

Gravitational Instantons,

Weyl Curvature, &

Conformally Kähler Geometry

Claude LeBrun

Stony Brook University

Differential Geometry Seminar
University of California, Berkeley

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Joint work with

Joint work with

Olivier Biquard

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Olivier Biquard
Sorbonne Université

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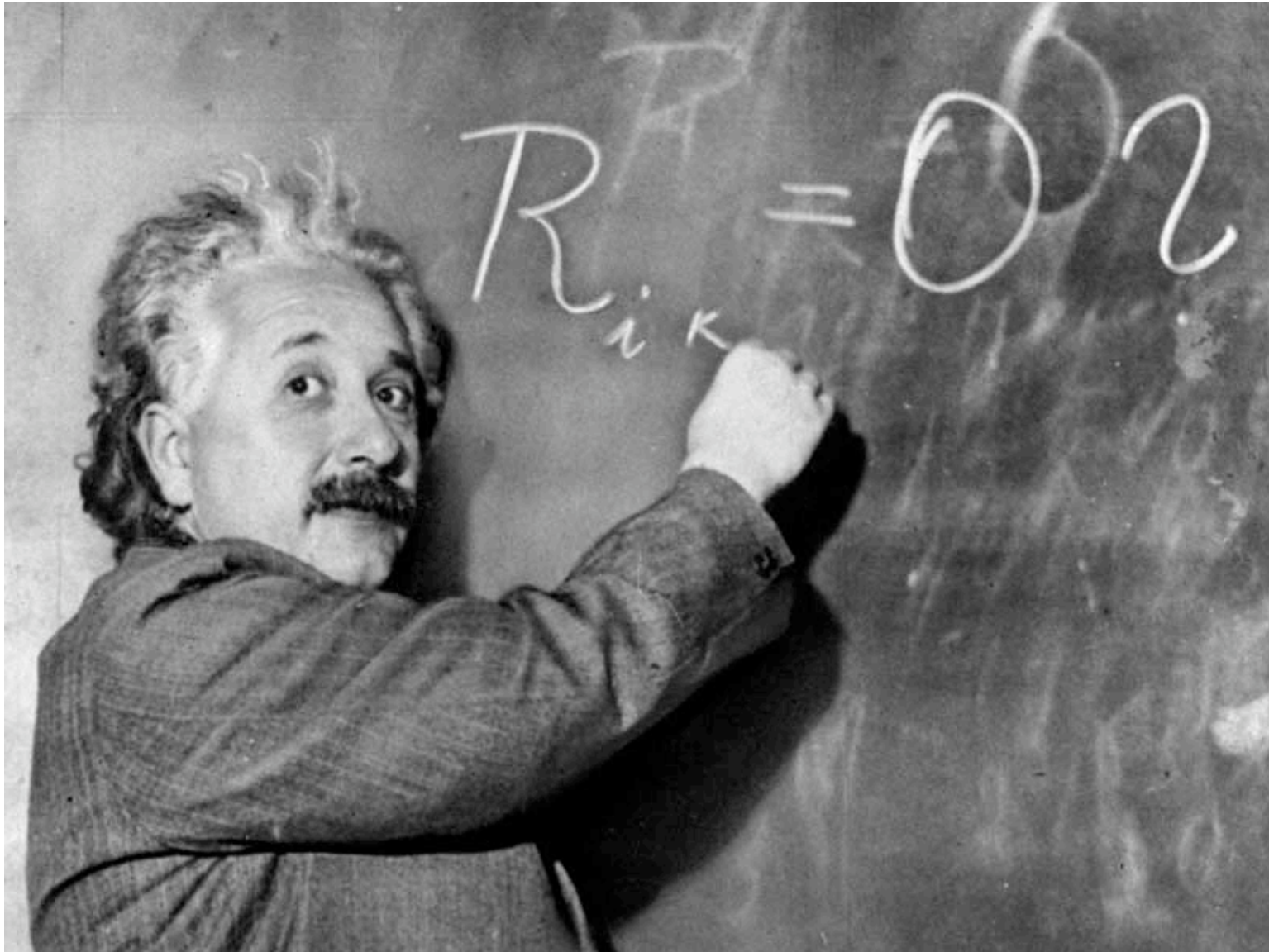
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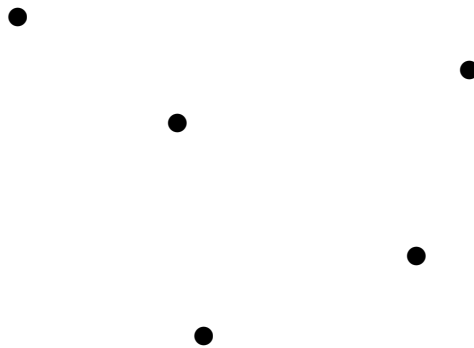
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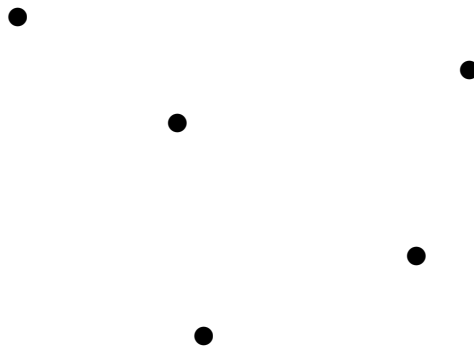
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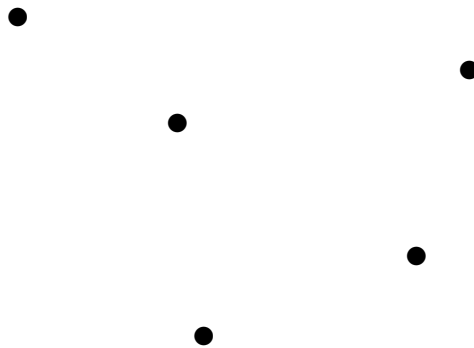
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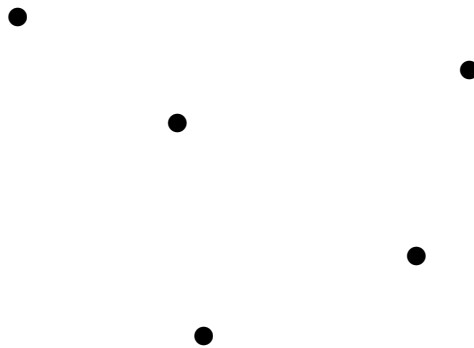


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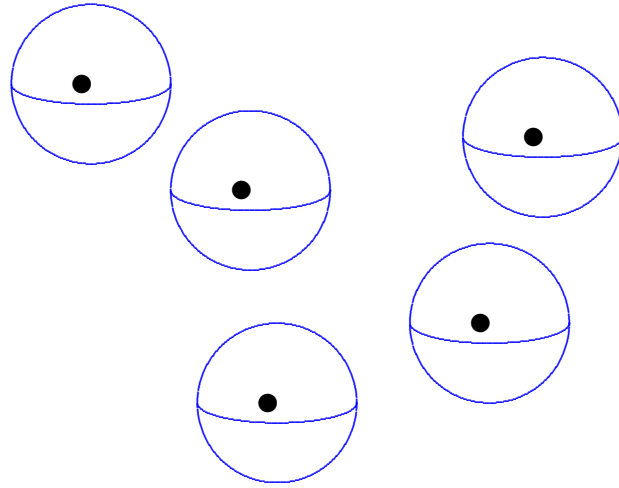
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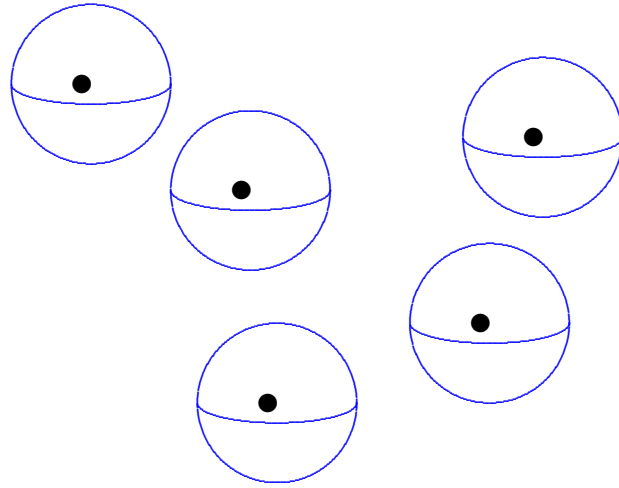
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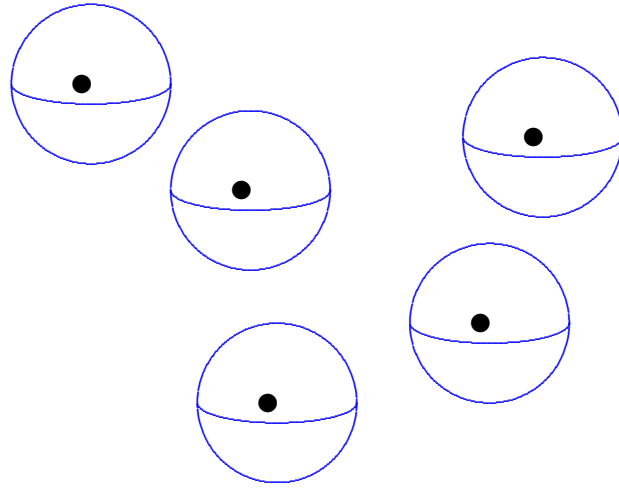
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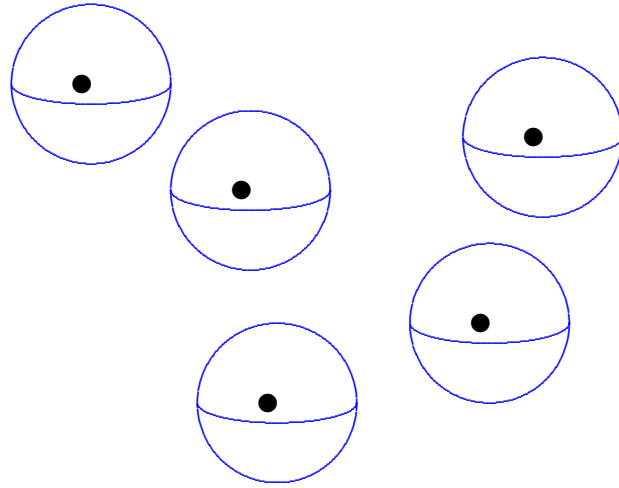
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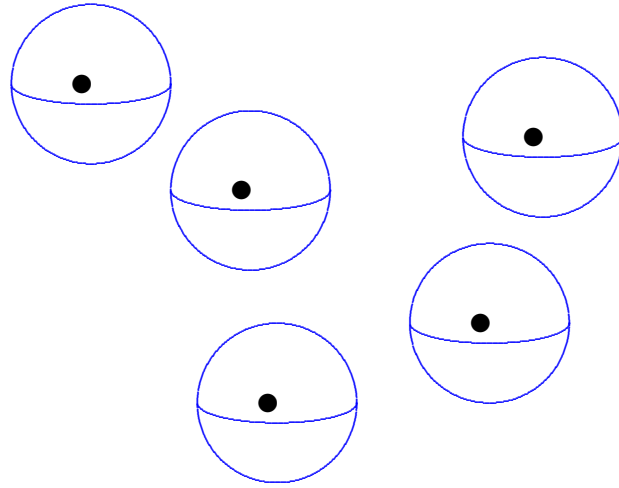
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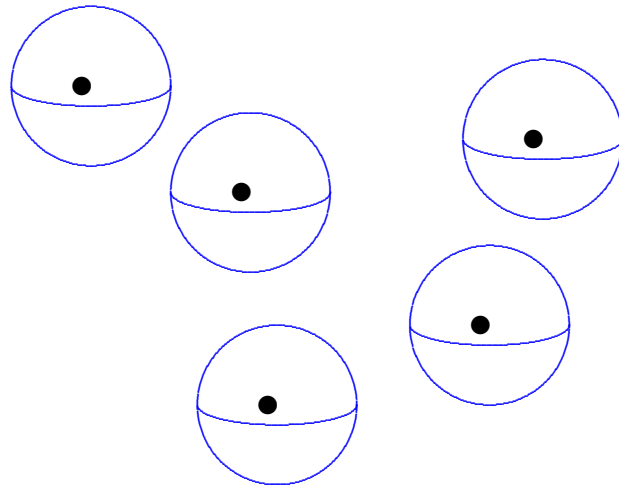
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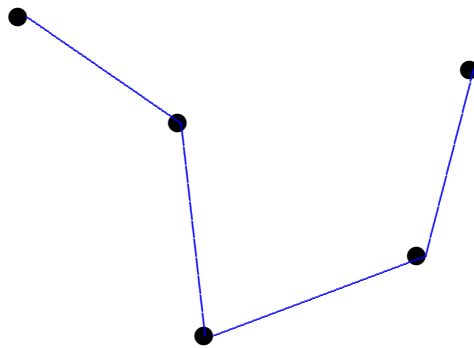
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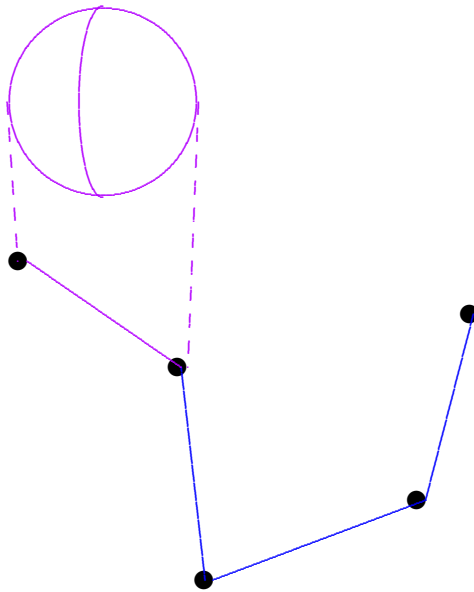
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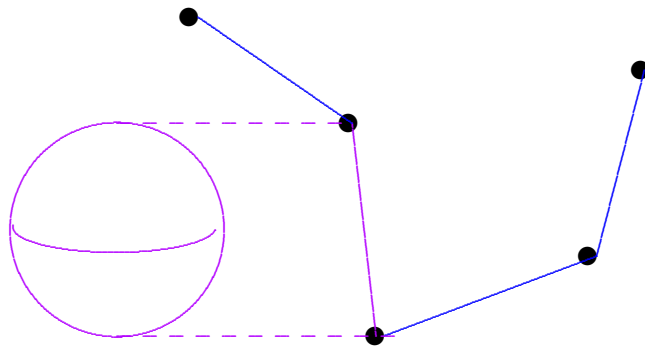
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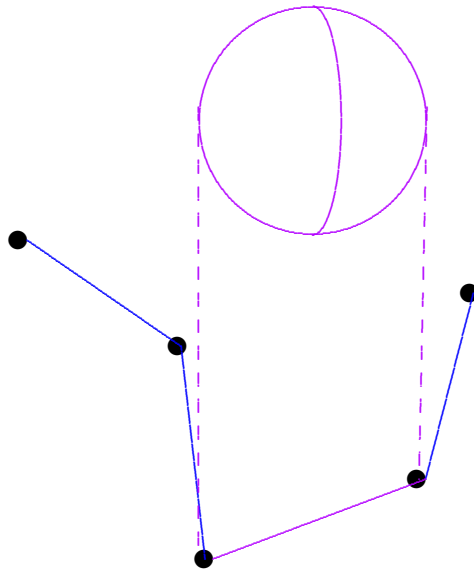
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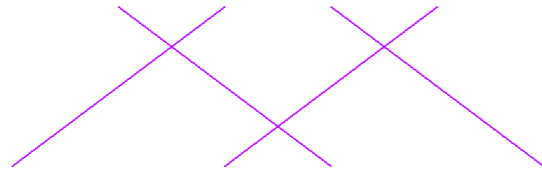
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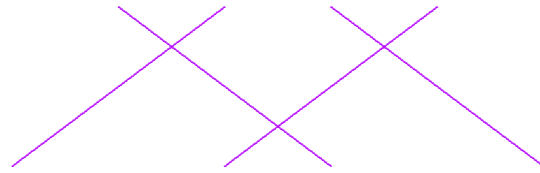
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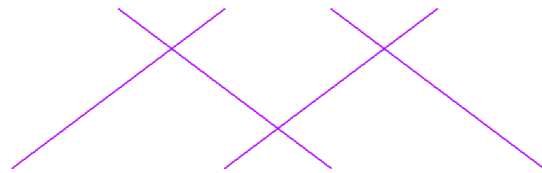


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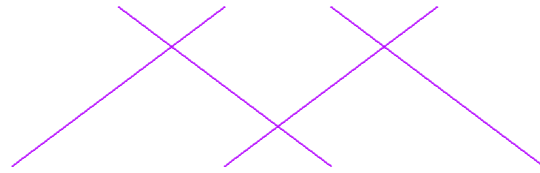
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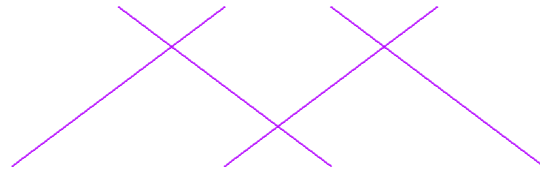


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cf. Bishop-Gromov inequality!

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This last property distinguishes the ALF spaces from other classes of gravitational instantons:

ALG, ALH, ALG*, ALH*, ...

Example.

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This J determines opposite orientation from the hyper-Kähler complex structures.

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Non-Kähler, but **conformally** Kähler!

Hawking also explored non-hyper-Kähler examples. . .

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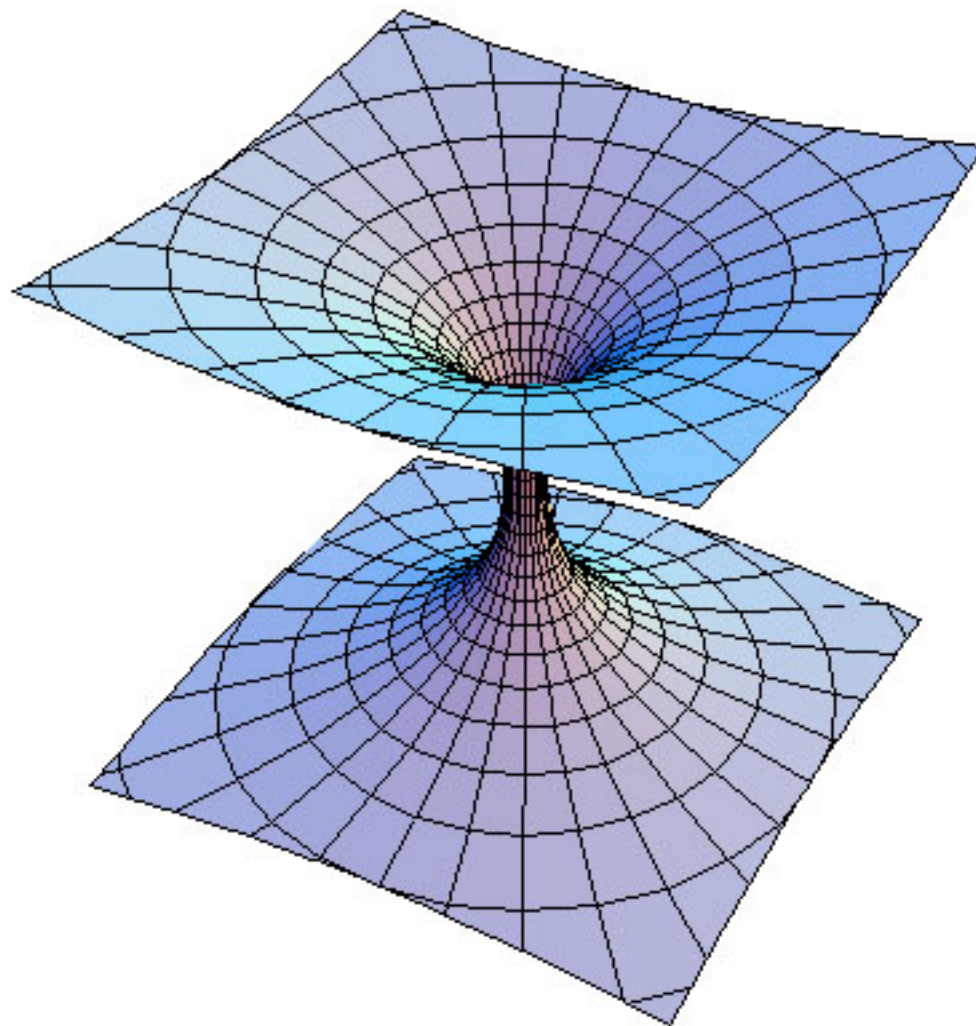
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$$\mathbb{R} \times S^2 \subset \mathbb{R}^2 \times S^2$$

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$$\mathcal{U} = O(\rho^{-1}), \quad \nabla\mathcal{U} = O(\rho^{-2}), \quad \dots \quad \nabla^3\mathcal{U} = O(\rho^{-4})$$

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$$\implies \text{Vol}(B_\rho) \sim \text{const} \cdot \rho^3$$

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Reversing orientation interchanges $\Lambda^+ \leftrightarrow \Lambda^-$.

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Equation $\delta W^+ = 0$ conformally invariant w/ weight.

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for $fW^+ \in \text{End}(\Lambda^+)$.

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Let $\alpha \geq \beta \geq \gamma$ be eigenvalues of W^+ :

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$$\alpha > 0, \quad \gamma < 0, \quad \text{if } W^+ \neq 0$$

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$$2\sqrt{6}|W^+| + |s| \geq |\omega \wedge \star d\omega|^2$$

everywhere on (M, h) .

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Then (M, h) is an extremal Kähler manifold with non-constant, positive scalar curvature.

Theorem A. *Let (M, g_0) be one of the ALF toric Hermitian gravitational instantons featured in Biquard-Gauduchon classification. Then any other Ricci-flat Riemannian metric g on M which is sufficiently C_1^3 -close to g_0*

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However, if g_0 is Kerr or Taub-bolt, we can prove a more definitive rigidity result, because g_0 then has both $\det(W^+) > 0$ and $\det(W^-) > 0$.

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Ambi-Kähler Einstein metrics are ambi-toric!

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Thank you for inviting me!





It's a pleasure to be here!