

Anti-Self-Dual 4-Manifolds,

Quasi-Fuchsian Groups, &

Almost-Kähler Geometry

Claude LeBrun

Stony Brook University

Brussels-Oxford-Warwick-London
Geometry Seminar, February 16, 2021

Today's main results joint with

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Christopher J. Bishop
Stony Brook University

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Comm. Analysis Geom. 28 (2020) 745-780.

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(in honor of Karen Uhlenbeck)

Depend on generalizing results from

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Scalar-Flat Kähler Metrics on
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Journal für die reine und
angewandte Mathematik
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Journal of Geometric Analysis
25 (2015) 1744–1772

Facets of Kähler Geometry:

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Complex Geometry

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Symplectic Geometry

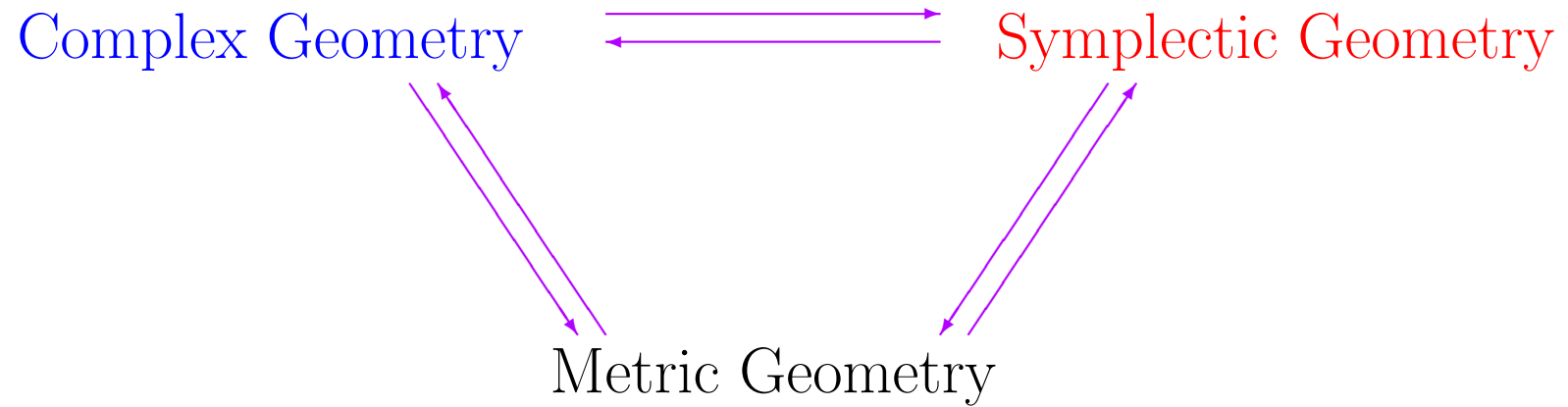
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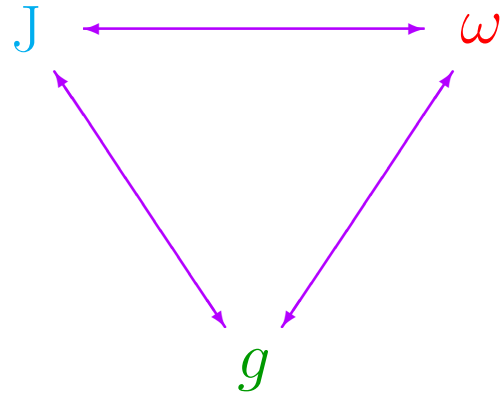
Symplectic Geometry

Metric Geometry

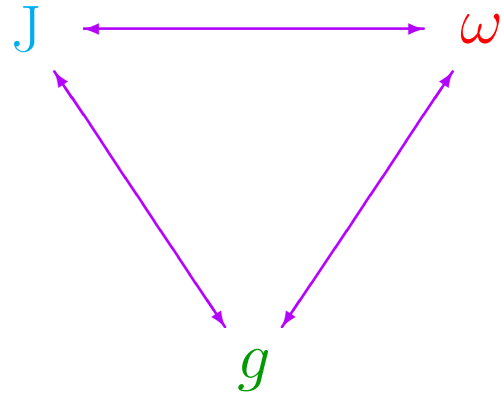
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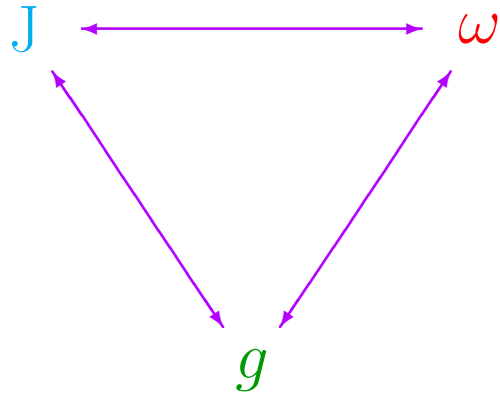
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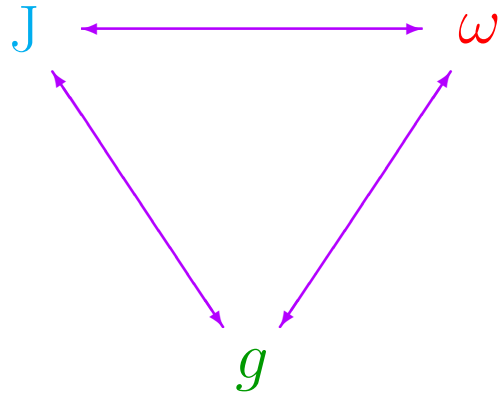


Almost-Kähler Geometry:



Drop demand that J be integrable.

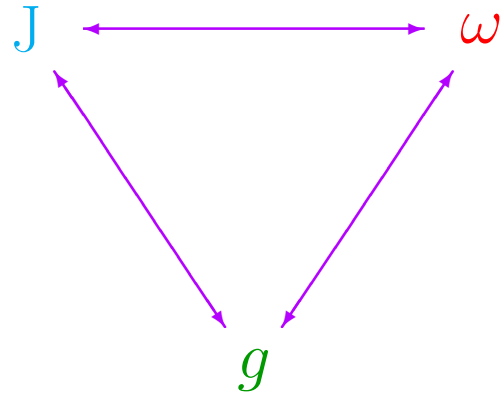
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Linked to conformal geometry in dimension 4.

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Higher dimensions are demonstrably different.

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Imitates Kähler geometry in a non-Kähler setting.

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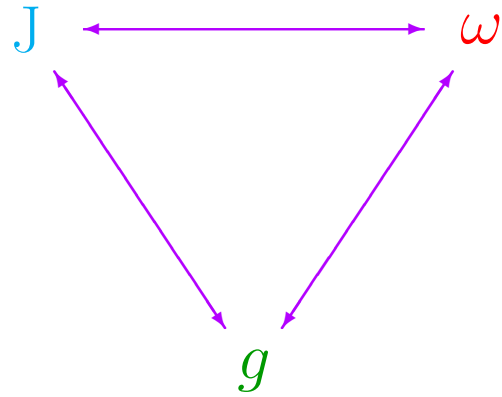
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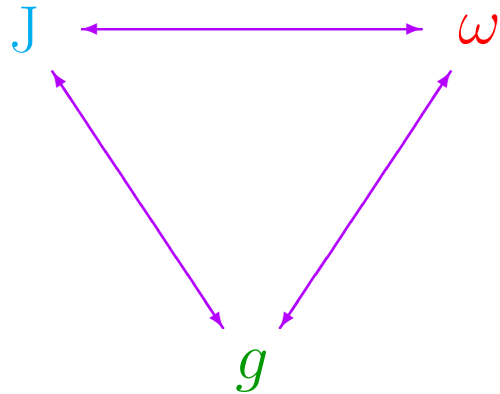
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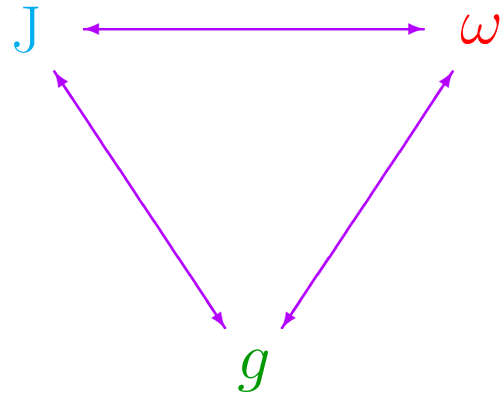


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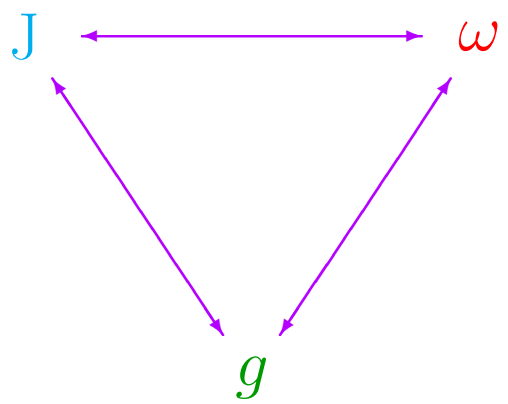
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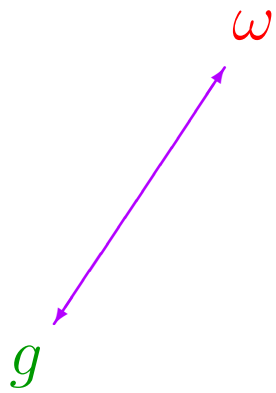
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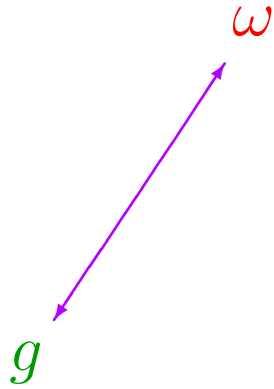


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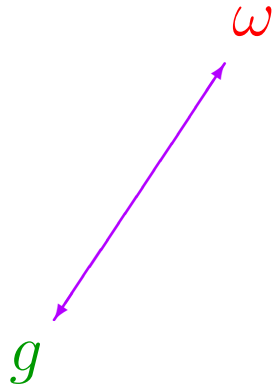
For example, can avoid explicitly mentioning J .



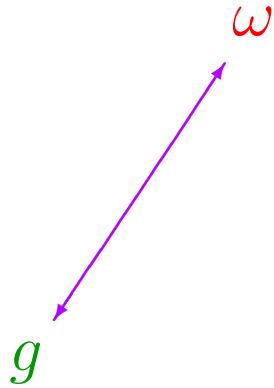




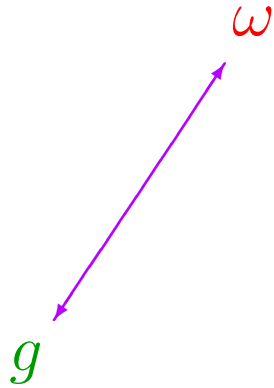
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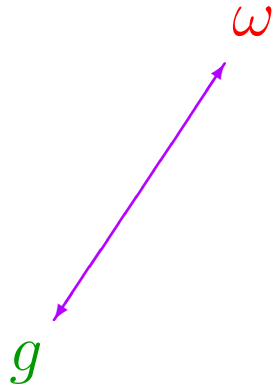
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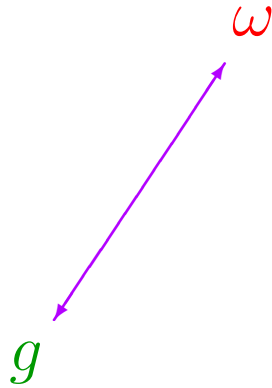


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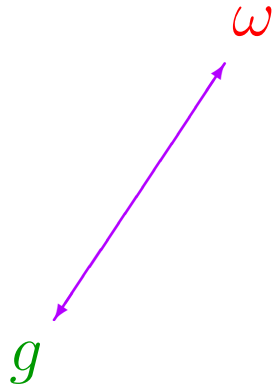
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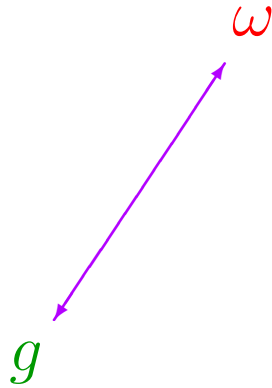
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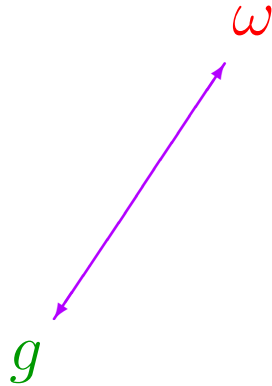
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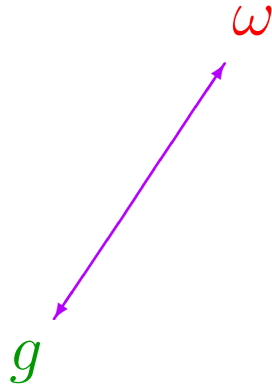
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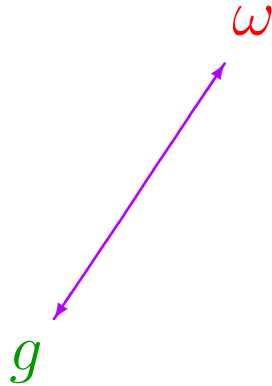
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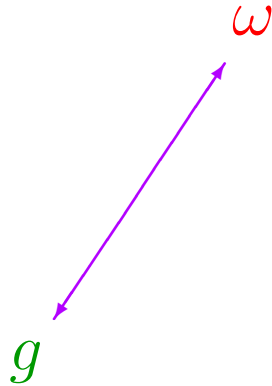
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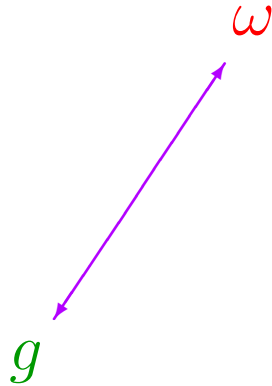
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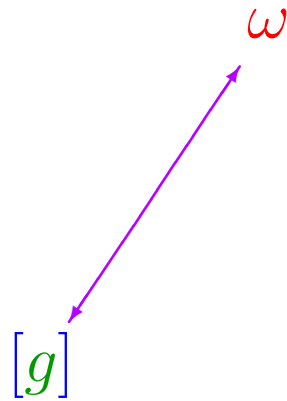
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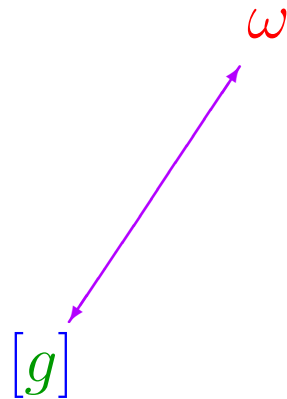


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Moreover, the set of conformal classes $[g]$ on M that carry such a harmonic form ω is open in the C^2 topology.

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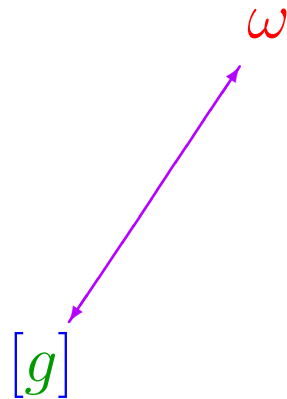
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In particular, the numbers

$$b_\pm(M) = \dim \mathcal{H}_g^\pm$$

are independent of g , and so are invariants of M .

$b_{\pm}(M)?$

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$$b_2(M) = b_+(M) + b_-(M)$$

Signature defined in terms of intersection pairing

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$$\tau(M) = b_+(M) - b_-(M)$$

Signature of M .

Signature defined in terms of intersection pairing

$$\begin{aligned} H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) &\longrightarrow \mathbb{R} \\ ([\varphi], [\psi]) &\longmapsto \int_M \varphi \wedge \psi \end{aligned}$$

$$\tau(M) = b_+(M) - b_-(M)$$

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but also expressible as a curvature integral:

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(Thom-Hirzebruch Signature Formula)

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Has major consequences in conformal geometry.

On oriented (M^4, g) ,

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

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Basic problems: For given smooth compact M^4 ,

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So Weyl functional is essentially equivalent to

$$[g] \longmapsto \int_M |W_+|^2 d\mu_g$$

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(ASD)

Twistor picture of anti-self-duality condition:

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Oriented $(M^4, g) \longleftrightarrow (Z, J)$.

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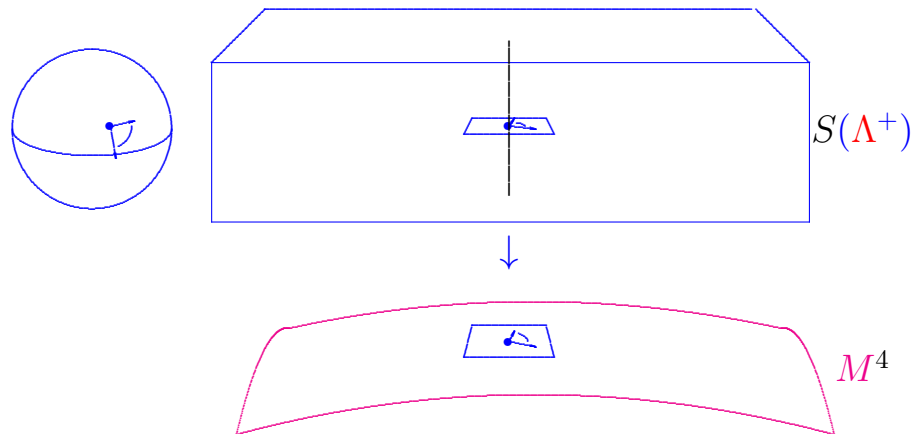
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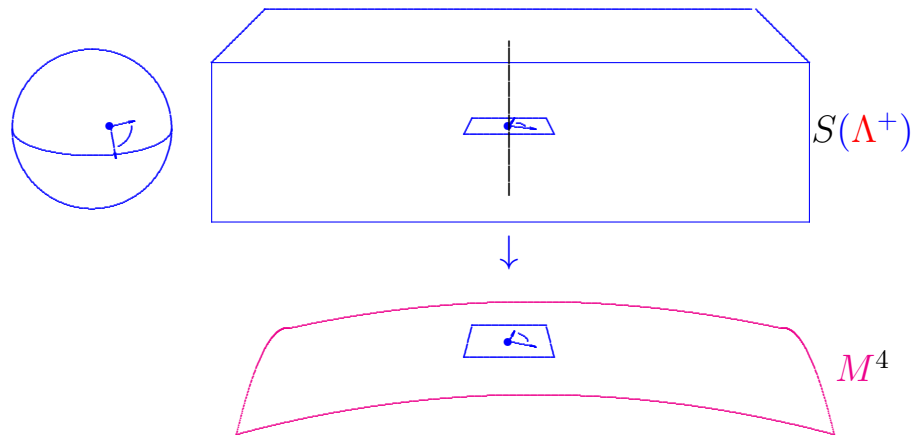
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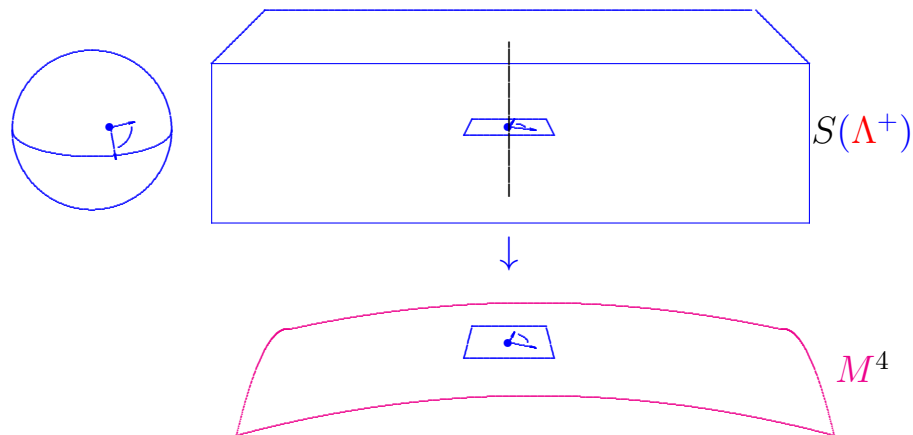


Theorem (Atiyah-Hitchin-Singer). (Z, J) is a complex 3-manifold iff $W_+ = 0$.

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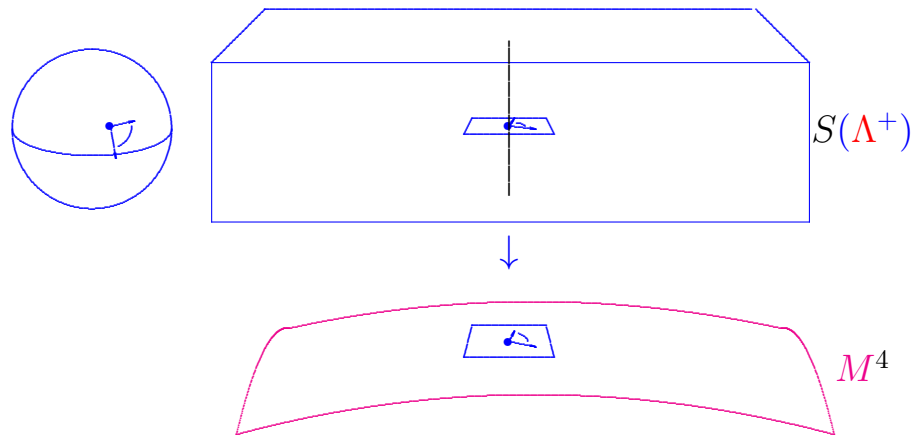
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Reconceptualizes earlier work by Penrose.

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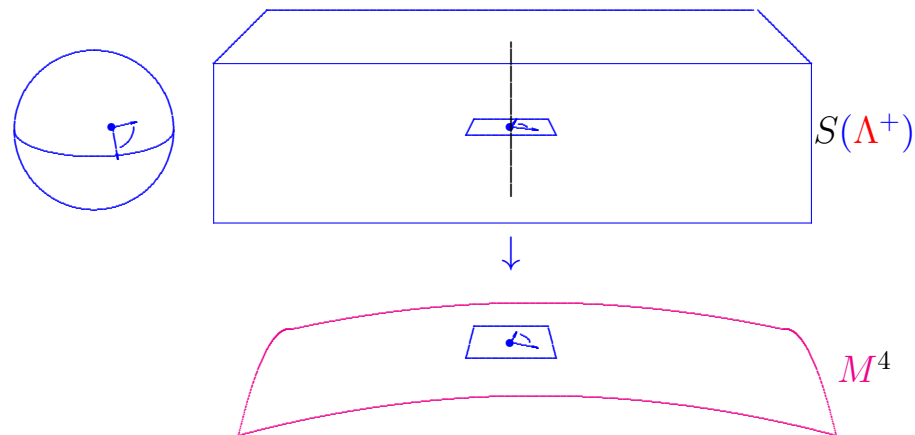


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Motivates study of ASD metrics,
and yields methods for constructing them.

So ASD metrics are linked to complex geometry. . .

A different link with complex geometry:

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If (M^4, g, J) is a Kähler surface,

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Results proved about SFK in '90s foreshadowed
many more recent results about general case.

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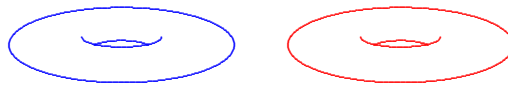
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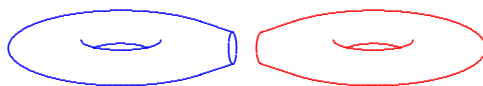
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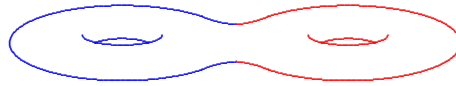
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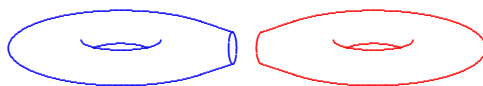
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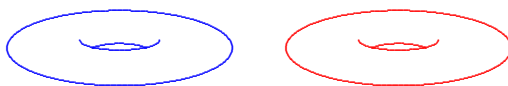
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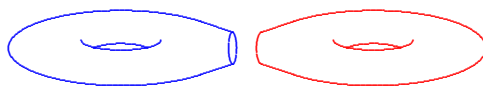
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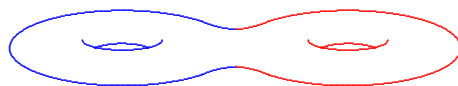
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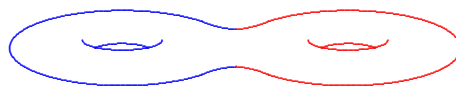
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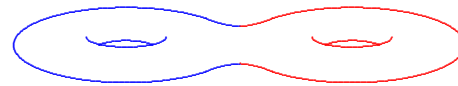
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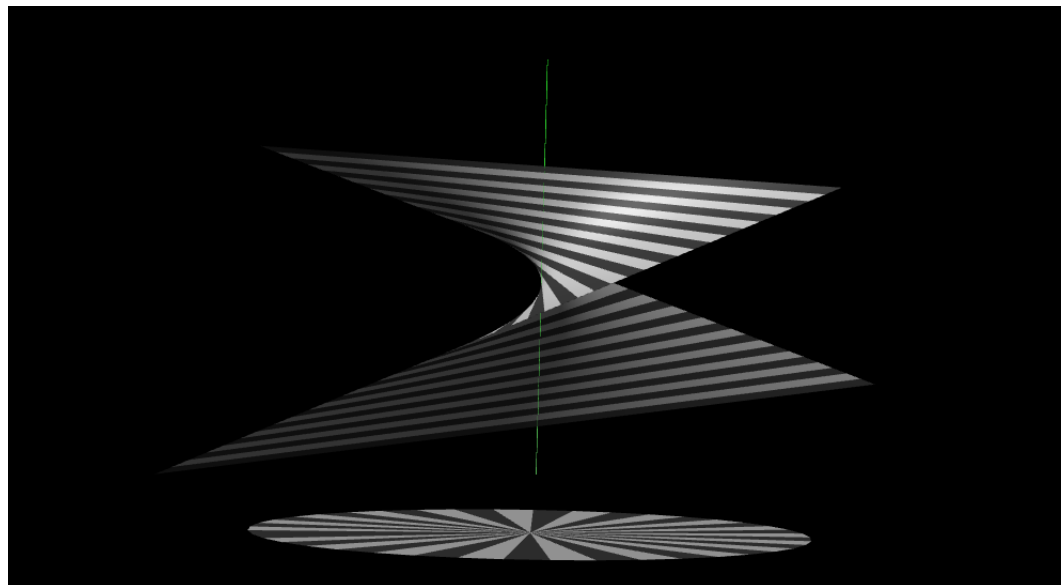
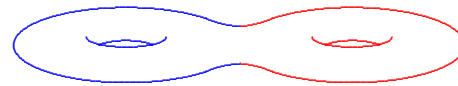


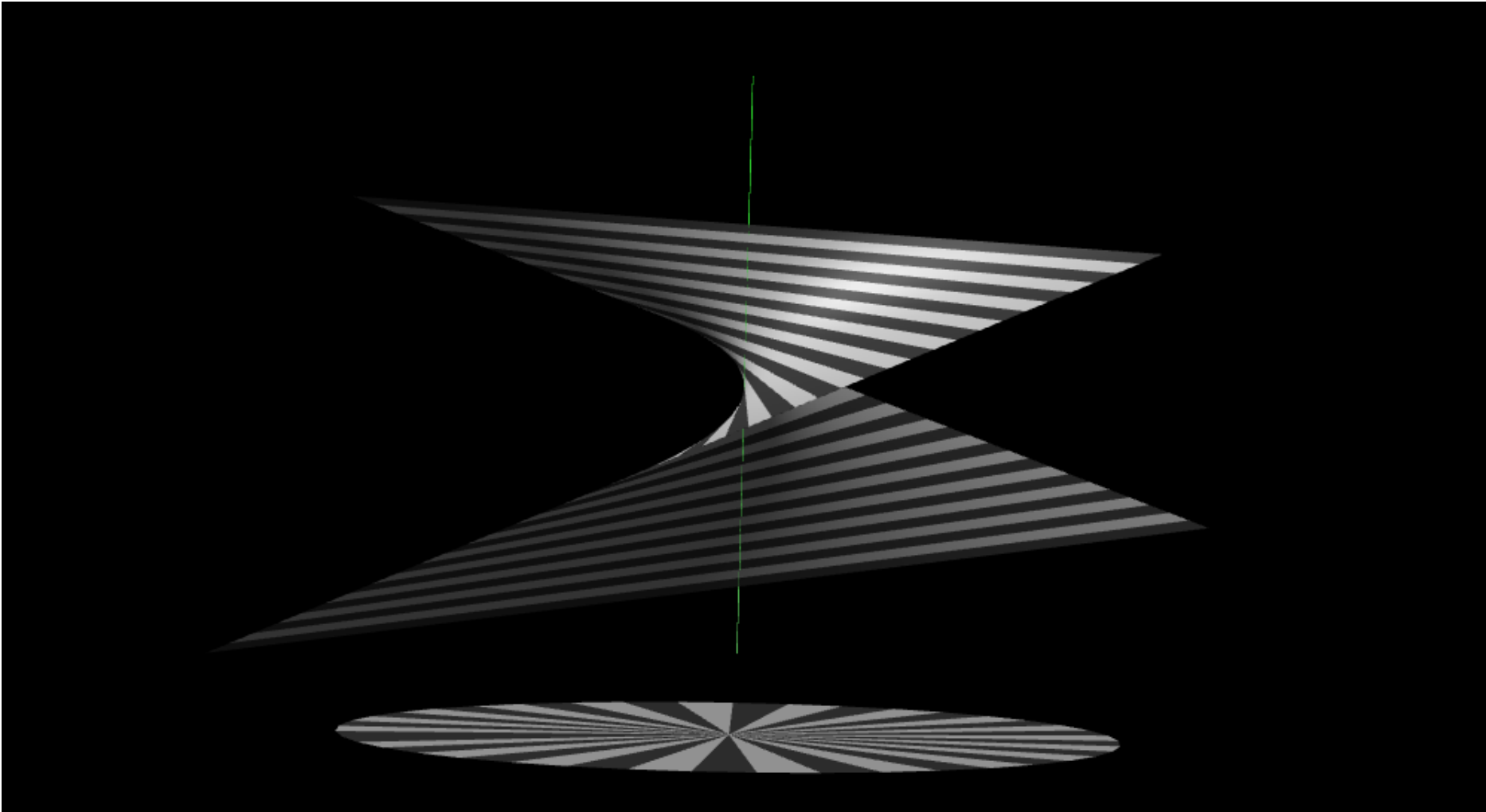
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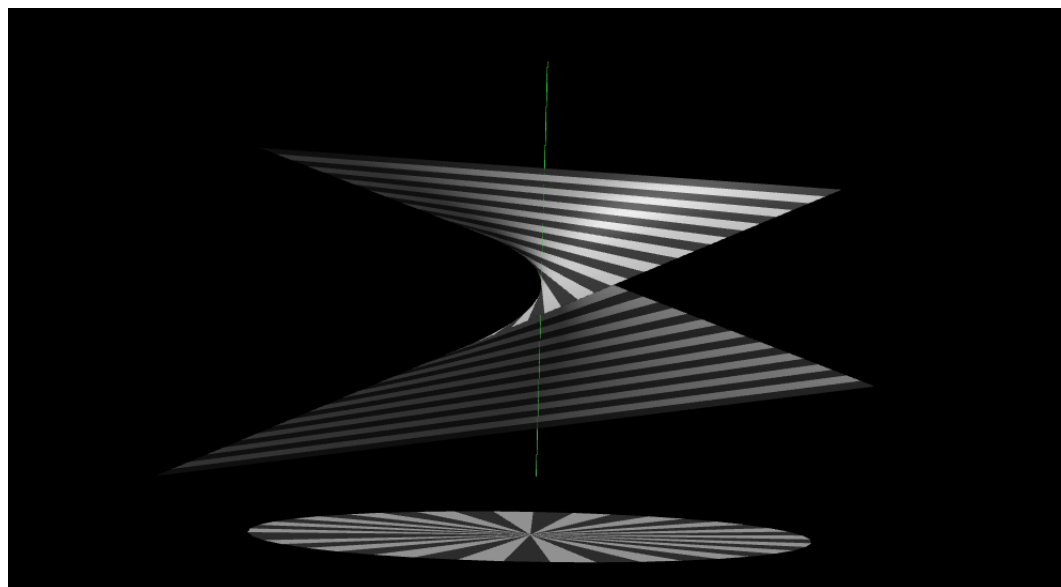
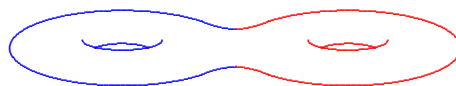




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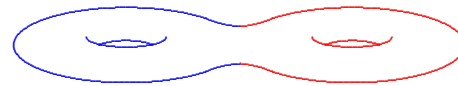
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If (M^4, g, J) is a Kähler surface, then $[g]$ is ASD
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Scalar-flat Kähler surfaces:

Classification up to diffeomorphism:

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any metric on one of these manifolds satisfies

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Notice that the 4-manifolds $\mathbb{C}P_2 \#^k \overline{\mathbb{C}P_2}$
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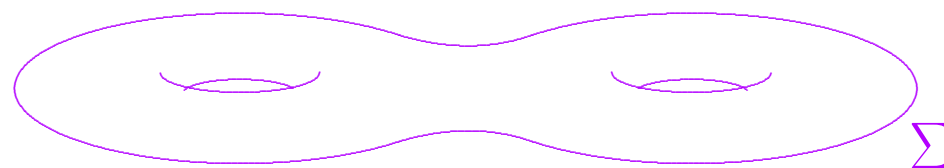
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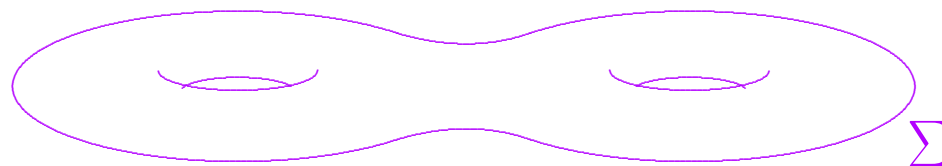
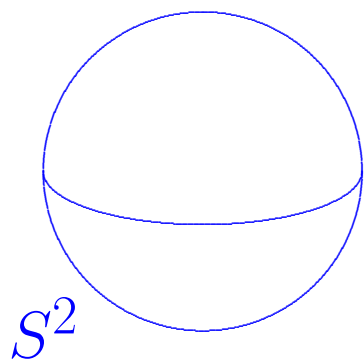
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Almost-Kähler ASD metrics sweep out an open set in the ASD moduli space.

Example.

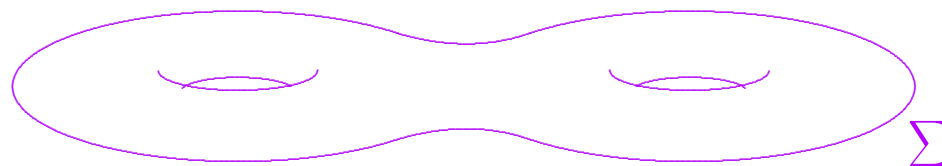
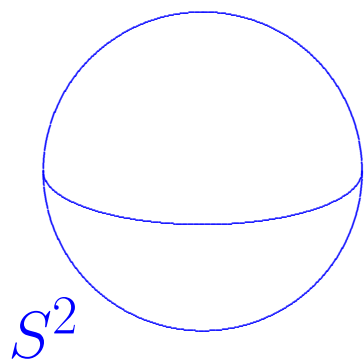


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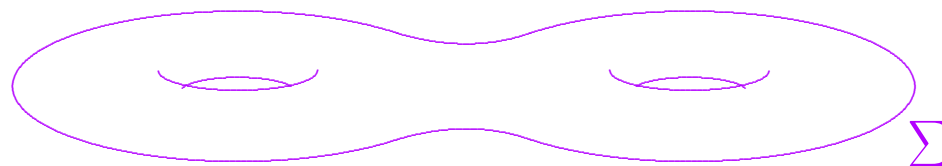
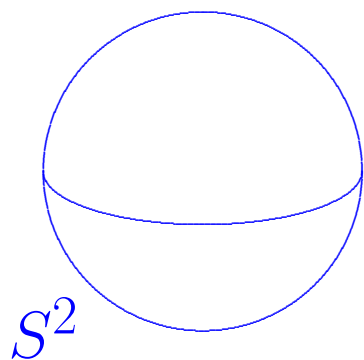
Example.

$$M = \Sigma \times S^2$$



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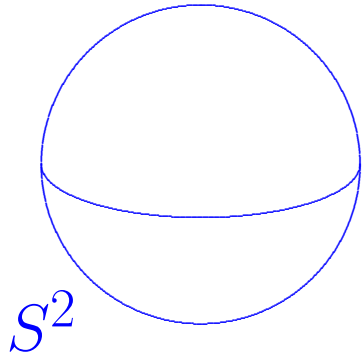
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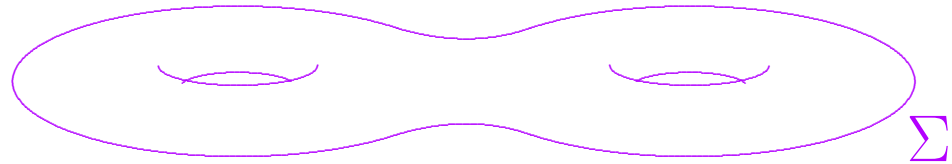
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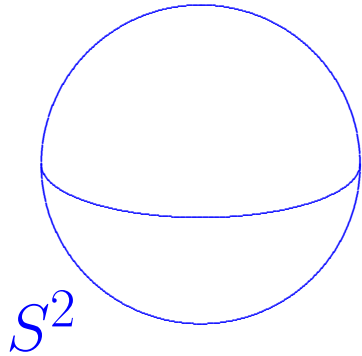


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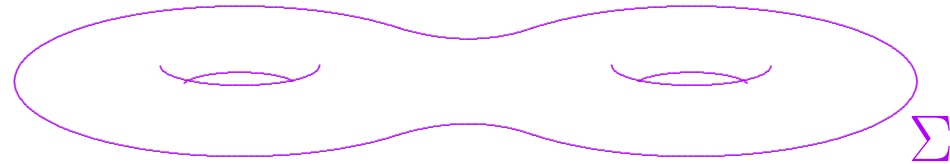
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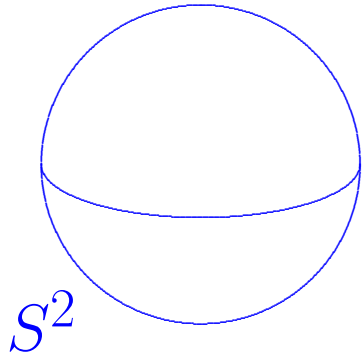
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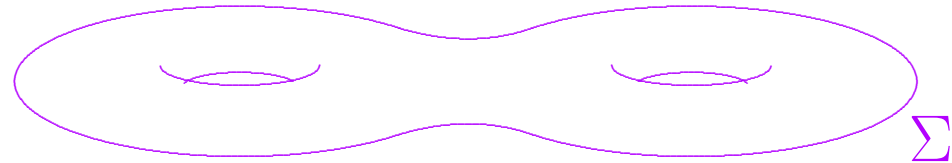
Product is scalar-flat

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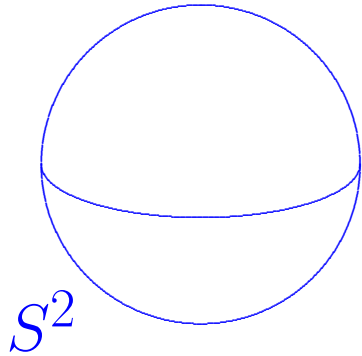


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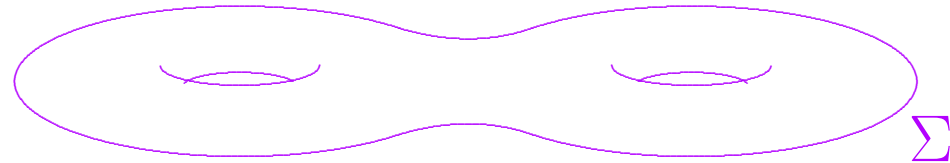
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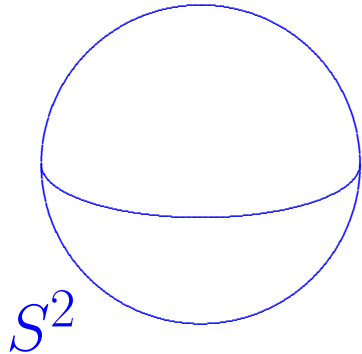
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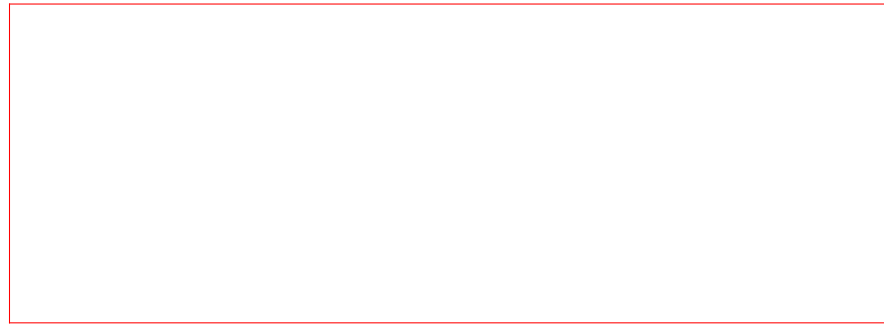
For both orientations!

Example.

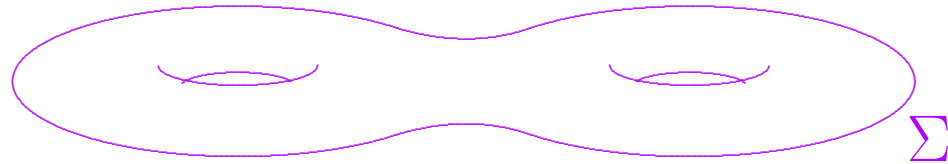
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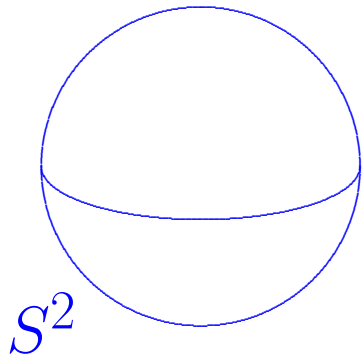
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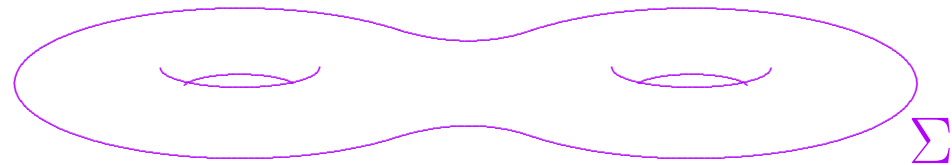
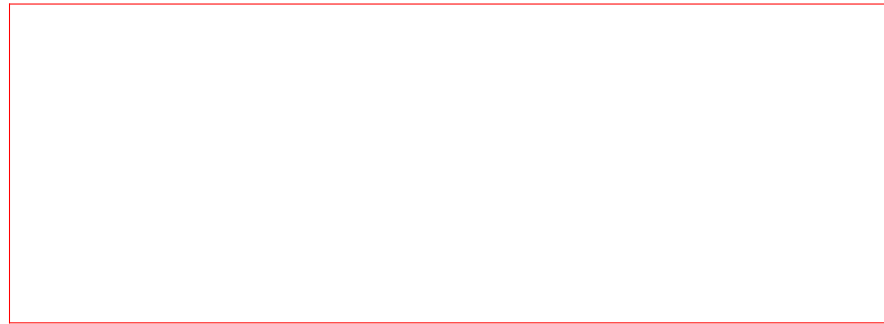
$$W_+ = 0.$$

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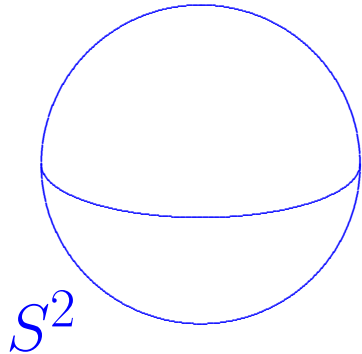
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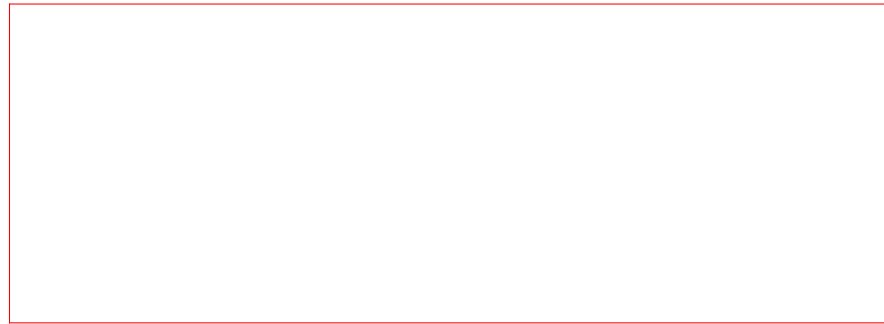
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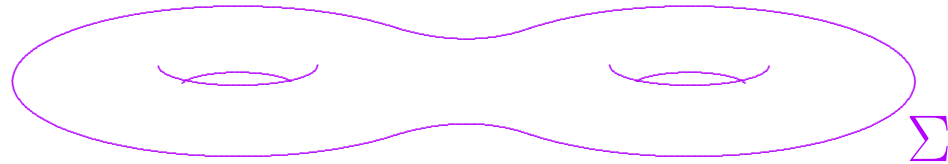
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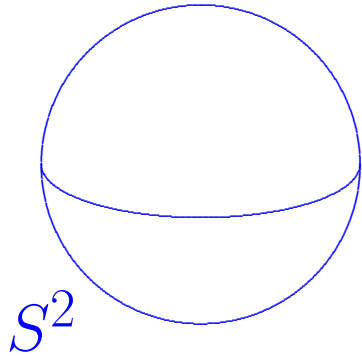
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Locally conformally flat!

Example.

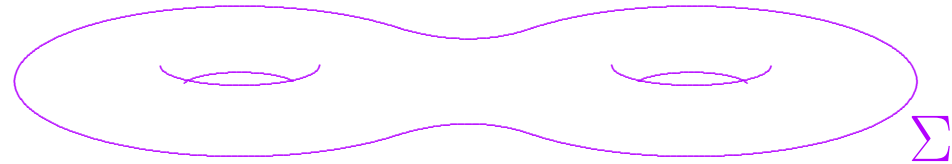
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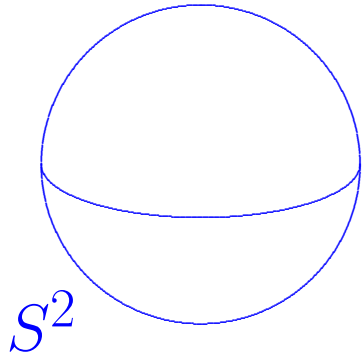
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$$\widetilde{M} = \mathcal{H}^2 \times S^2$$

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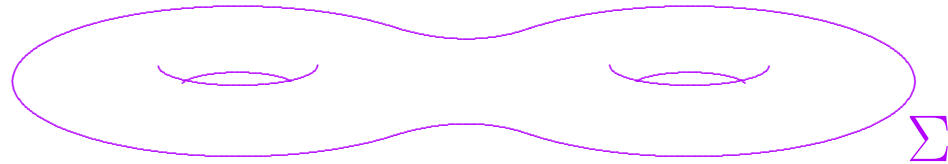
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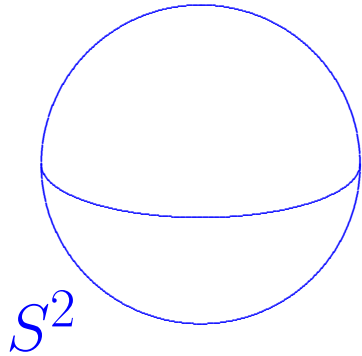
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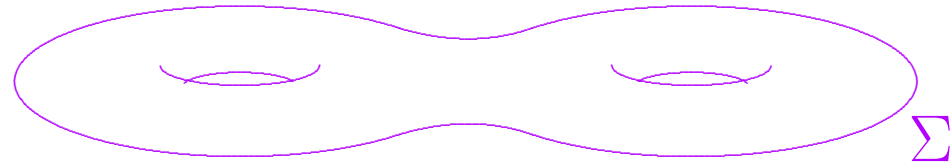
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Example.

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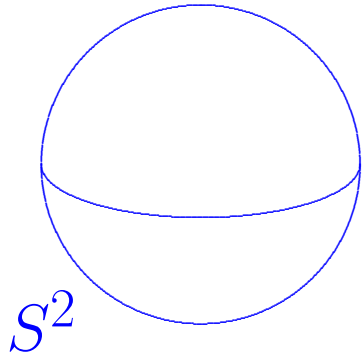
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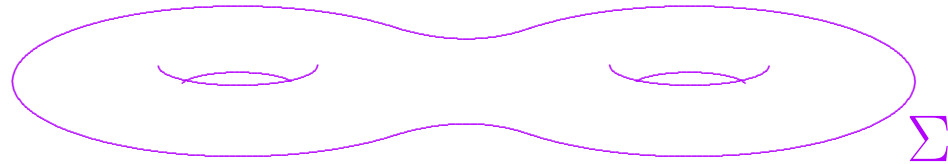
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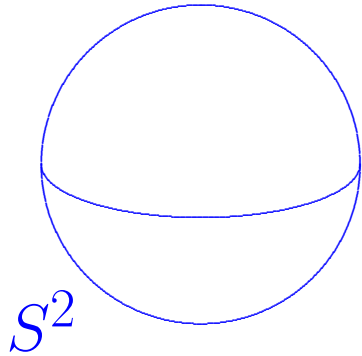


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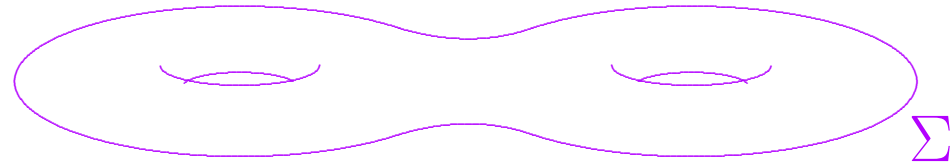
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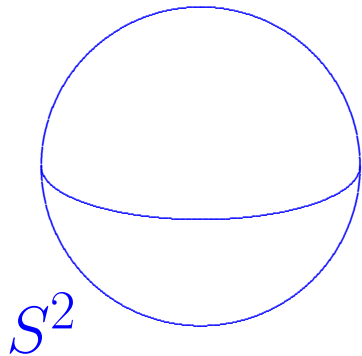
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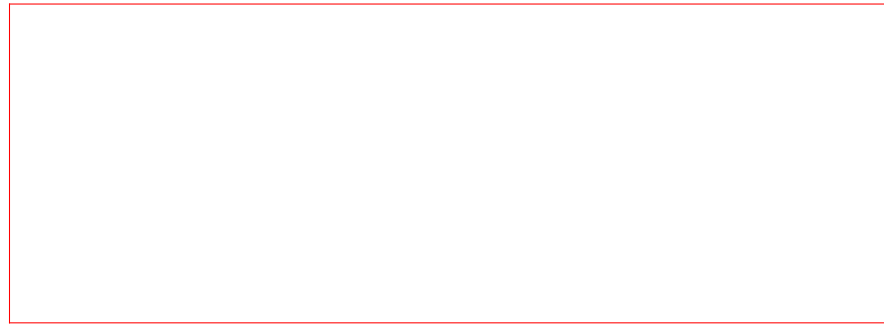
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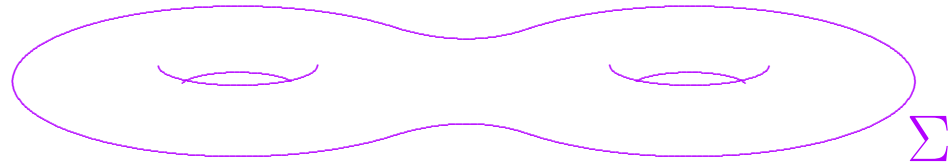
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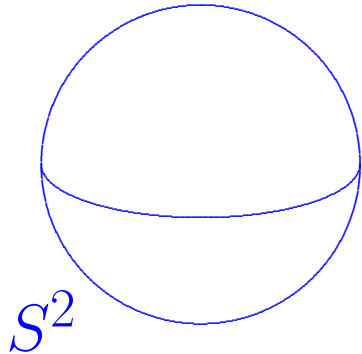
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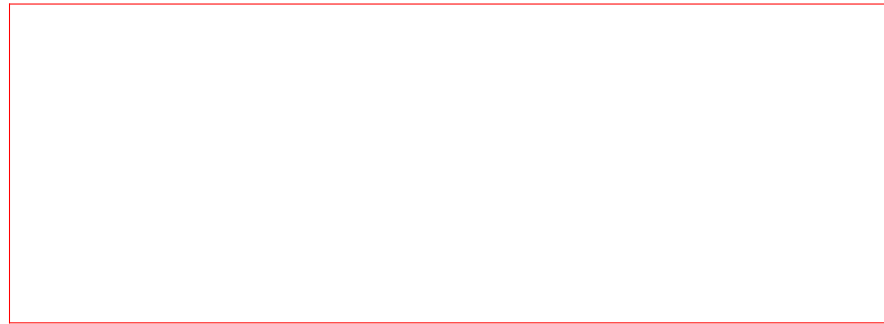
Scalar-flat Kähler deformations: $12(g - 1)$ moduli

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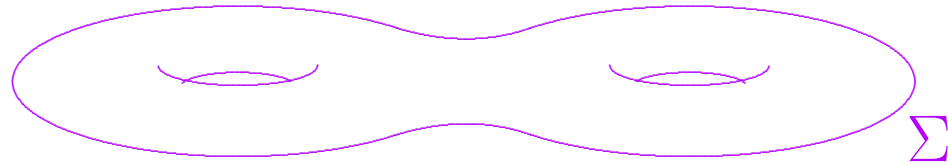
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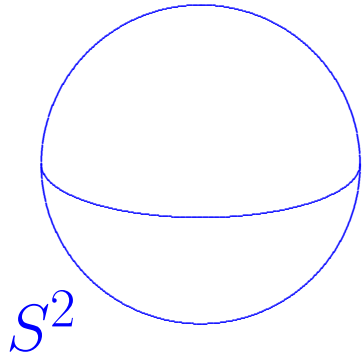
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Scalar-flat Kähler deformations: $12(g - 1)$ moduli
Locally conformally flat def'ms: $30(g - 1)$ moduli

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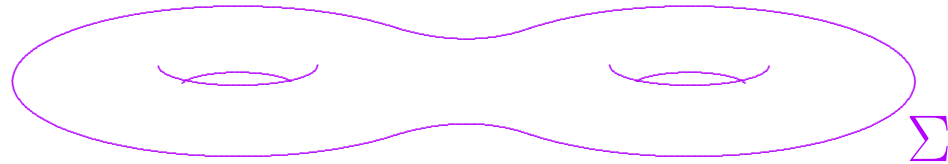
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Scalar-flat Kähler deformations: $12(g-1)$ moduli
almost-Kähler ASD deformations: $30(g-1)$ moduli

Last result indicates that almost-Kähler condition gives extra control on ASD conformal geometry.

Inyoung Kim '16: classification of almost-Kähler ASD roughly the same as in scalar-flat Kähler case.

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Alas, **No!**

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Same method simultaneously proves...

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Proof hinges on a construction of hyperbolic 3-manifolds.

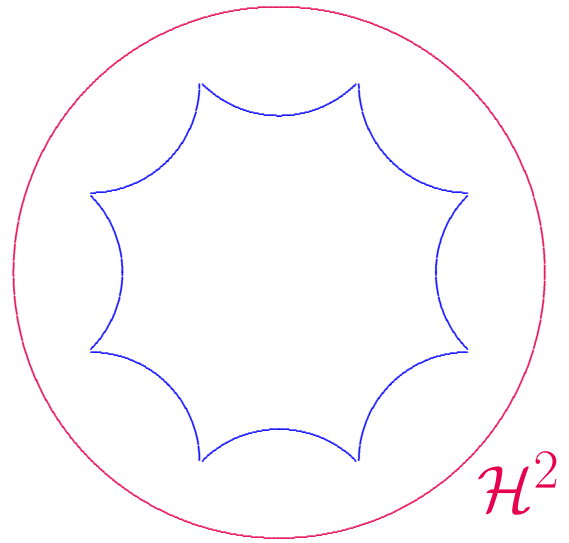
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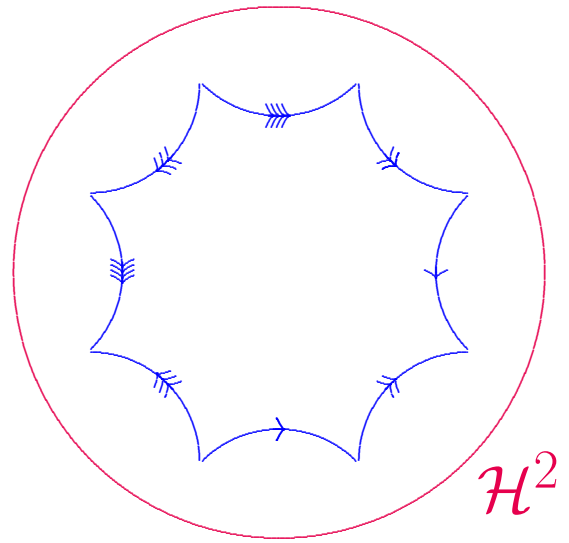
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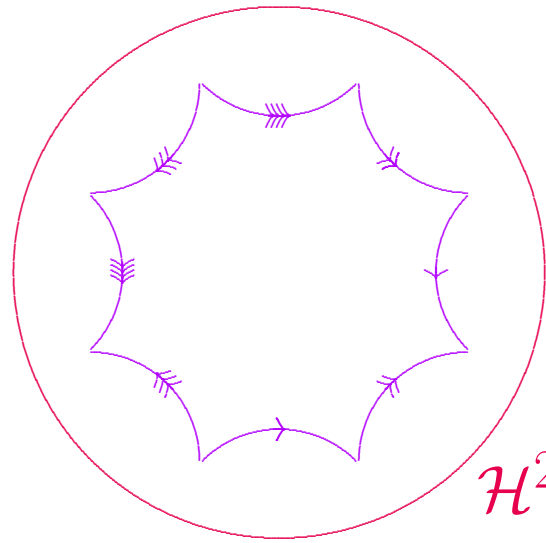
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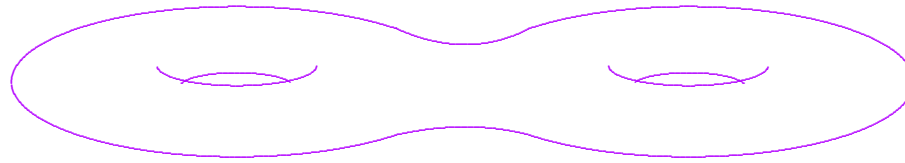
We begin by revisiting hyperbolic metrics on Σ .

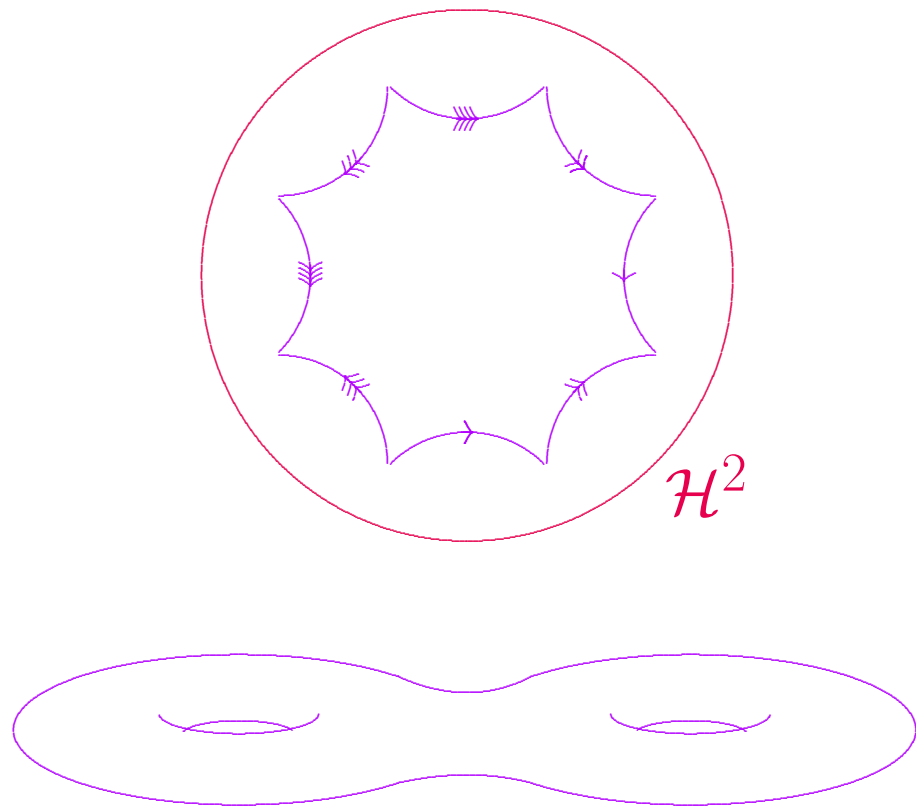




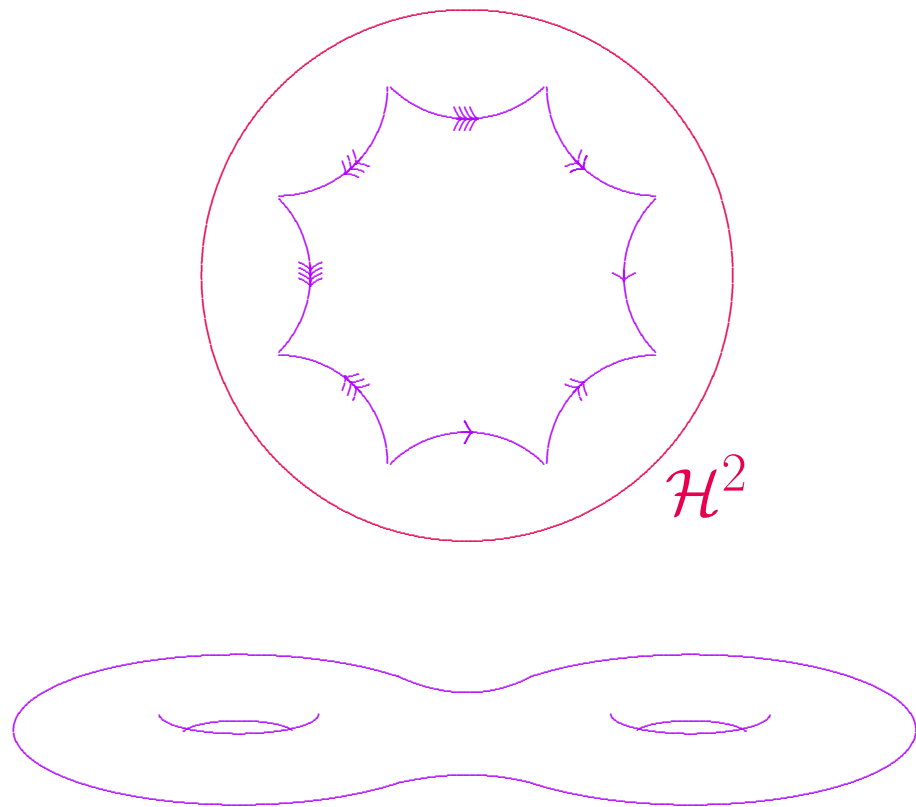


\mathcal{H}^2

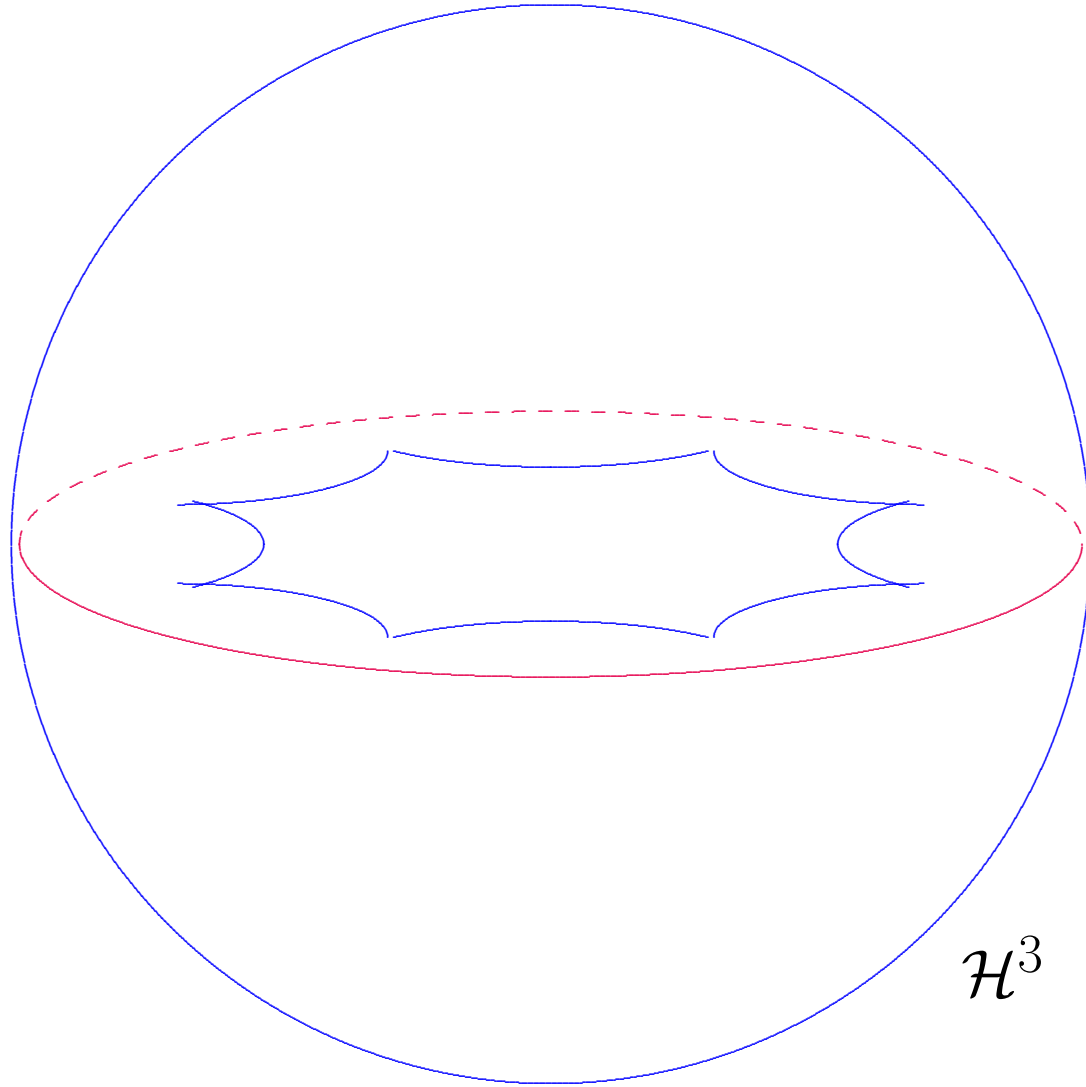




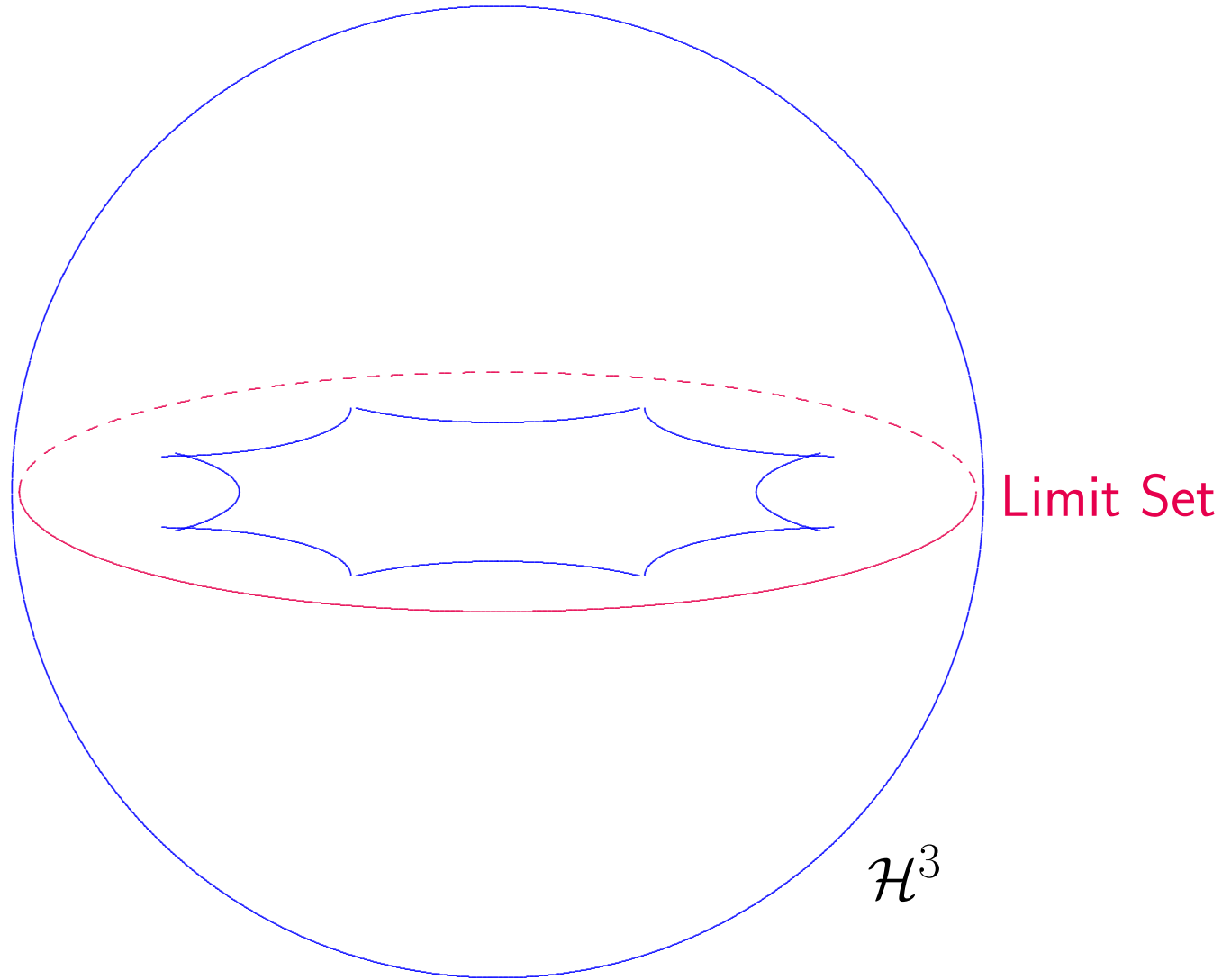
$$\pi_1(\Sigma) \hookrightarrow \mathbf{SO}_+(1, 2) = \mathbf{PSL}(2, \mathbb{R})$$

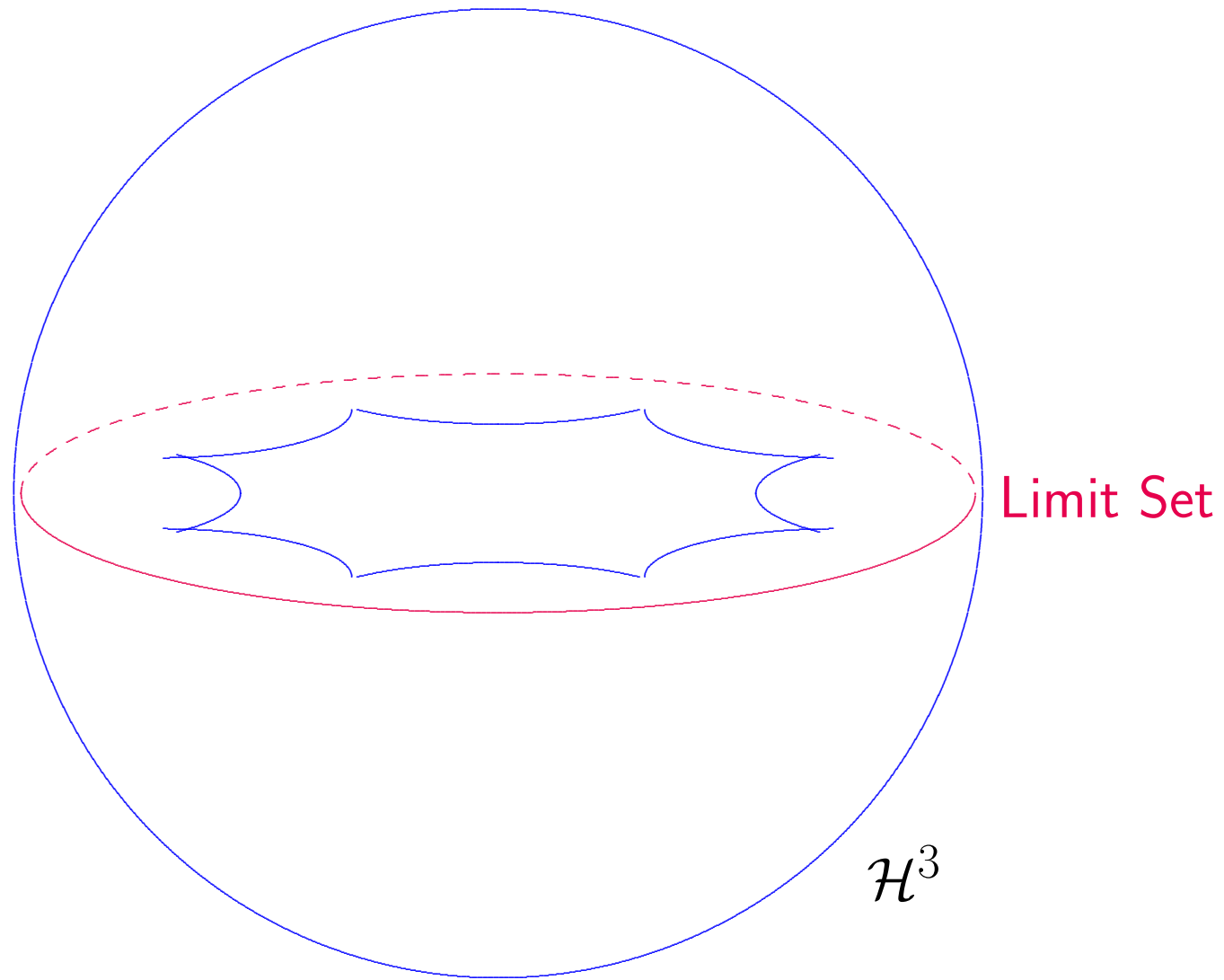


$$\begin{array}{ccc}
 \pi_1(\Sigma) & \hookrightarrow & \mathbf{SO}_+(1, 2) = \mathbf{PSL}(2, \mathbb{R}) \\
 & & \cap \qquad \qquad \cap \\
 & & \mathbf{SO}_+(1, 3) = \mathbf{PSL}(2, \mathbb{C})
 \end{array}$$

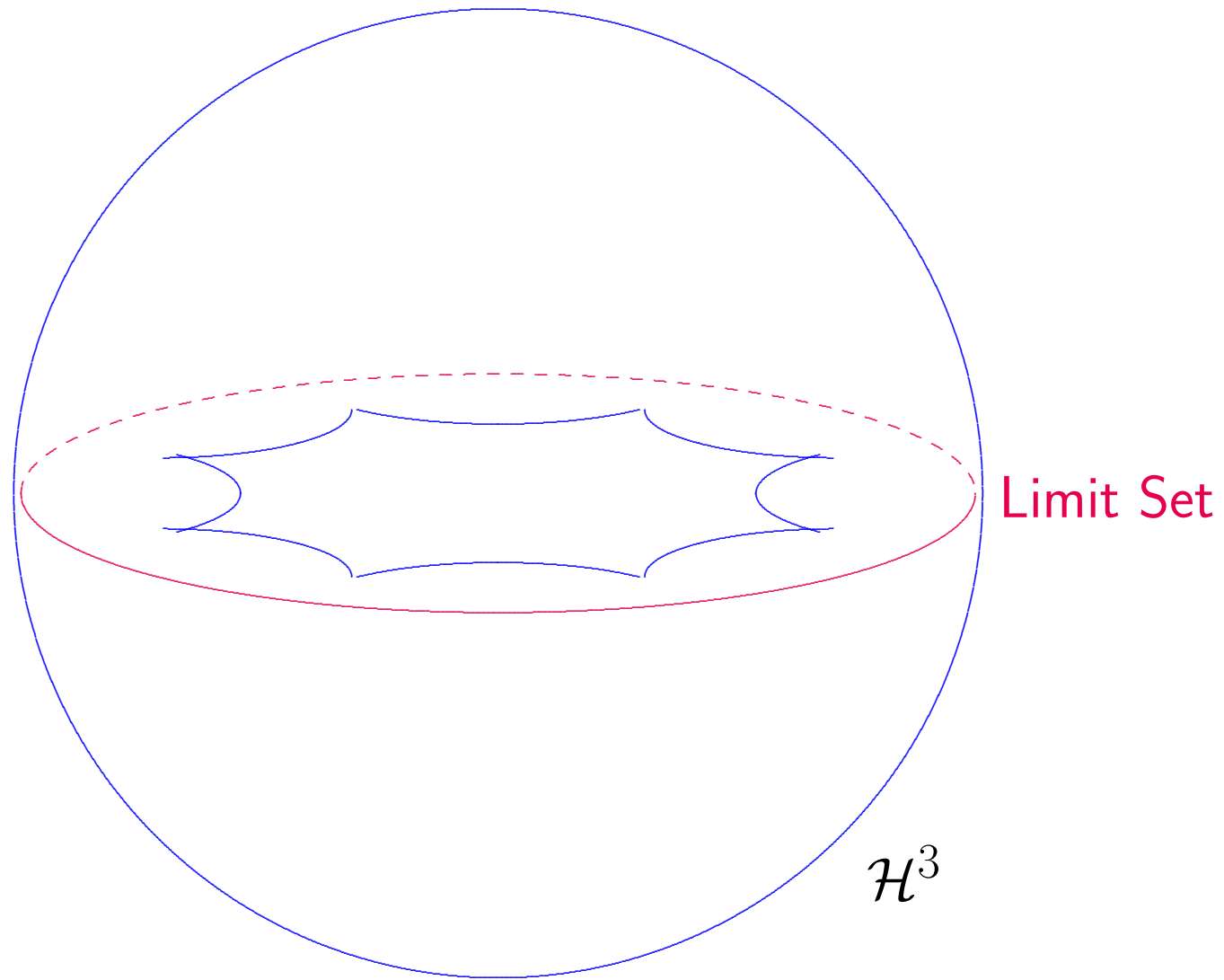


\mathcal{H}^3

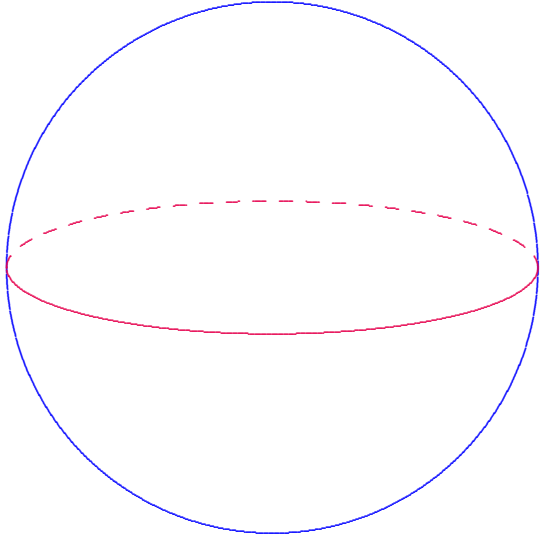


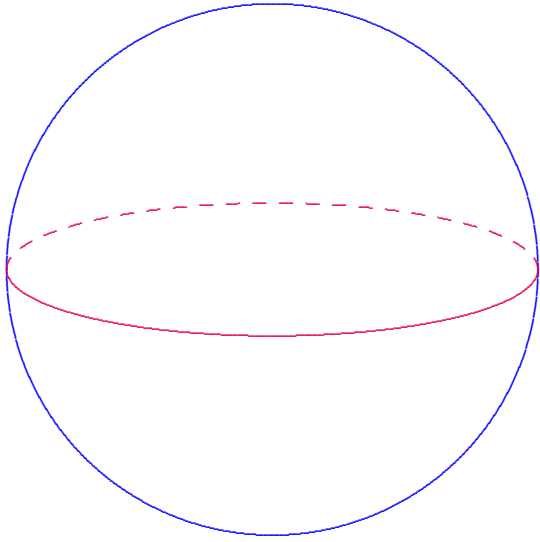


$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{R}) \text{ Fuchsian group}$$

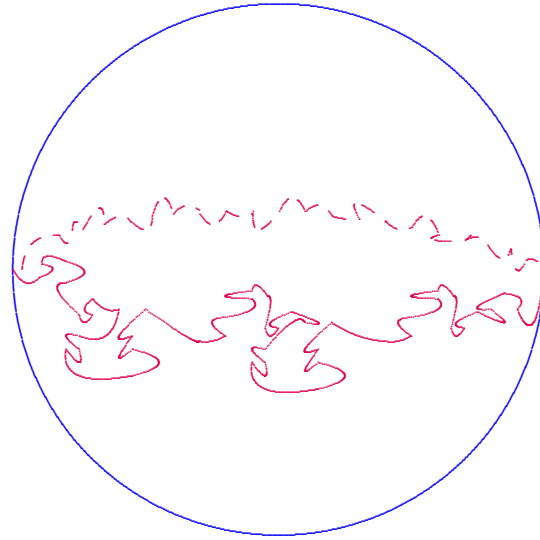
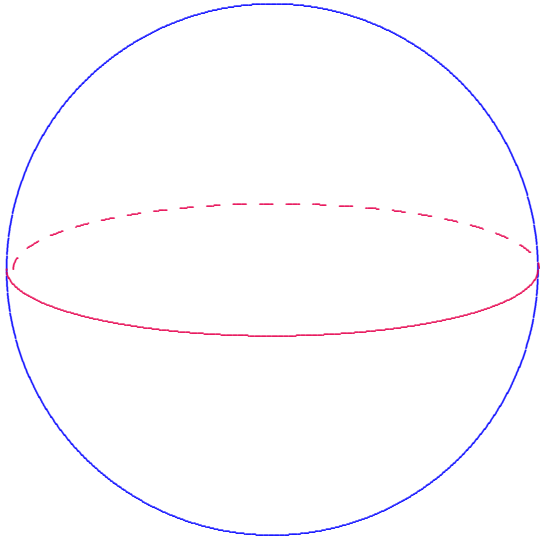


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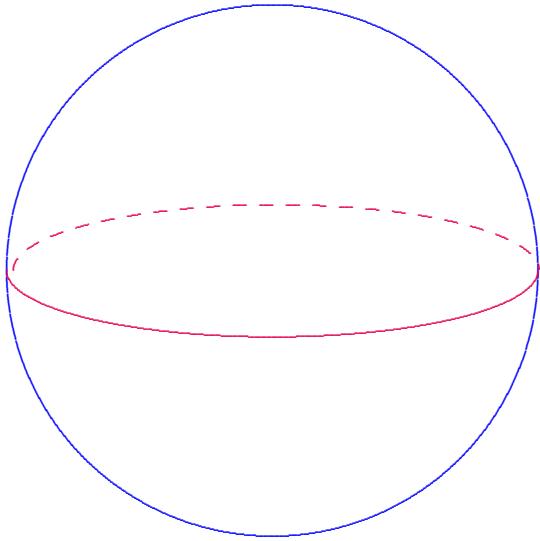




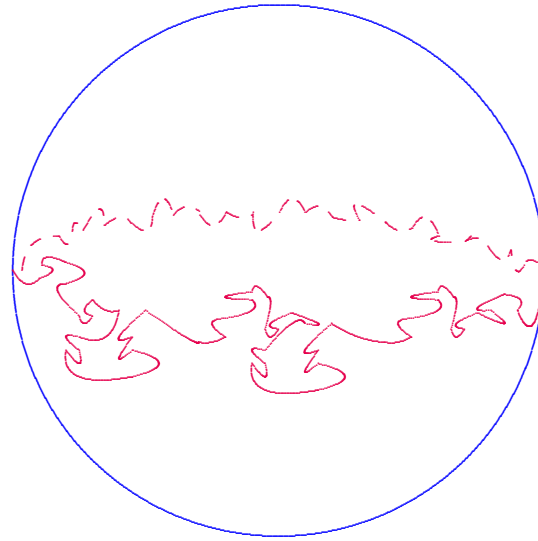
Fuchsian



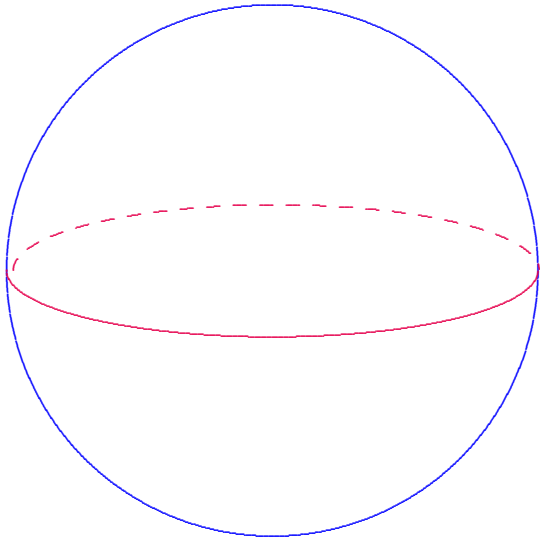
Fuchsian



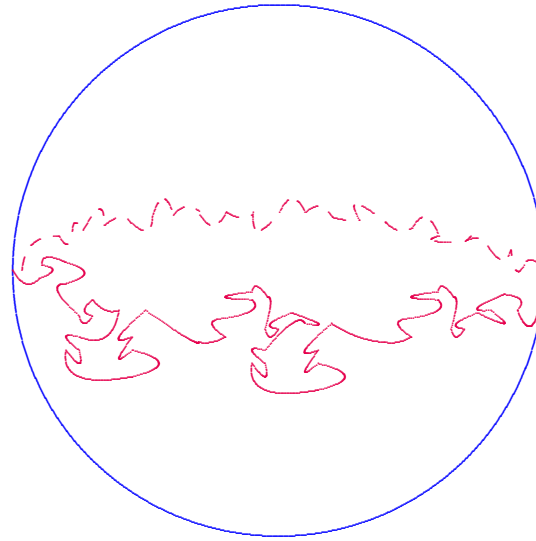
Fuchsian



quasi-Fuchsian

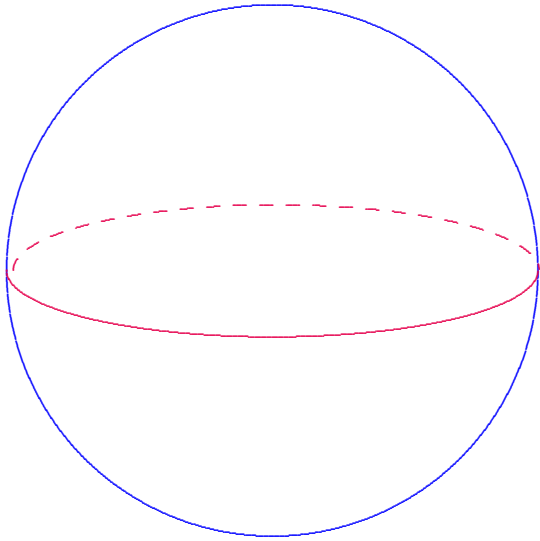


Fuchsian

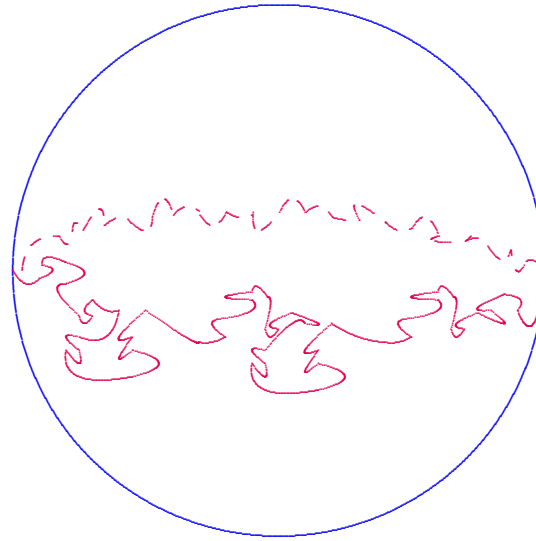


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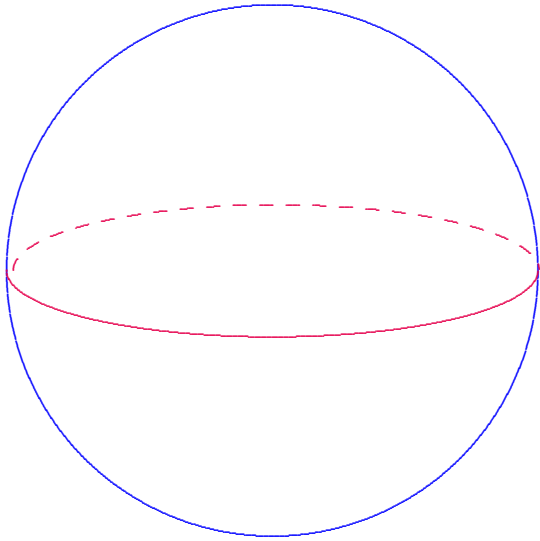


Fuchsian

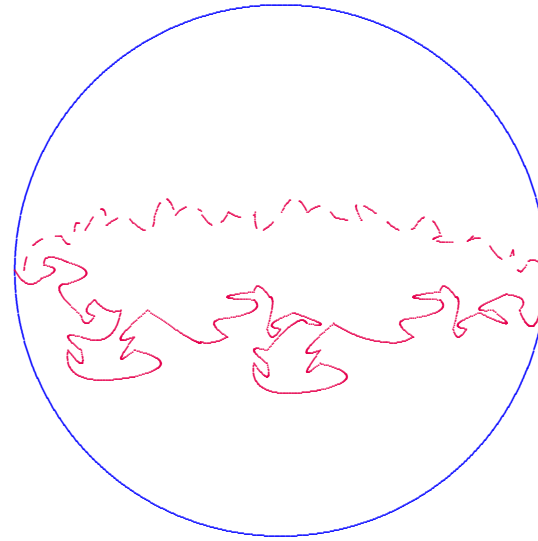


quasi-Fuchsian

$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{C}) \text{ quasi-Fuchsian group of Bers type}$$



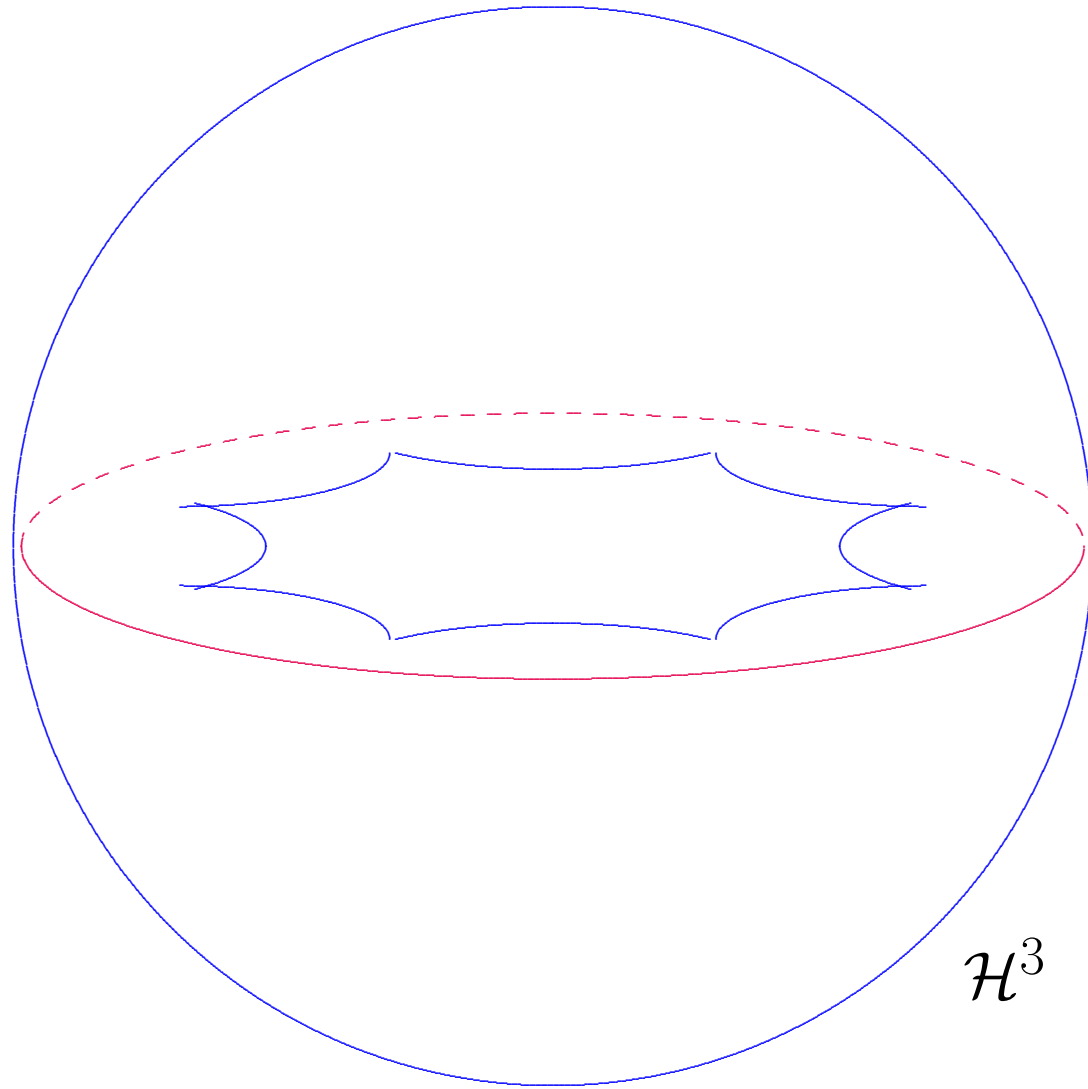
Fuchsian



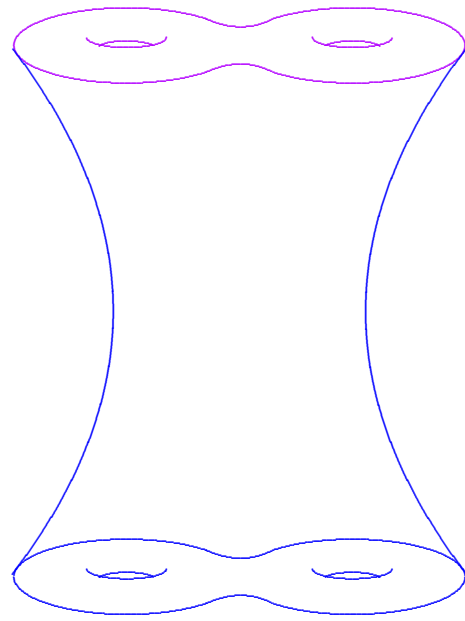
quasi-Fuchsian

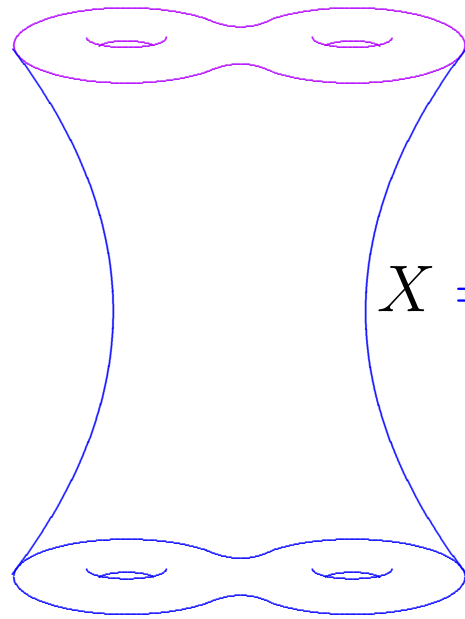
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Quasi-conformally conjugate to Fuchsian.

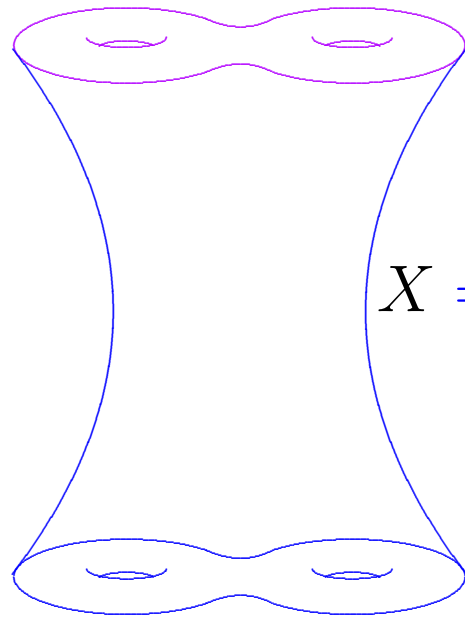


\mathcal{H}^3



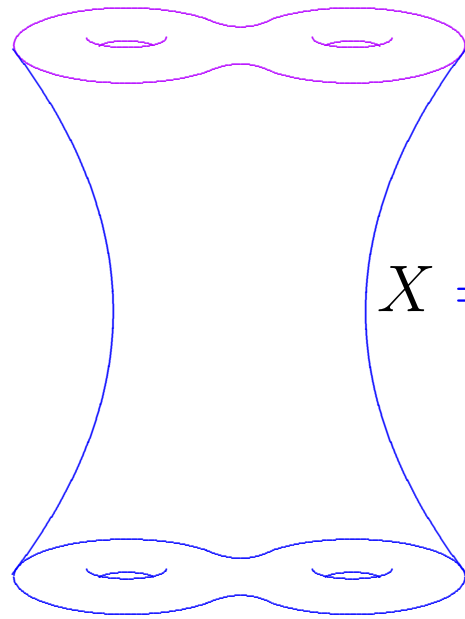


$$X = \mathcal{H}^3 / \Gamma$$



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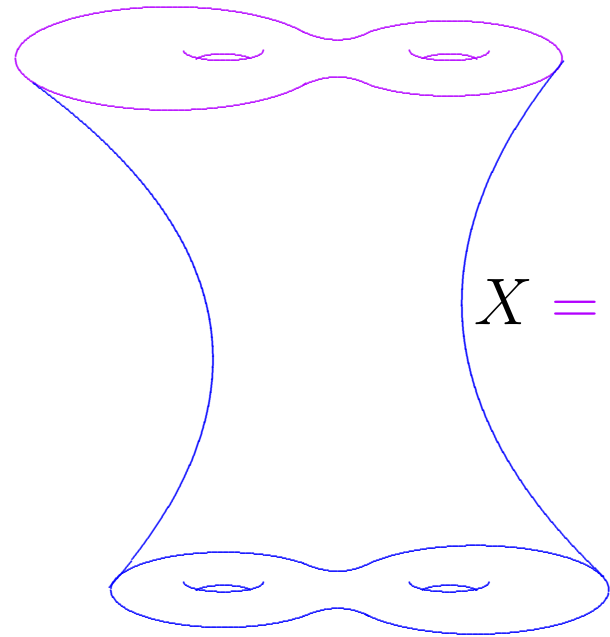
Γ Fuchsian



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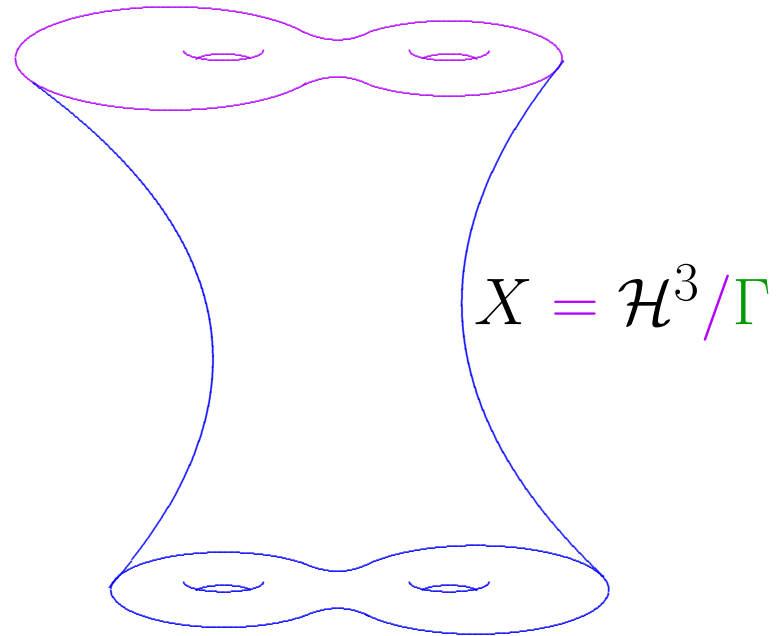
$$X \approx \Sigma \times \mathbb{R}$$



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Γ quasi-Fuchsian

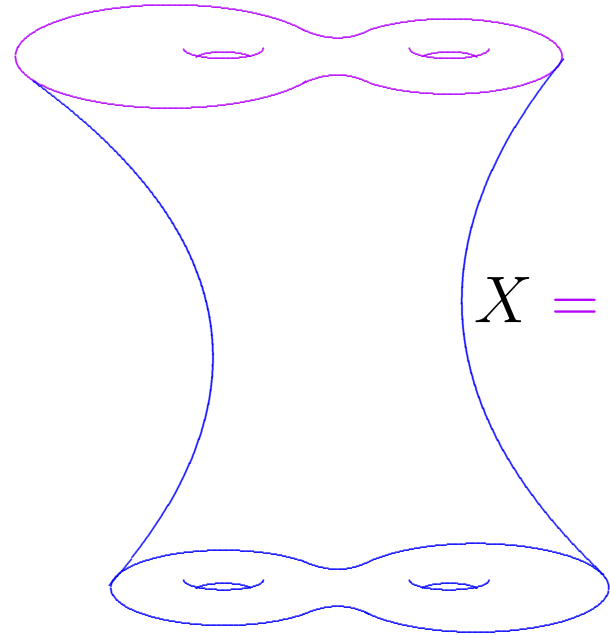
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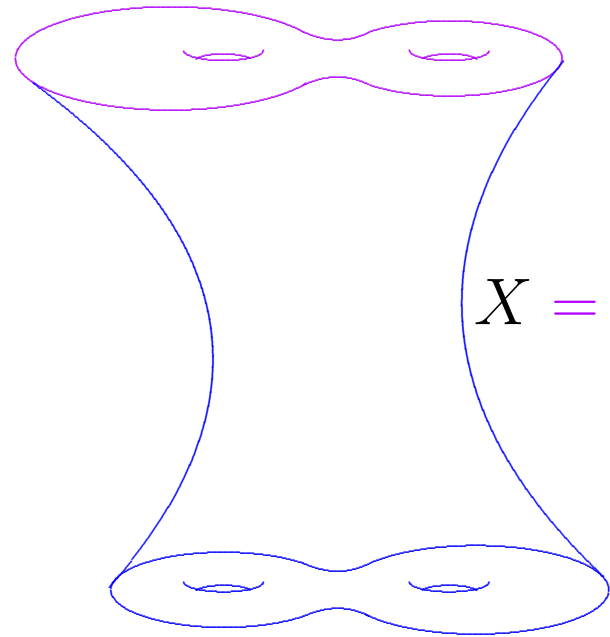
Freedom: two points in Teichmüller space.



$$X = \mathcal{H}^3 / \Gamma$$

Γ quasi-Fuchsian

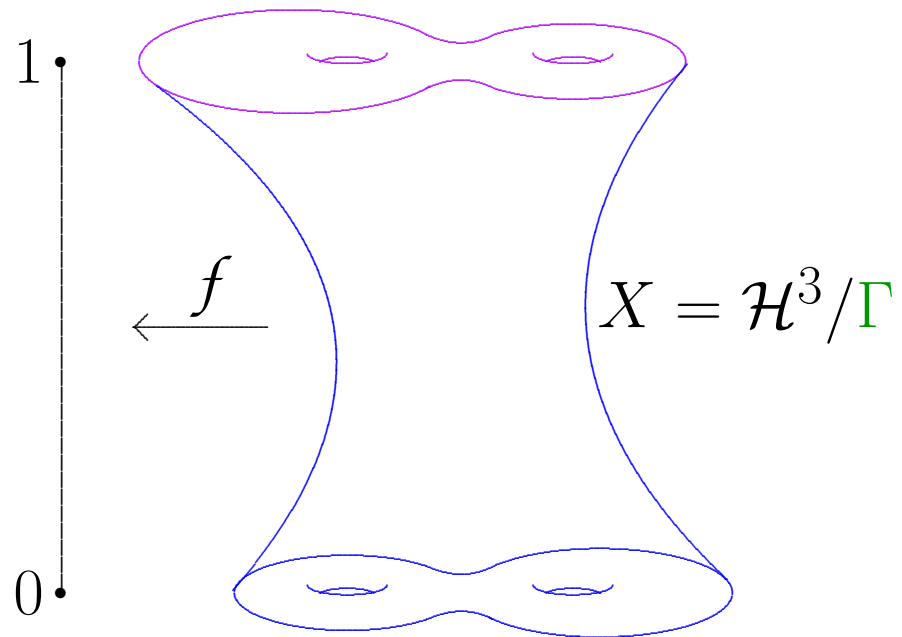
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$$X = \mathcal{H}^3 / \Gamma$$

Γ quasi-Fuchsian

$$\overline{X} \approx \Sigma \times [0, 1]$$

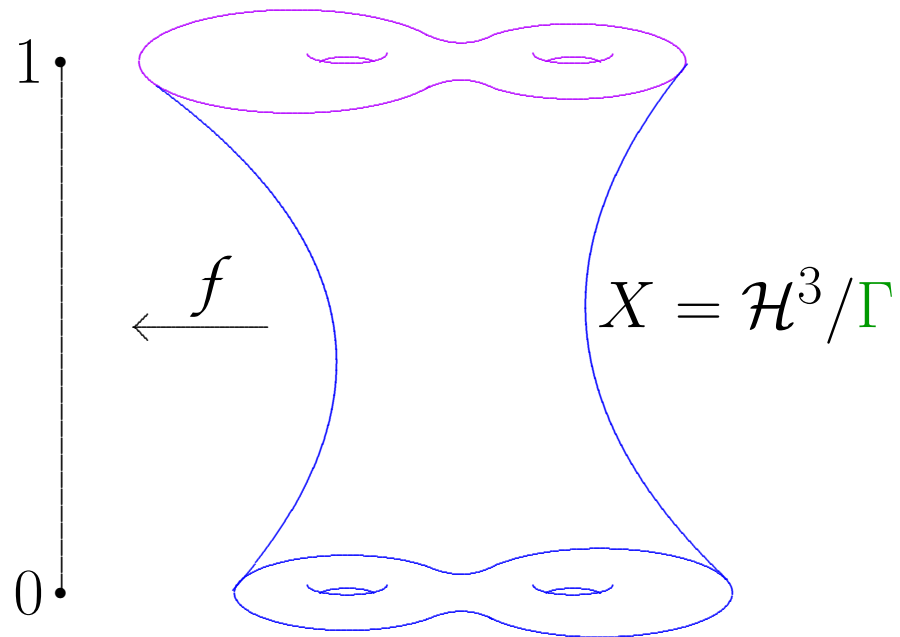


Γ quasi-Fuchsian

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Tunnel-Vision function:

$$f : \overline{X} \rightarrow [0, 1]$$



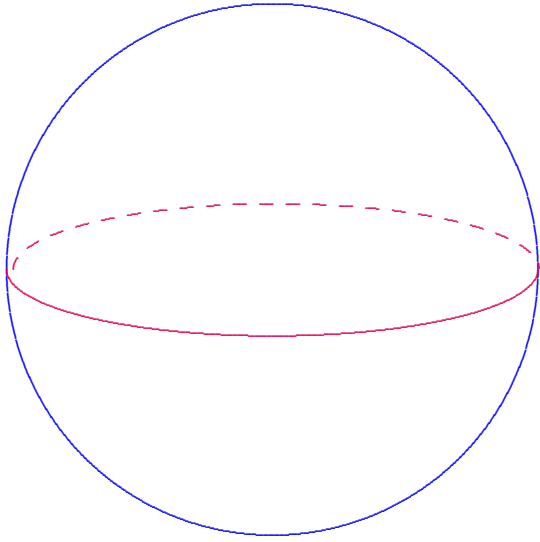
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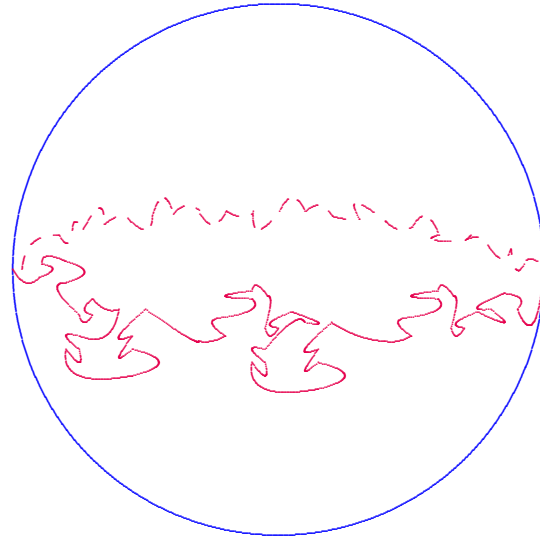
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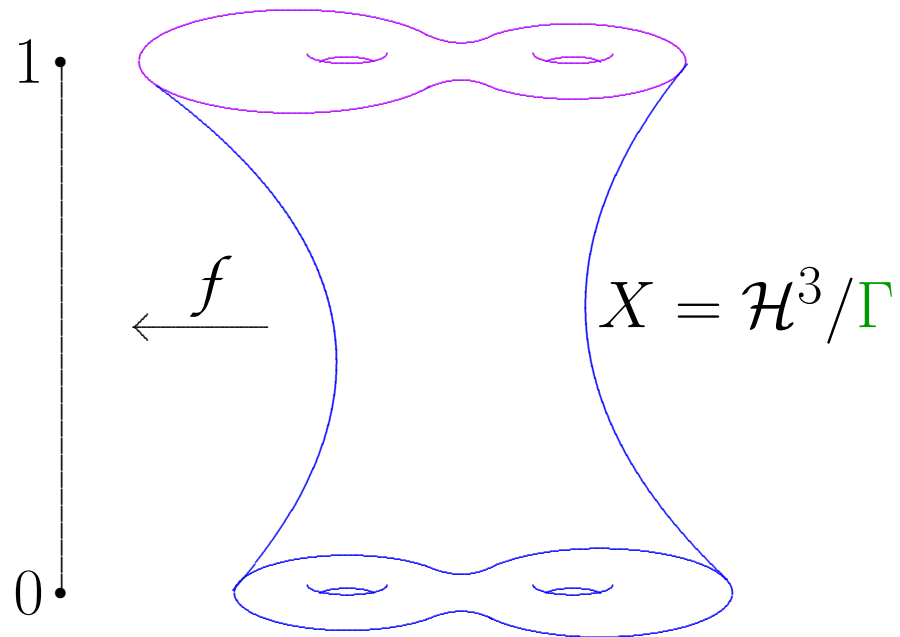
$$\Delta f = 0$$



Fuchsian



quasi-Fuchsian



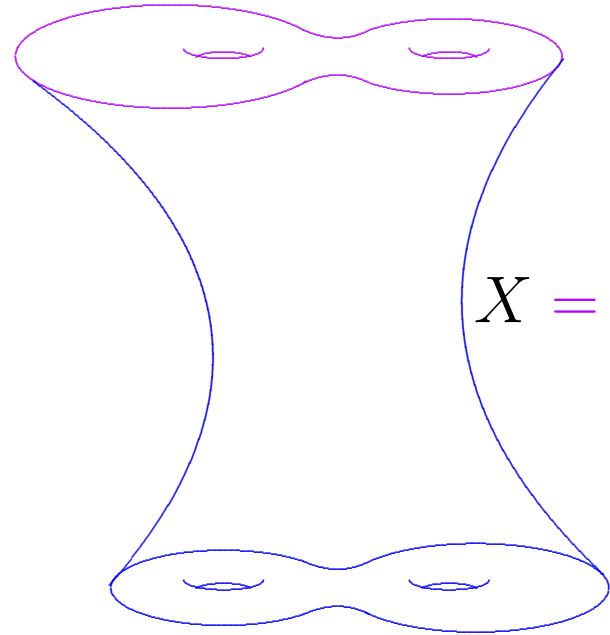
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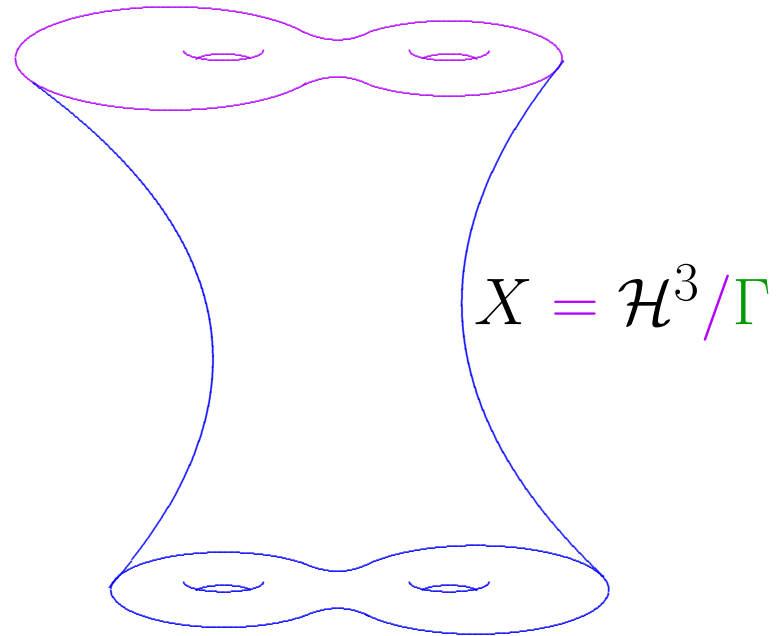
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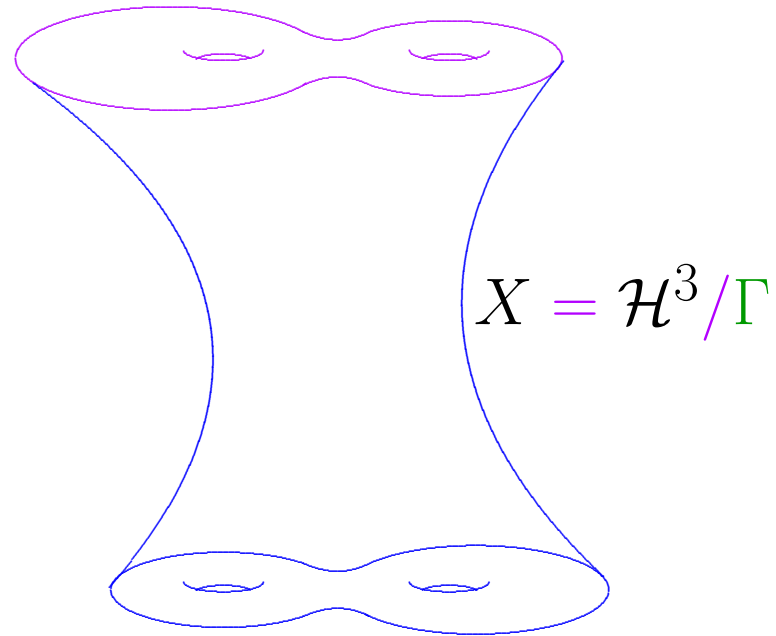
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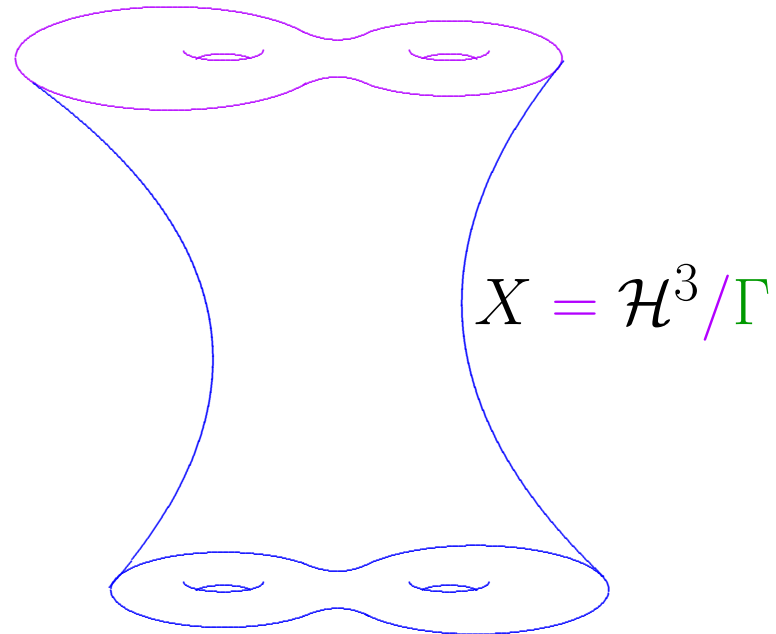


Construction of conformally flat 4-manifolds:



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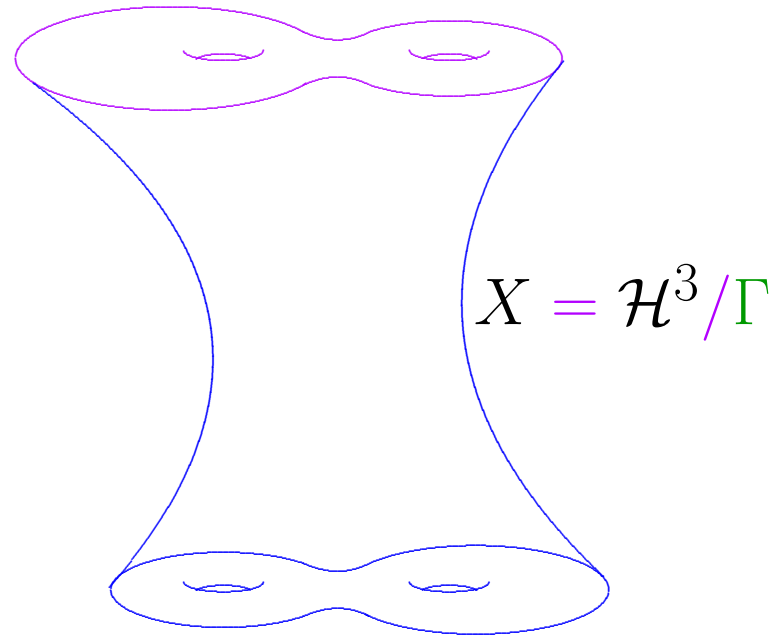
$$M = [\bar{X} \times S^1] / \sim$$



Construction of conformally flat 4-manifolds:

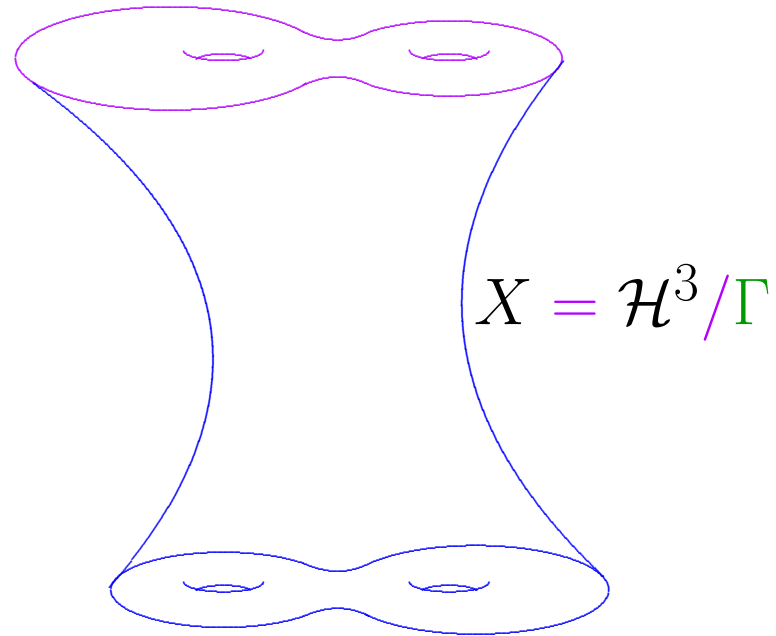
$$M = [\bar{X} \times S^1] / \sim$$

\sim : crush $\partial\bar{X} \times S^1$ to $\partial\bar{X}$.



Construction of conformally flat 4-manifolds:

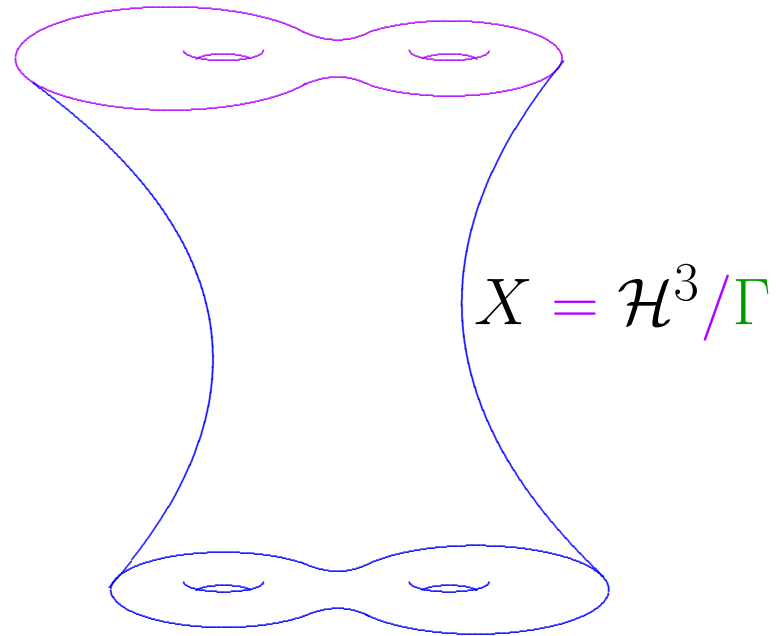
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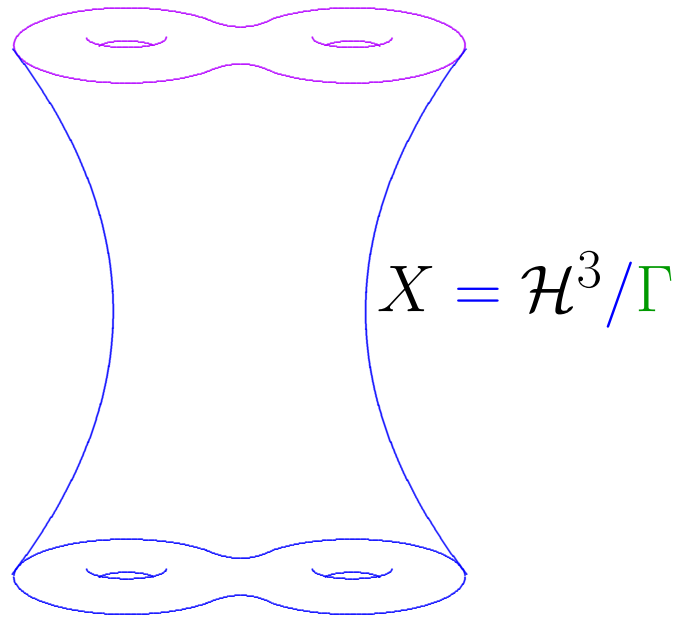
$$g = h + dt^2$$



Construction of conformally flat 4-manifolds:

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$$g = f(1 - f)[h + dt^2]$$

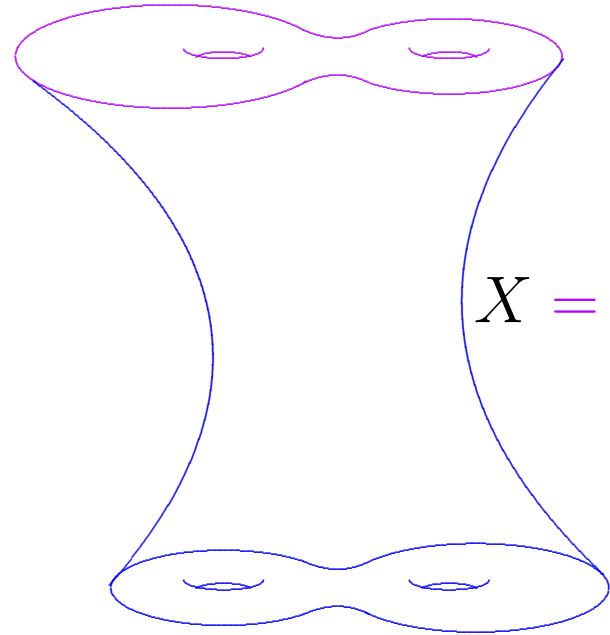


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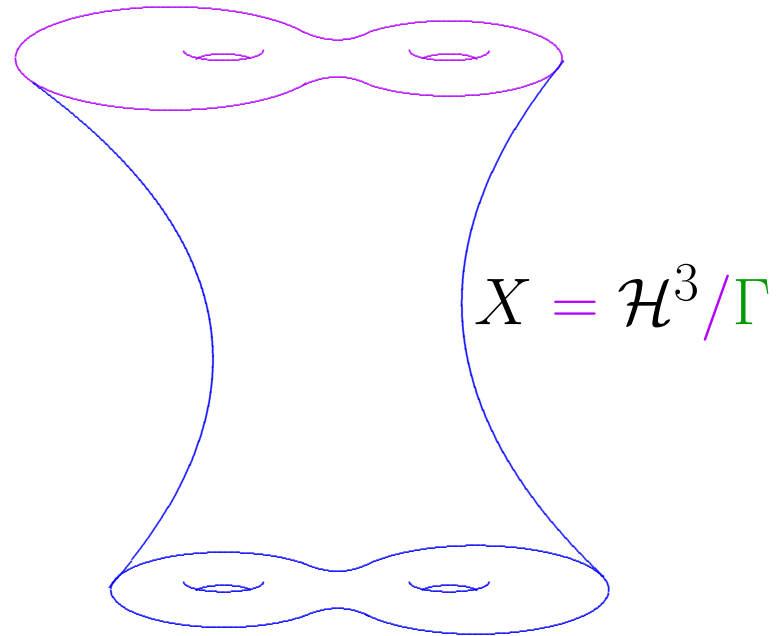
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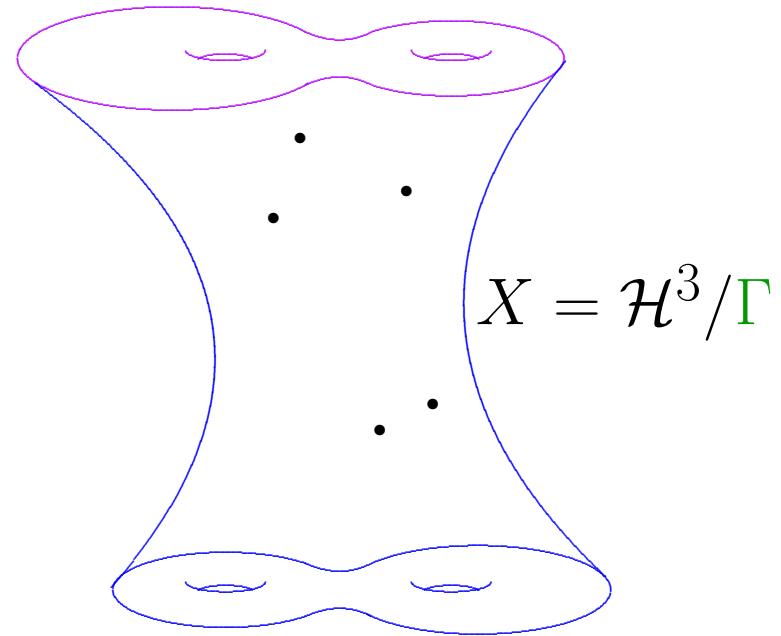
Fuchsian case: $\Sigma \times S^2$ scalar-flat Kähler.



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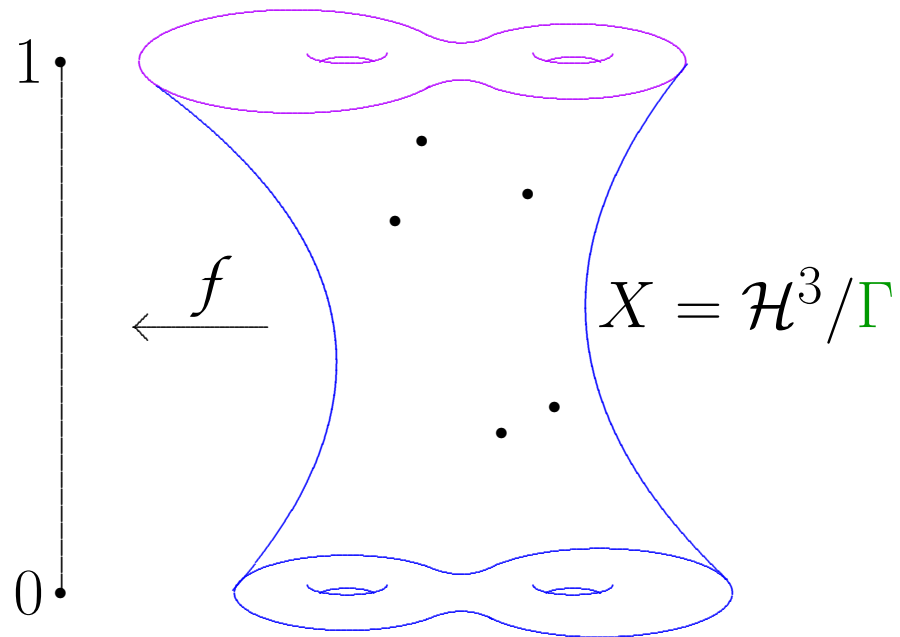


Construction of ASD 4-manifolds:



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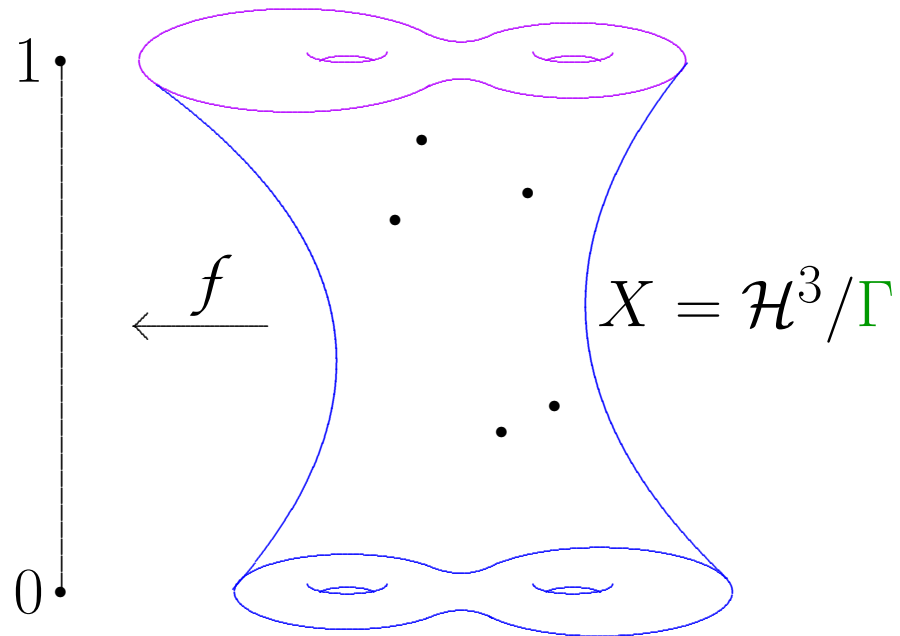
Choose k points $p_1, \dots, p_k \in X$



Construction of ASD 4-manifolds:

Choose k points $p_1, \dots, p_k \in X$

satisfying $\sum_{j=1}^k f(p_j) \in \mathbb{Z}$.

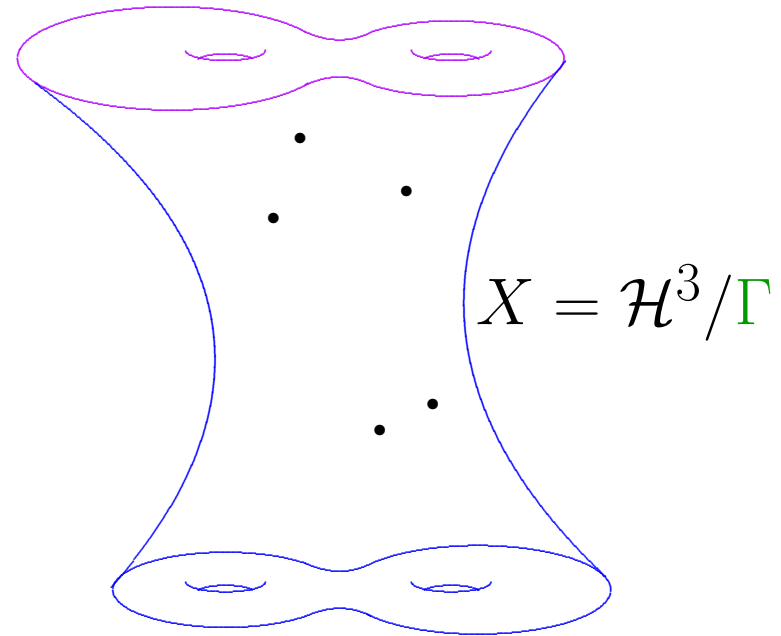


Construction of ASD 4-manifolds:

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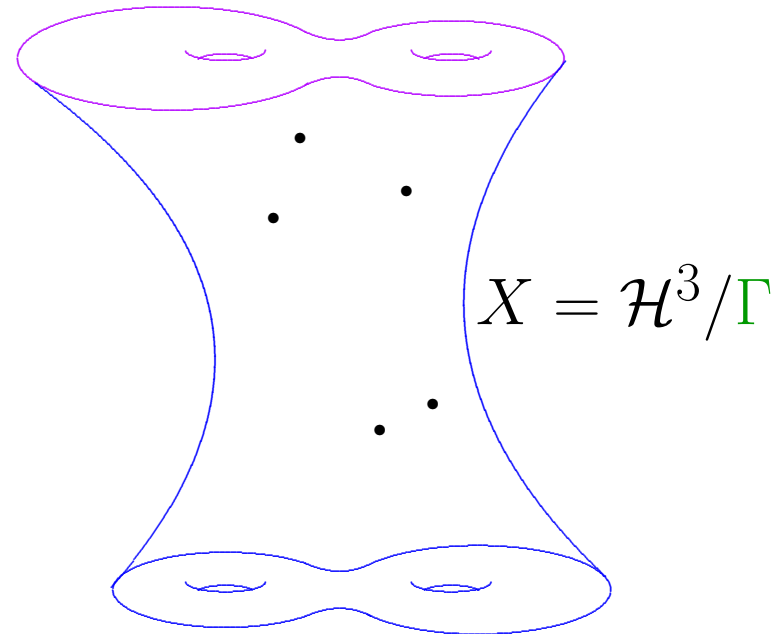
satisfying $\sum_{j=1}^k f(p_j) \in \mathbb{Z}$.

Can do if $k \neq 1$.



Construction of ASD 4-manifolds:

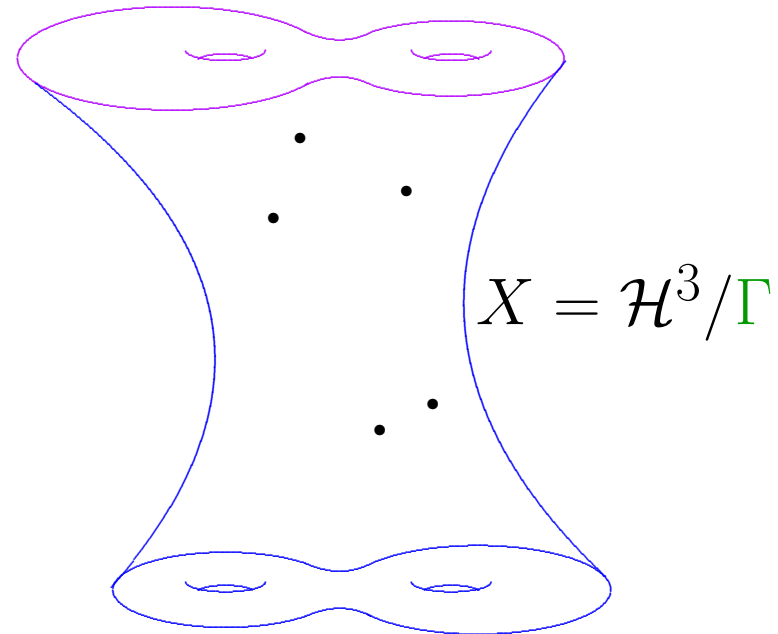
Let G_j be the Green's function of p_j :



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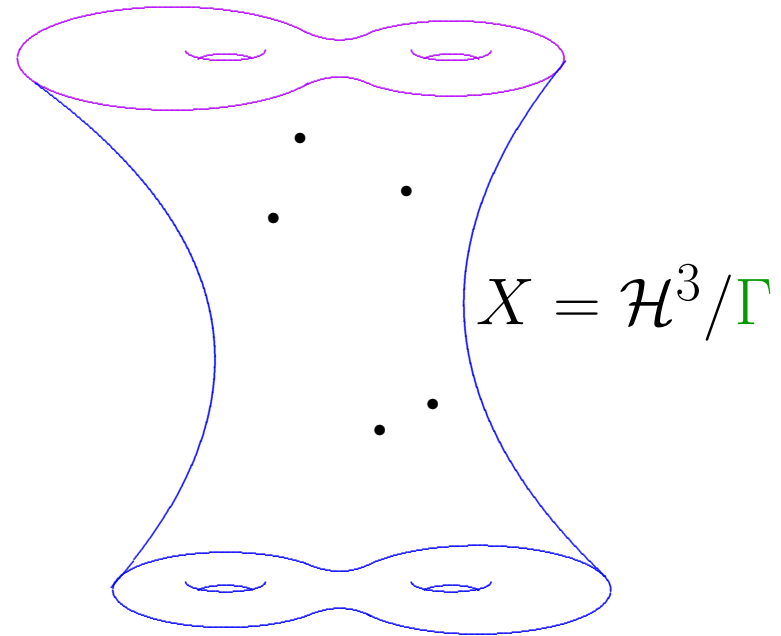
$$\Delta G_j = 2\pi\delta_{p_j}, \quad G_j \rightarrow 0 \text{ at } \partial\bar{X}$$



Construction of ASD 4-manifolds:

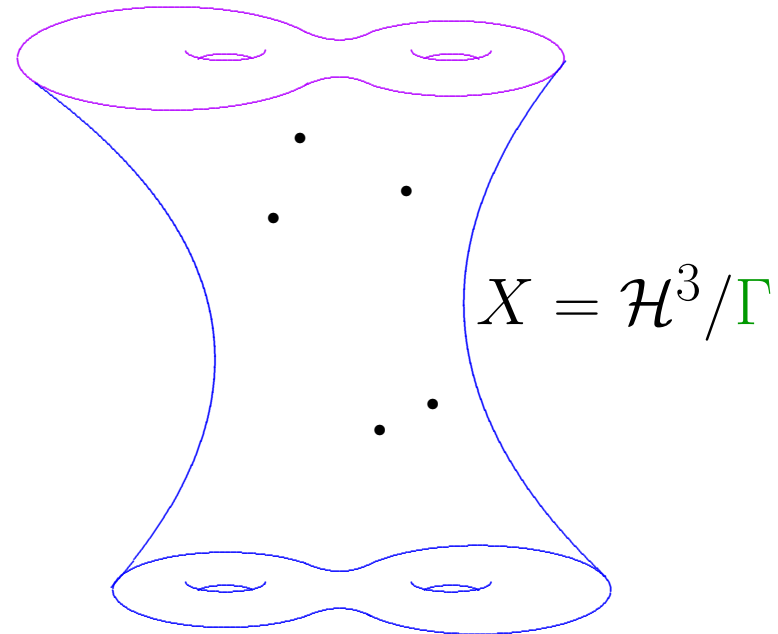
Let G_j be the Green's function of p_j , and set

$$V = 1 + \sum_{j=1}^k G_j.$$



Construction of ASD 4-manifolds:

$$V = 1 + \sum_{j=1}^k G_j.$$

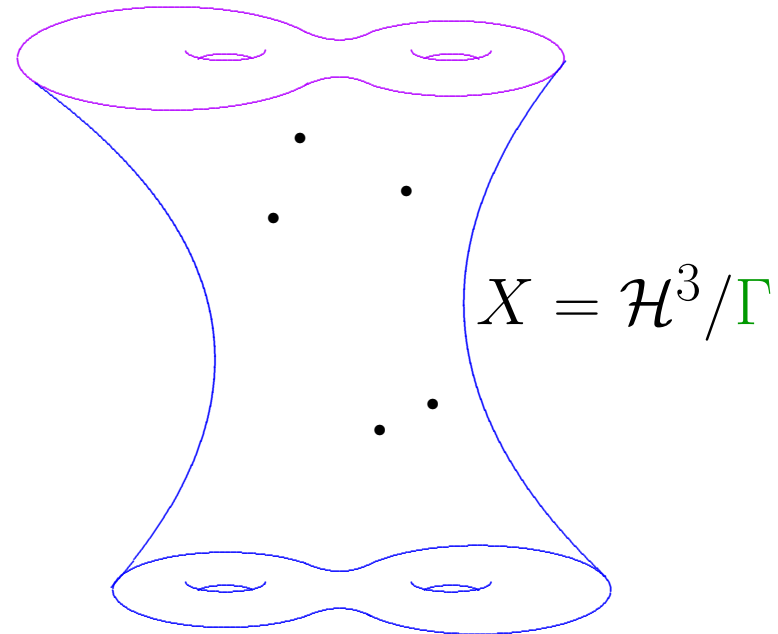


Construction of ASD 4-manifolds:

$$V = 1 + \sum_{j=1}^k G_j.$$

Choose $P \rightarrow (X - \{p_1, \dots, p_k\})$ circle bundle with connection form θ such that

$$d\theta = \star dV.$$

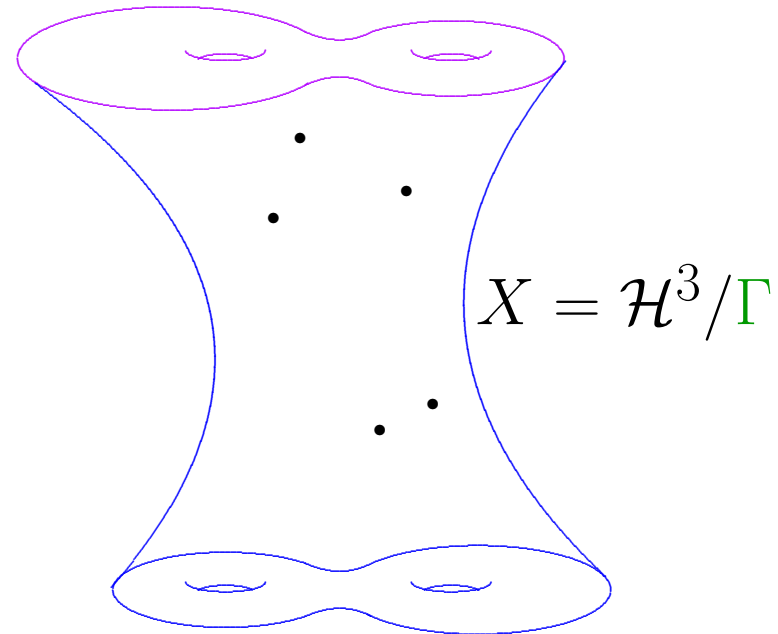


Construction of ASD 4-manifolds:

$$g = Vh + V^{-1}\theta^2$$

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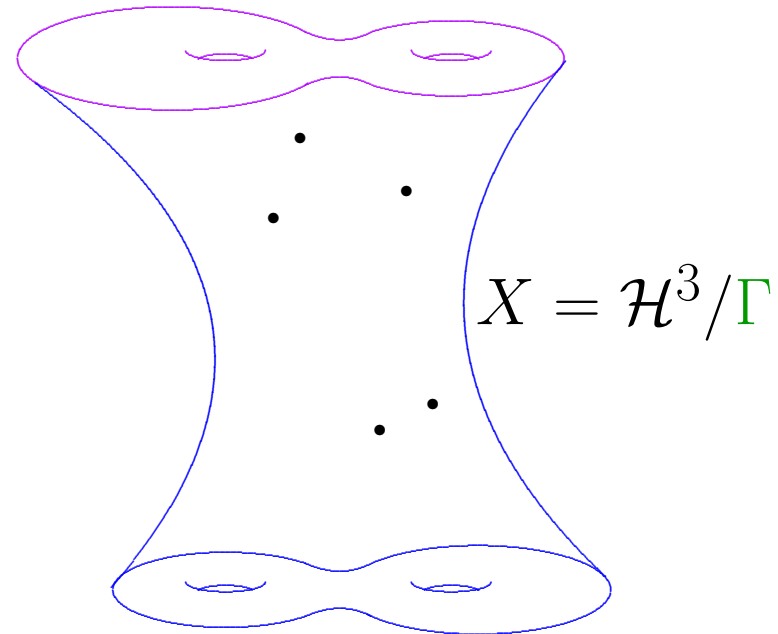


Construction of ASD 4-manifolds:

$$g = f(1 - f)[Vh + V^{-1}\theta^2]$$

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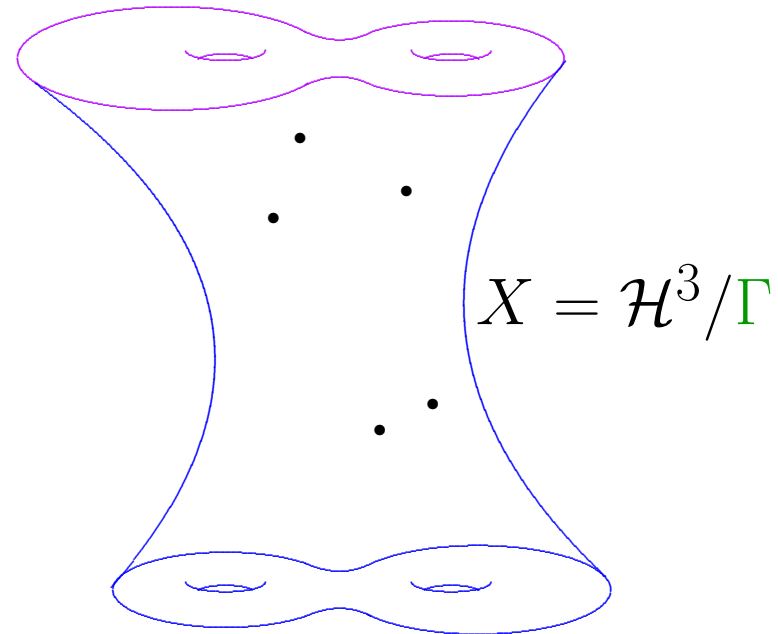
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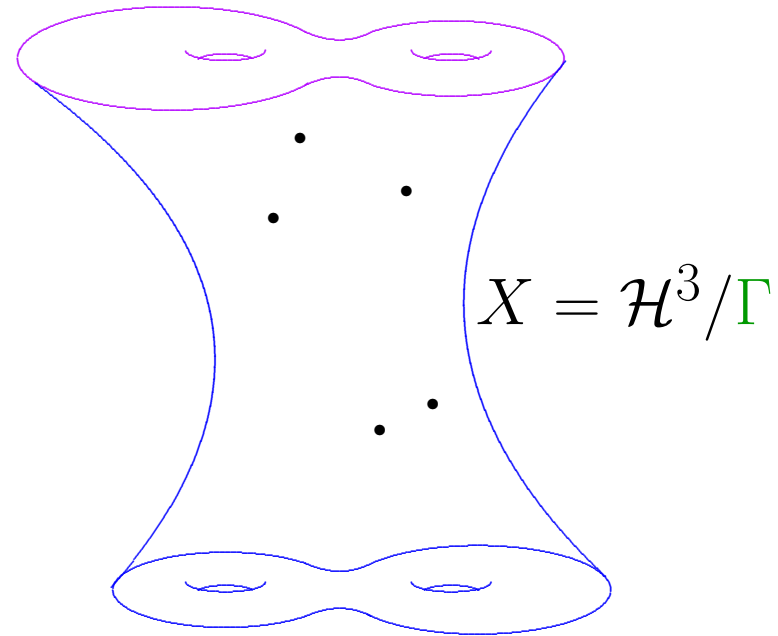
$$M = P \cup \{\hat{p}_1, \dots, \hat{p}_k\} \cup \partial\bar{X}$$



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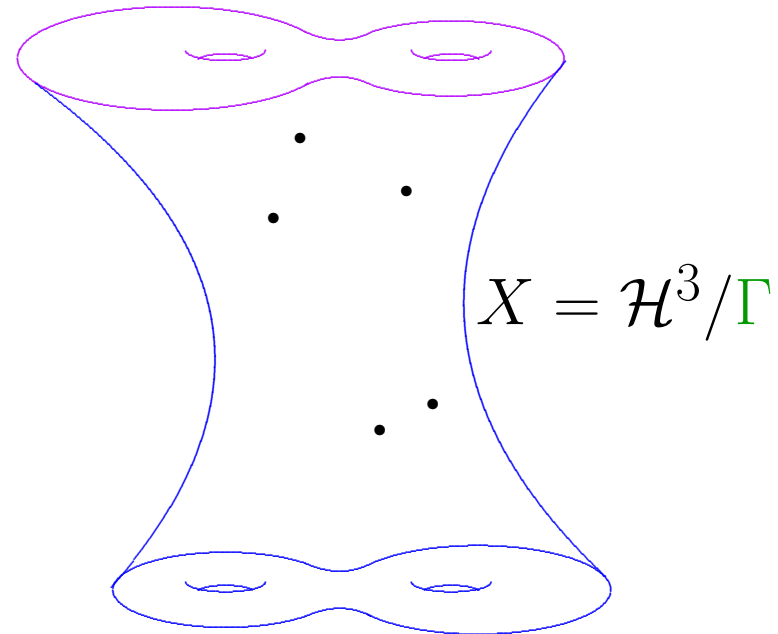
$$\begin{array}{ccccccc}
 M & = & P & \cup & \{\hat{p}_1, \dots, \hat{p}_k\} & \cup & \partial \bar{X} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \bar{X} & = & X - \{p_1, \dots, p_k\} & \cup & \{p_1, \dots, p_k\} & \cup & \partial \bar{X}
 \end{array}$$



Construction of ASD 4-manifolds:

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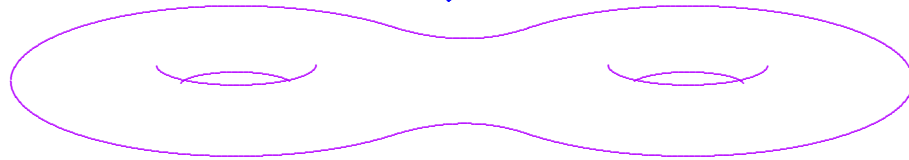
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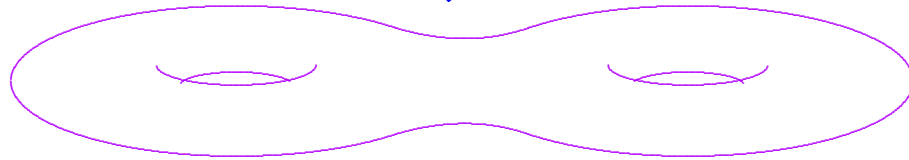
$$\approx (\Sigma \times S^2) \# k\bar{\mathbb{C}P}_2$$

$$\overline{X} = \Sigma \times [0, 1]$$



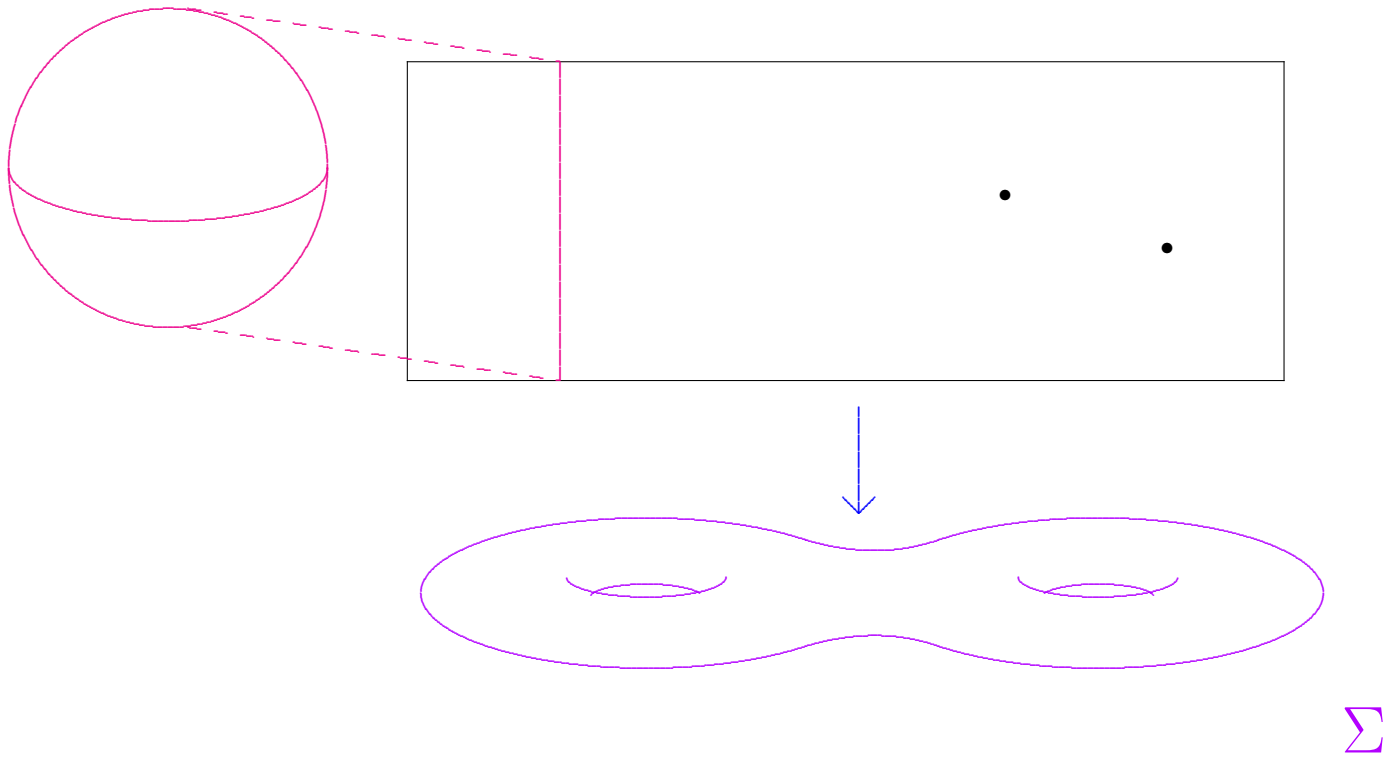
Σ

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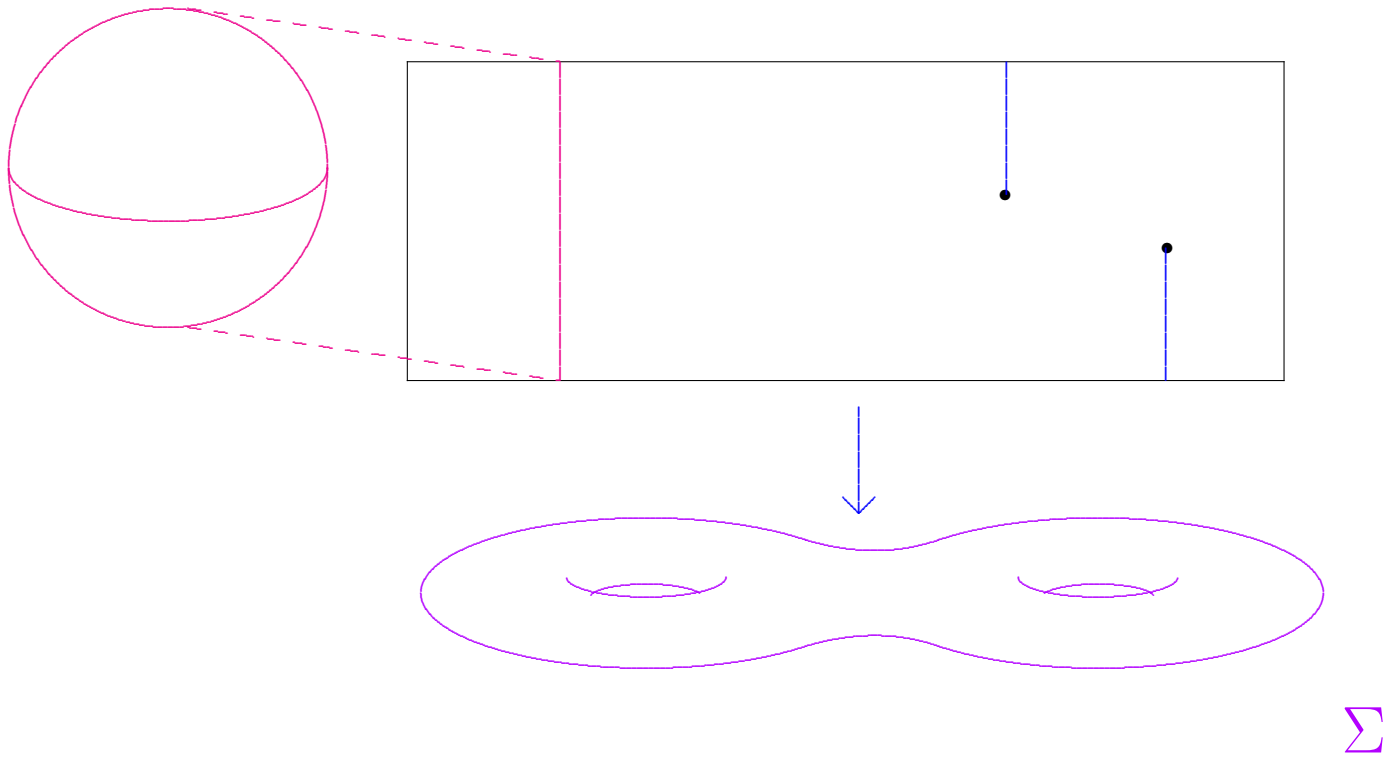


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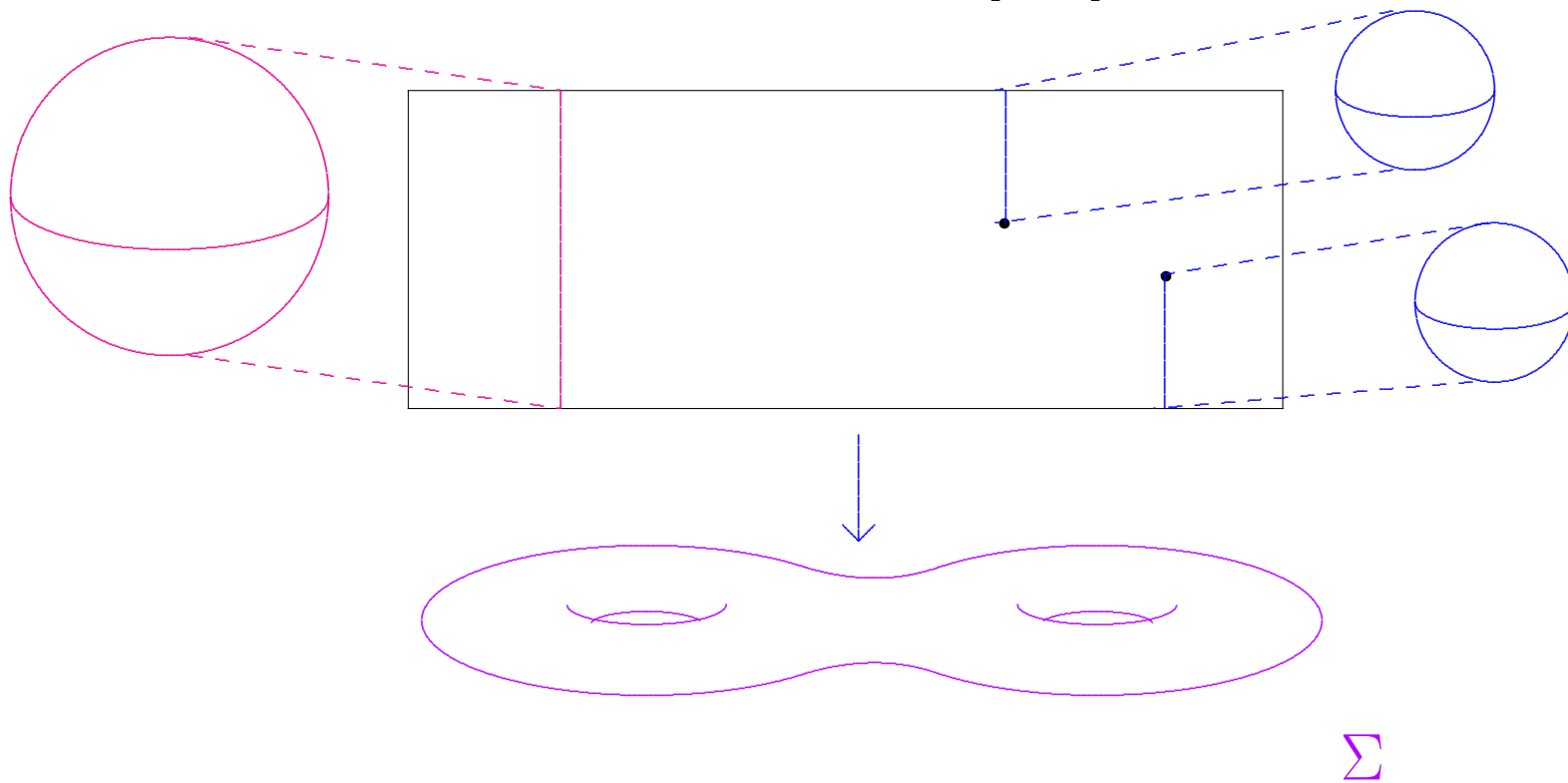
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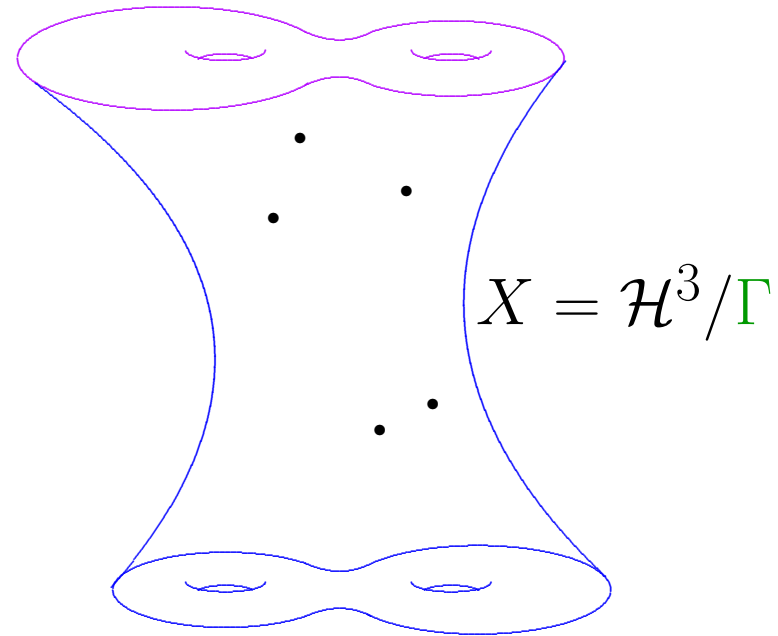


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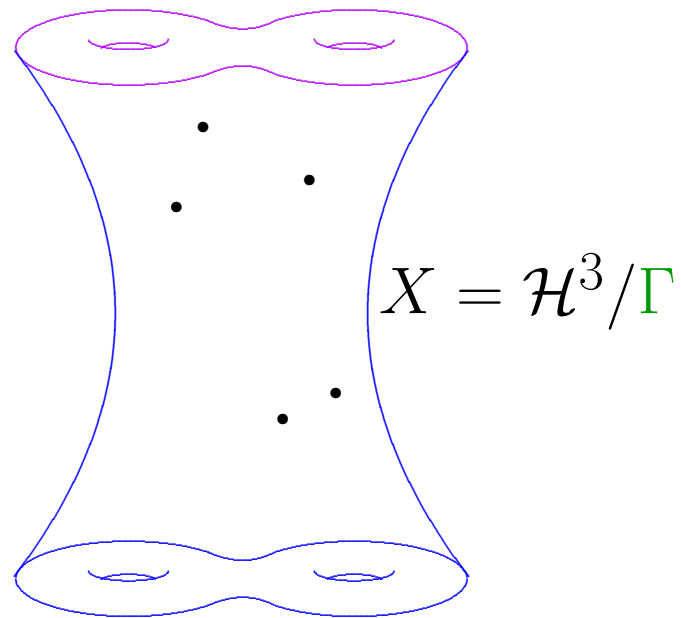


Construction of ASD 4-manifolds:

$$g = f(1 - f)[Vh + V^{-1}\theta^2]$$

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$$\approx (\Sigma \times S^2) \# k\overline{\mathbb{C}\mathbb{P}_2}$$

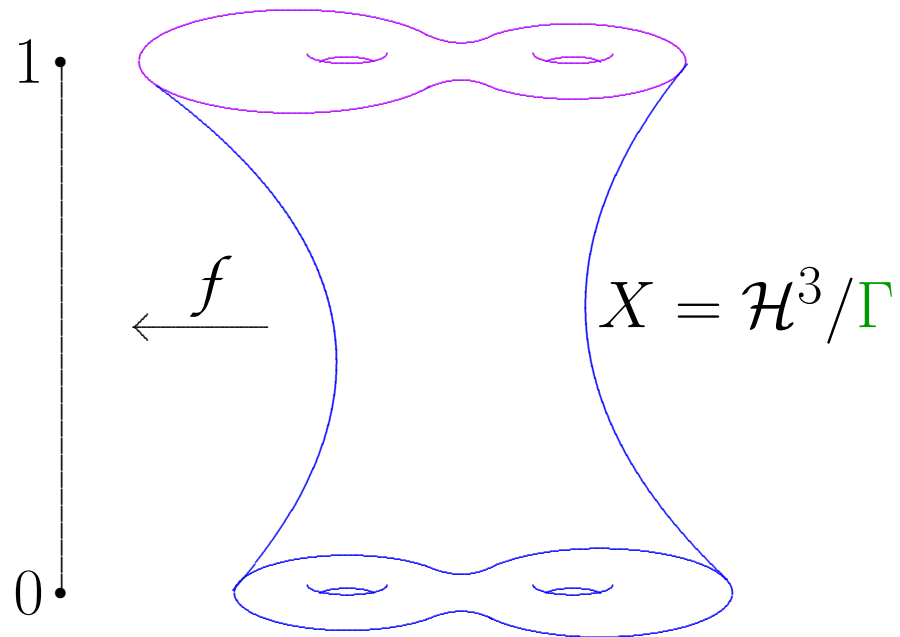


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Fuchsian case: $(\Sigma \times S^2) \# k\overline{\mathbb{C}\mathbb{P}}_2$ scalar-flat Kähler



Γ quasi-Fuchsian

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Tunnel-Vision function:

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$$\Delta f = 0$$

Theorem.

Theorem. *Let $(M, [g])$ be ASD manifold*

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$$b_+[(\Sigma \times S^2) \# k \overline{\mathbb{C}P}_2] = 1.$$

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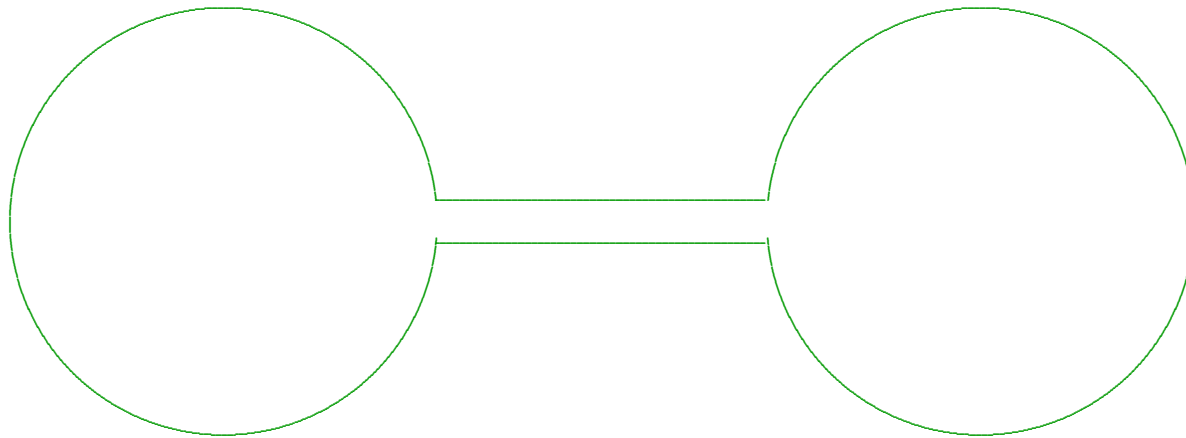
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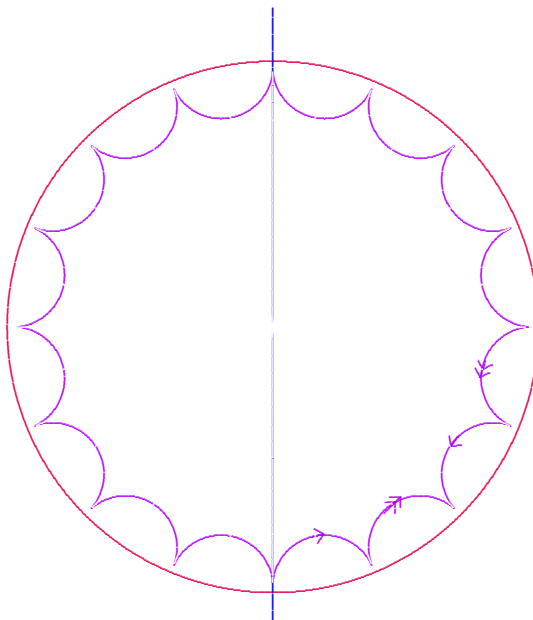
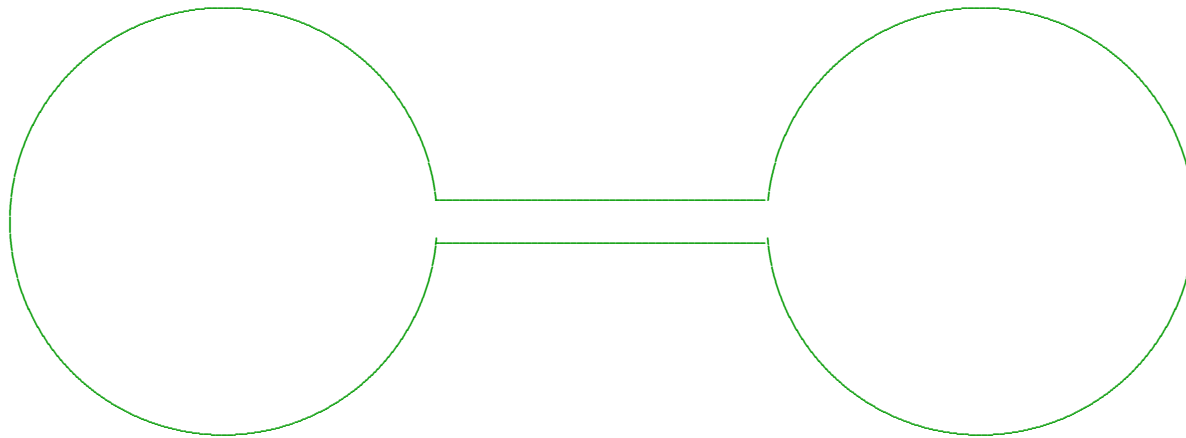
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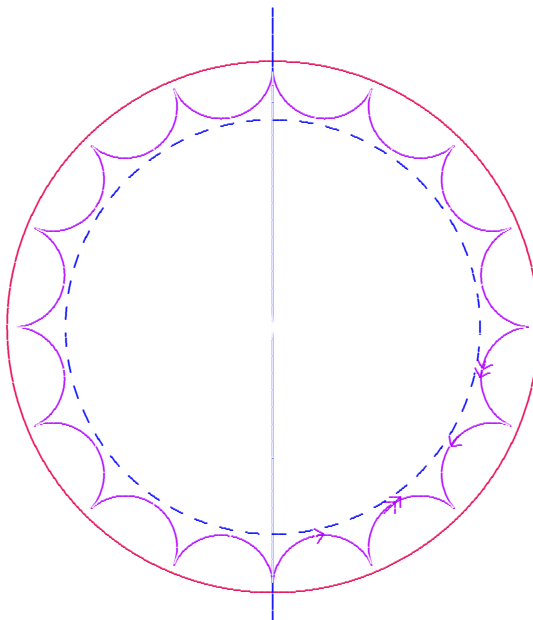
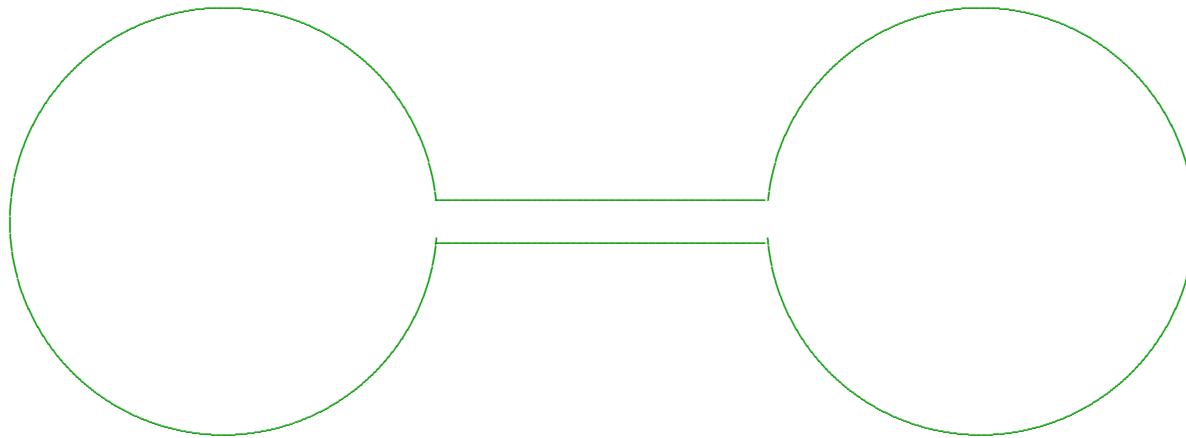
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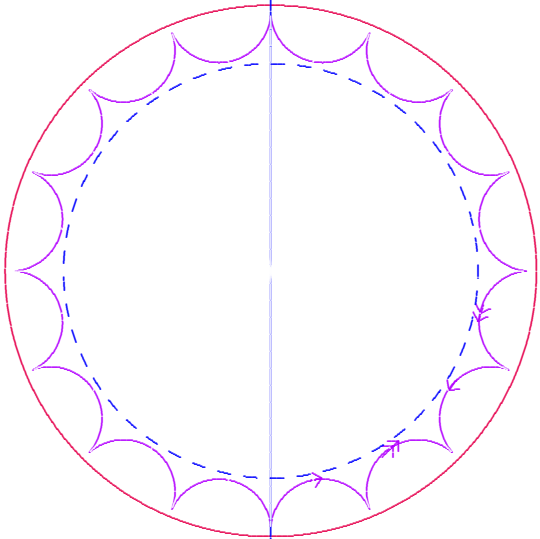
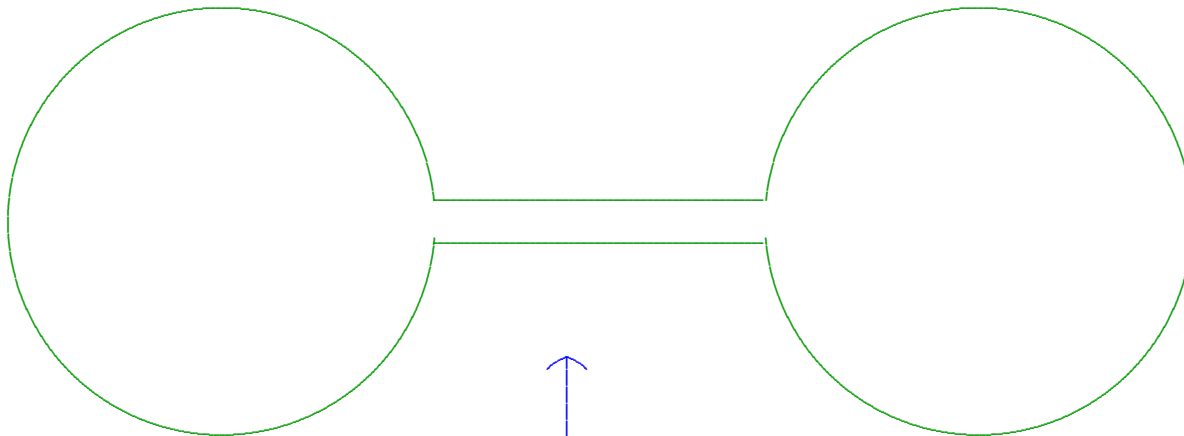
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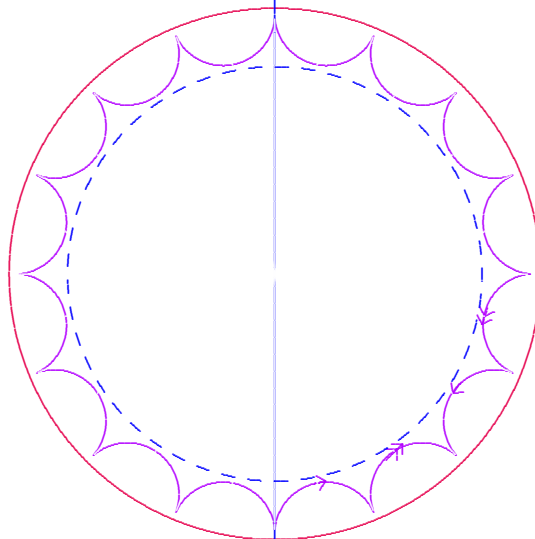
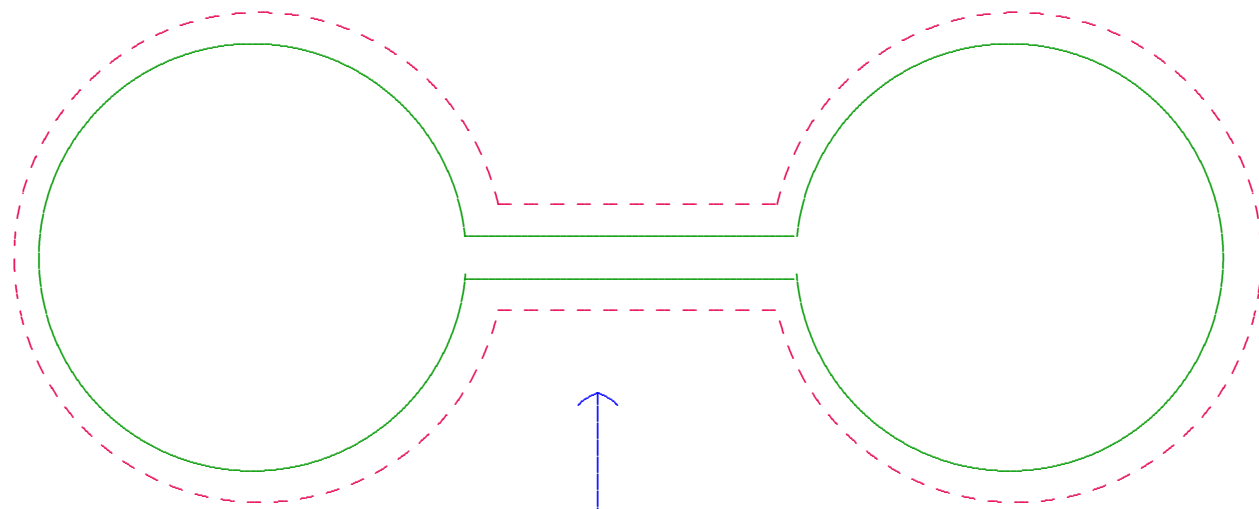
Ahlfors-Bers: Quasi-conformal mappings

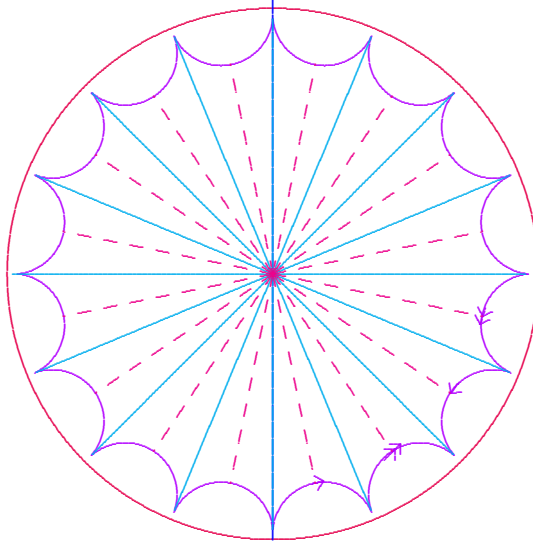
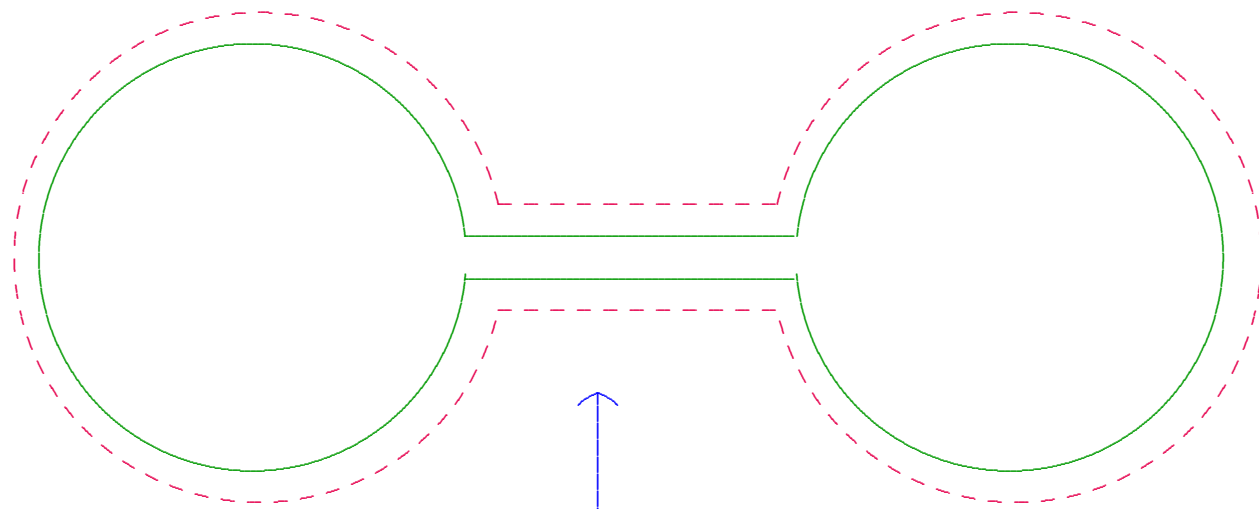


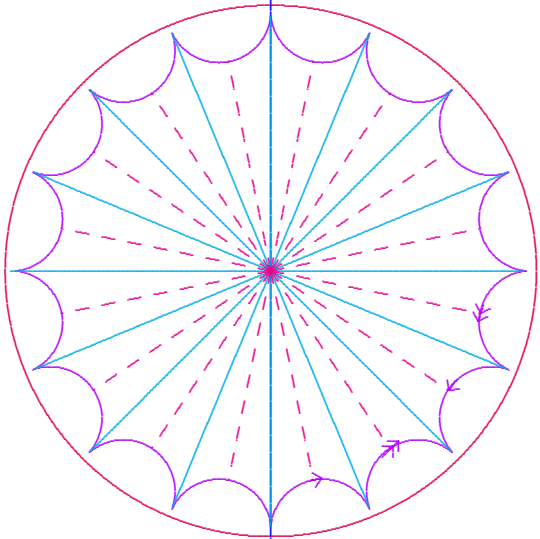
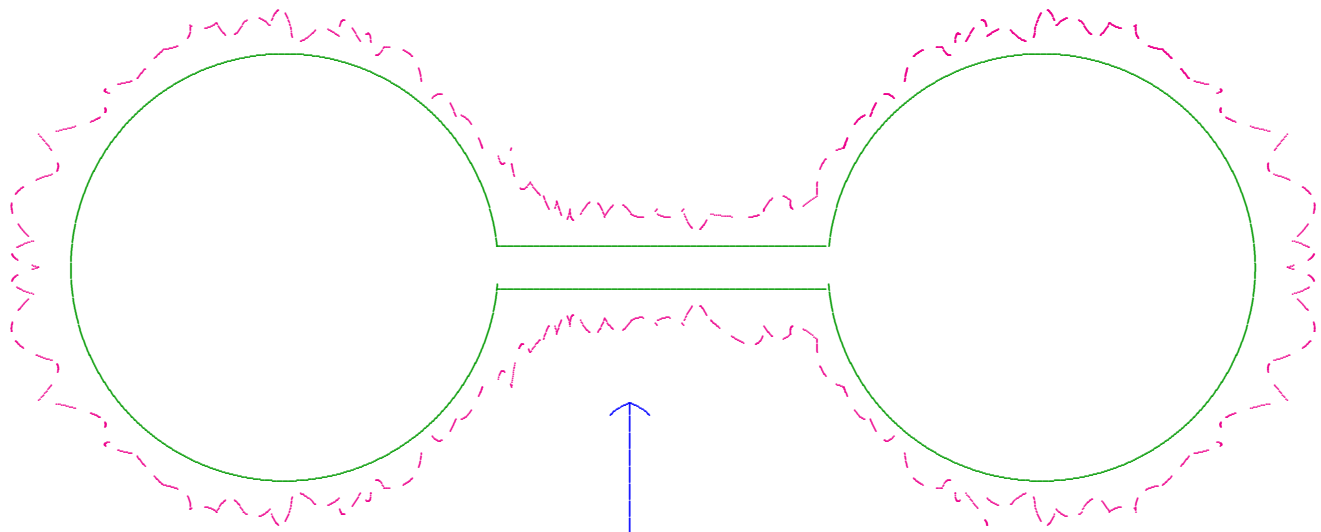


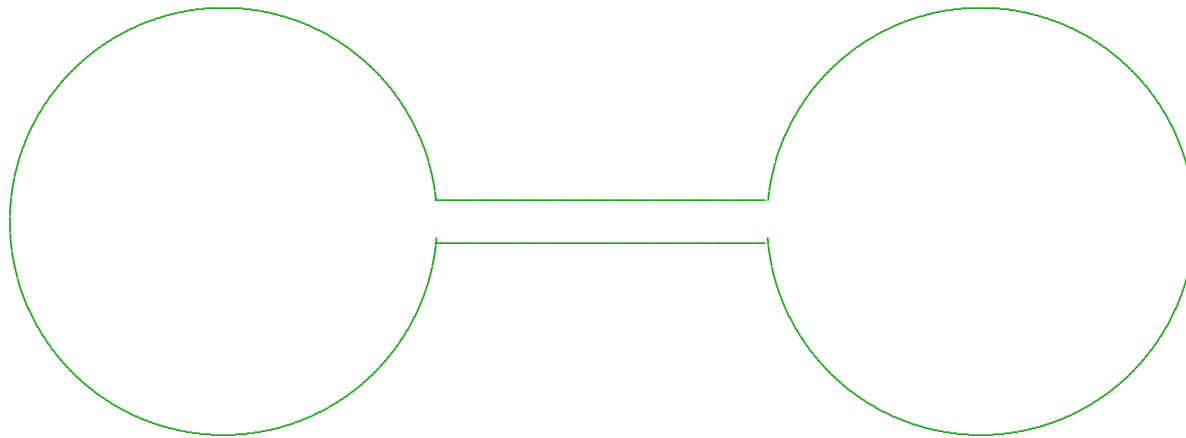


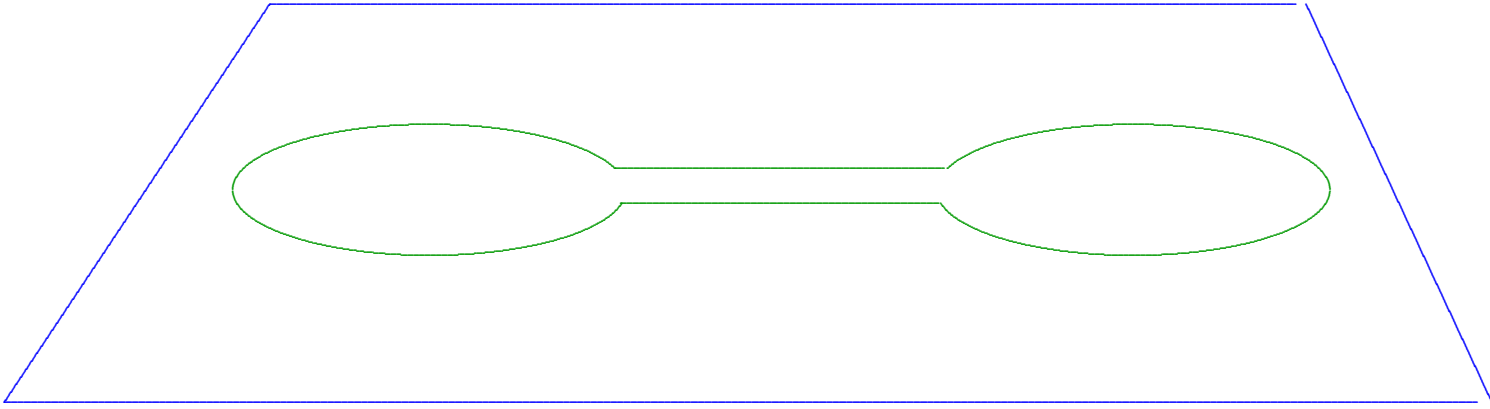


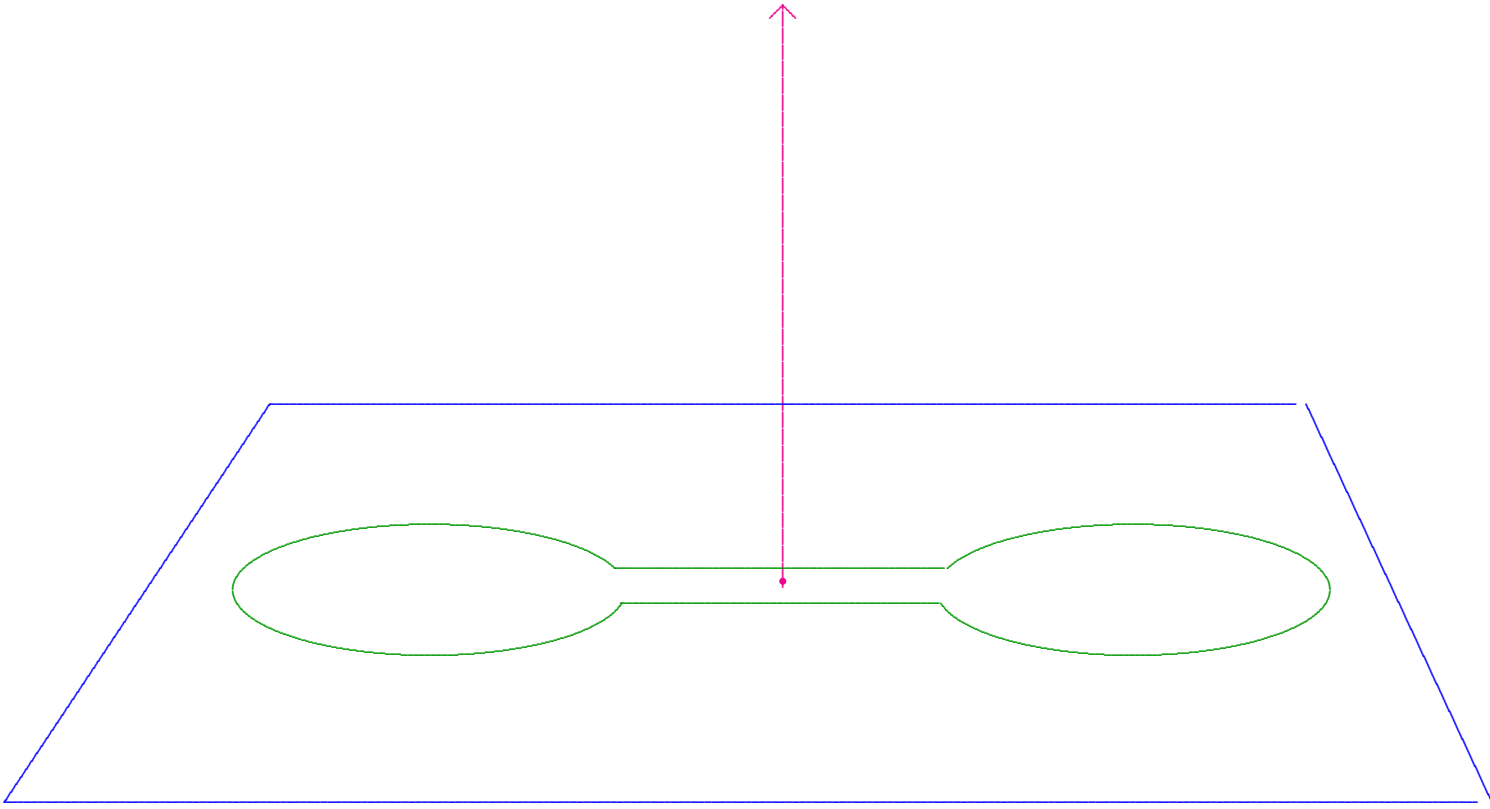


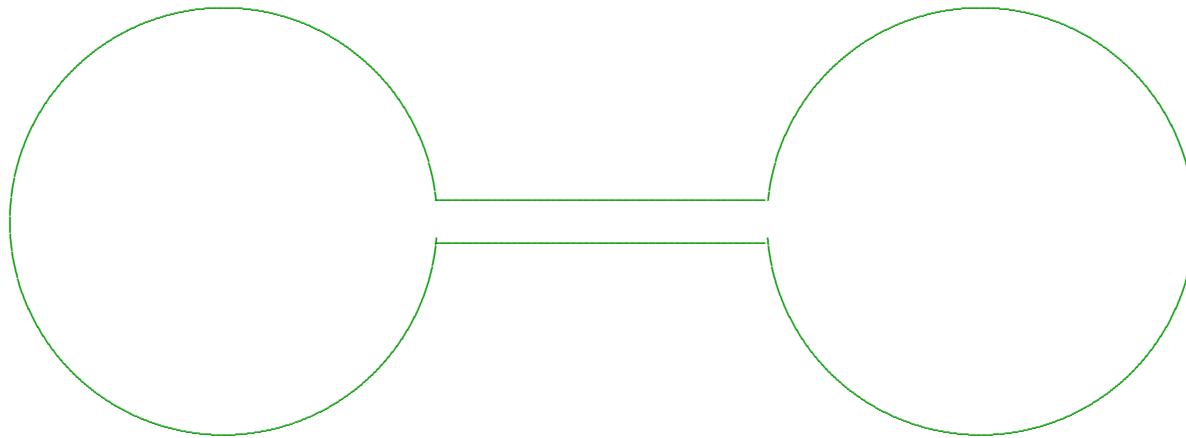


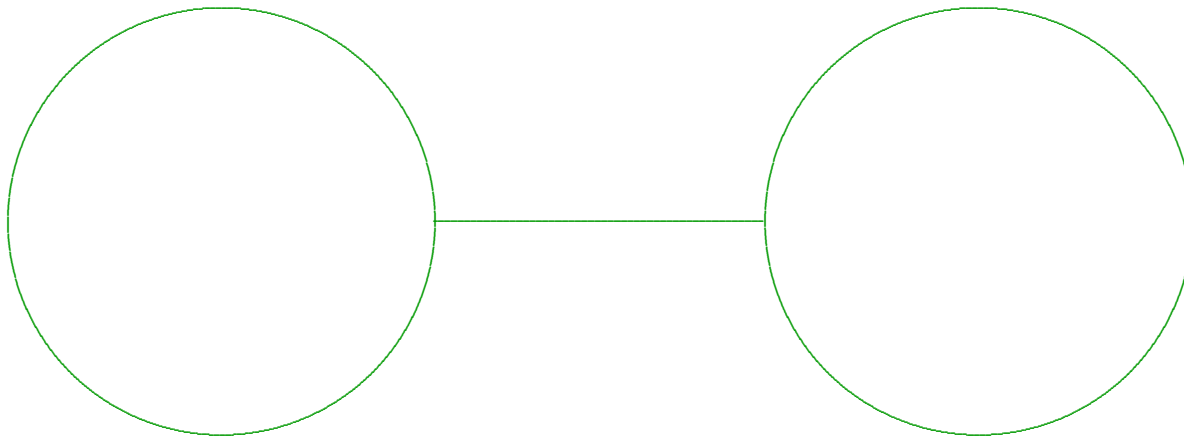


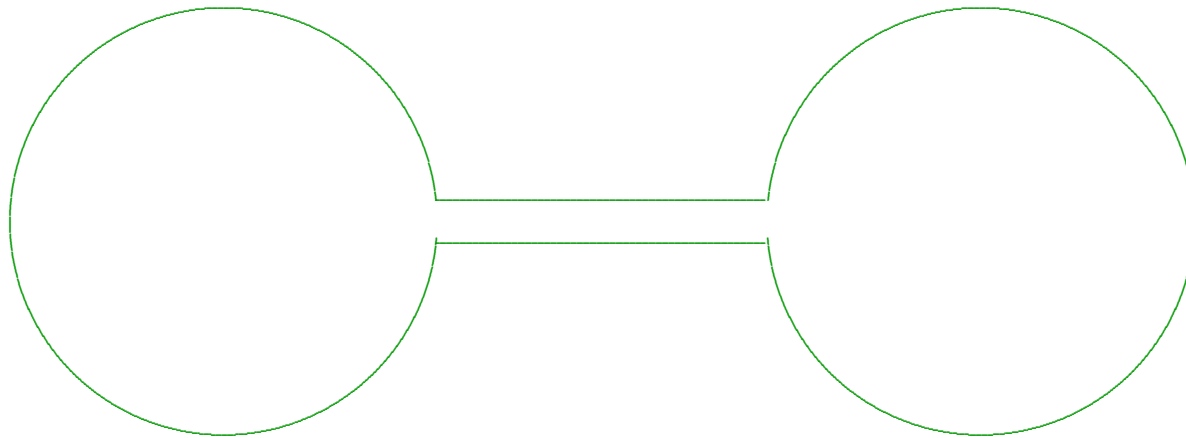


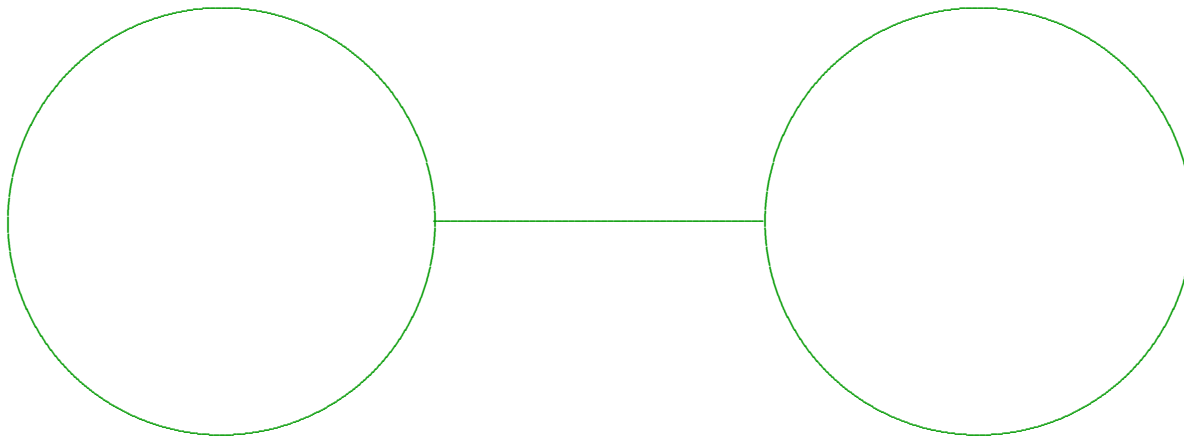


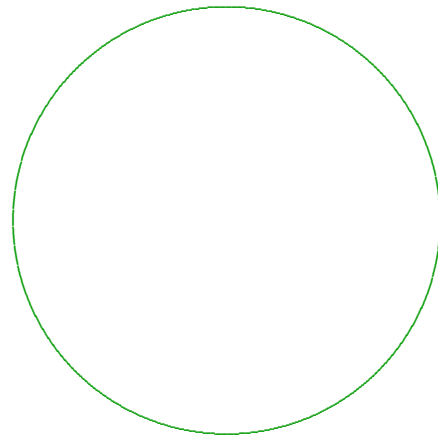
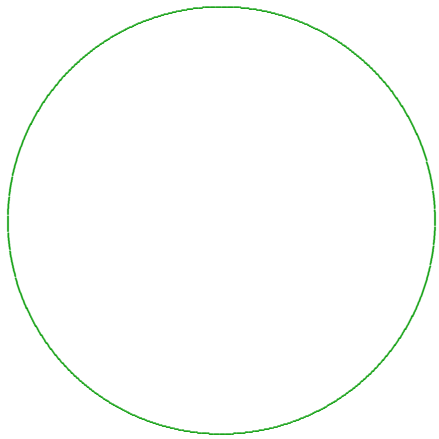


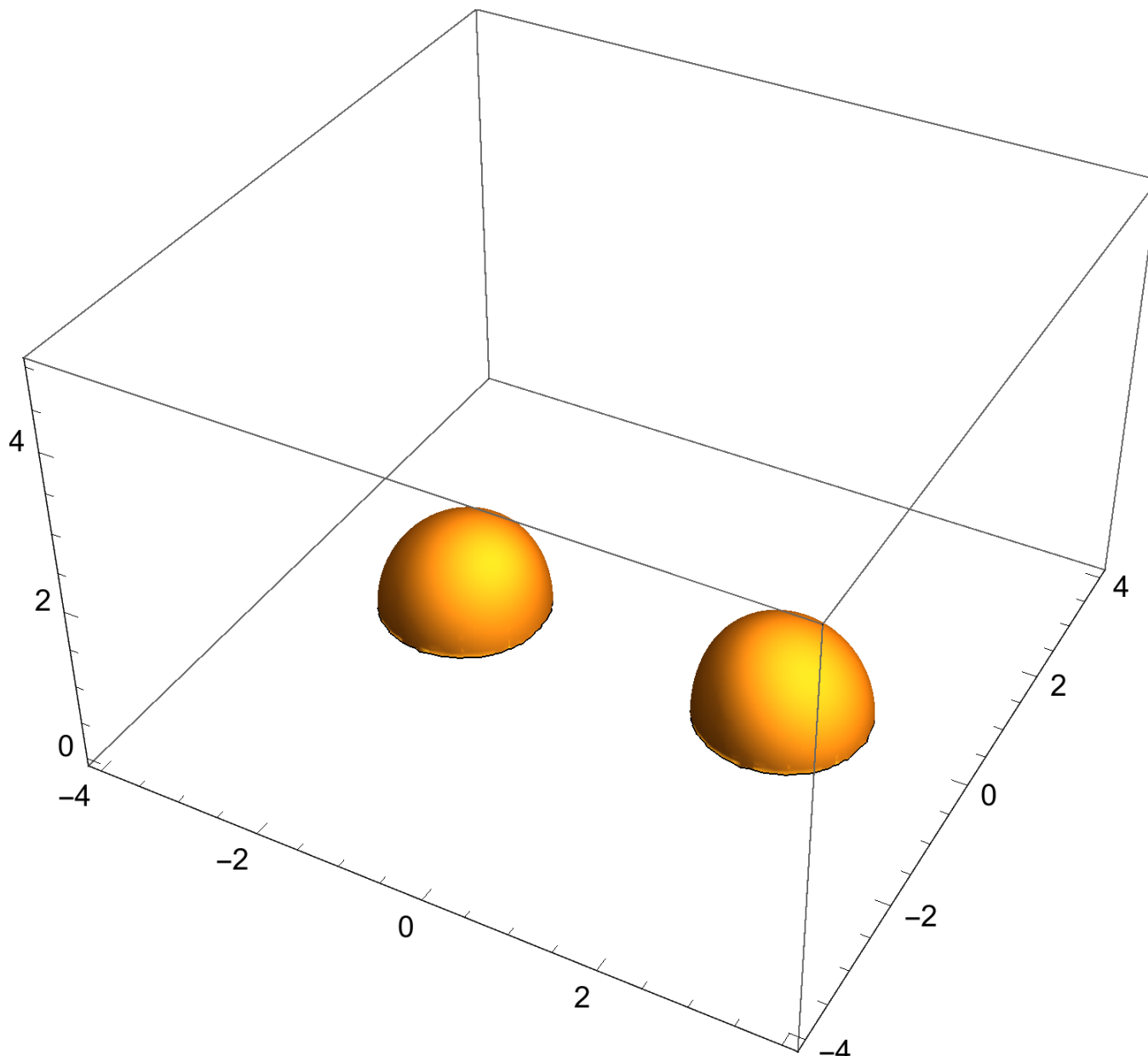


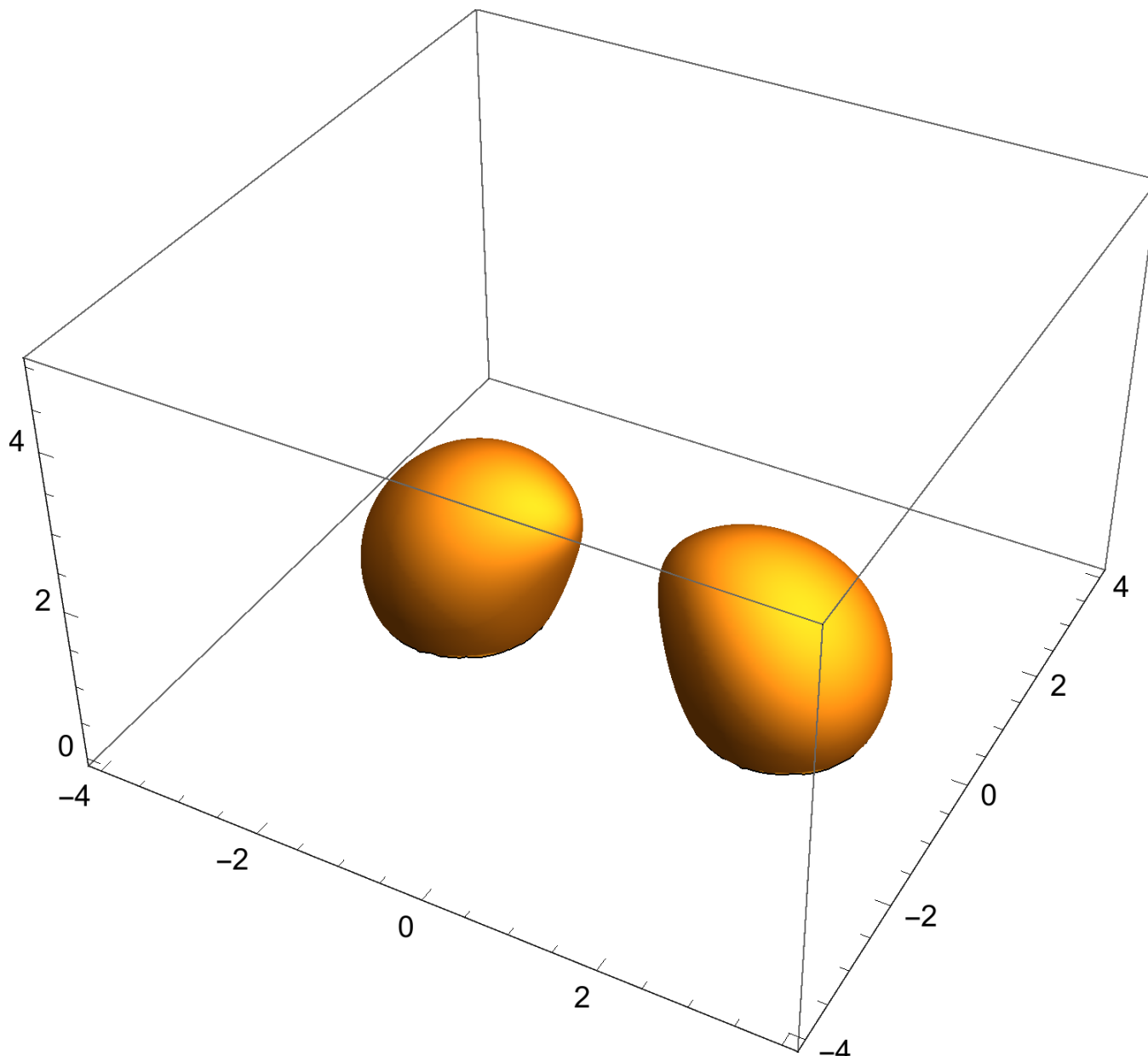


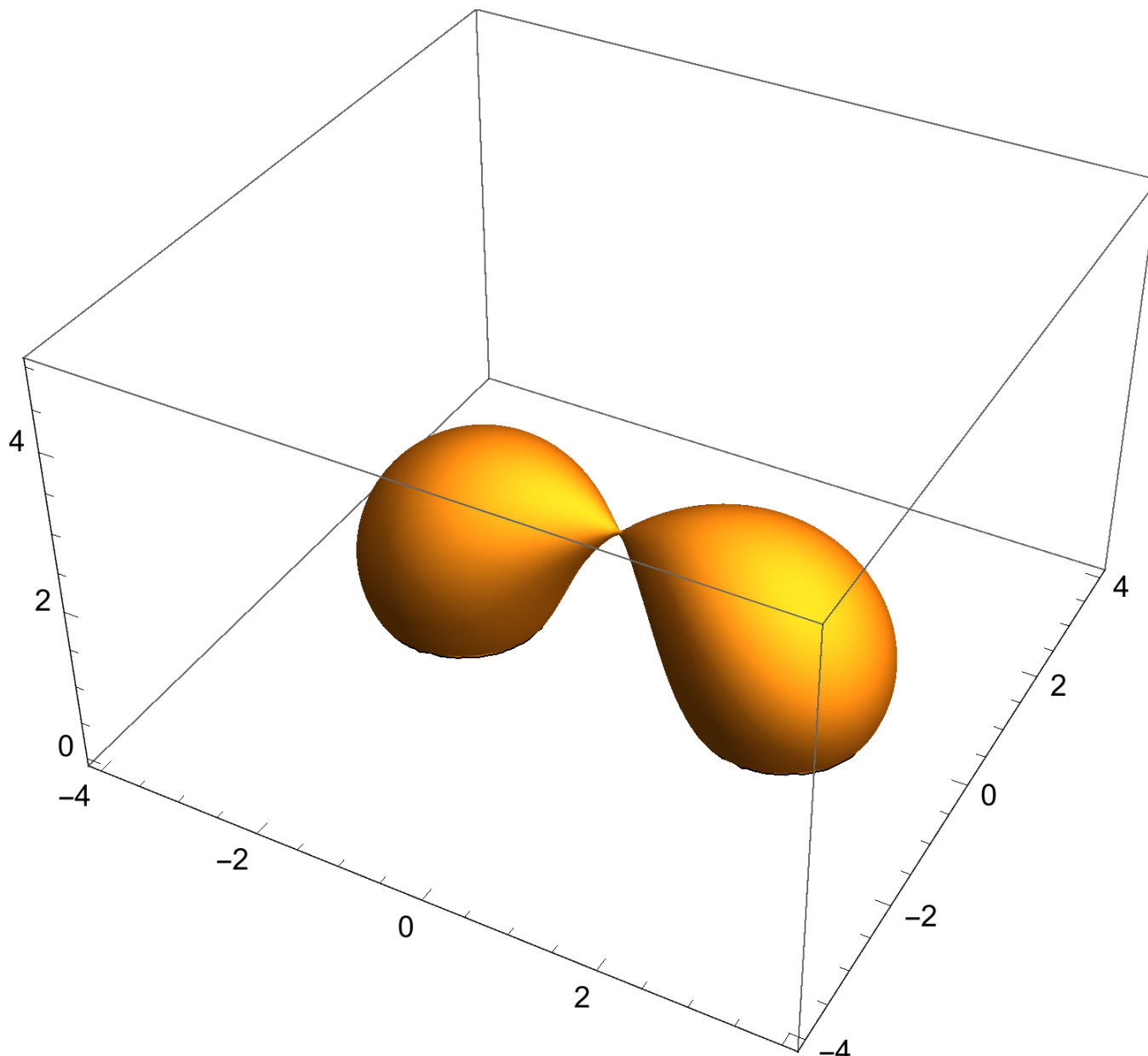


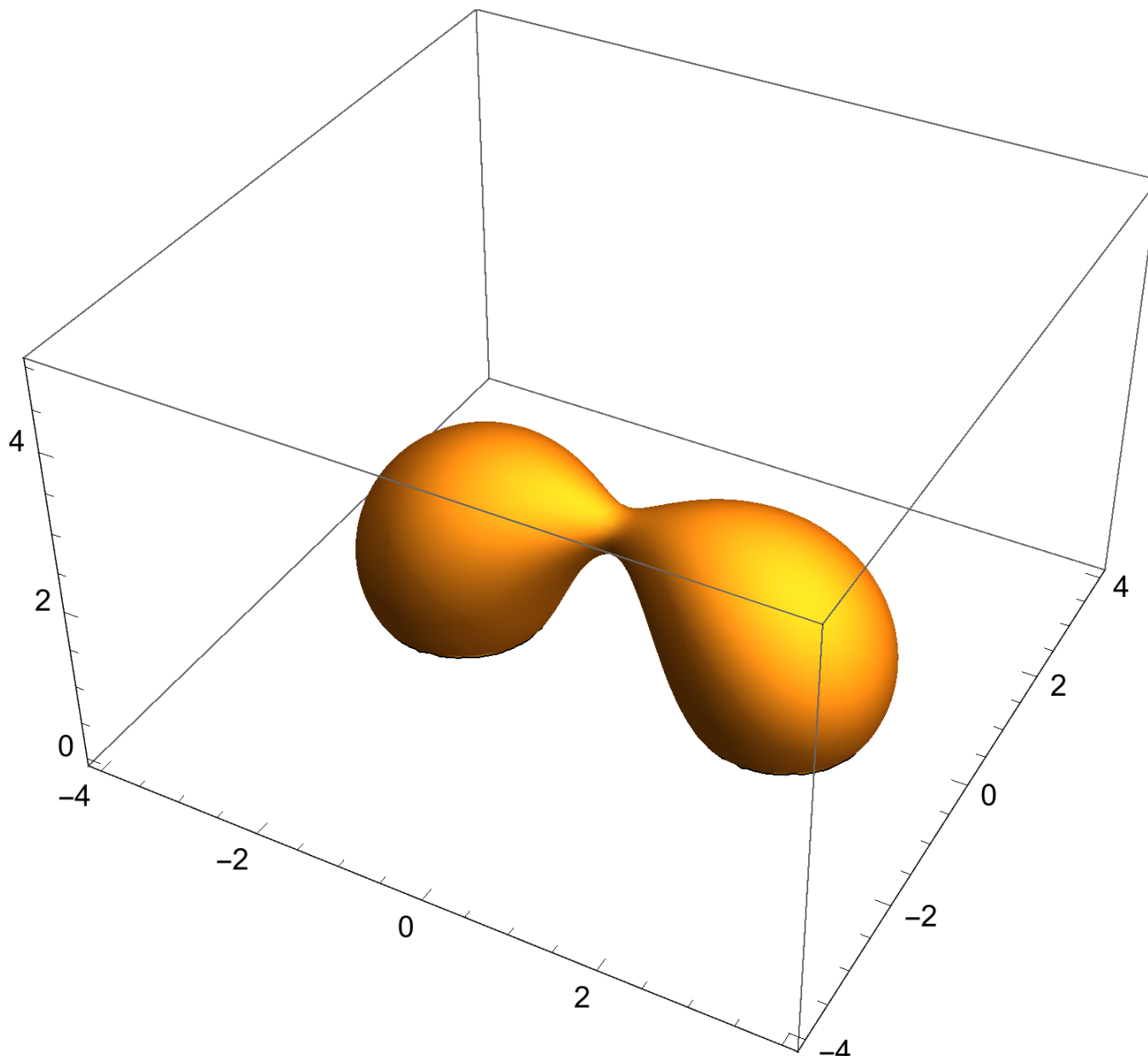


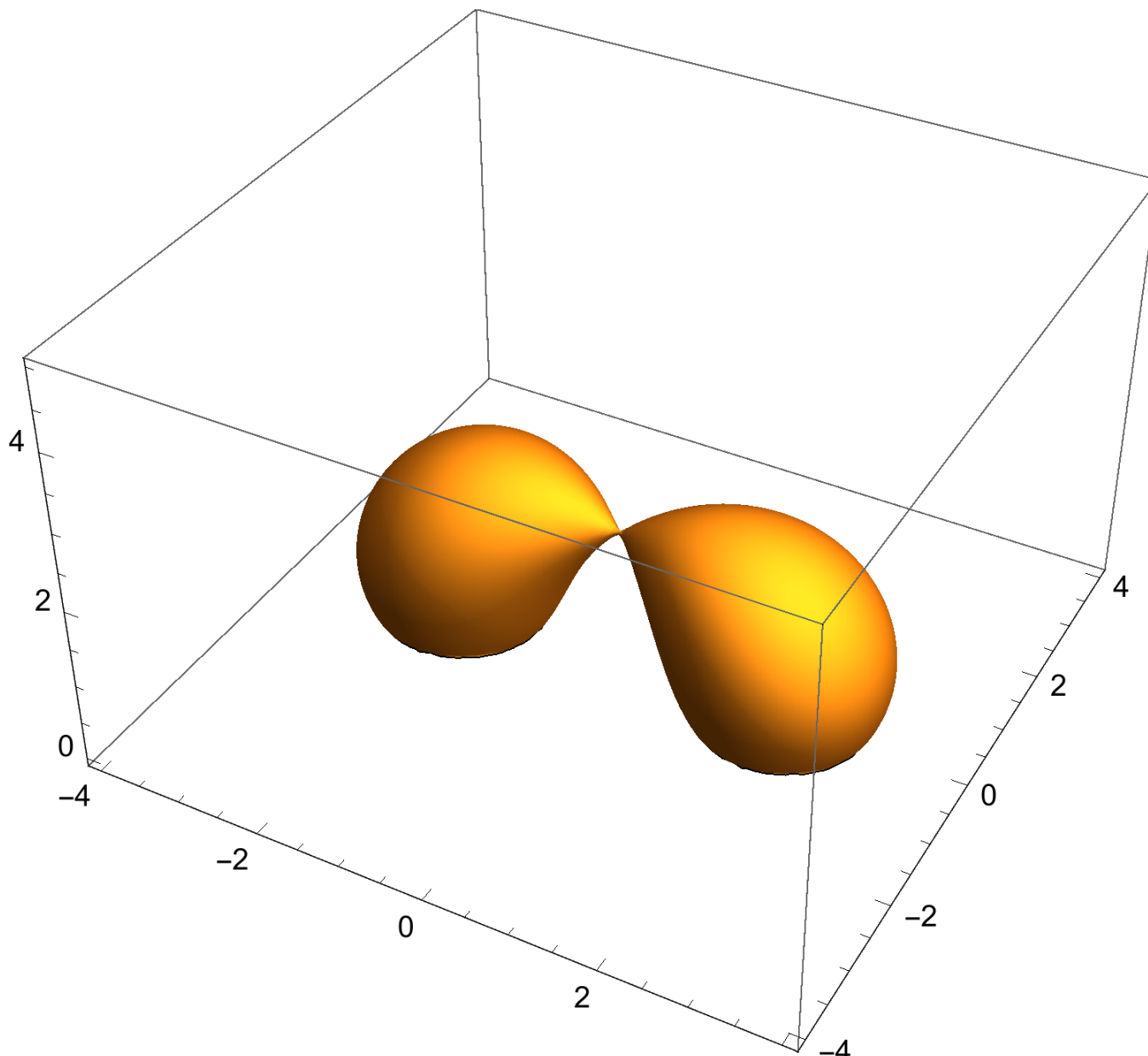


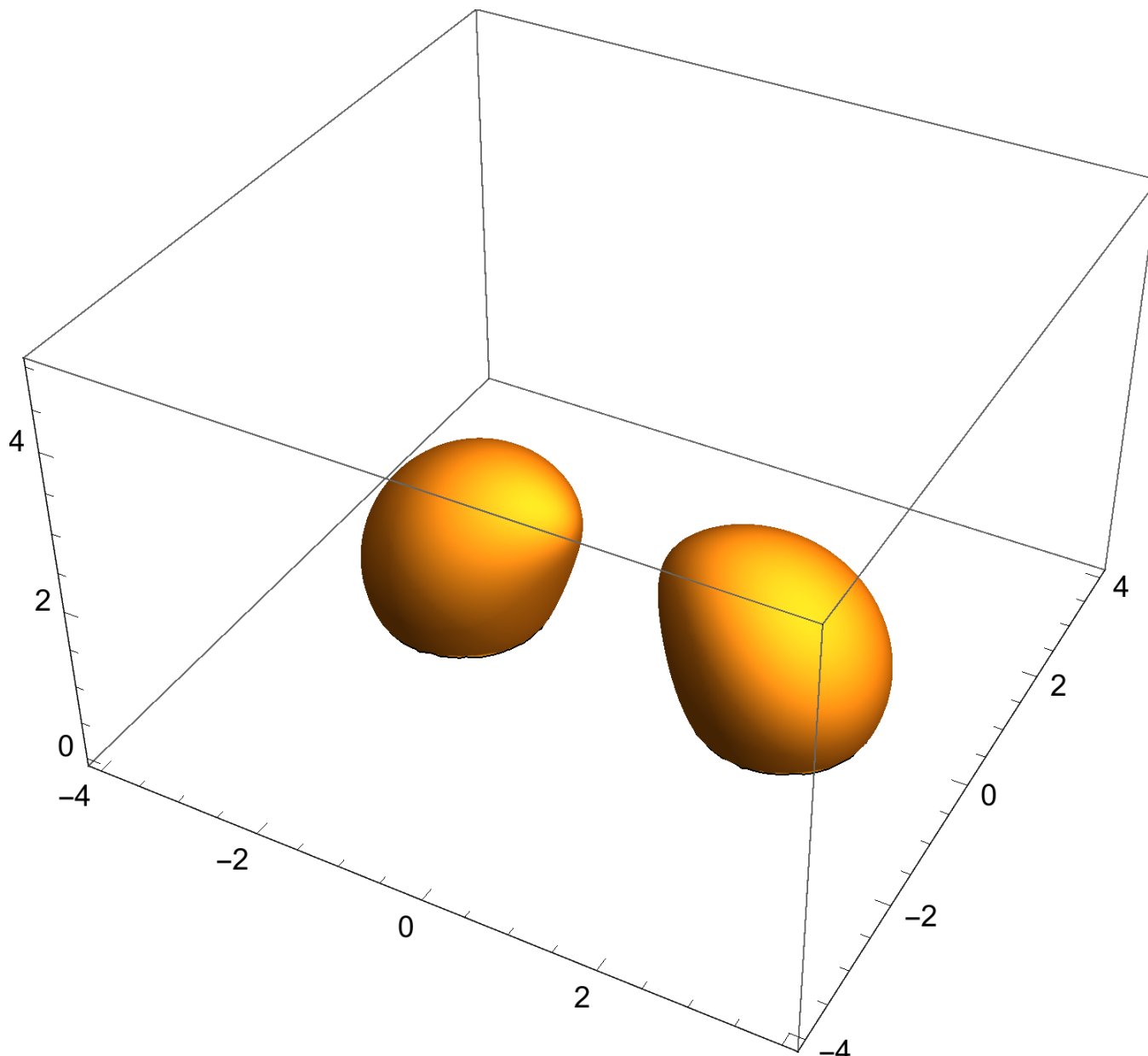


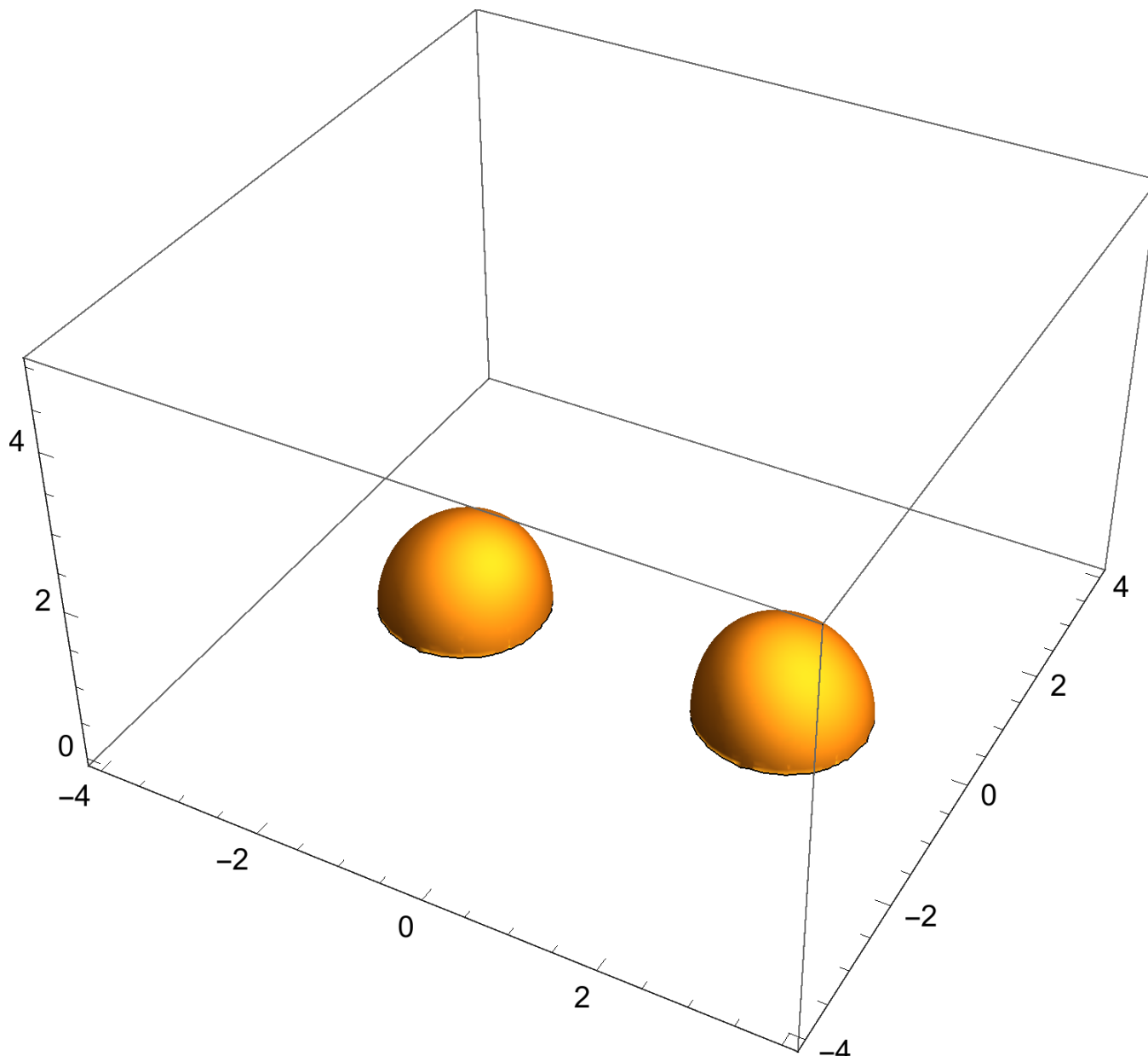


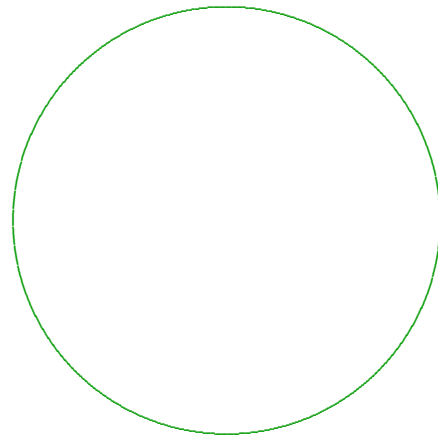
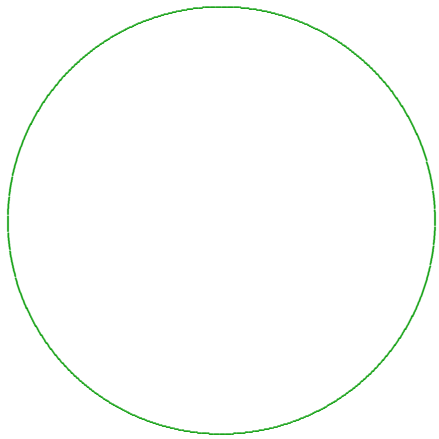


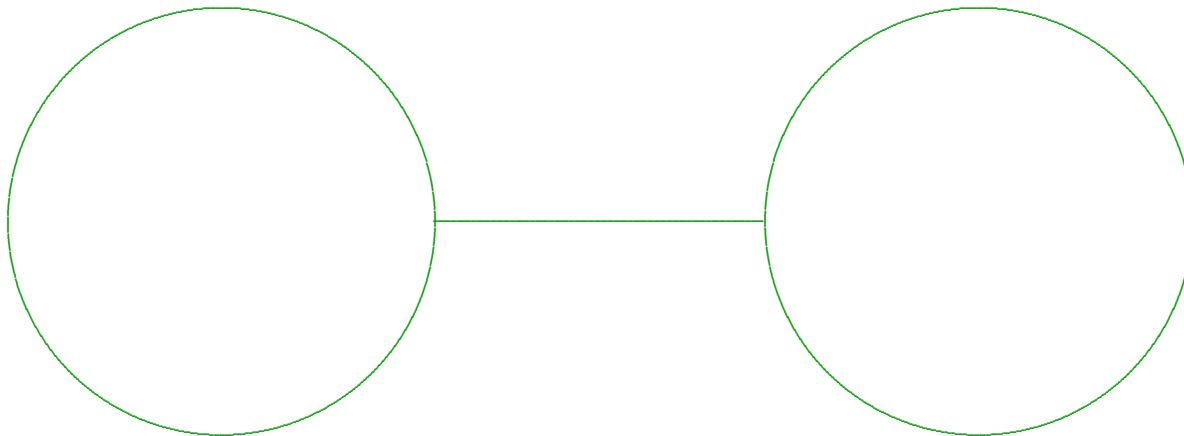


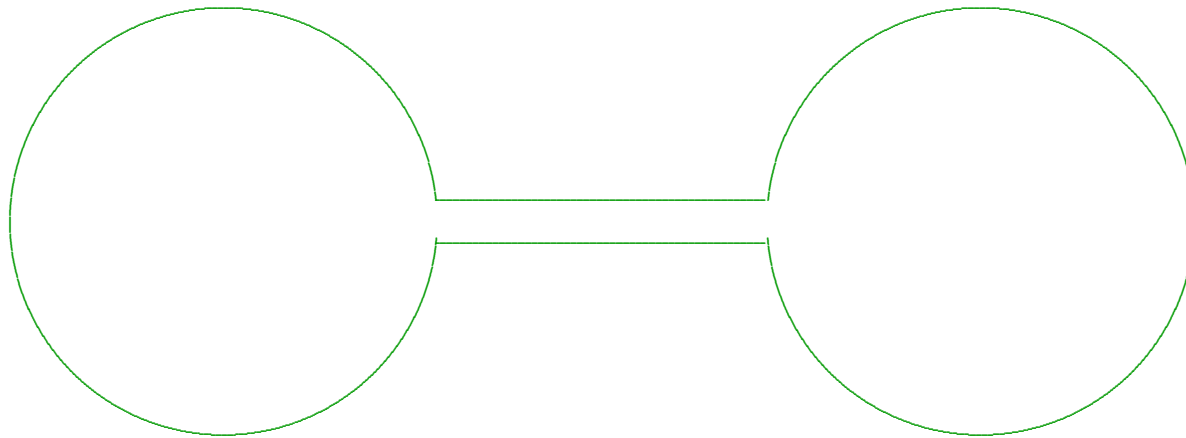


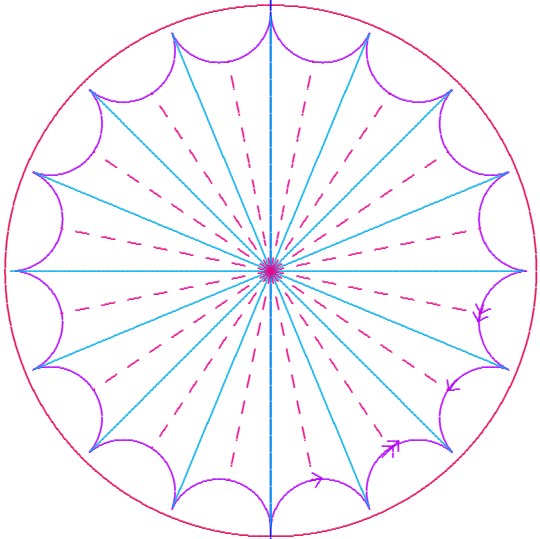
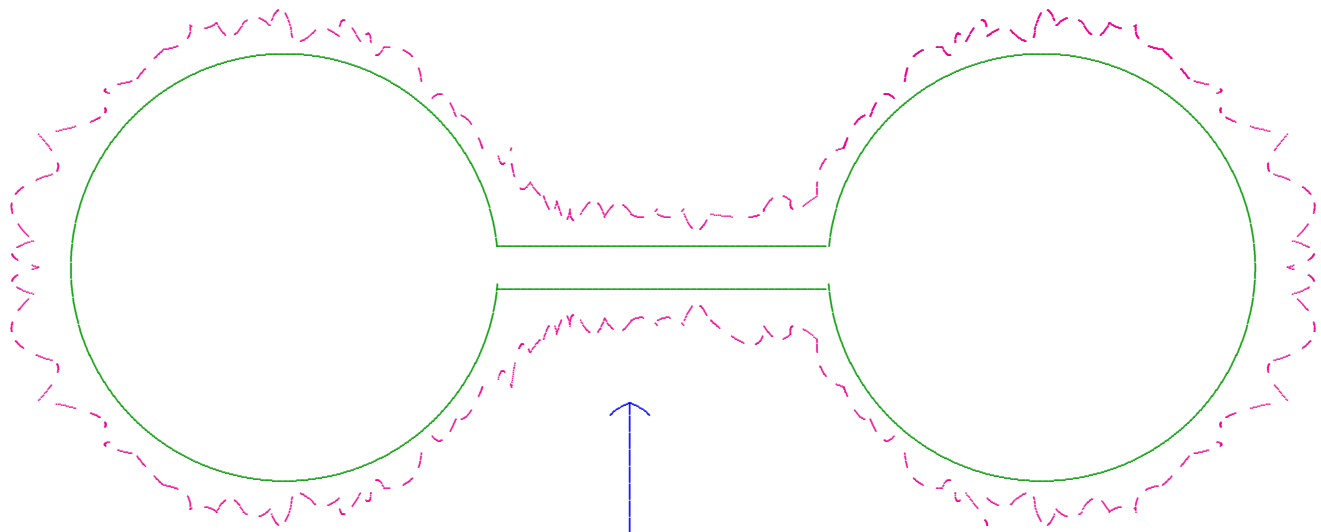








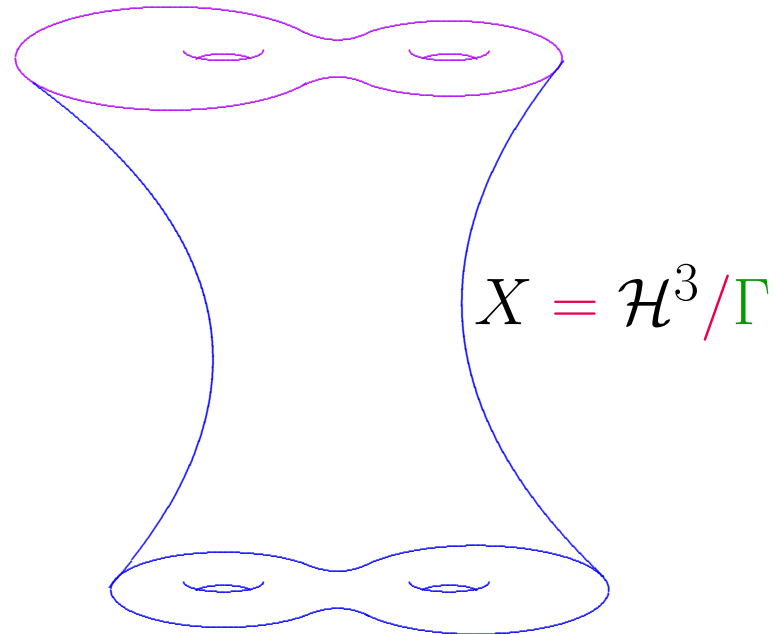




Theorem A. Consider 4-manifolds $M = \Sigma \times S^2$, where Σ compact Riemann surface of genus g .

Then \forall even $g \gg 0$, \exists family $[g_t]$, $t \in [0, 1]$, of locally-conformally-flat classes on M , such that

- \exists scalar-flat Kähler metric $g_0 \in [g_0]$; but
- \nexists almost-Kähler metric $g \in [g_1]$.



Construction of conformally flat 4-manifolds:

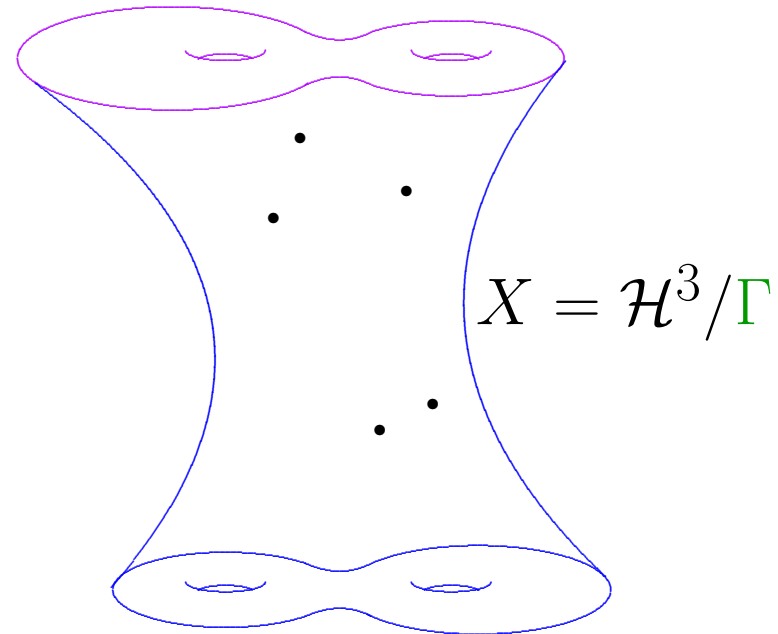
$$M = [\bar{X} \times S^1] / \sim$$

$$g = f(1 - f)[h + dt^2]$$

Theorem B. Fix an integer $k \geq 2$, and then consider the 4-manifolds $M = (\Sigma \times S^2) \#^k \overline{\mathbb{C}\mathbb{P}}_2$, where Σ compact Riemann surface of genus g .

Then \forall even $g \gg 0$, \exists family $[g_t]$, $t \in [0, 1]$, of anti-self-dual conformal classes on M , such that

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Construction of ASD 4-manifolds:

$$g = f(1 - f)[Vh + V^{-1}\theta^2]$$

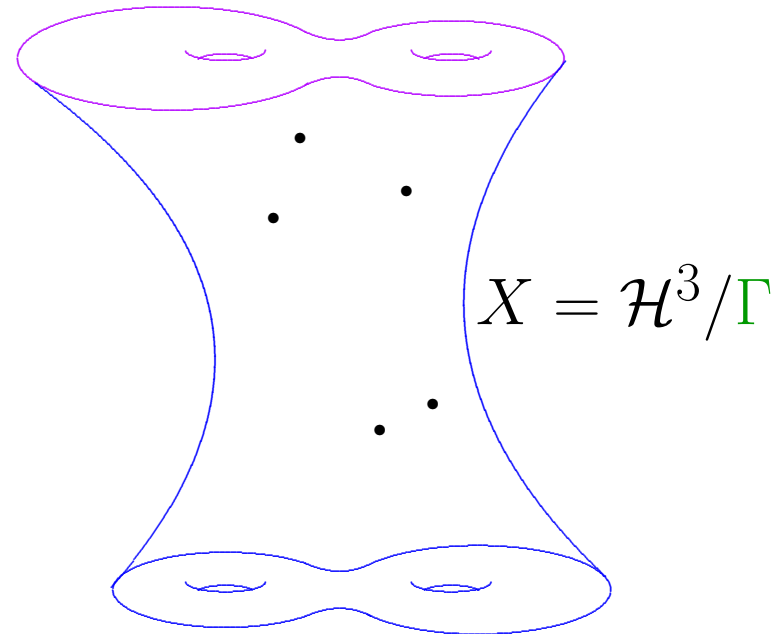
$$V = 1 + \sum_{j=1}^k G_j$$

$$d\theta = \star dV$$

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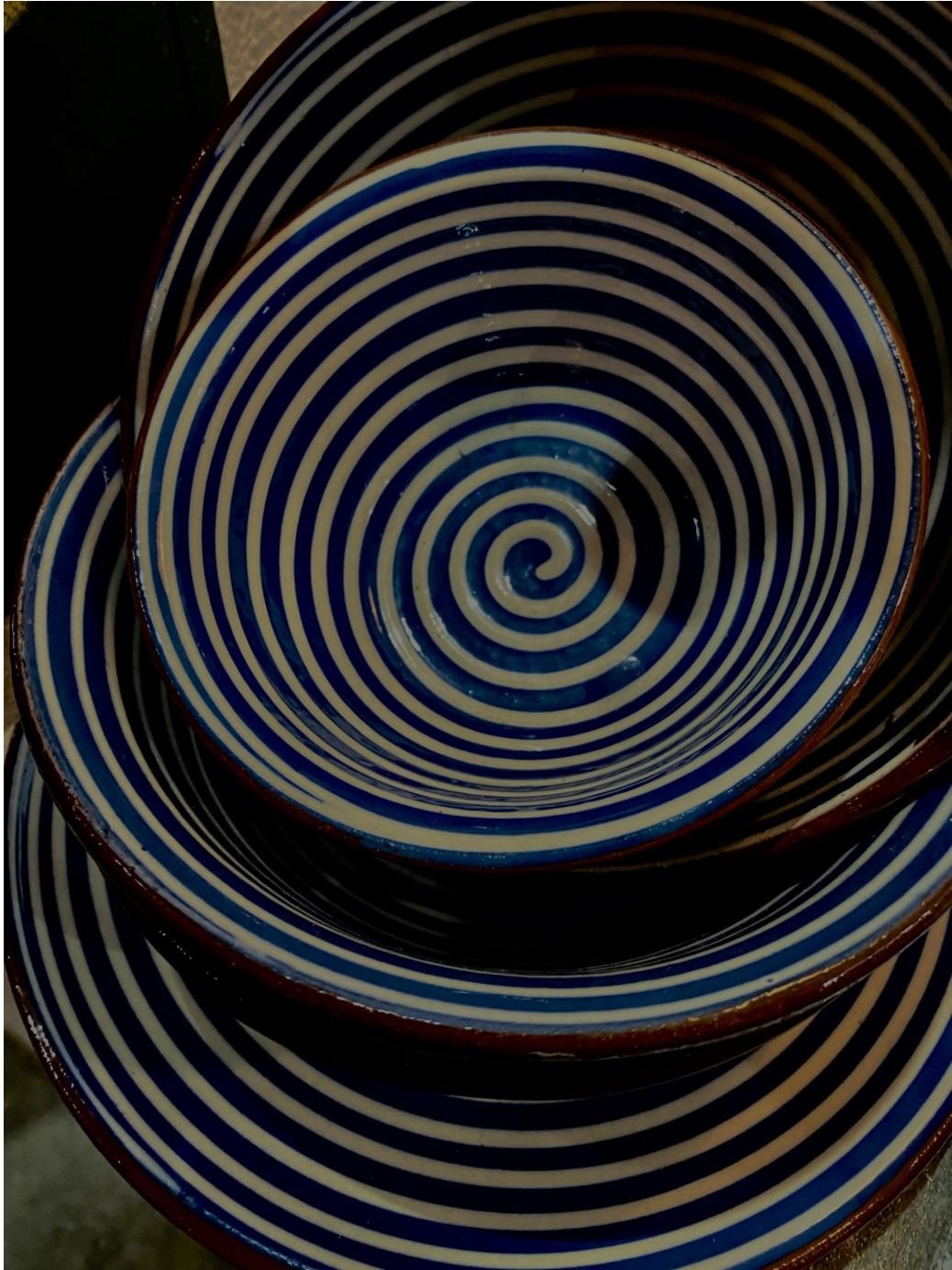


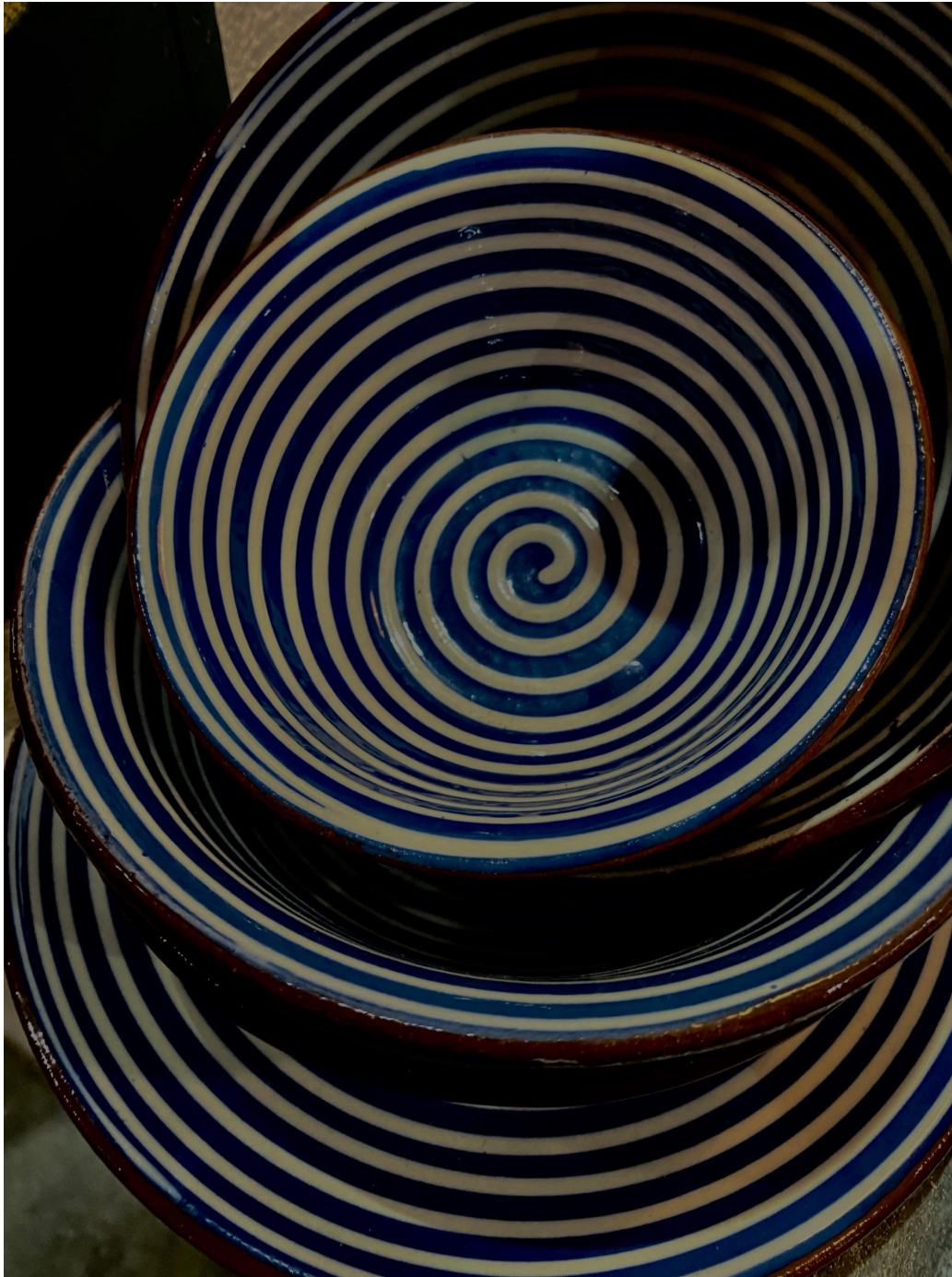
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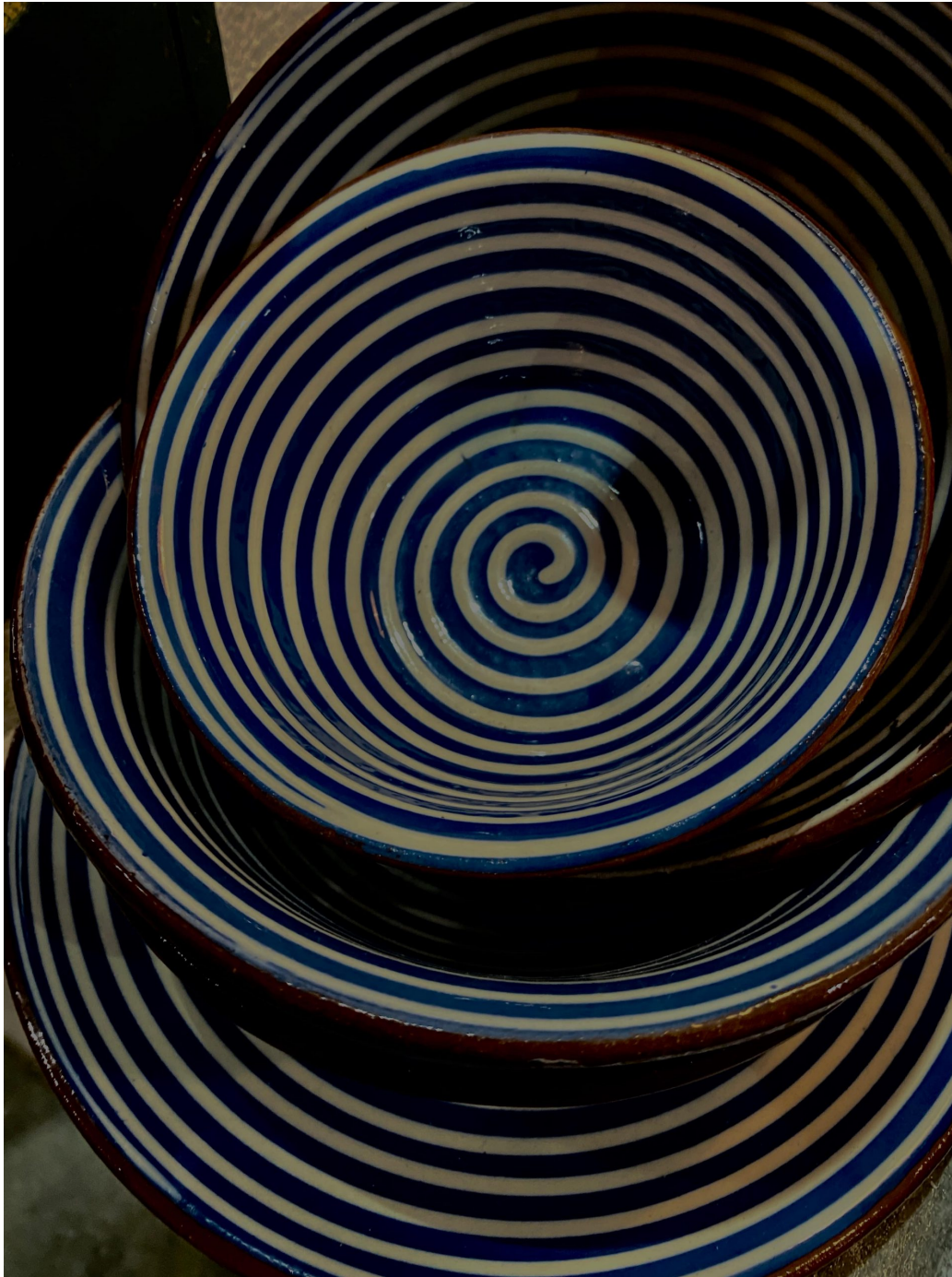
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Thanks for
inviting me!



Joyeux Mardi Gras!