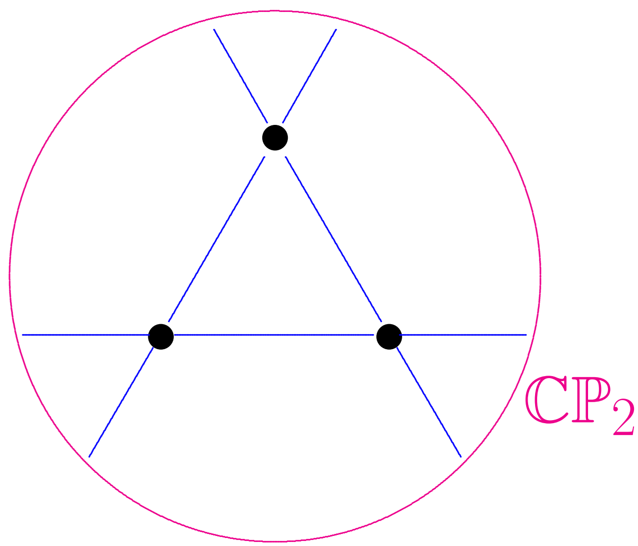
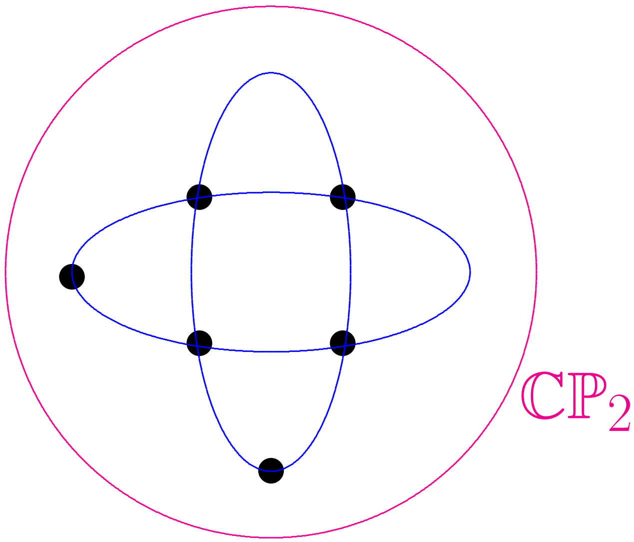


No 3 on a line,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

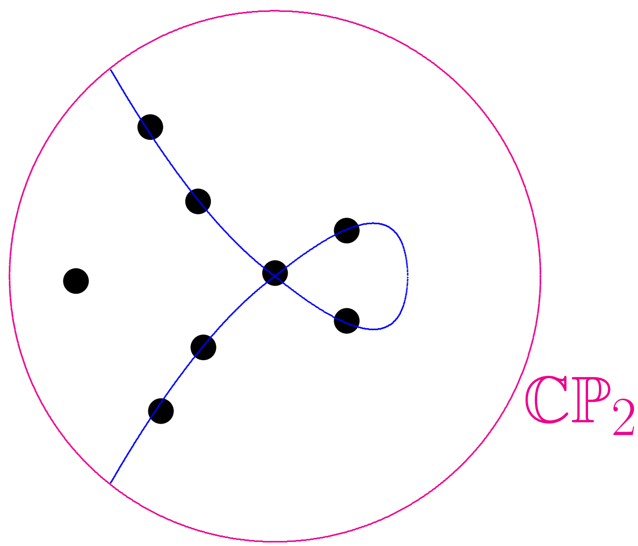


No 3 on a line, no 6 on conic,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



No 3 on a line, no 6 on conic, no 8 on nodal cubic.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

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Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo* (M^4, J) *admits a* J*-compatible*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo* (M^4, J) *admits a* J*-compatible conformally Kähler,*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric,*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is unique*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is unique up to complex automorphisms and constant rescalings.*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

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Conformally Kähler:

$$g = u^2 h$$

$\exists$ some Kähler metric h & some smooth function u.

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where Kähler metric h is extremal & $u = s_h^{-1}$.

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Existence: Page

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Above the line:

Moduli space $\mathscr{E}(M) \neq \varnothing$. But is it connected?

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

$$K3,$$
$$K3/\mathbb{Z}_2,$$
$$T^4,$$
$$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$$
$$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4).$$

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

One fundamental open problem:

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Understand all Einstein metrics on del Pezzos.

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Nice characterizations of known Einstein metrics.

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Exactly one connected component of moduli space!

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Corollary. *These known Einstein metrics on any del Pezzo* M^4 *sweep out exactly one connected component of the Einstein moduli space* $\mathscr{E}(M)$*.*

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Kähler $\Longrightarrow \Lambda^+ = \mathbb{R}\omega \oplus \Re e \Lambda^{2,0}$

$$W^+ = \text{trace-free part of } \begin{bmatrix} 0 & & \\ & 0 & \\ & & \frac{s}{4} \end{bmatrix}$$

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for these metrics & conformal rescalings:

$$g \rightsquigarrow h = u^2 g \quad \Longrightarrow \quad \det(W^+) \rightsquigarrow u^{-6} \det(W^+).$$

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method also proves more general results.

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L (2021b): related classification result.

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Corollary. *Every simply-connected compact oriented Einstein (M^4, g) with $\det(W^+) > 0$ is diffeomorphic to a del Pezzo surface. Conversely, every del Pezzo M^4 carries Einstein g with $\det(W^+) > 0$, and these sweep out exactly one connected component of moduli space $\mathscr{E}(M)$.*

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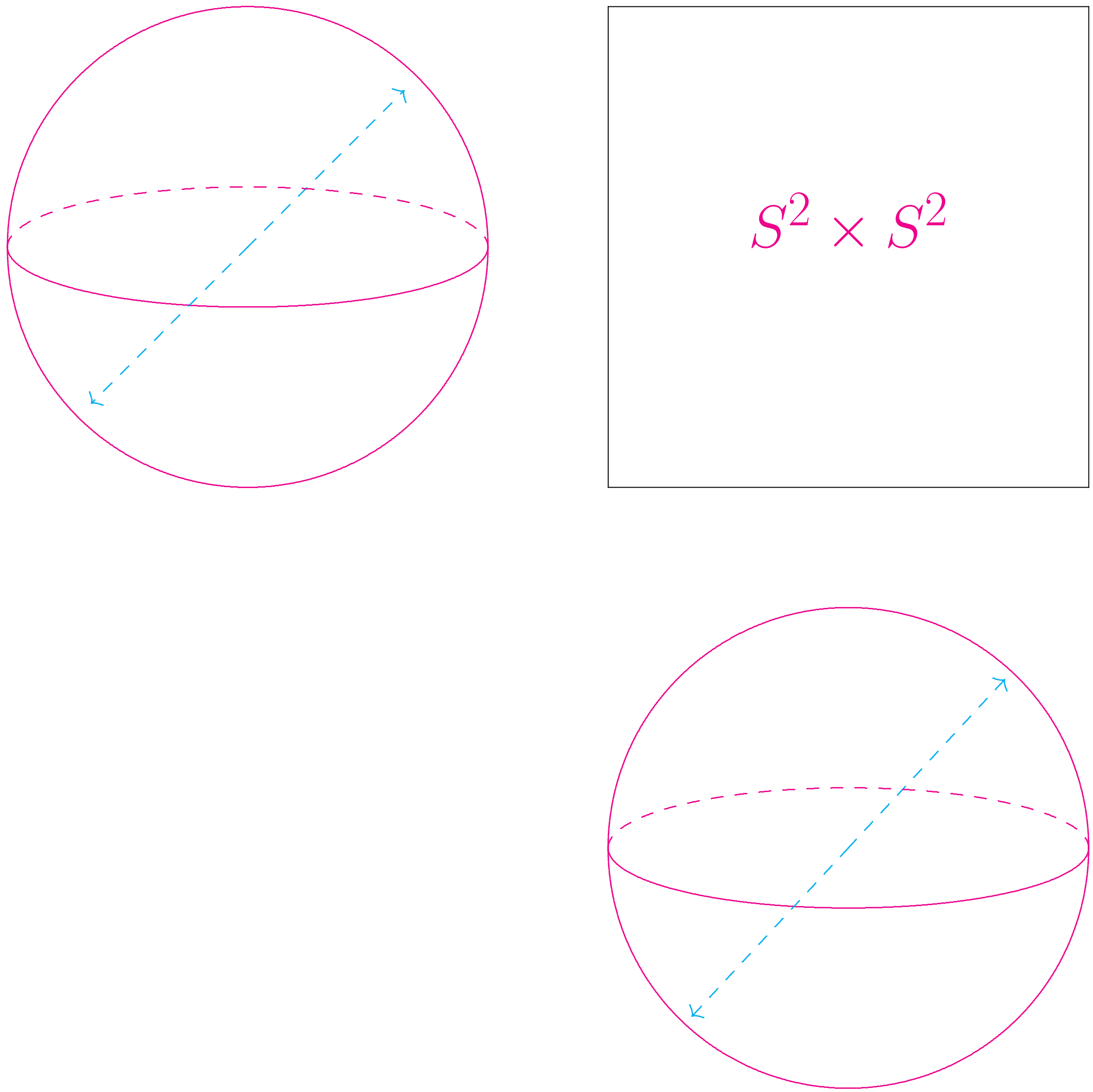
Excludes 5 types with $\pi_1 = \mathbb{Z}_2$ and $b_+(M) = 0$.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Similar results govern moduli spaces in these cases.

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Proof once again based on Wu's criterion.

Thanks for the invitation!

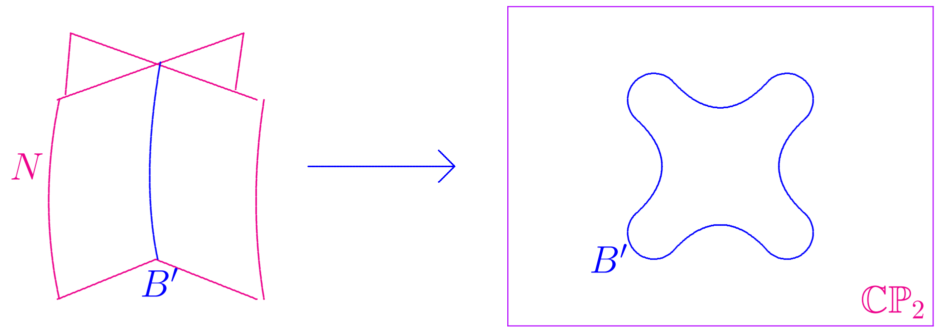
It's a pleasure to be here!

Supplementary material:

The $\lambda < 0$ case.

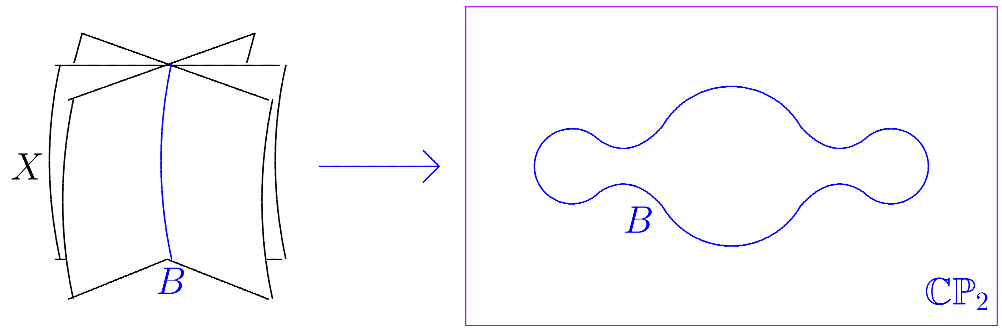
Theorem. *Let* M *be the* 4*-manifold underlying a compact complex surface. Suppose that* M *an Einstein metric* g*. Then either* M *appears on list for* $\lambda \geq 0$*, or else* M *is a surface of general type, and is not too non-minimal, in the sense that it is obtained from its minimal model* X *by blowing up at* $k < {c_1}^2(X)/3$ *points.*

Example. Let N be double branched cover $\mathbb{CP}_2$, ramified at a smooth octic:



Aubin/Yau $\Longrightarrow$ N carries Einstein metric.

Now let X be a triple cyclic cover $\mathbb{CP}_2$, ramified at a smooth sextic



and set

$$M = X \# \overline{\mathbb{CP}_2}.$$

Then $\alpha^2(M) = c_1^2(X) = 3$,

$(2\chi + 3\tau)(M) = c_1^2(M) = 2$.

Theorem **??** $\Longrightarrow$ *no* Einstein metric on M.