Curvature,

Cones, and

Characteristic Numbers

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Joint work with

Michael Atiyah
University of Edinburgh

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Has same sign as the *scalar curvature*

$$s = r_j^j = \mathcal{R}^{ij}{}_{ij}.$$

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Proposition. *If* $n \geq 3$*, A Riemannian* n*-manifold* (M^n, g) *is Einstein iff the trace-free part of its Ricci tensor vanishes:*

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Proof. Bianchi identity $\Longrightarrow \nabla \cdot \mathring{r} = (\frac{1}{2} - \frac{1}{n}) ds$.

Two homotopy invariants

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Theorem. *If smooth compact oriented M^4 admits Einstein metric g, then*

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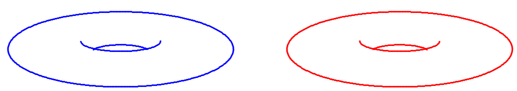

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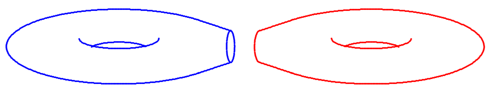

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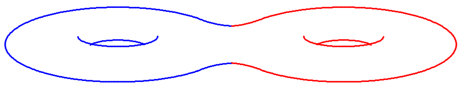

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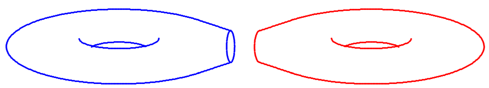

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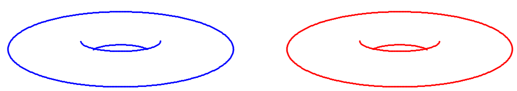

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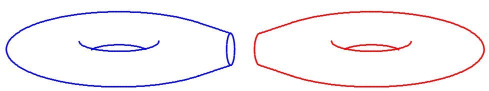

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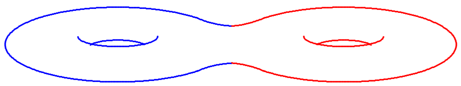

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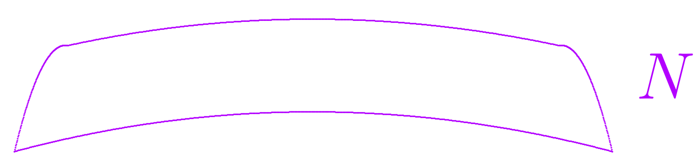

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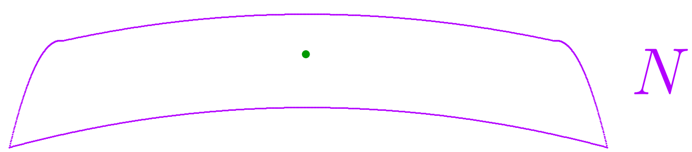

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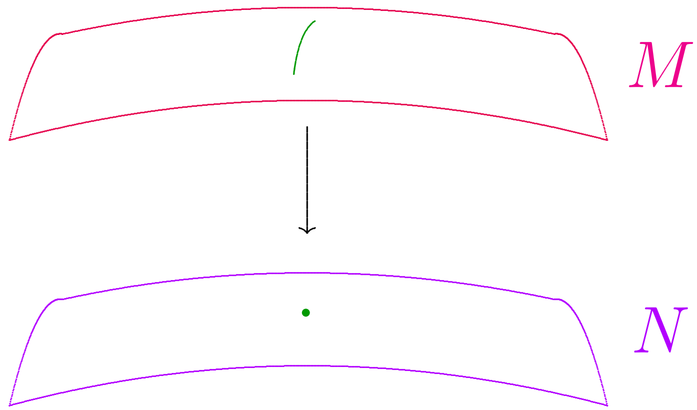

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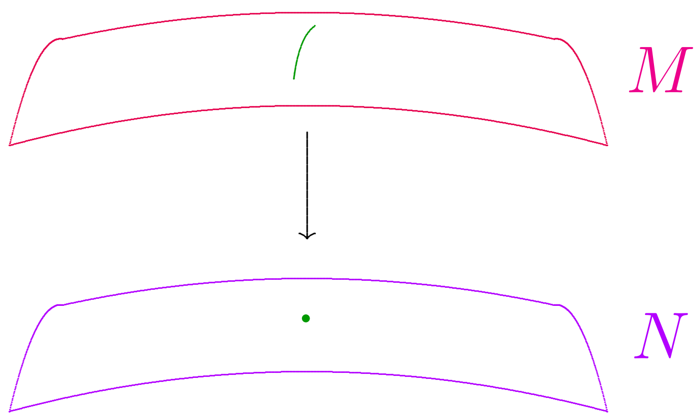

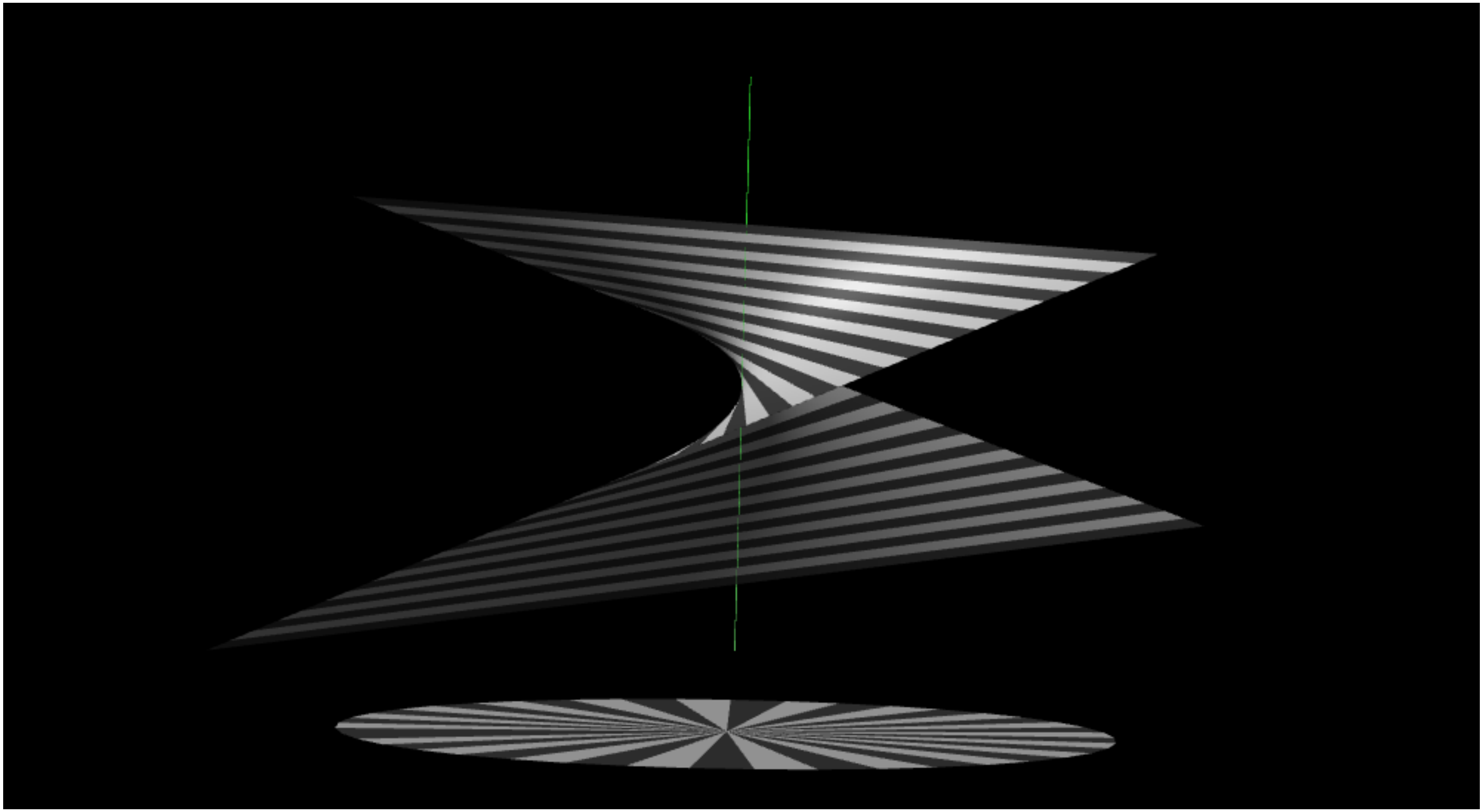

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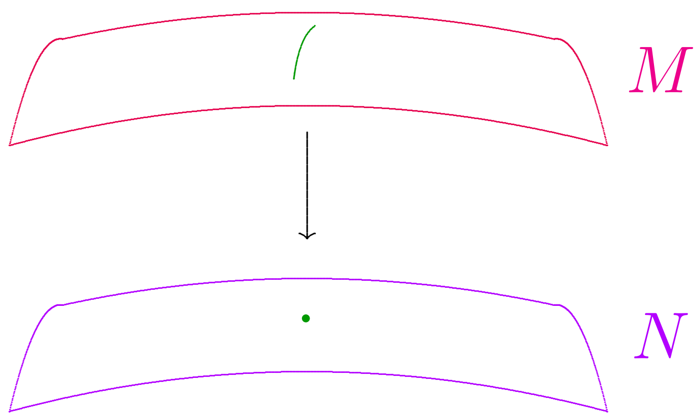

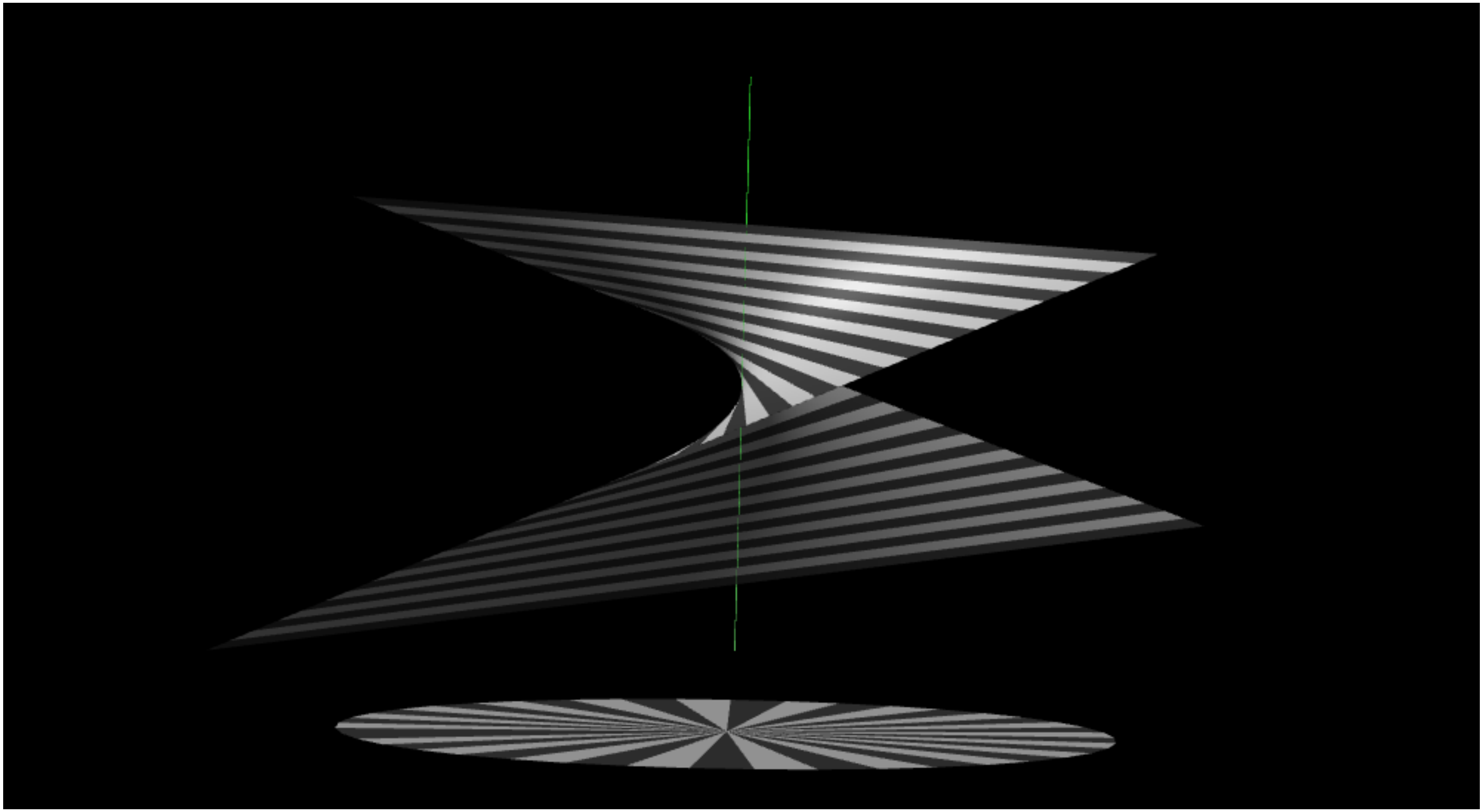

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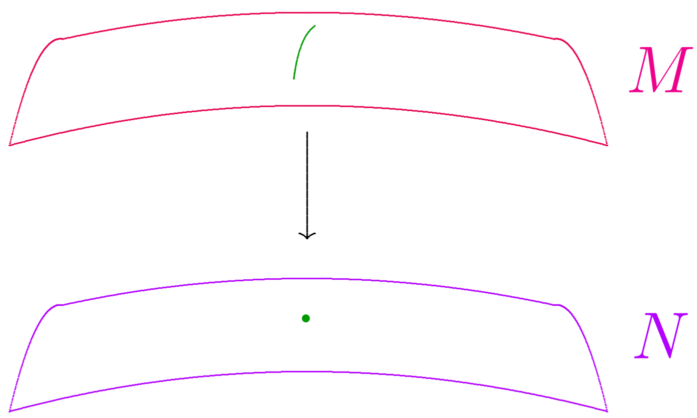

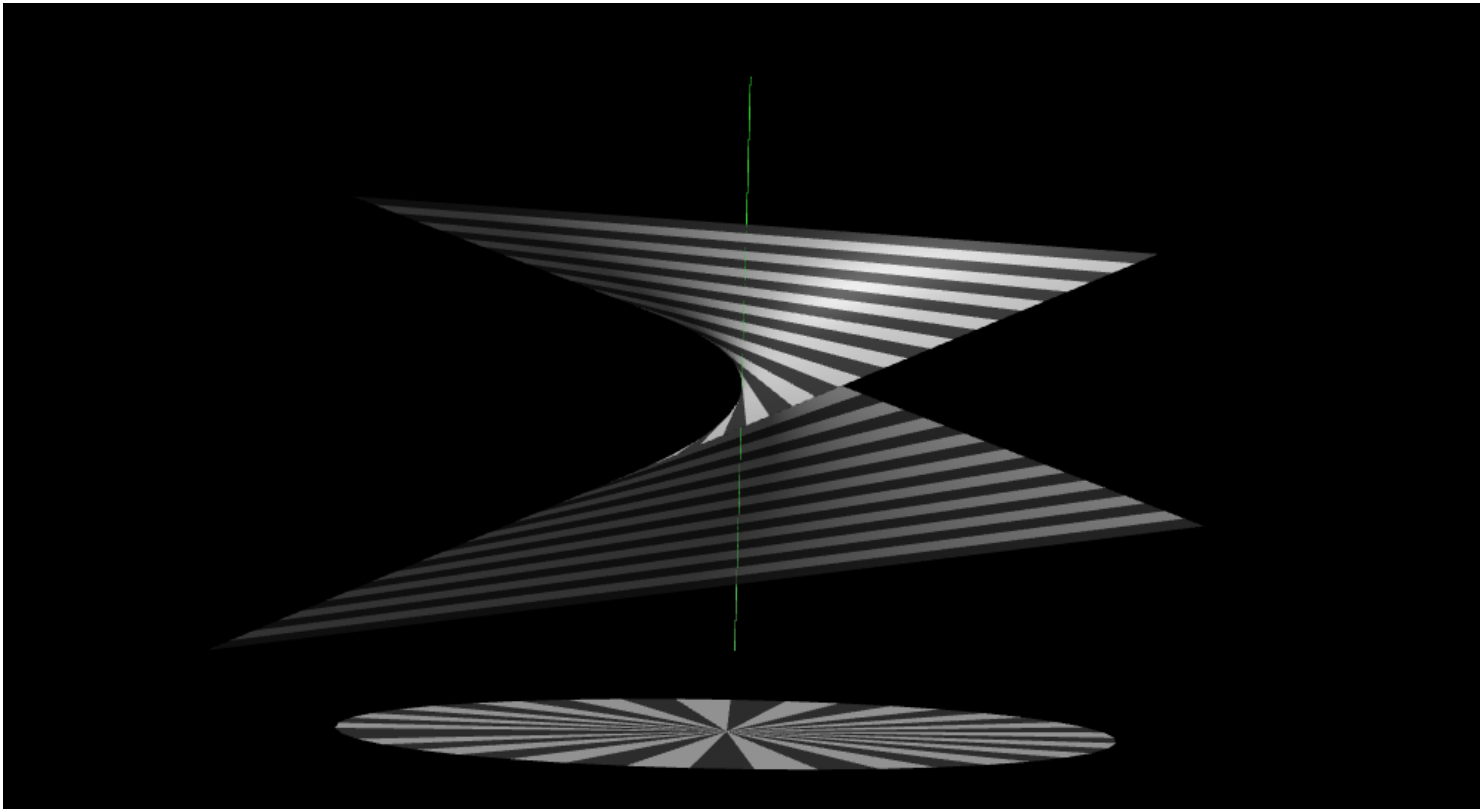

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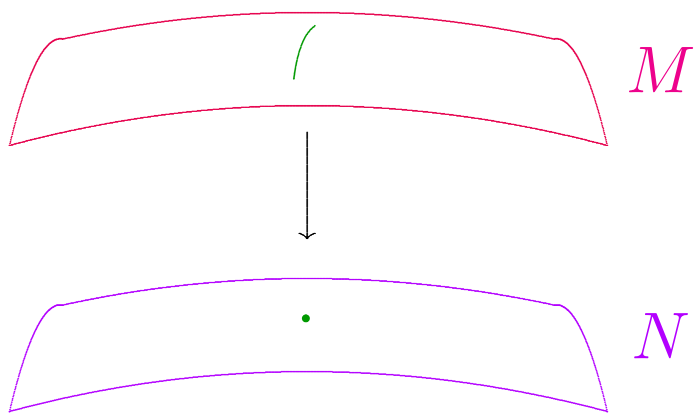

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Yau, Tian-Yau, Chen-LeBrun-Weber…

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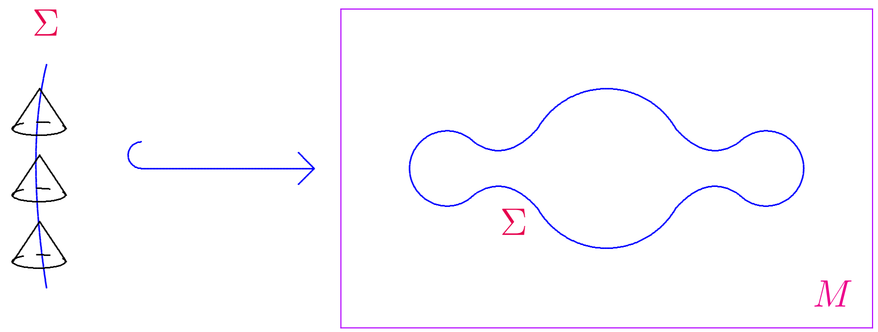

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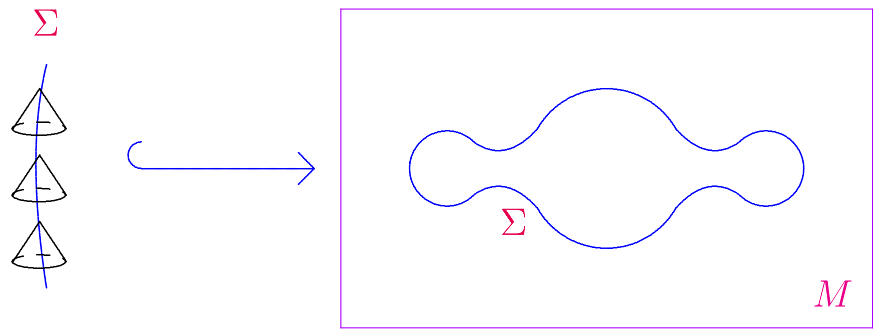

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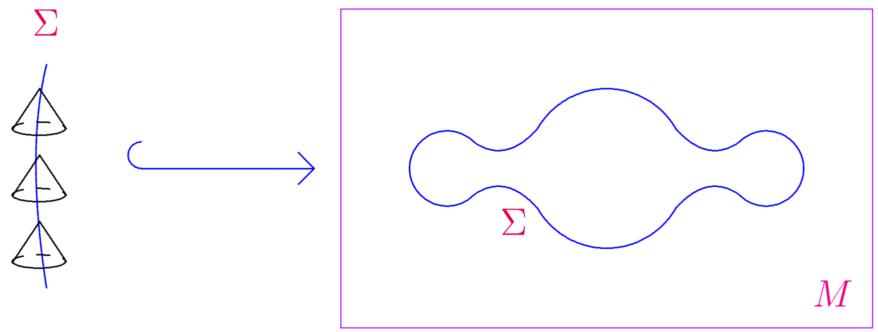

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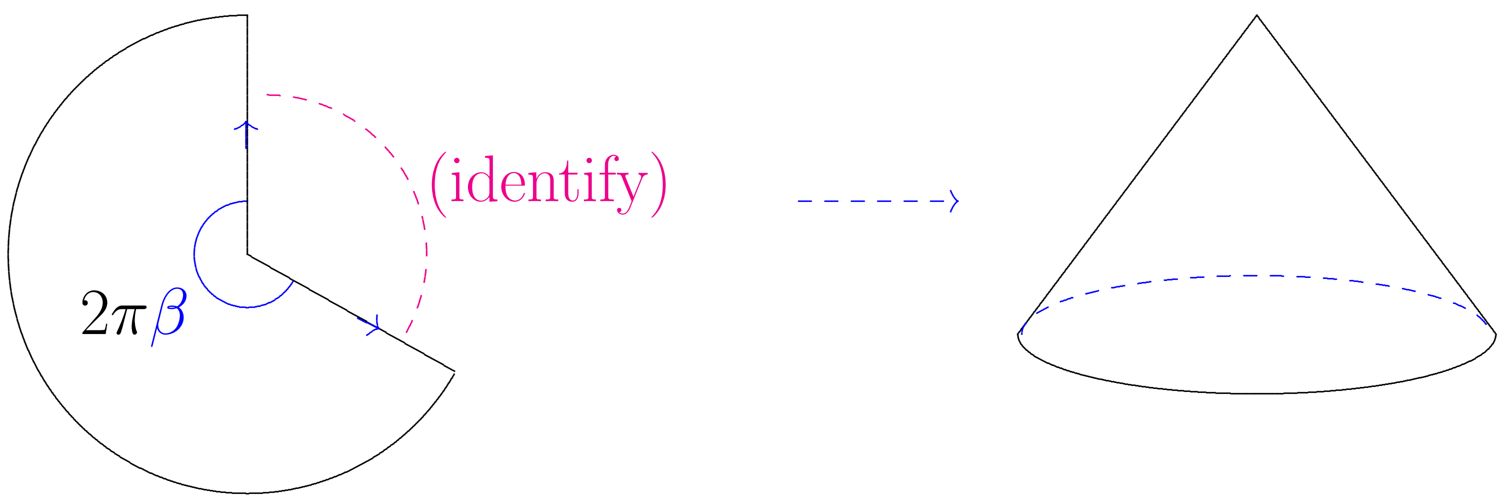

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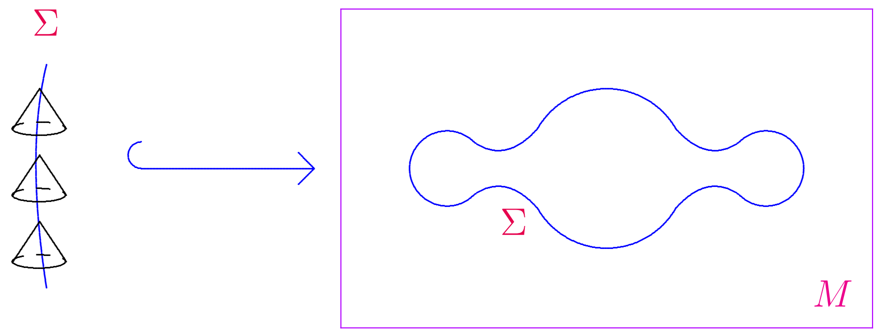

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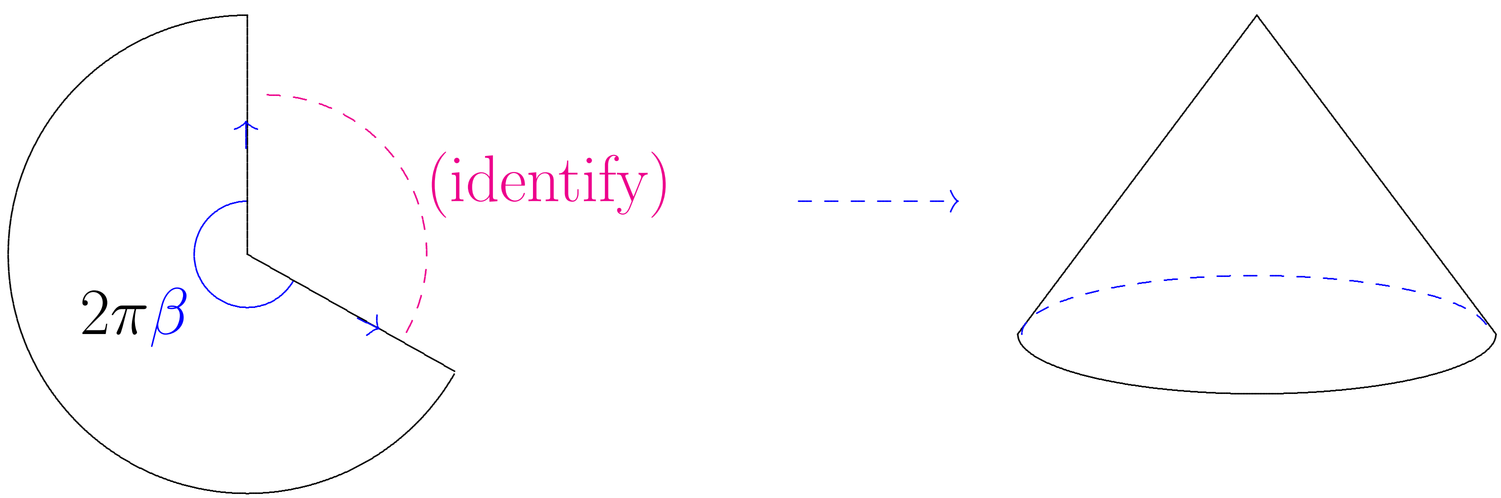

Brendle, Chen, Jeffres, Li, Mazzeo, Rubinstein, Sun. . .

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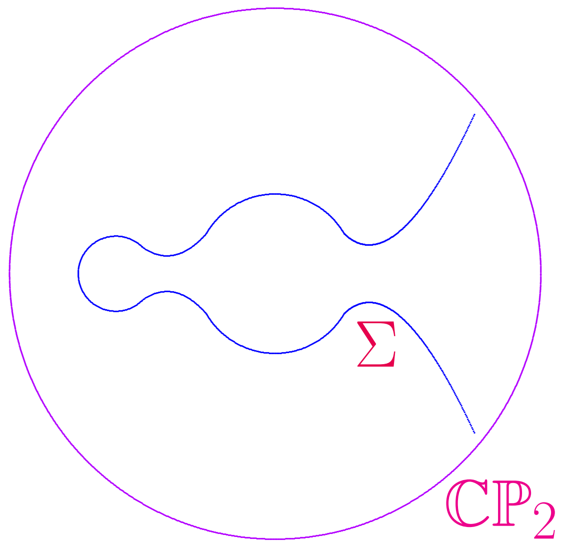

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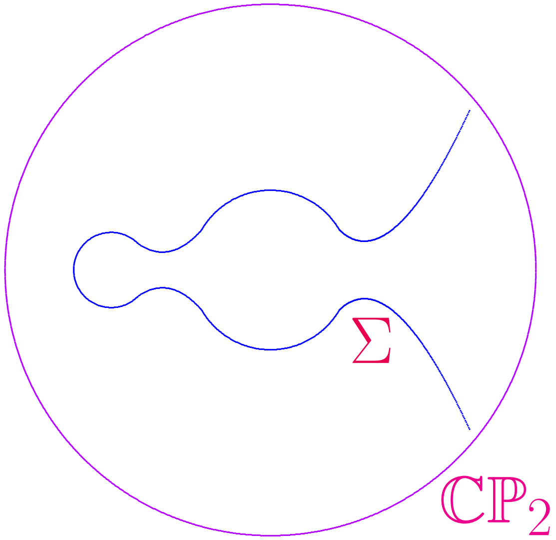

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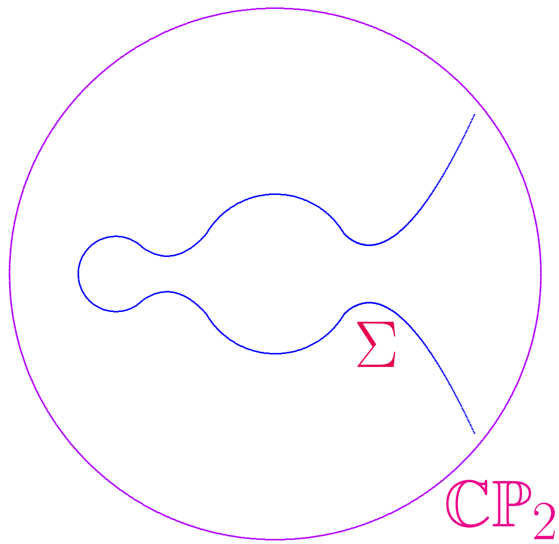

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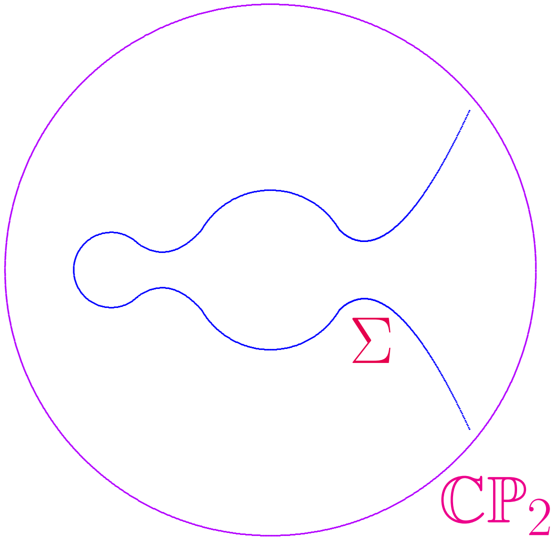

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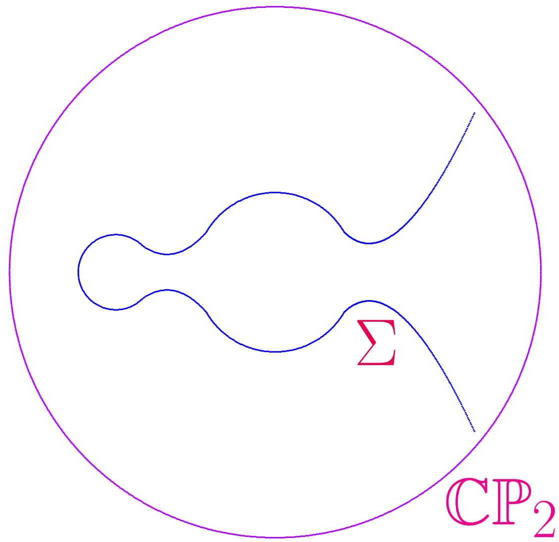

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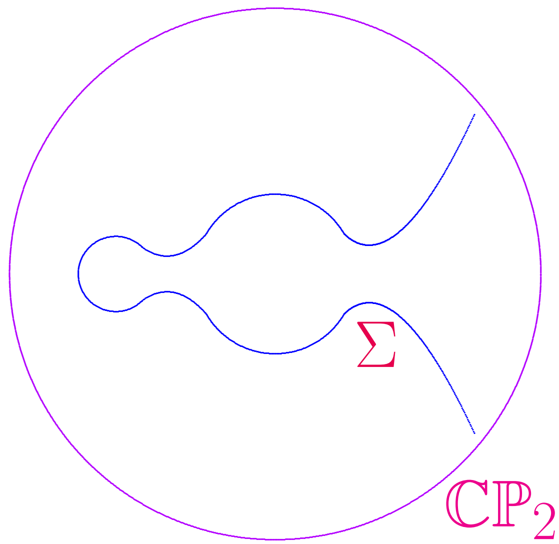

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Theorem $\Longrightarrow$ unique Ricci-flat metric, up to scale.

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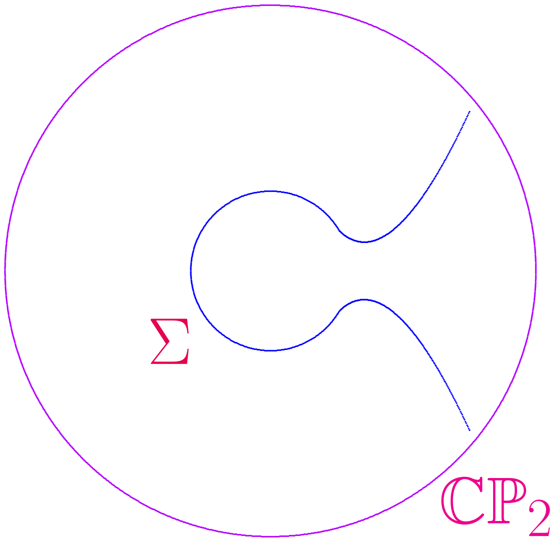

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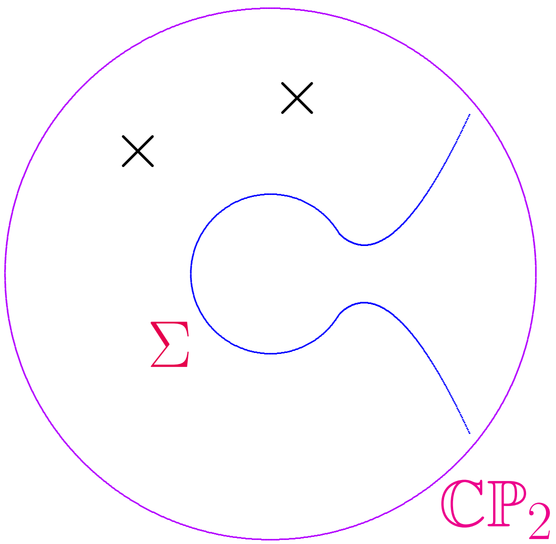

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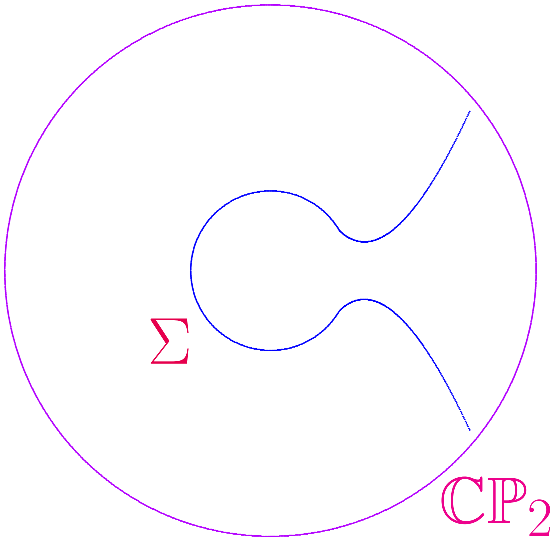

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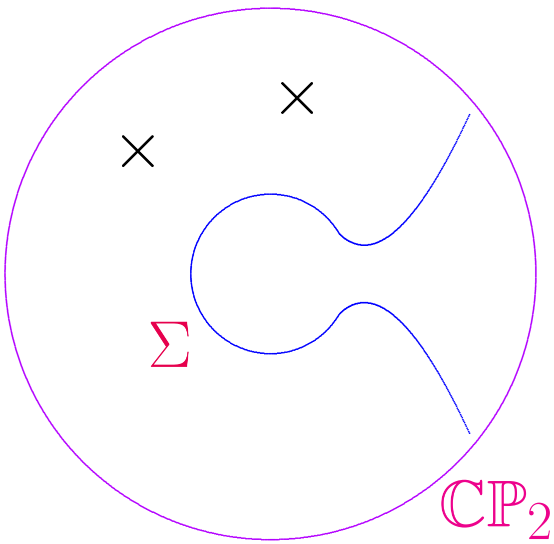

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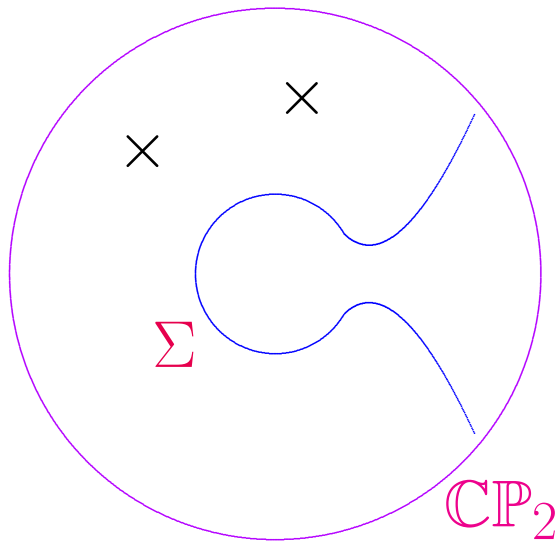

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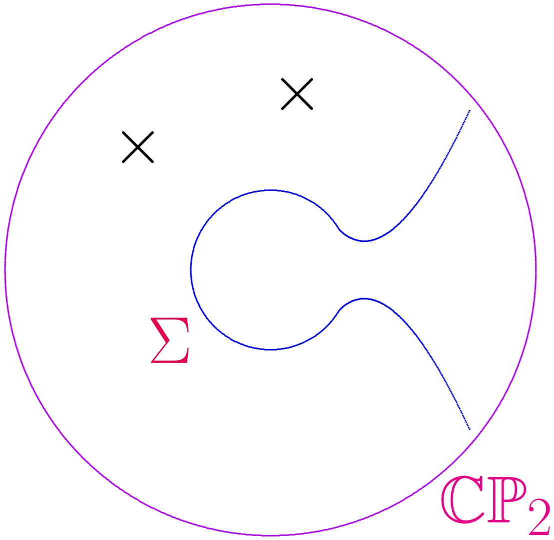

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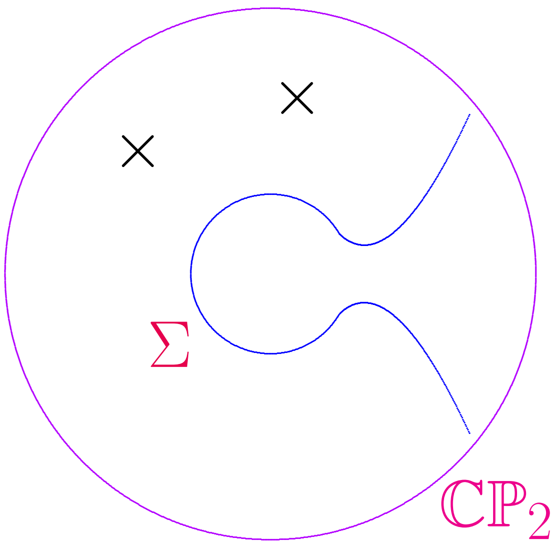

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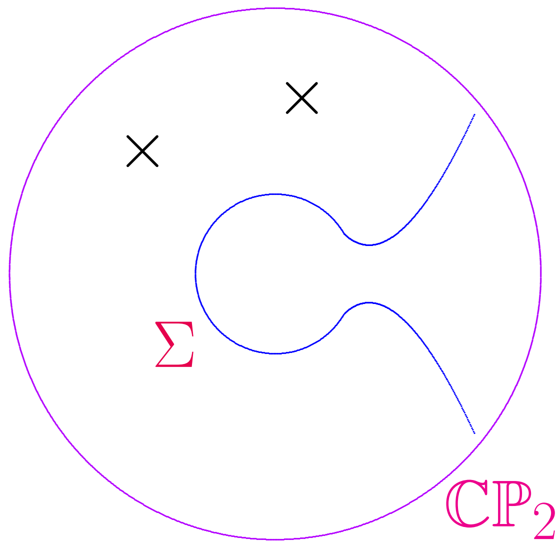

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For small β, Theorem $\Longrightarrow$ (M, Σ) never Einstein.

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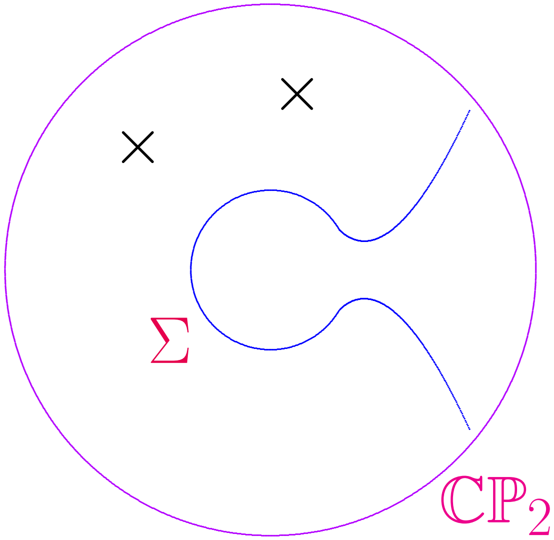

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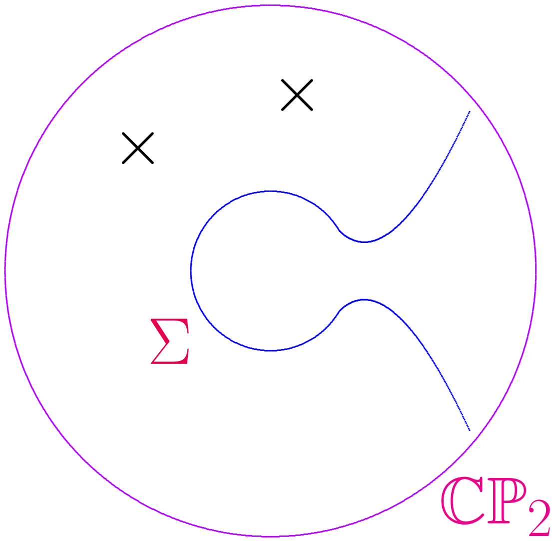

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Berman, Li-Sun, . . .

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Nothing analogous known in other dimensions.

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Λ^+ self-dual 2-forms.

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where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

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What about edge-cone metrics?

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Second is just orientation of first.

Taking linear combinations, $\Longleftrightarrow$ two formulæ

$$(2\chi + 3\tau)(M) \; + \; 2(\beta - 1)\chi(\Sigma) + (\beta^2 - 1)[\Sigma]^2$$

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So suffices to prove first.

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Lemma. *If this formula holds for one edge-cone metric on* (M, Σ) *of cone angle* $2\pi\beta$,

$$(2\chi + 3\tau)(M) \; + \; 2(\beta - 1)\chi(\Sigma) + (\beta^2 - 1)[\Sigma]^2$$

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Lemma. *If this formula holds for one edge-cone metric on* (M, Σ) *of cone angle* $2\pi\beta$*, it also holds for every other edge-cone metric on* (M, Σ) *of the same cone angle.*

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For family g_t of edge-cone metrics,

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If g is edge-cone metric, g_0 is smooth metric, want

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$$- \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24}+2|W_+|^2-\frac{|\mathring{r}|^2}{2}\right)d\mu_{g_0}$$

If g is edge-cone metric, g_0 is smooth metric, want

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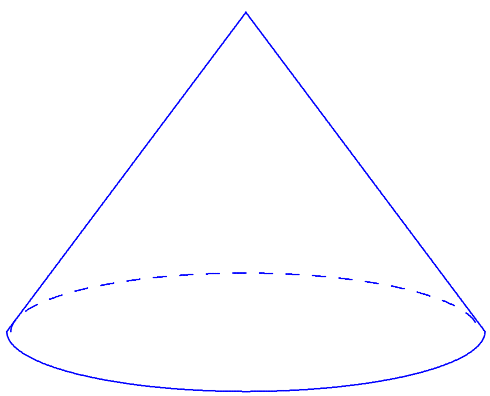

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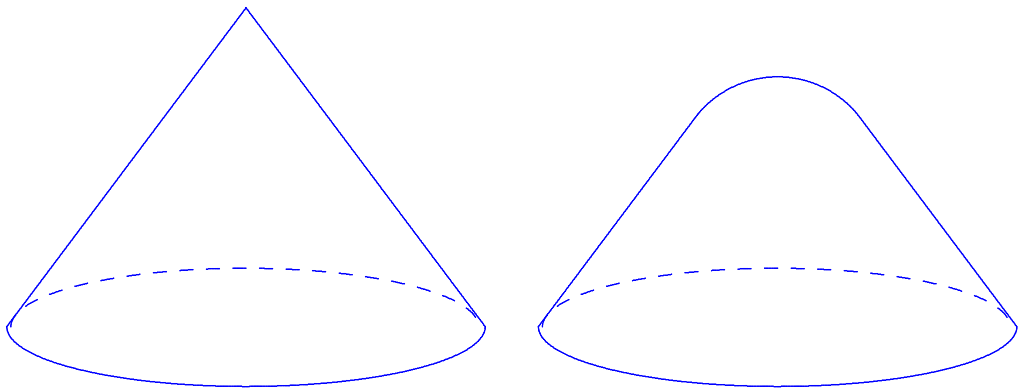

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If g and g_0 are Kähler on V, both satisfy

$$\frac{s^2}{24}=|W_+|^2$$

on this tubular neighborhood of Σ.

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provided $g=g_0$ on $M-V$, where $V\supset\Sigma$.

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Poincaré-Lelong: For edge Kähler metric,

$$\frac{1}{2\pi}[\varrho_g] = c_1 + (\beta - 1)[\Sigma] .$$

If g is edge-cone metric, g_0 is smooth metric, want

$$(\beta - 1)[\Sigma] \cdot (2c_1 + (\beta - 1)[\Sigma])$$

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$$\frac{1}{2\pi}[\varrho_g - \varrho_{g_0}] = (\beta - 1)[\Sigma] \ .$$

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= \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu_g
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Einstein case:

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Theorem (A-L). *Let* (M, Σ) *be smooth compact* 4*-manifold with smoothly embedded compact oriented surface. If* (M, Σ) *admits Einstein edge-cone metric* g *of cone angle* $2\pi\beta$*, then*

$$(2\chi + 3\tau)(M) \geq (1-\beta)\left(2\chi(\Sigma) + (1+\beta)[\Sigma]^2\right)$$

with equality $\iff$ g *is Ricci-flat Kähler up to covers.*

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$$(2\chi - 3\tau)(M) \geq (1-\beta)\left(2\chi(\Sigma) - (1+\beta)[\Sigma]^2\right)$$

with equality $\iff$ g *is Ricci-flat and, up to coverings, is reverse-oriented Kähler.*