Anti-Self-Dual 4-Manifolds,

Quasi-Fuchsian Groups, &

Almost-Kähler Geometry

Claude LeBrun
Stony Brook University

Conformal and Symplectic Geometry
University of Auckland, 7 February, 2018

Most recent results joint with

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Christopher J. Bishop
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e-print: `arXiv:1708.03824 [math.DG]`

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To appear in Comm. An. Geom.

Themes of this conference:

Conformal Geometry

Symplectic Geometry

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Higher dimensions are demonstrably different.

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Imitates Kähler geometry in a non-Kähler setting.

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$$\text{Kähler} \Longleftrightarrow J \text{ integrable.}$$

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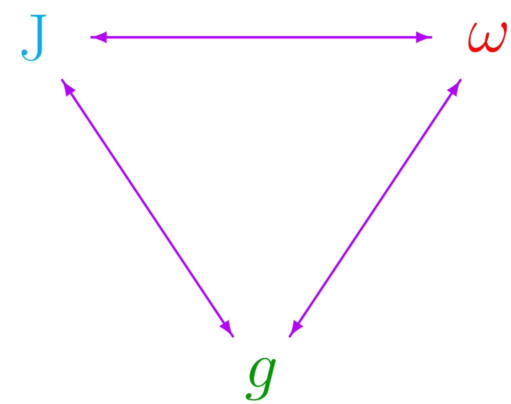
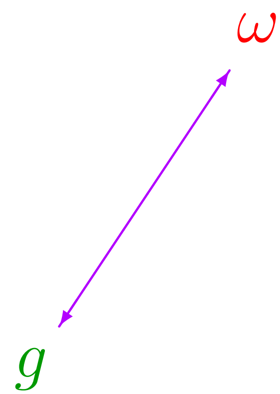
For example, can avoid explicitly mentioning J.


J
ω
g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$$\updownarrow$$

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Proposition. *A conformal class $[g]$ on a smooth compact oriented* 4*-manifold M is represented by an almost-Kähler* metric *g iff it carries a self-dual* harmonic 2-*form* ω *that is $\neq 0$ everywhere.*

$$[g] \longleftrightarrow \omega$$

Proposition. *A conformal class* $[g]$ *on a smooth compact oriented* 4*-manifold* M *is represented by an almost-Kähler metric* g *iff it carries a self-dual harmonic* 2*-form* ω *that is* $\neq 0$ *everywhere.*

Moreover, the set of conformal classes $[g]$ *on* M *that carry such a harmonic form* ω *is open in the* C^2 *topology.*

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“Conformal classes of symplectic type”

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In particular, the numbers

$$b_{\pm}(M) = \dim \mathcal{H}^{\pm}_g$$

are independent of g, and so are invariants of M.

$b_{\pm}(M)$?

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Diagonalize:

$$\begin{bmatrix} +1 & & & & & \\ & \ddots & & & & \\ & & +1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}.$$

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$$b_2(M) = b_+(M) + b_-(M)$$

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\tau(M) = b_+(M) - b_-(M)
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"Signature" of M.

Signature defined in terms of intersection pairing

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Signature defined in terms of intersection pairing,

but also expressible as a curvature integral:

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$$\begin{aligned}\tau(M) &= \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu \\ &= \langle \frac{1}{3}p_1(M), [M]\rangle\end{aligned}$$

(Thom-Hirzebruch Signature Formula)

Signature defined in terms of intersection pairing,

but also expressible as a curvature integral:

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

Has major consequences in conformal geometry.

On oriented (M^4, g),

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where

$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W_+ &= \text{self-dual Weyl curvature} \\ W_- &= \text{anti-self-dual Weyl curvature} \end{aligned}$$

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s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature *(conformally invariant)*

W_- = anti-self-dual Weyl curvature ″

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measures the deviation from conformal flatness, because (M^4, g) is locally conformally flat $\Longleftrightarrow$ its Weyl curvature $W = W_+ + W_-$ vanishes.

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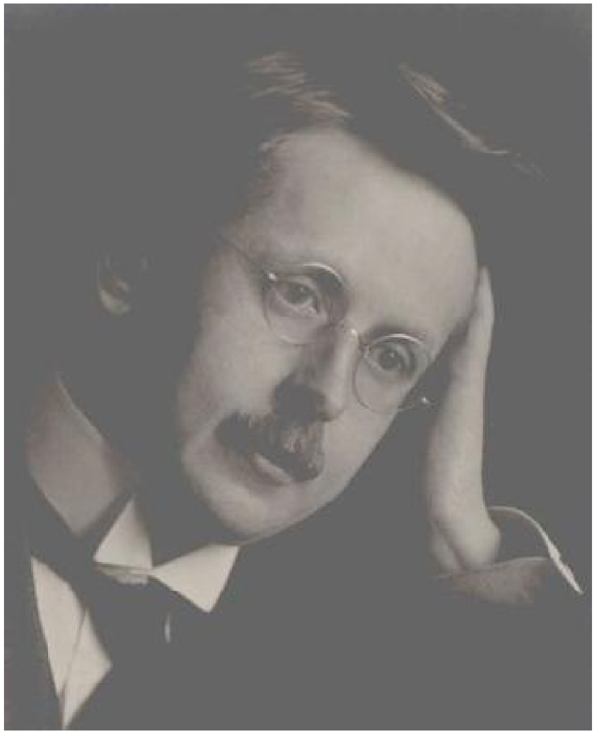


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Basic problems: For given smooth compact M^4,

- What is $\inf \mathscr{W}$?
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(ASD)

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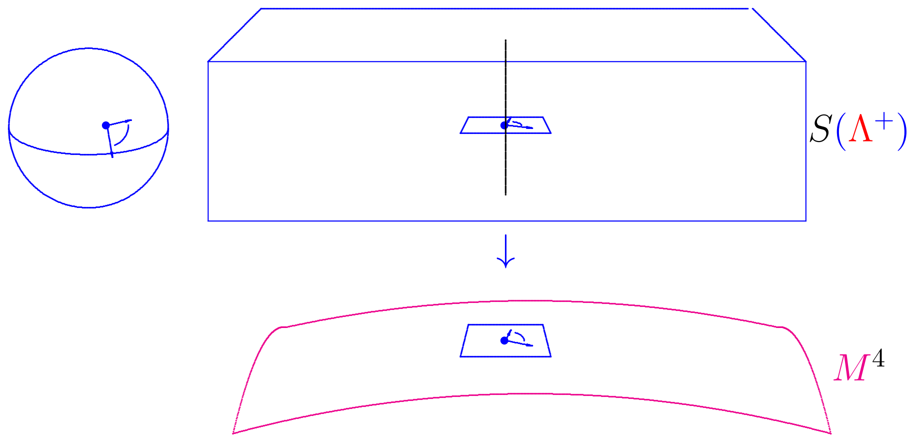

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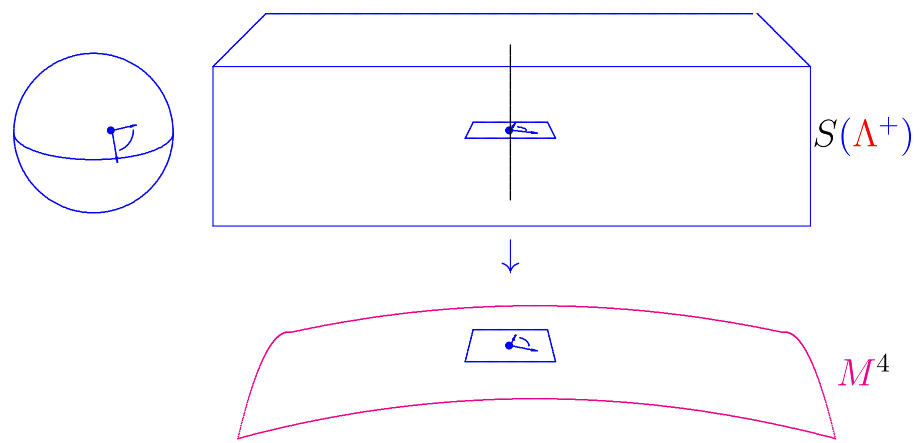

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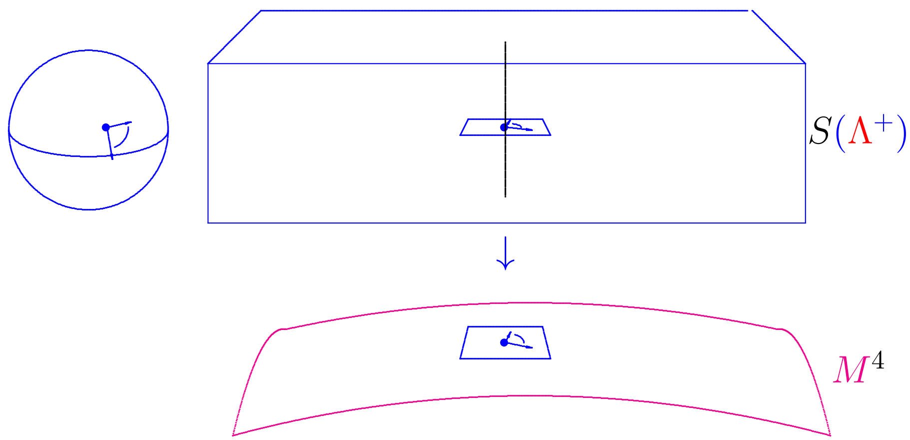

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Reconceptualizes earlier work by Penrose.

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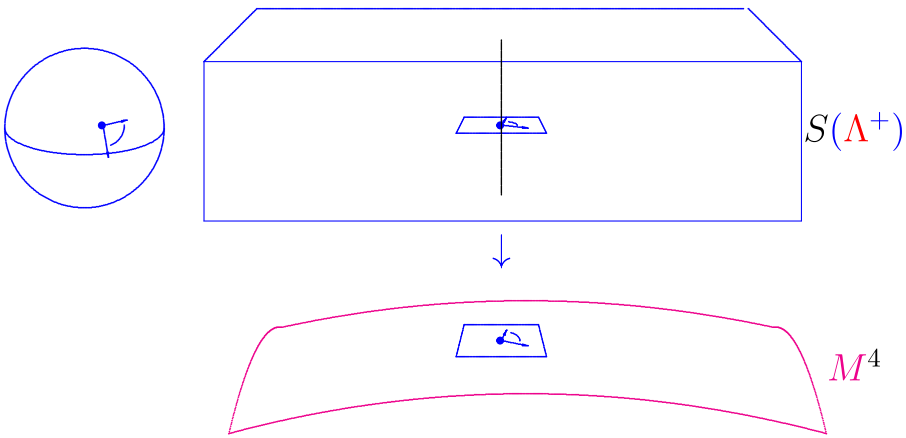

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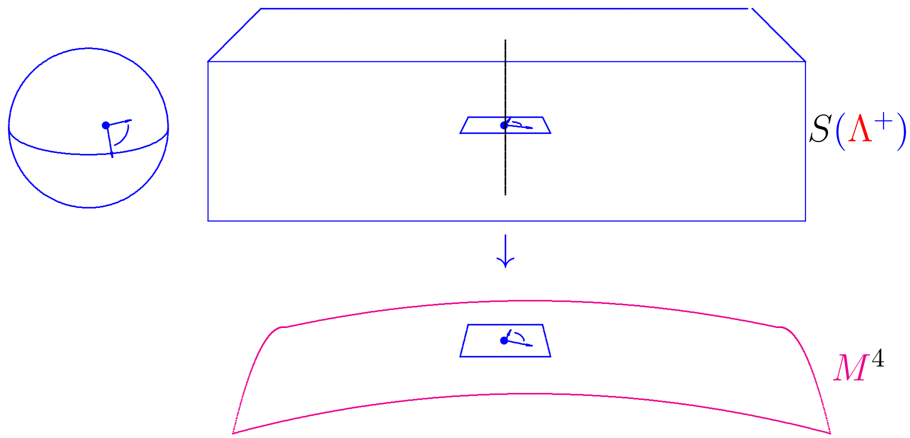

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Motivates study of ASD metrics,
and yields methods for constructing them.

So ASD metrics are linked to complex geometry. . .

A different link with complex geometry:

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Results proved about SFK in '90s foreshadowed
many more recent results about general case.

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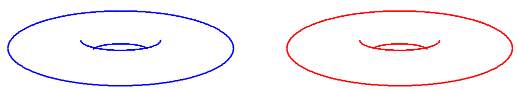

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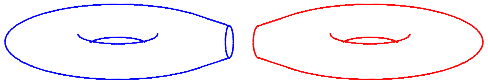

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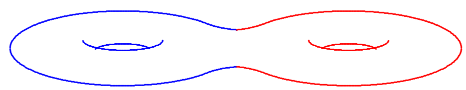

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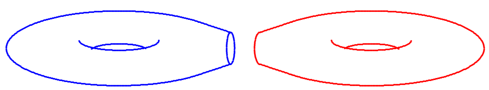

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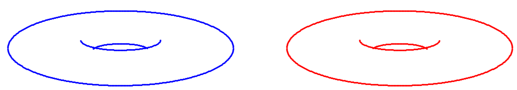

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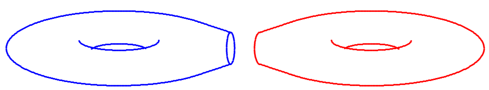

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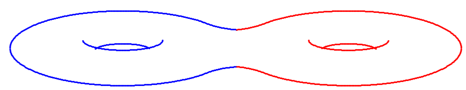

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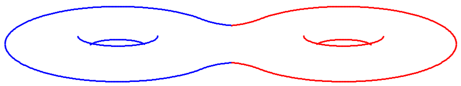

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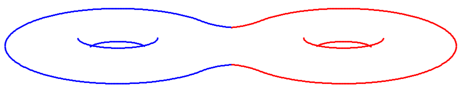

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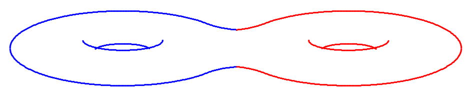

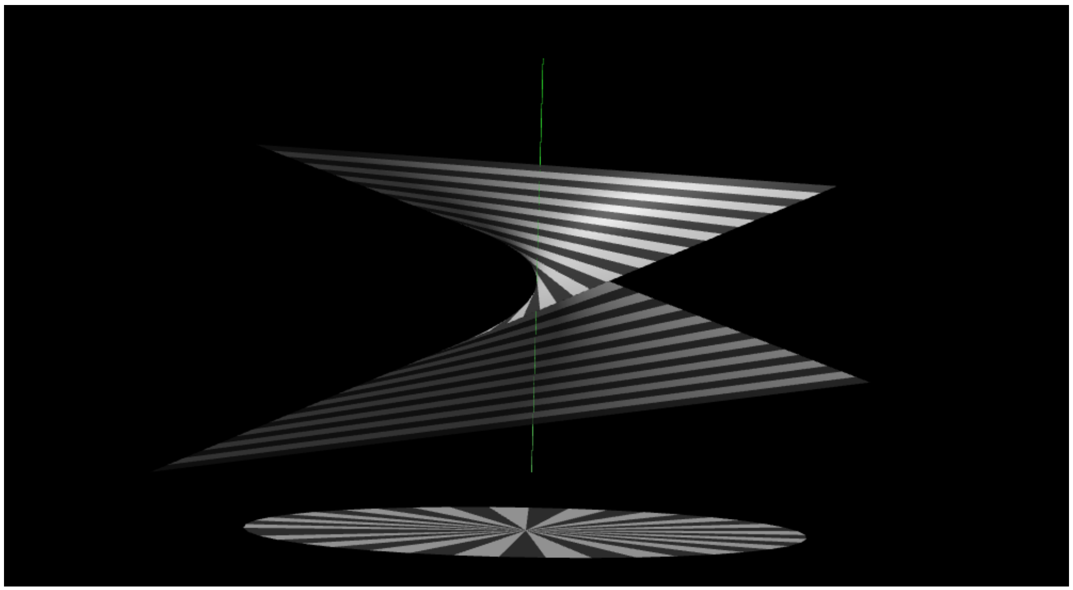

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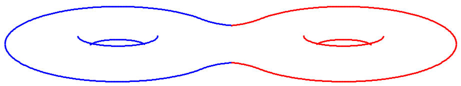

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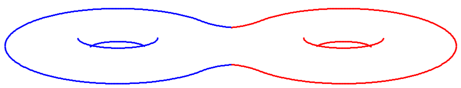

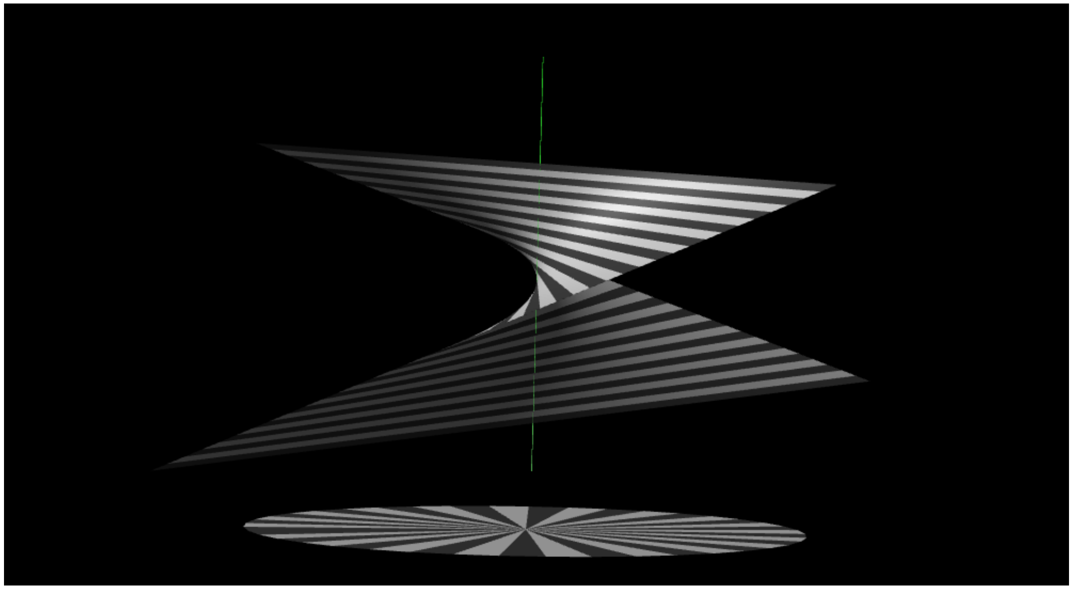

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Theorem (L '15). *True for conformal classes of symplectic type.*

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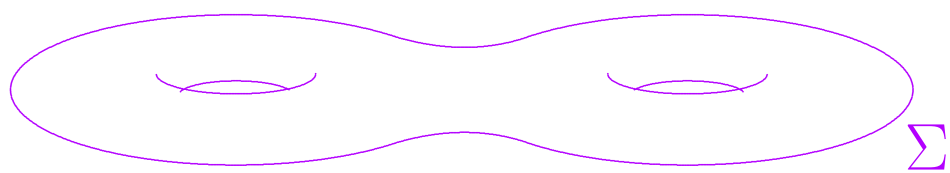
Almost-Kähler ASD metrics sweep out an open set in the ASD moduli space.

Example.



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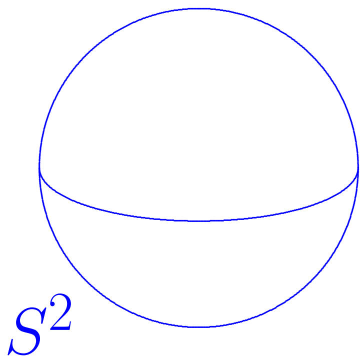
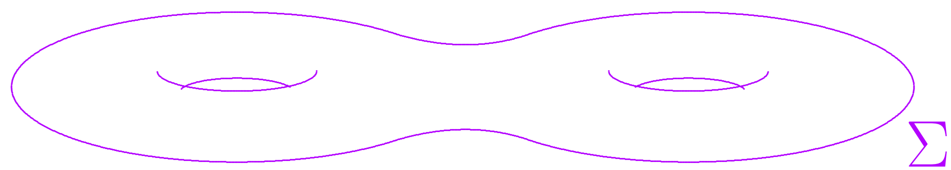
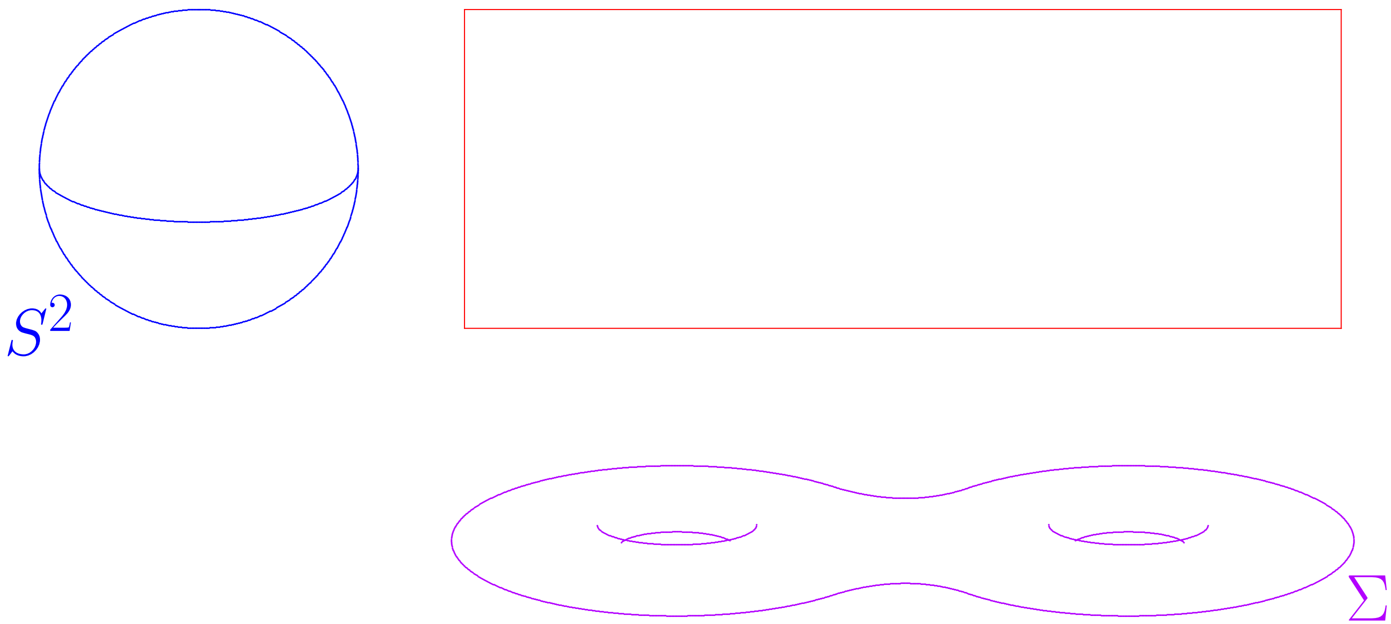




Example.

$$M = \Sigma \times S^2$$



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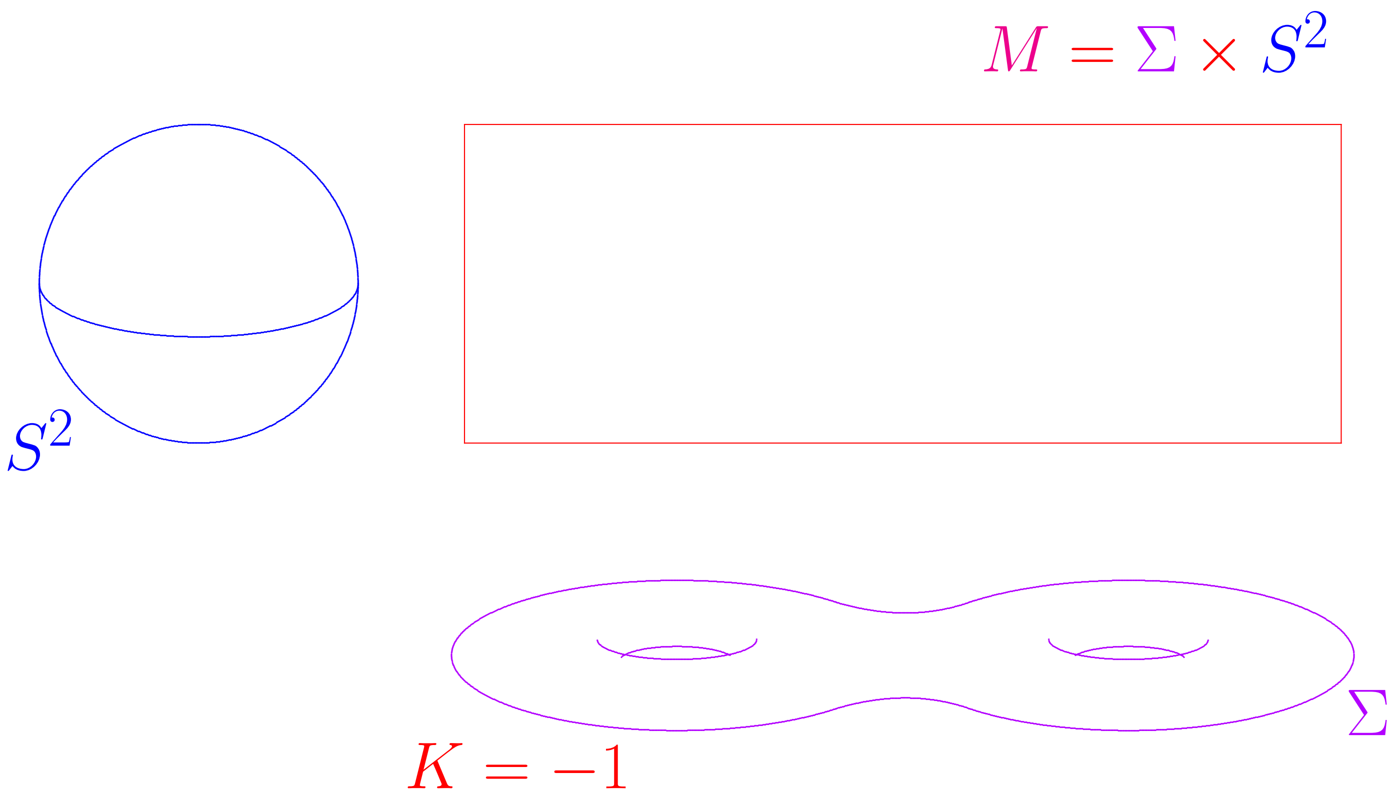

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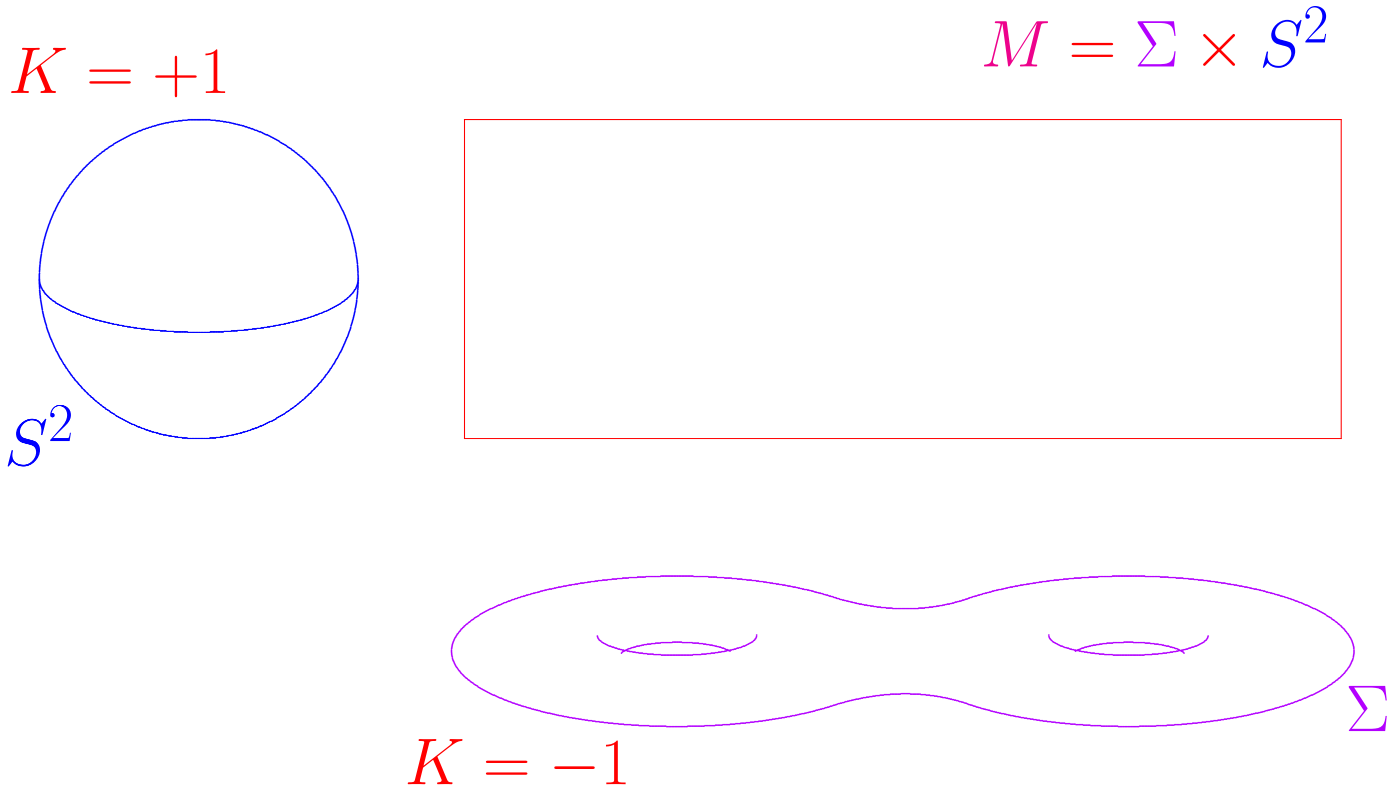

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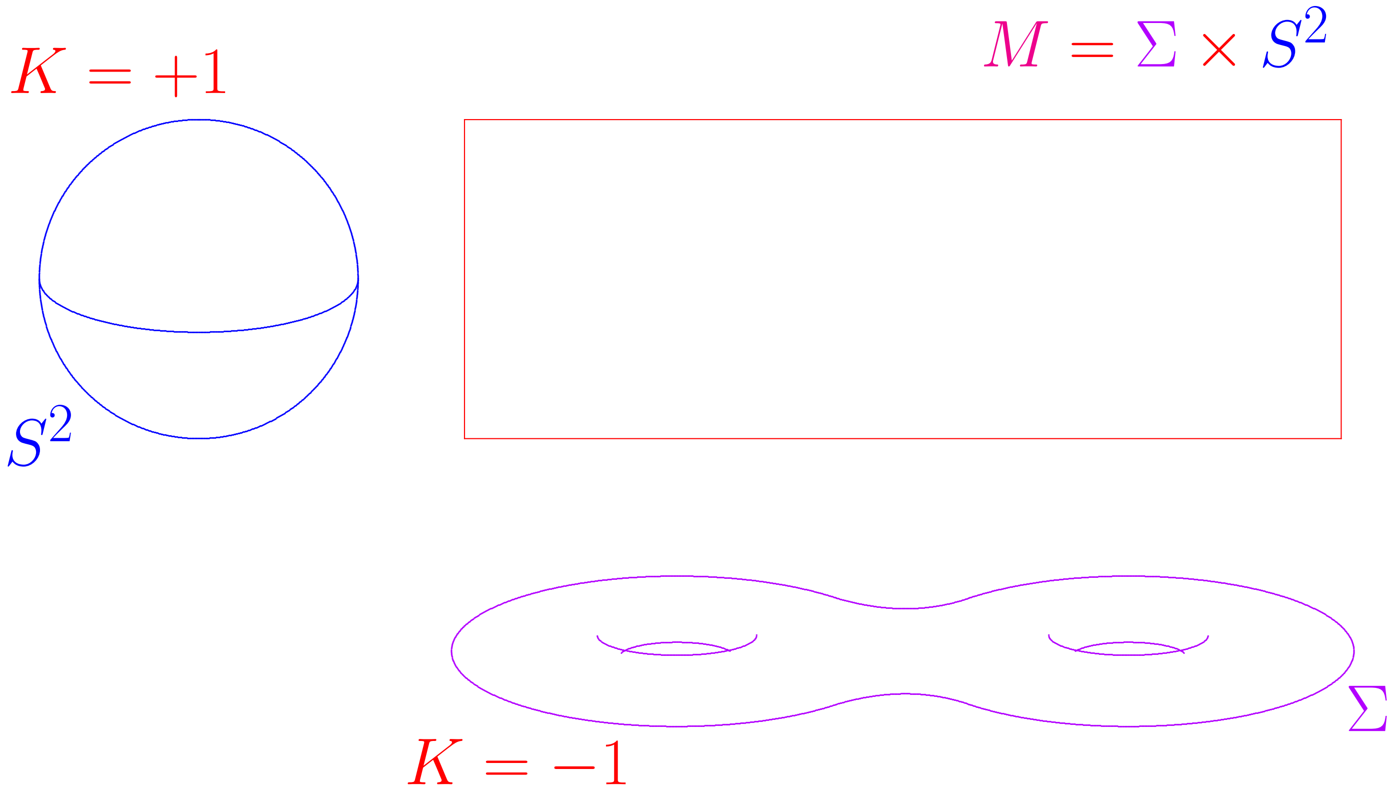
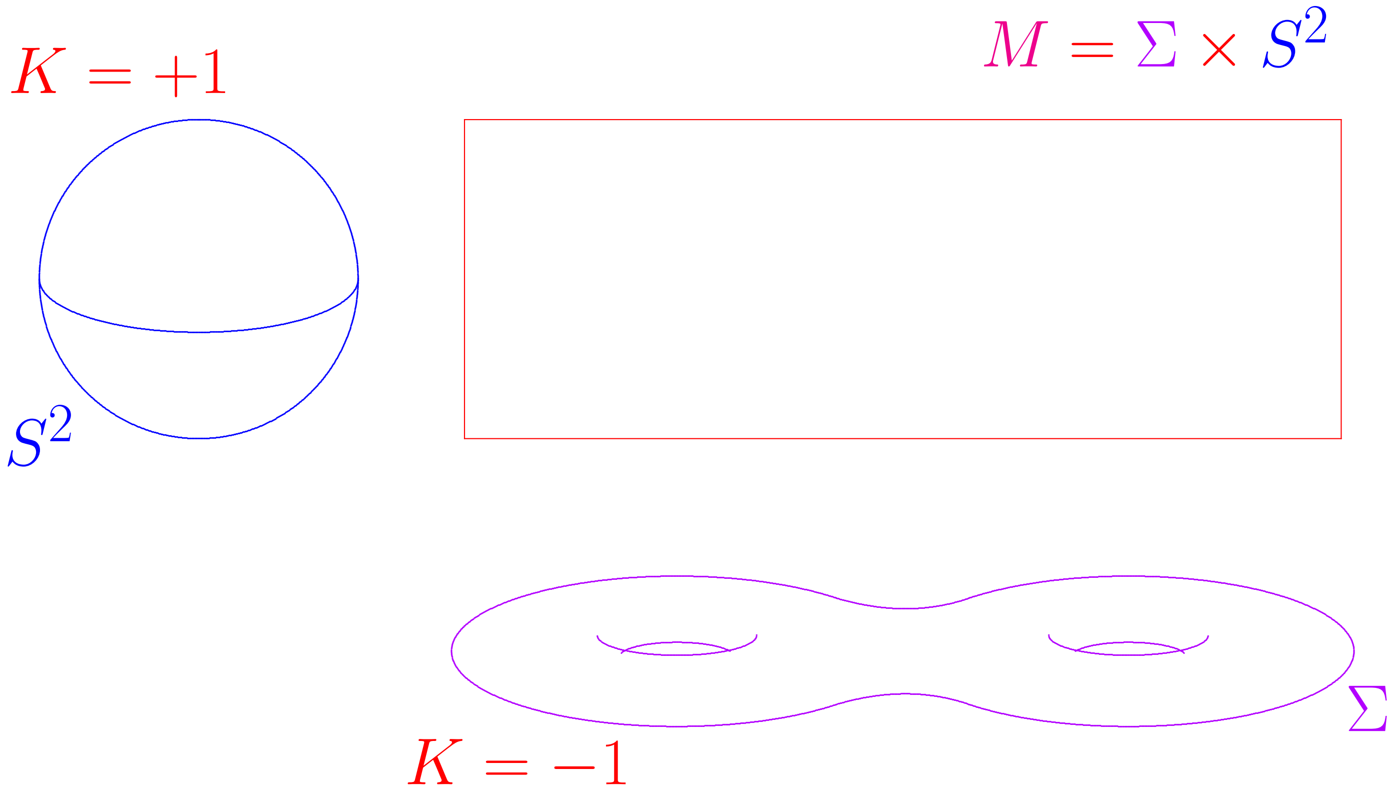


Product is scalar-flat

Example.



Product is scalar-flat Kähler.

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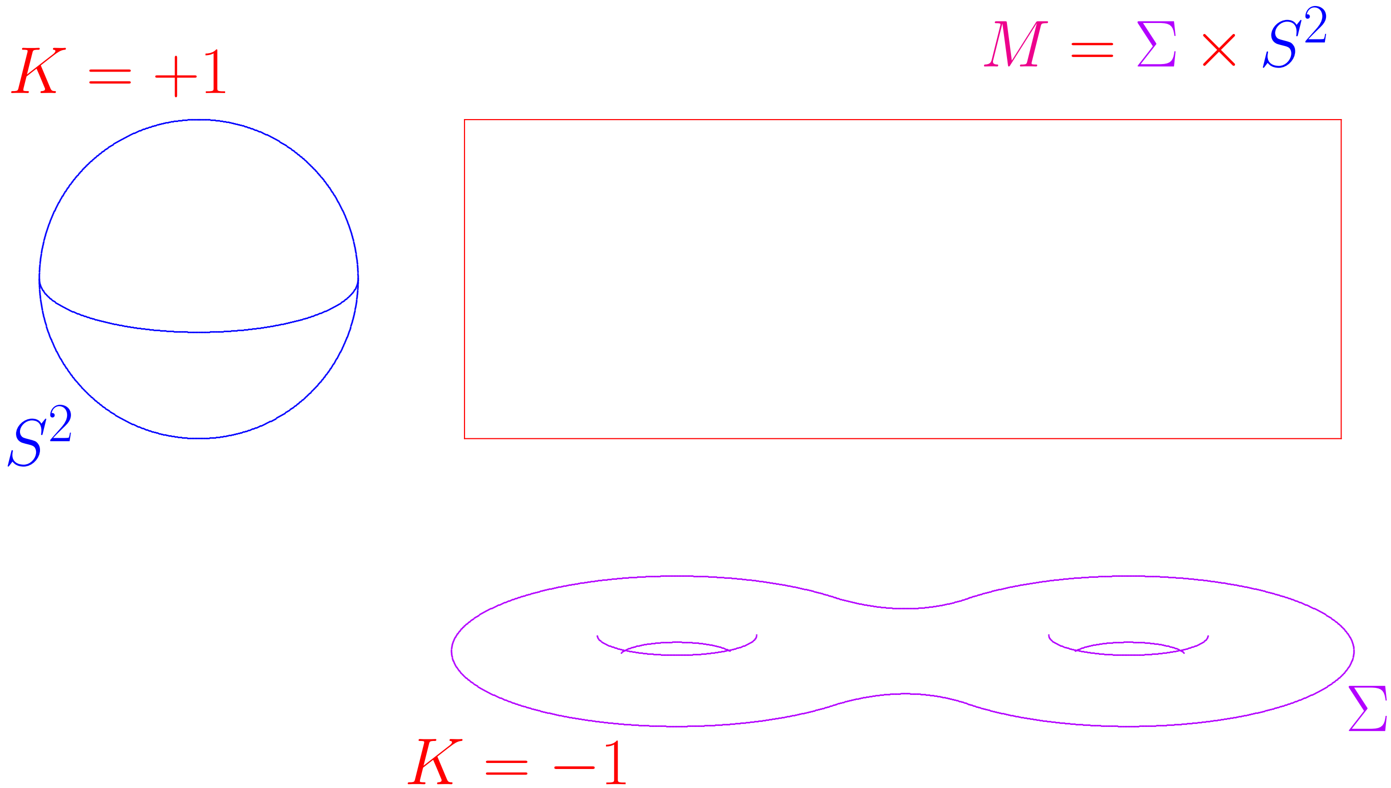

Product is scalar-flat Kähler.

For both orientations!

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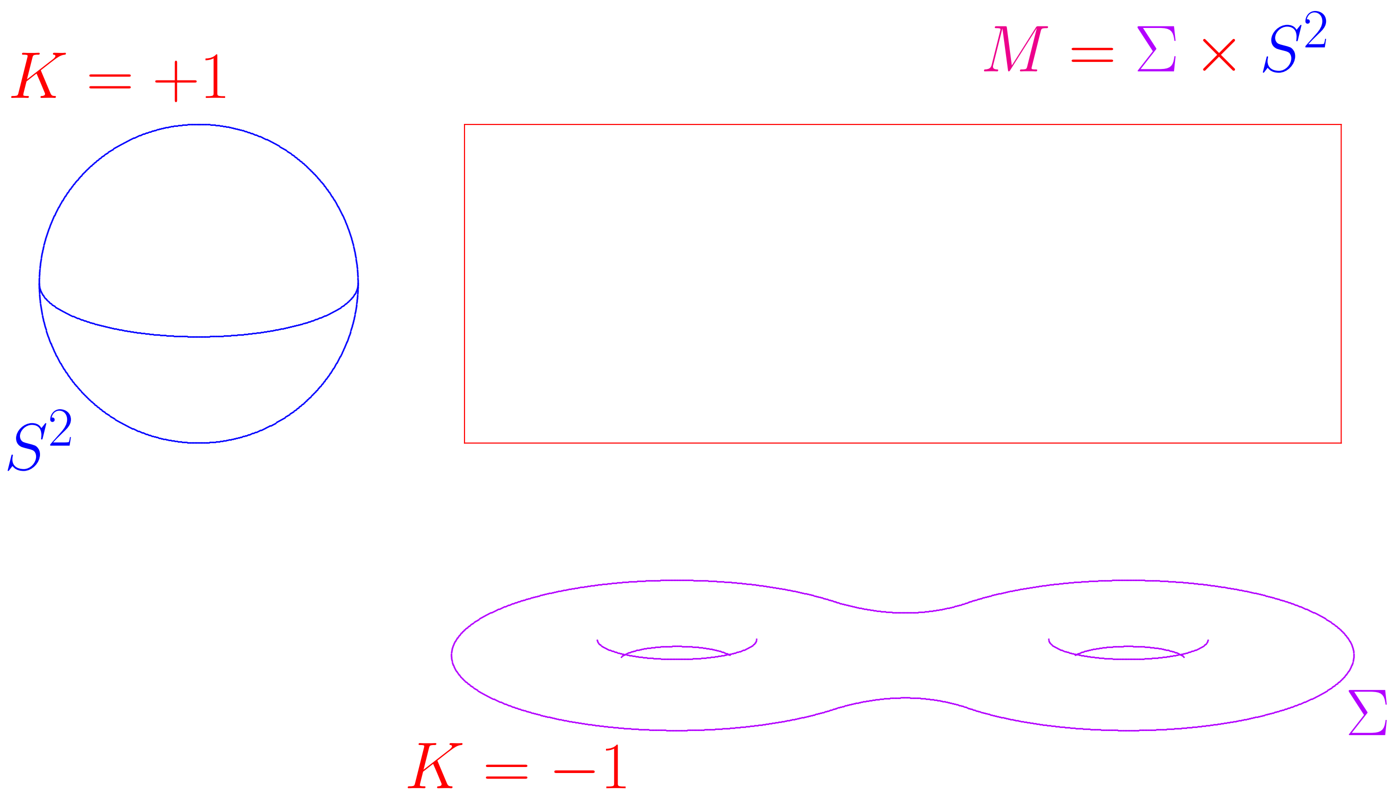

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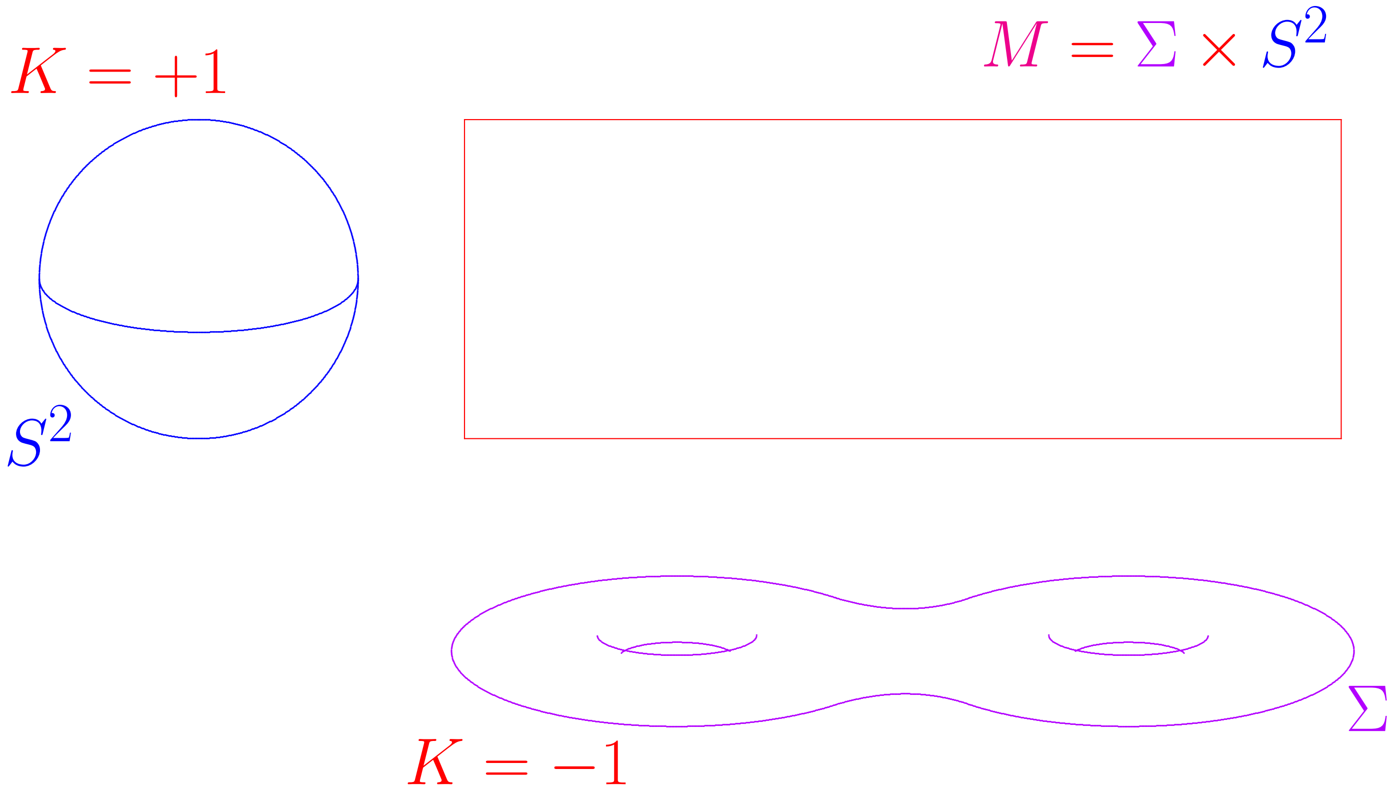

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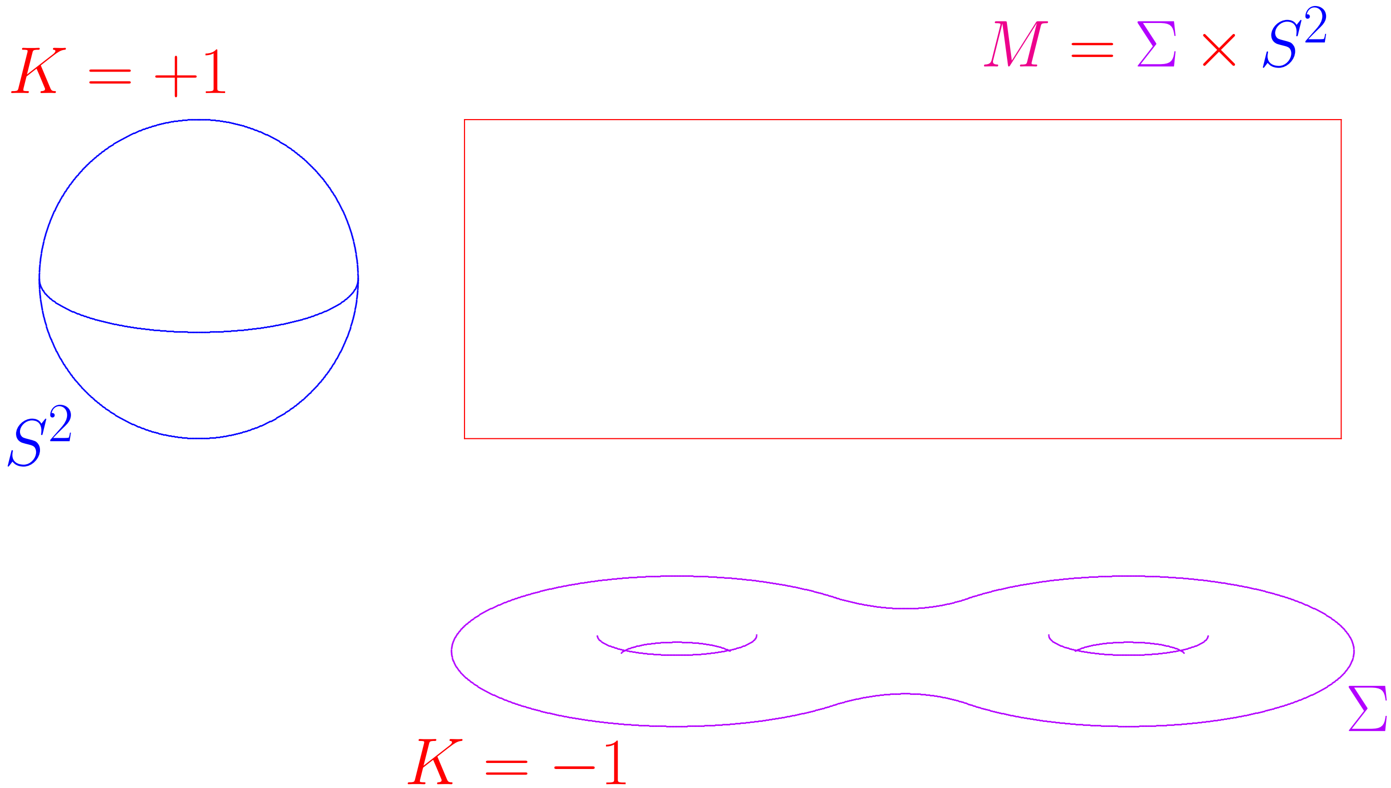

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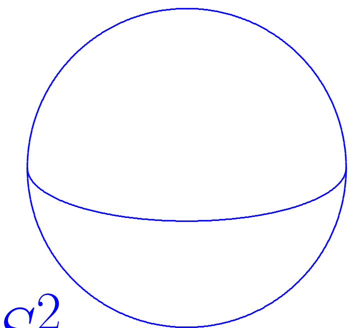
Locally conformally flat!

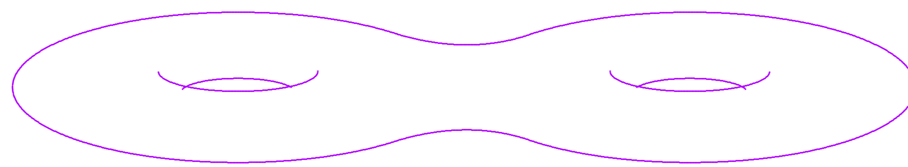
Example.

$K = +1$

$M = \Sigma \times S^2$



S^2

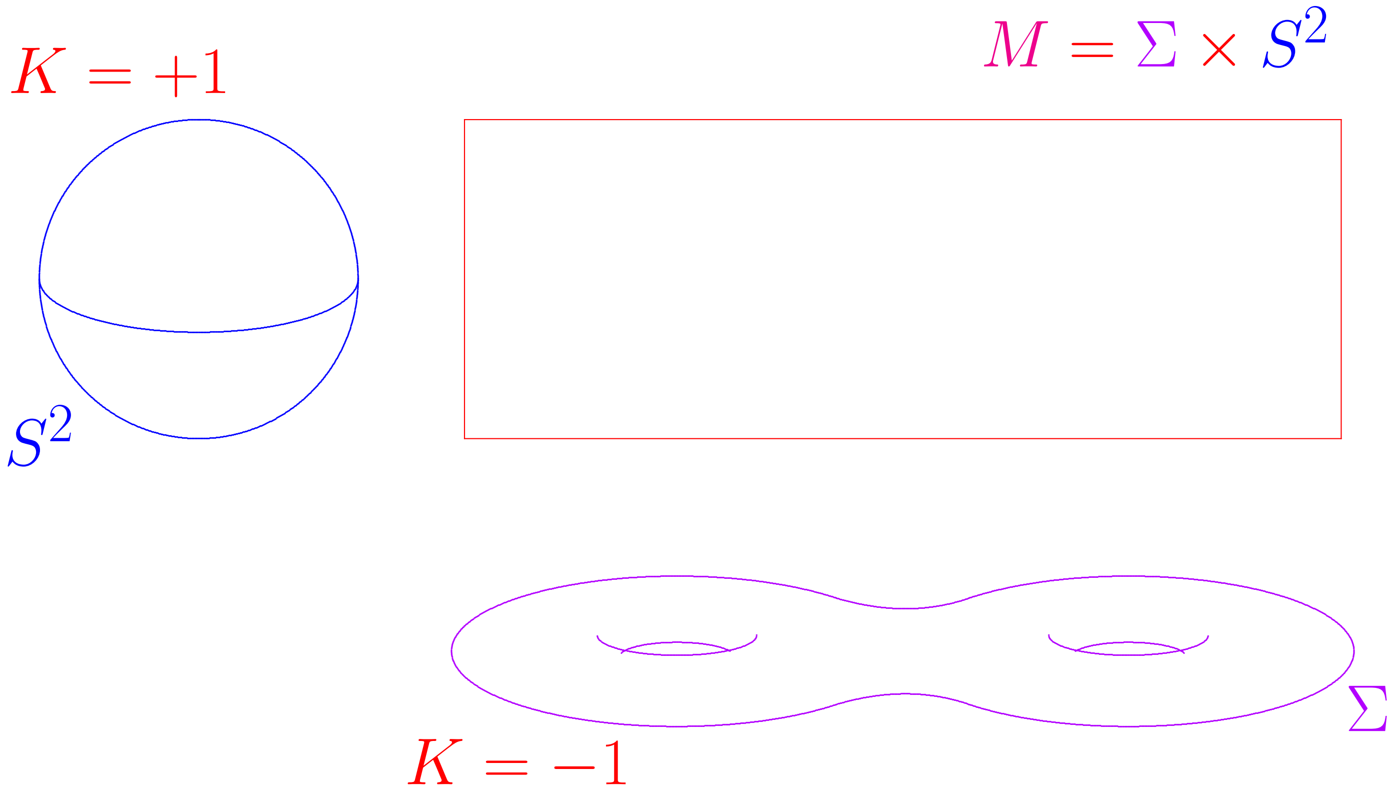


Σ

$K = -1$

$$\widetilde{M} = \mathcal{H}^2 \times S^2$$

Example.



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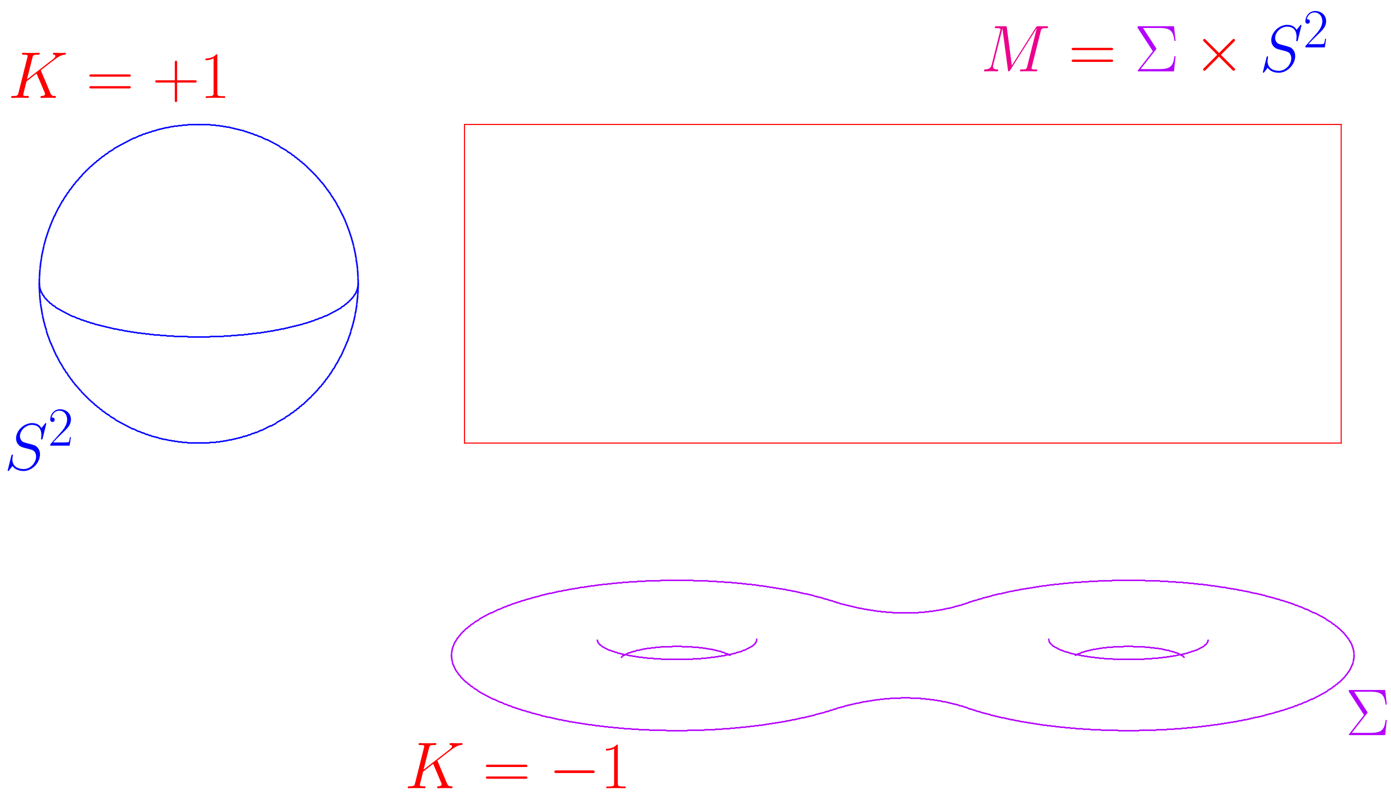

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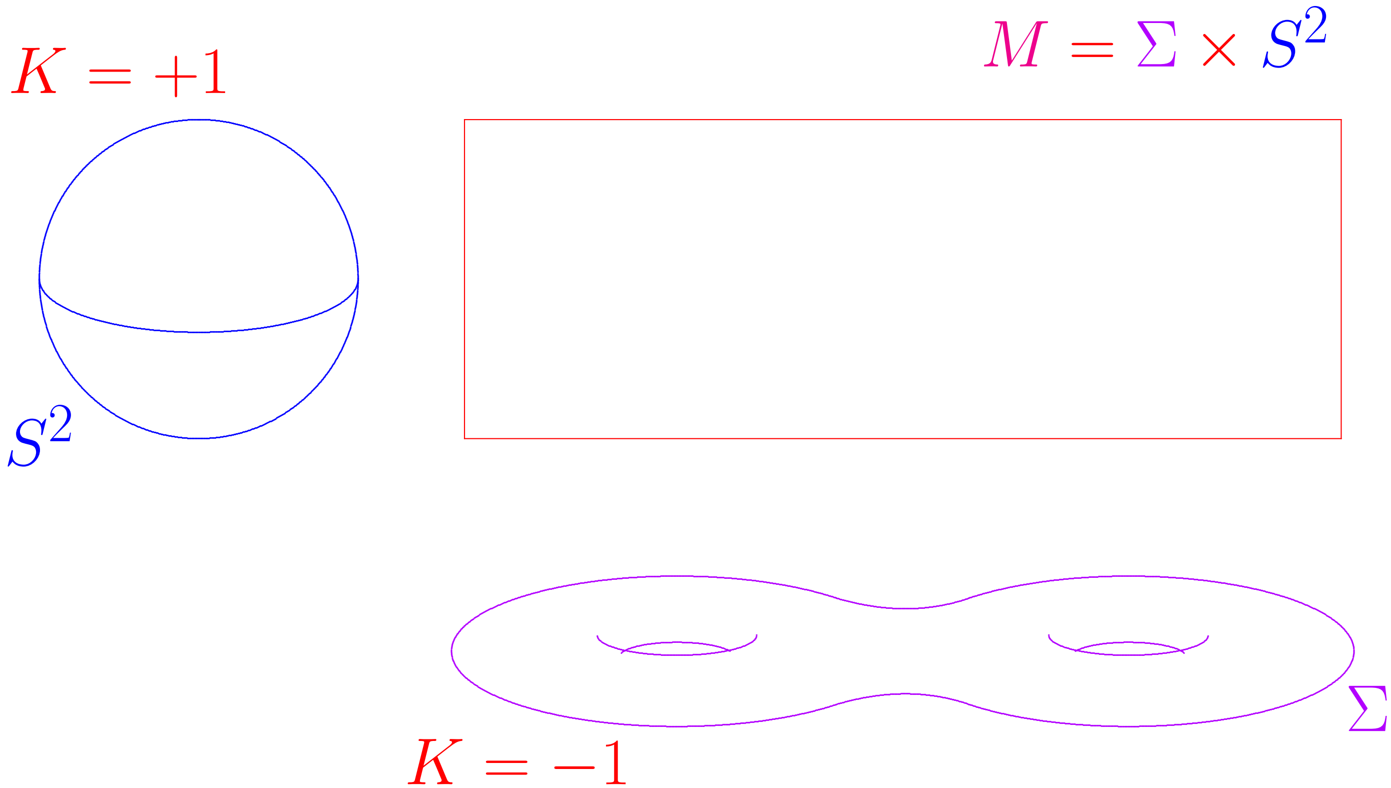

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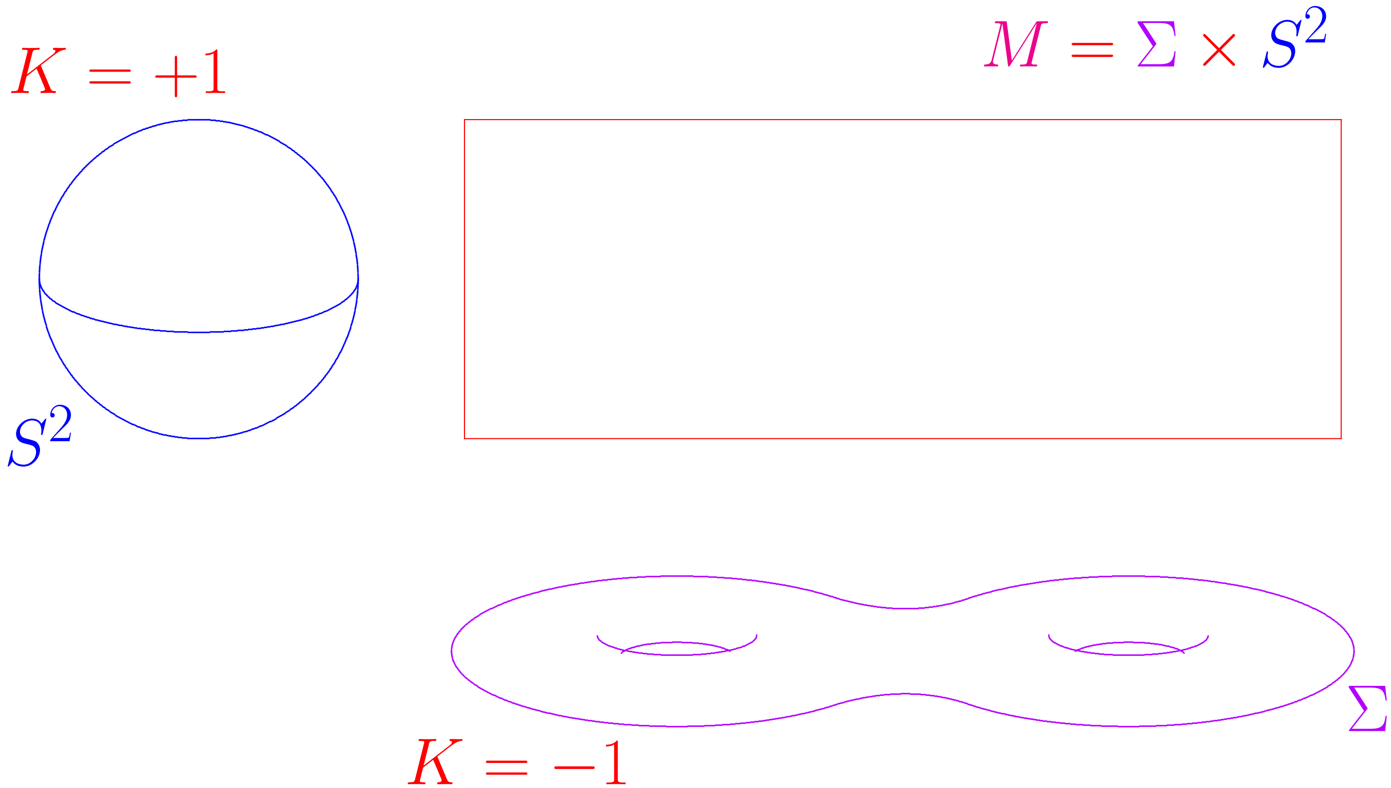

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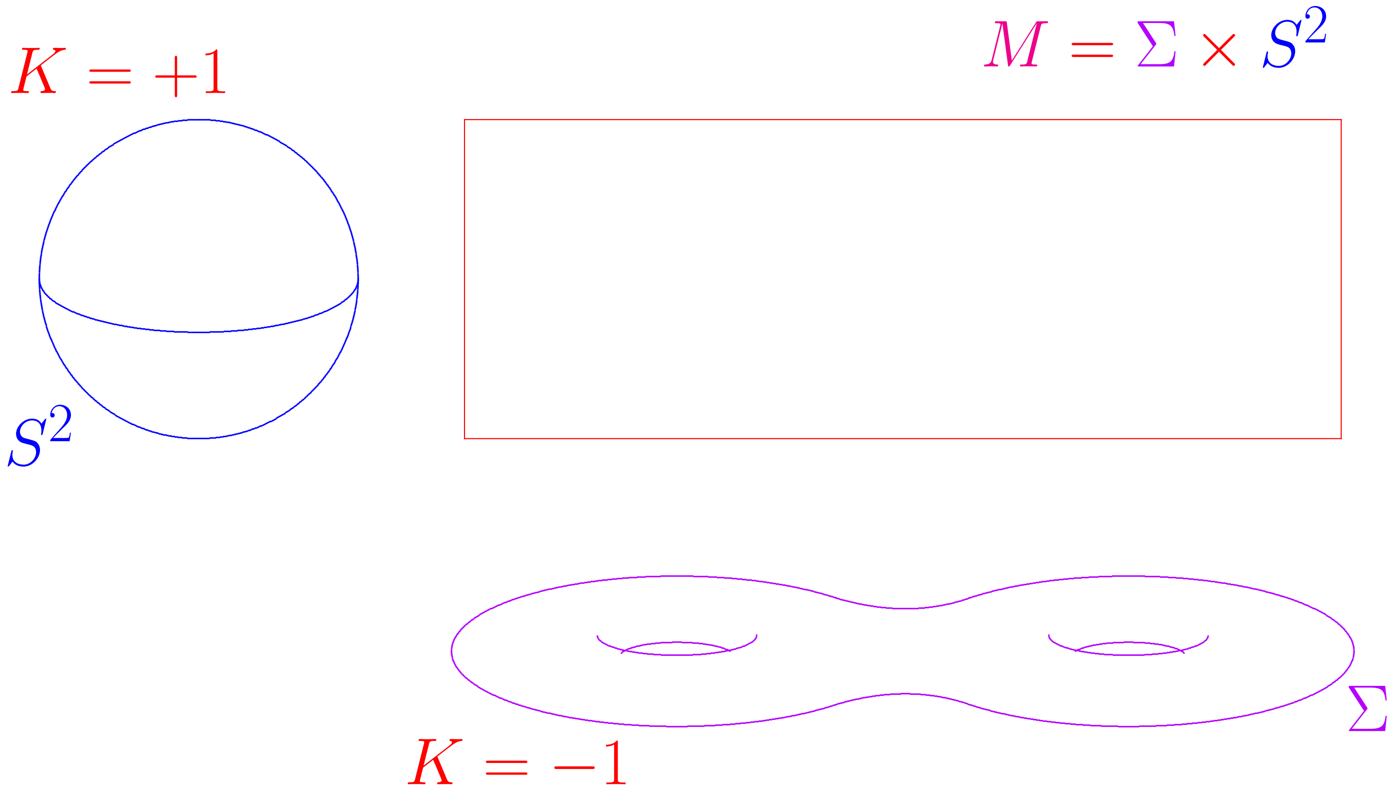

Scalar-flat Kähler deformations: $12(\mathbf{g} - 1)$ moduli

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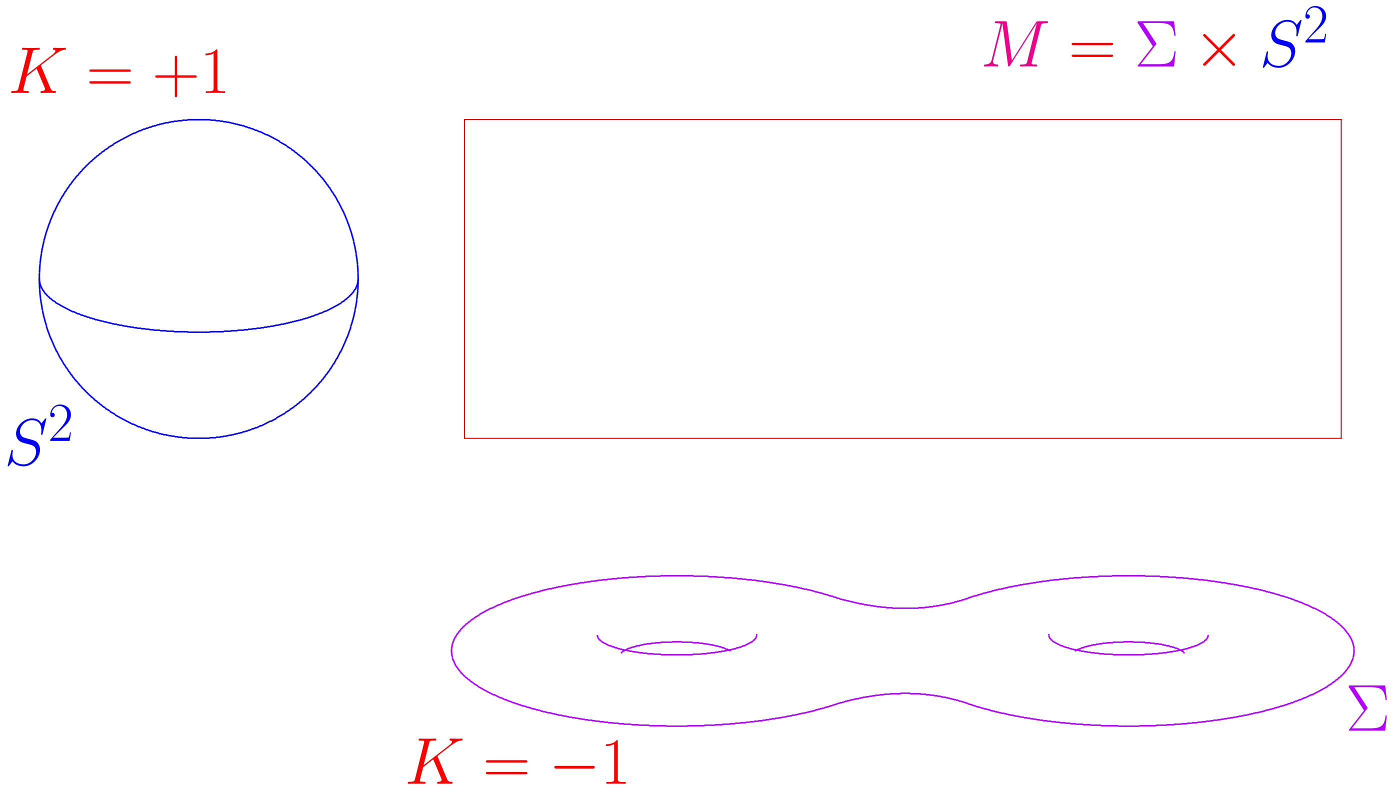
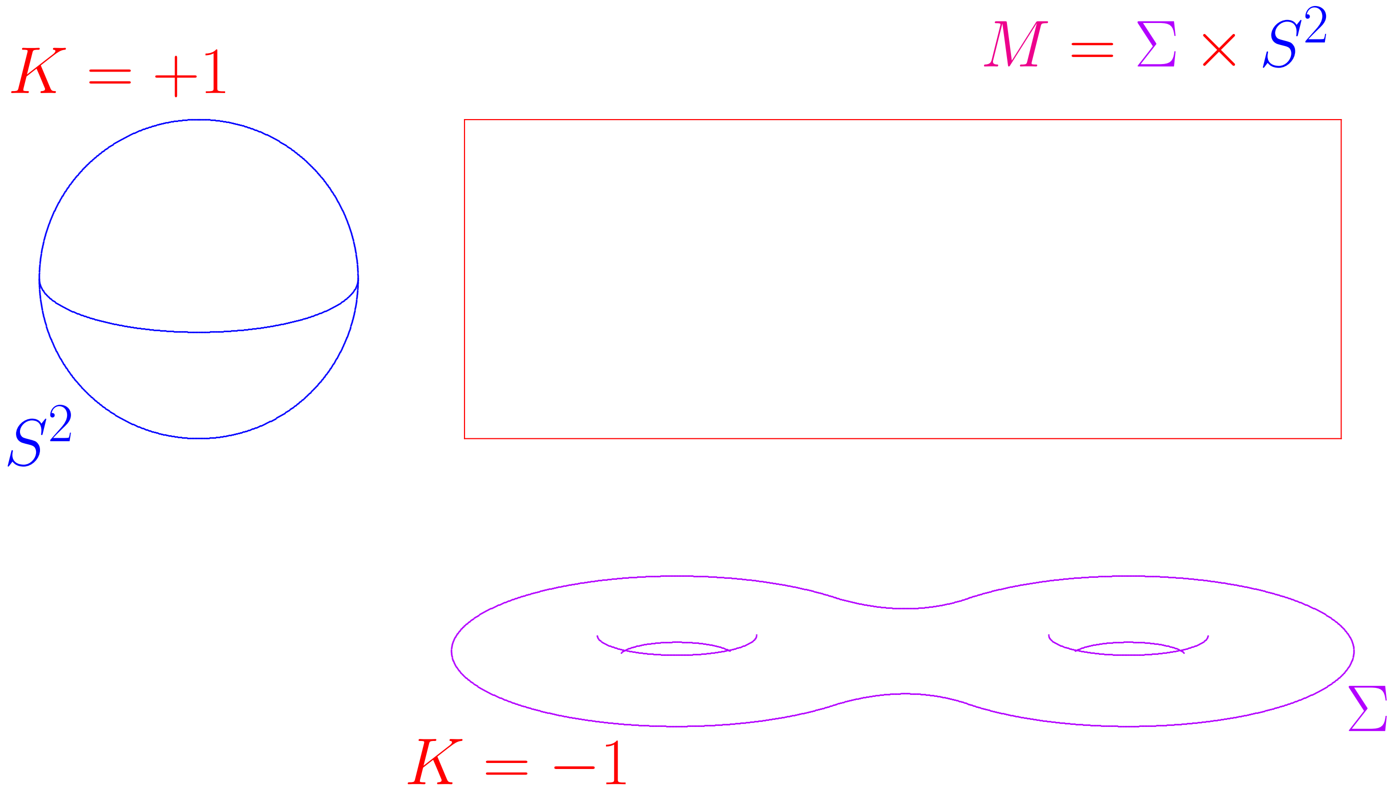


Scalar-flat Kähler deformations: $12(g-1)$ moduli

Locally conformally flat def'ms: $30(g-1)$ moduli

Example.



Scalar-flat Kähler deformations: $12(\mathfrak{g}-1)$ moduli

almost-Kähler ASD deformat'ns: $30(\mathfrak{g}-1)$ moduli

Last result indicates that almost-Kähler condition gives extra control on ASD conformal geometry.

Inyoung Kim '16: classification of almost-Kähler ASD roughly the same as in scalar-flat Kähler case.

Does this say anything about general ASD metrics?

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Alas, **No!**

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Same method simultaneously proves. . .

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Proof hinges on a construction of hyperbolic 3-manifolds.

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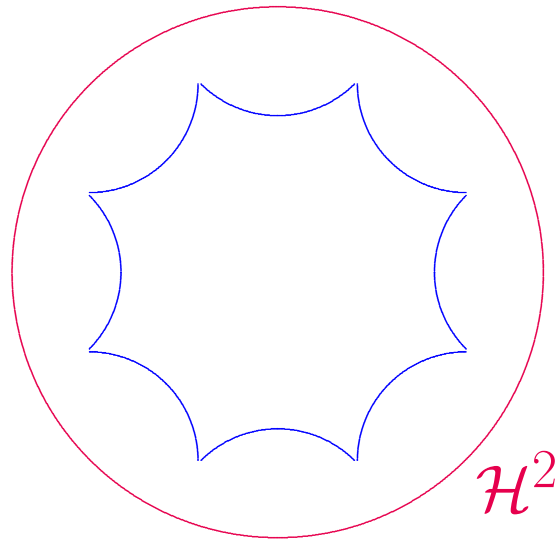
Proof hinges on a construction of hyperbolic 3-manifolds.

We begin by revisiting hyperbolic metrics on Σ.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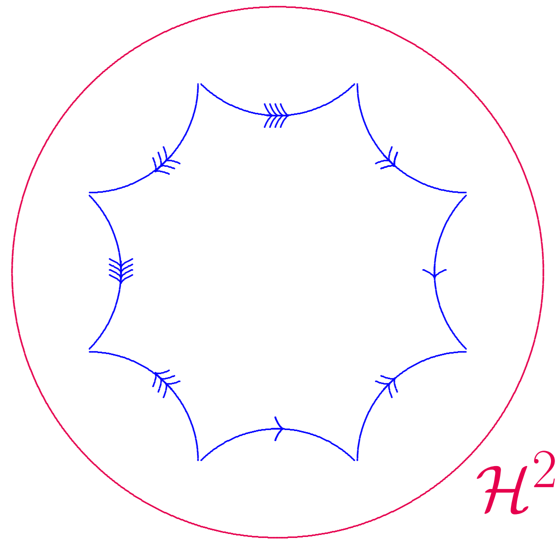
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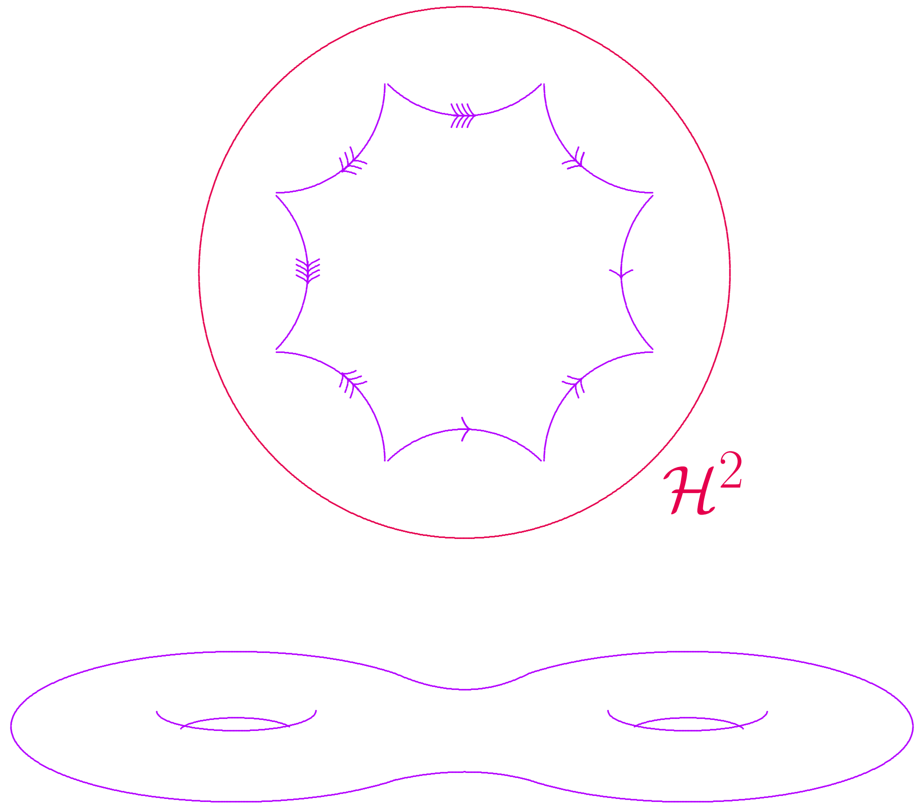
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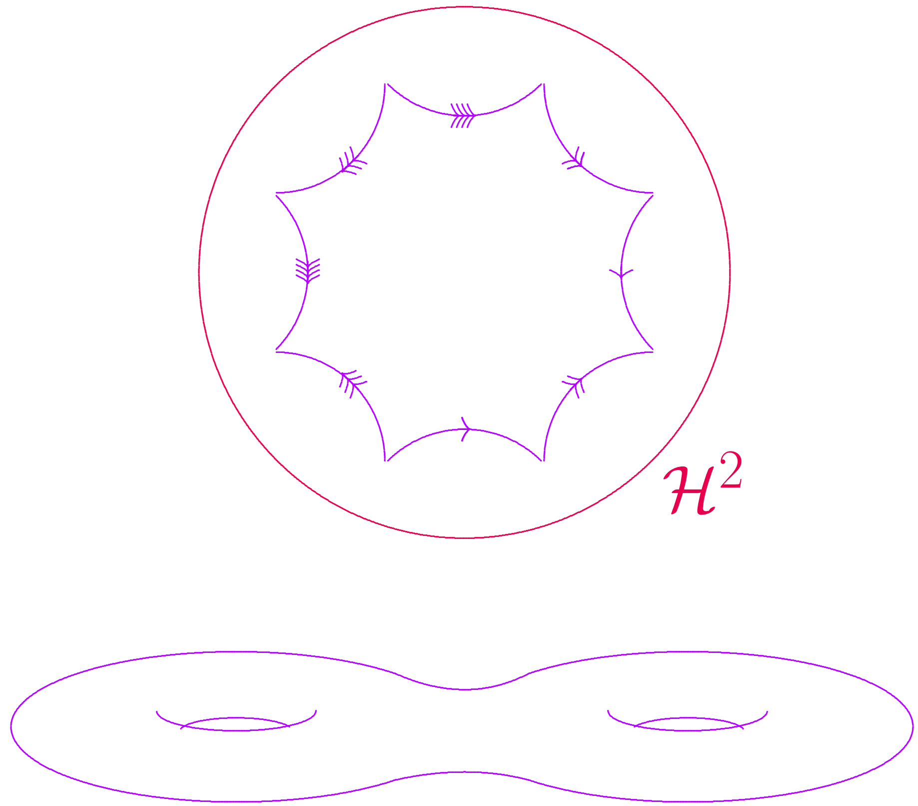
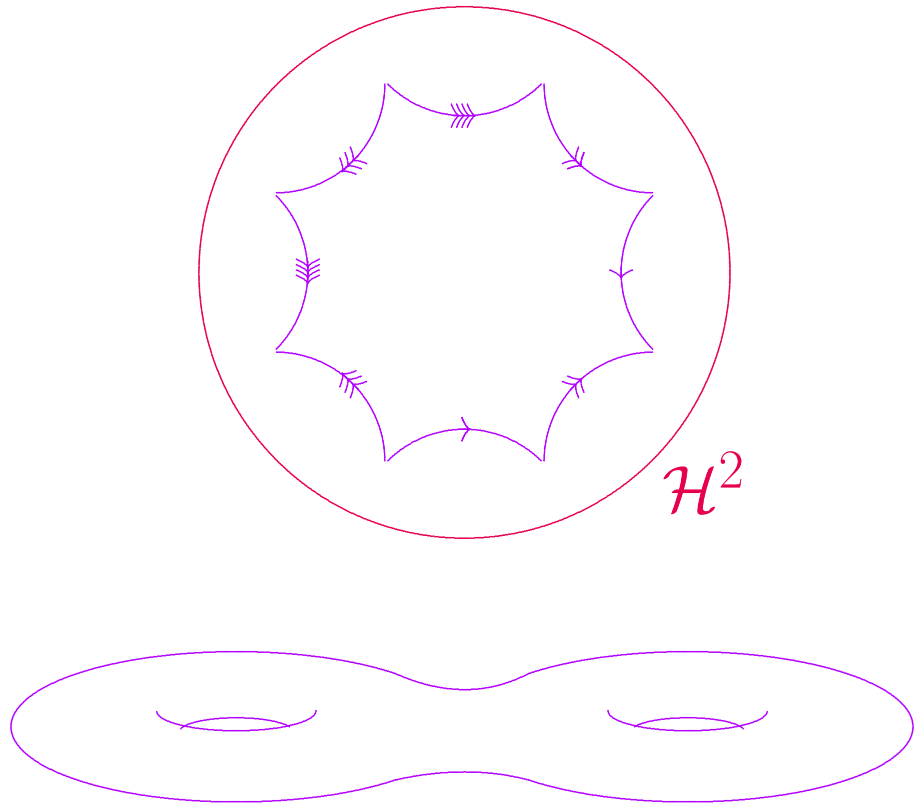


$$\pi_1(\Sigma) \hookrightarrow \mathbf{SO}_+(1,2) = \mathbf{PSL}(2,\mathbb{R})$$



$$\begin{array}{ccccc}
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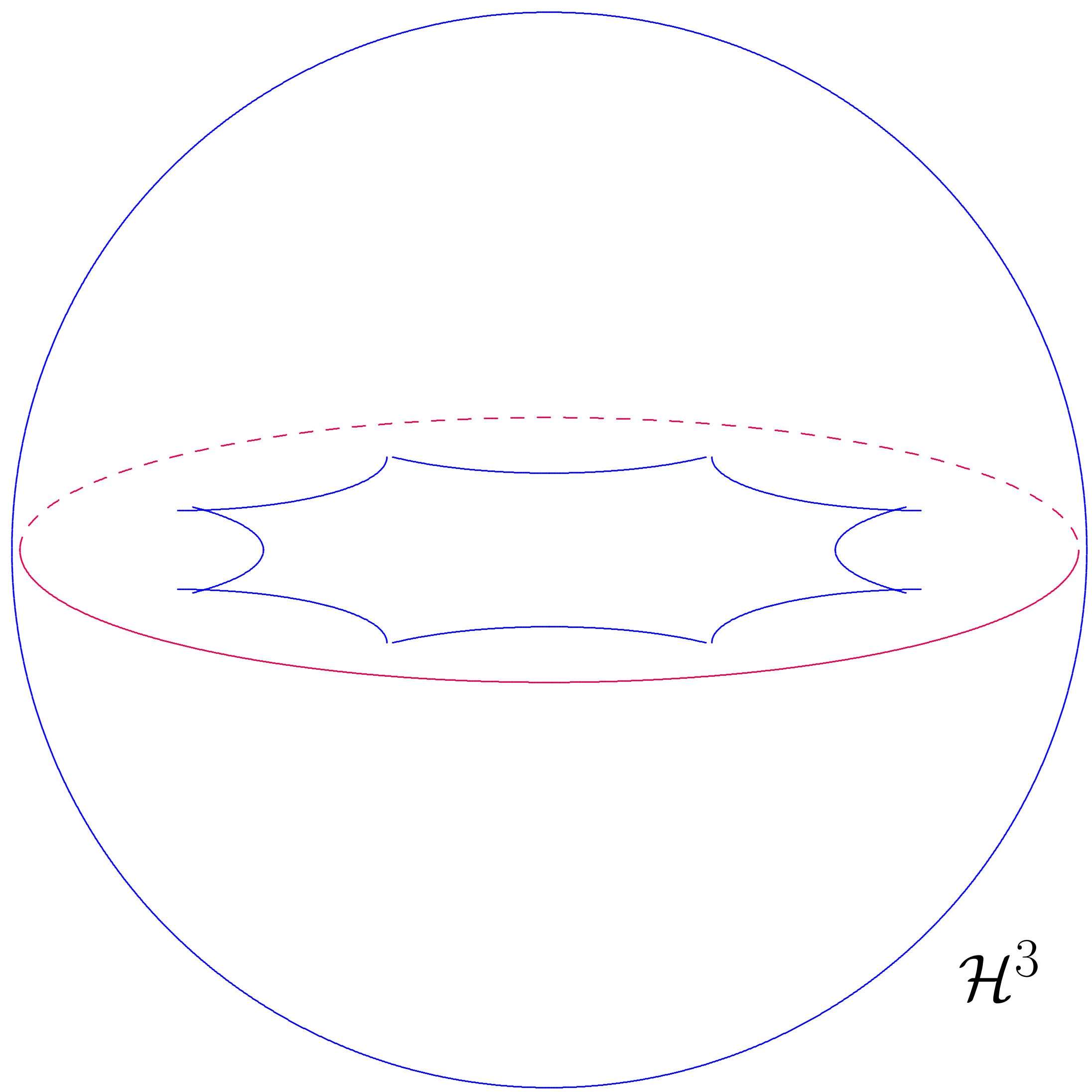
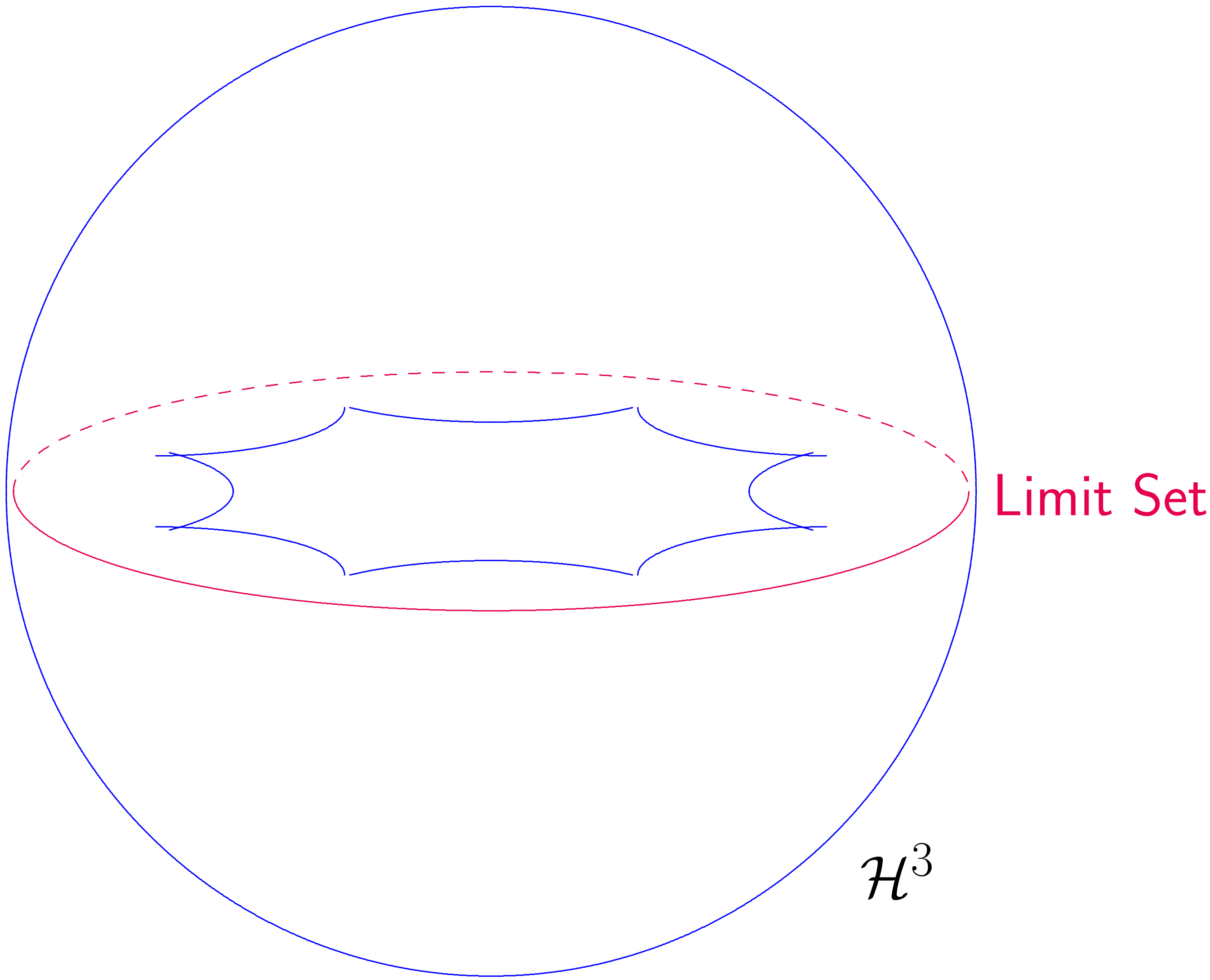
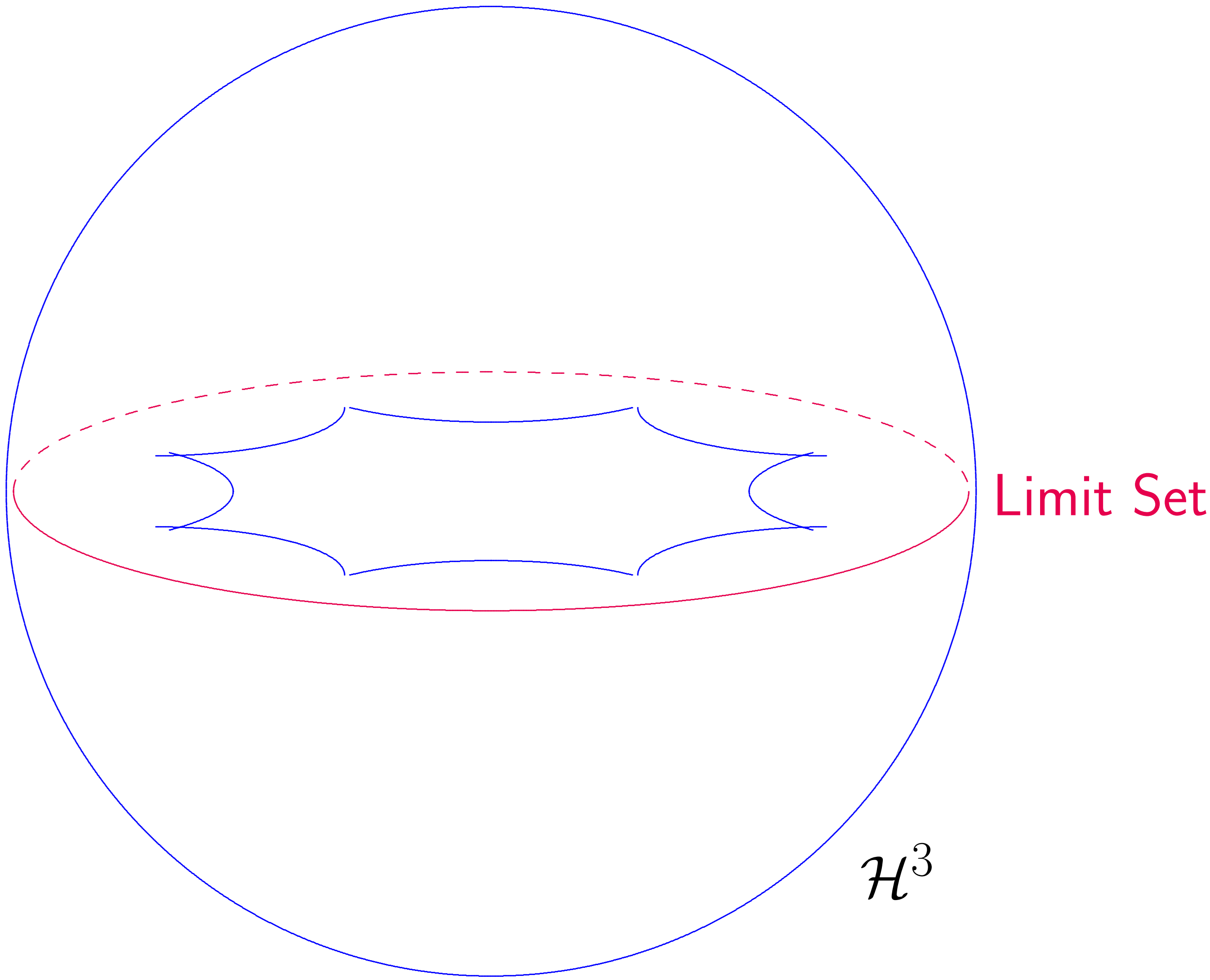
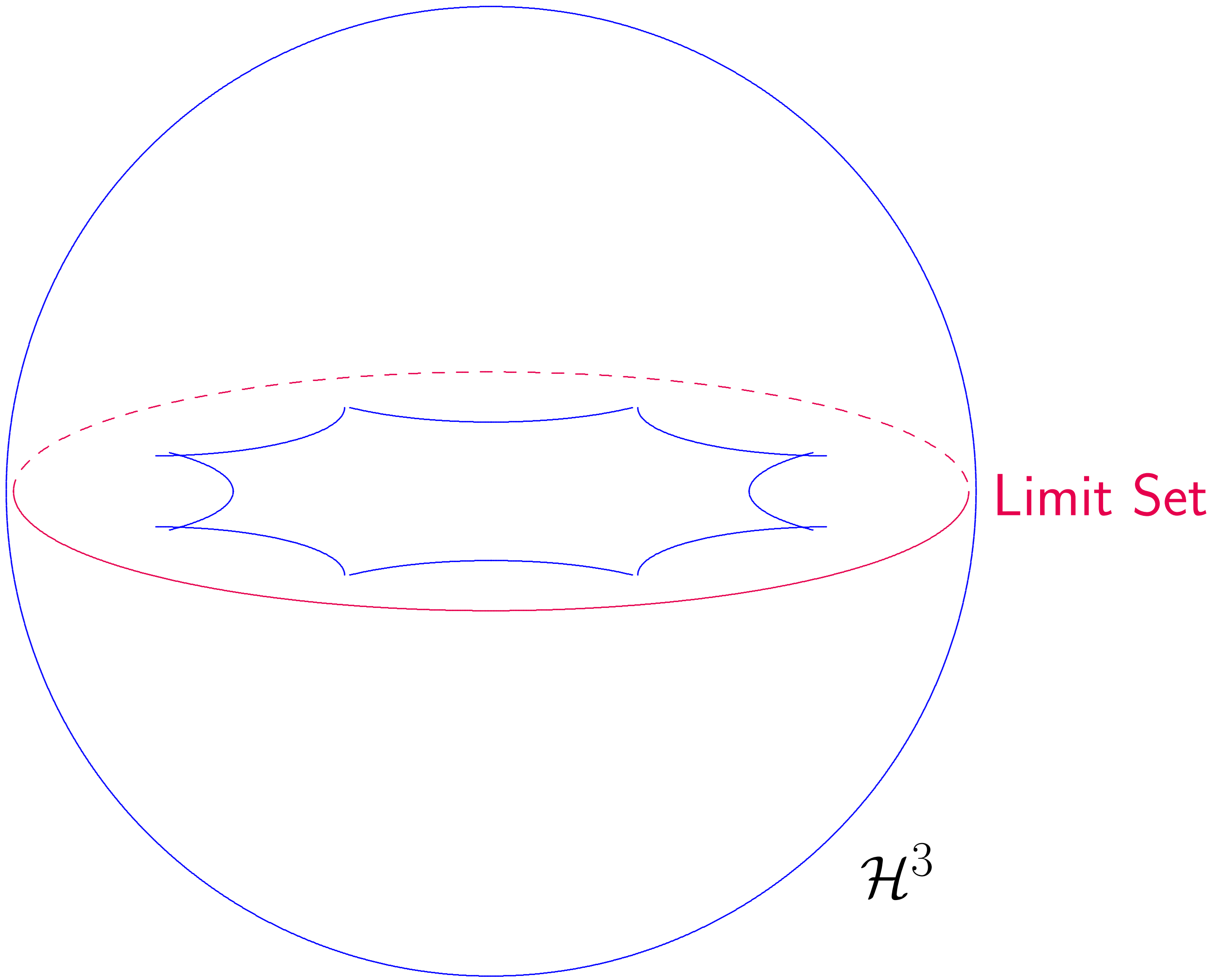
 & & \cap & & \cap \\
 & & \mathbf{SO}_+(1,3) & = & \mathbf{PSL}(2,\mathbb{C})
\end{array}$$


$\mathcal{H}^3$


Limit Set
$\mathcal{H}^3$



$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2,\mathbb{R}) \text{ Fuchsian group}$$



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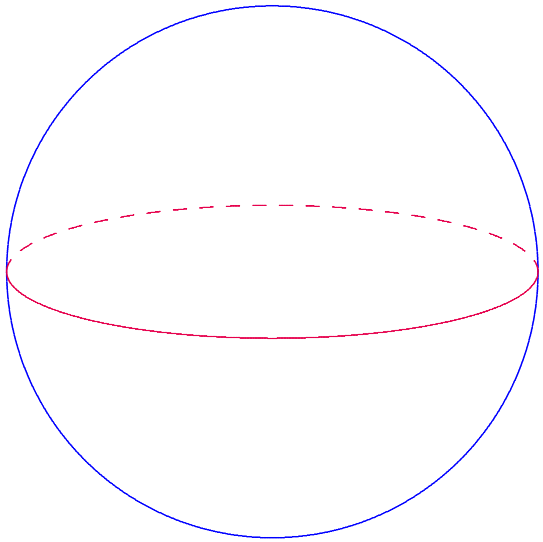

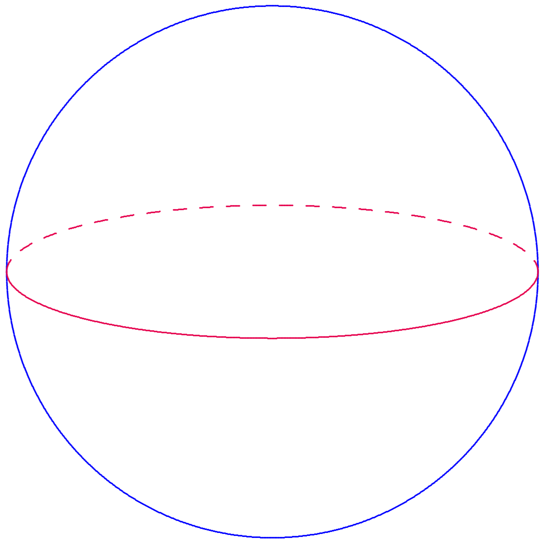

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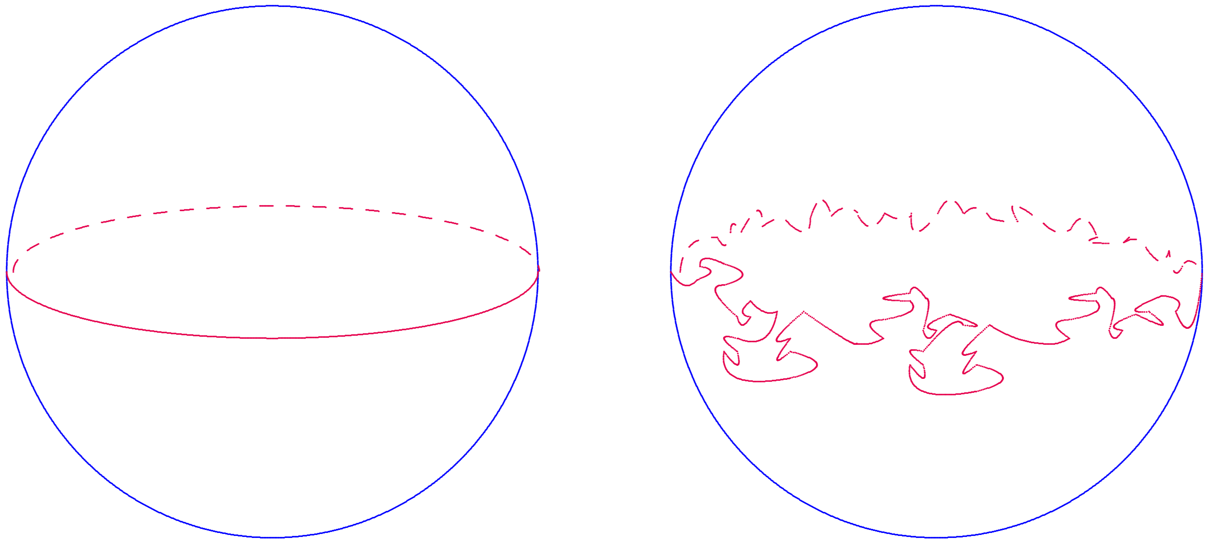

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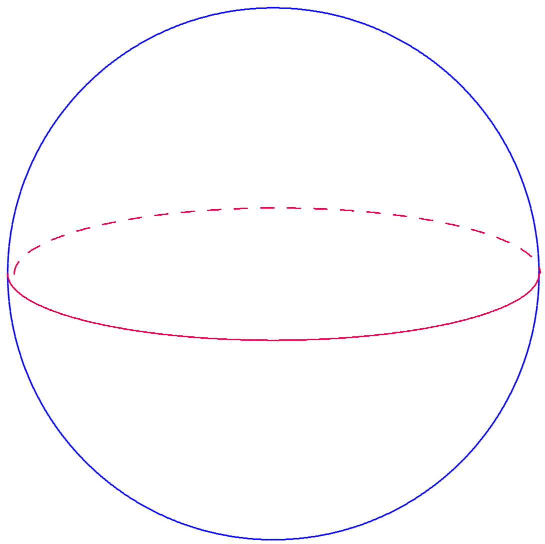

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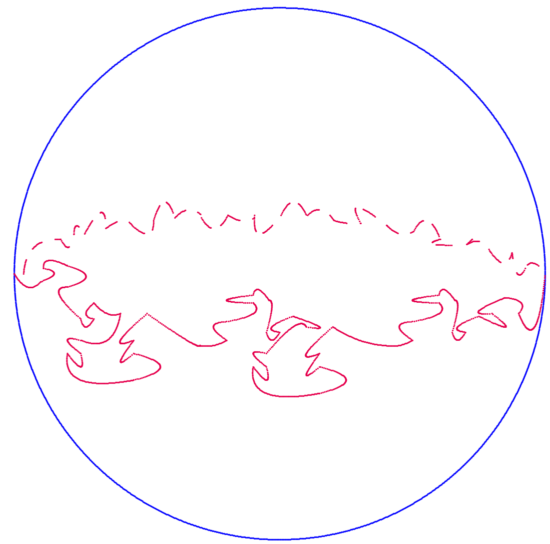
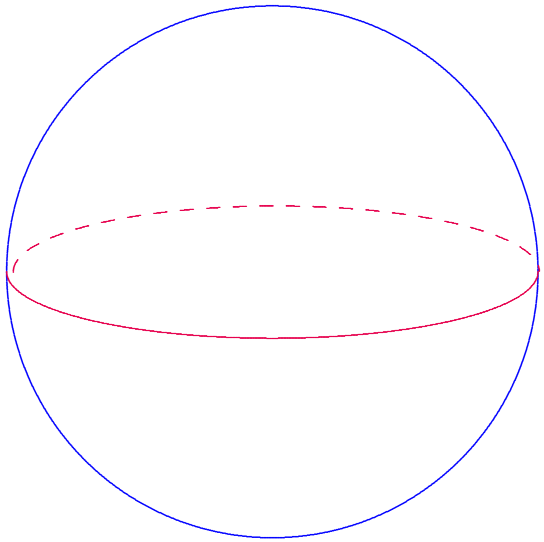


quasi-Fuchsian



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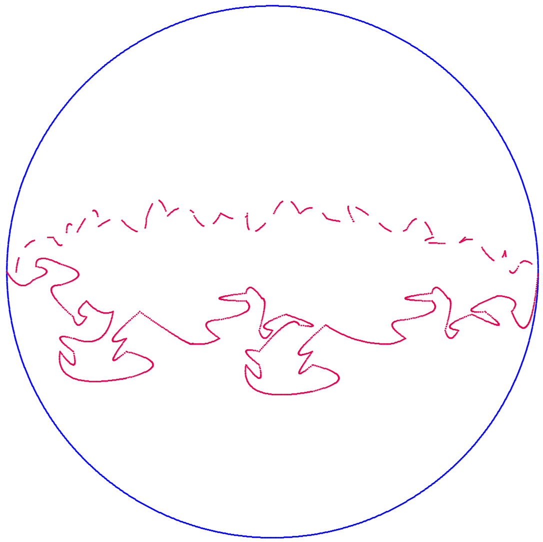
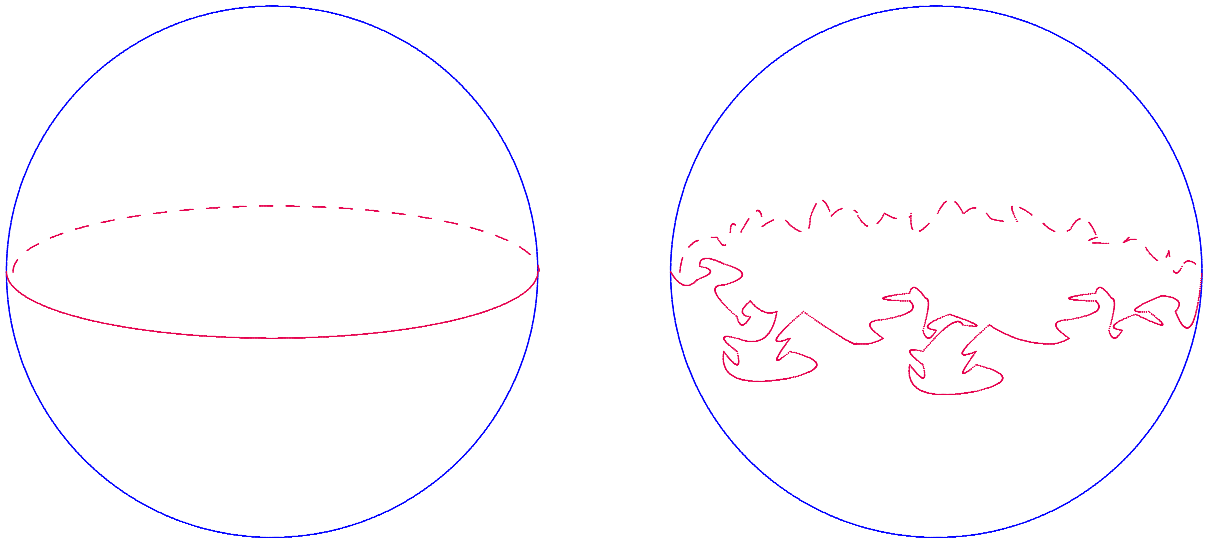


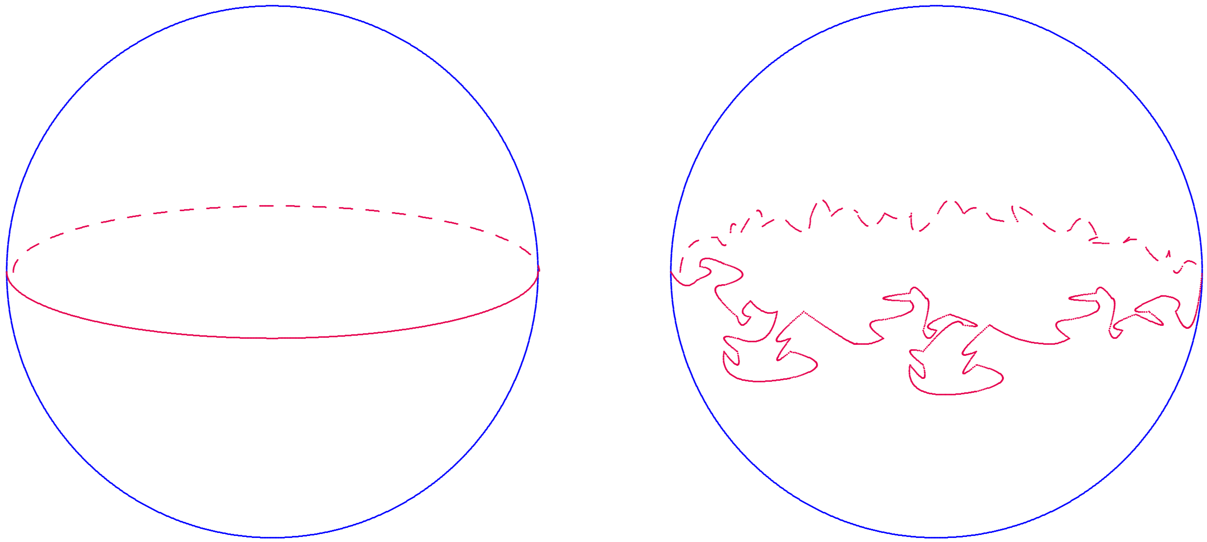
quasi-Fuchsian

$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{C})$ quasi-Fuchsian group



Fuchsian quasi-Fuchsian

$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2,\mathbb{C})$$ quasi-Fuchsian group of Bers type



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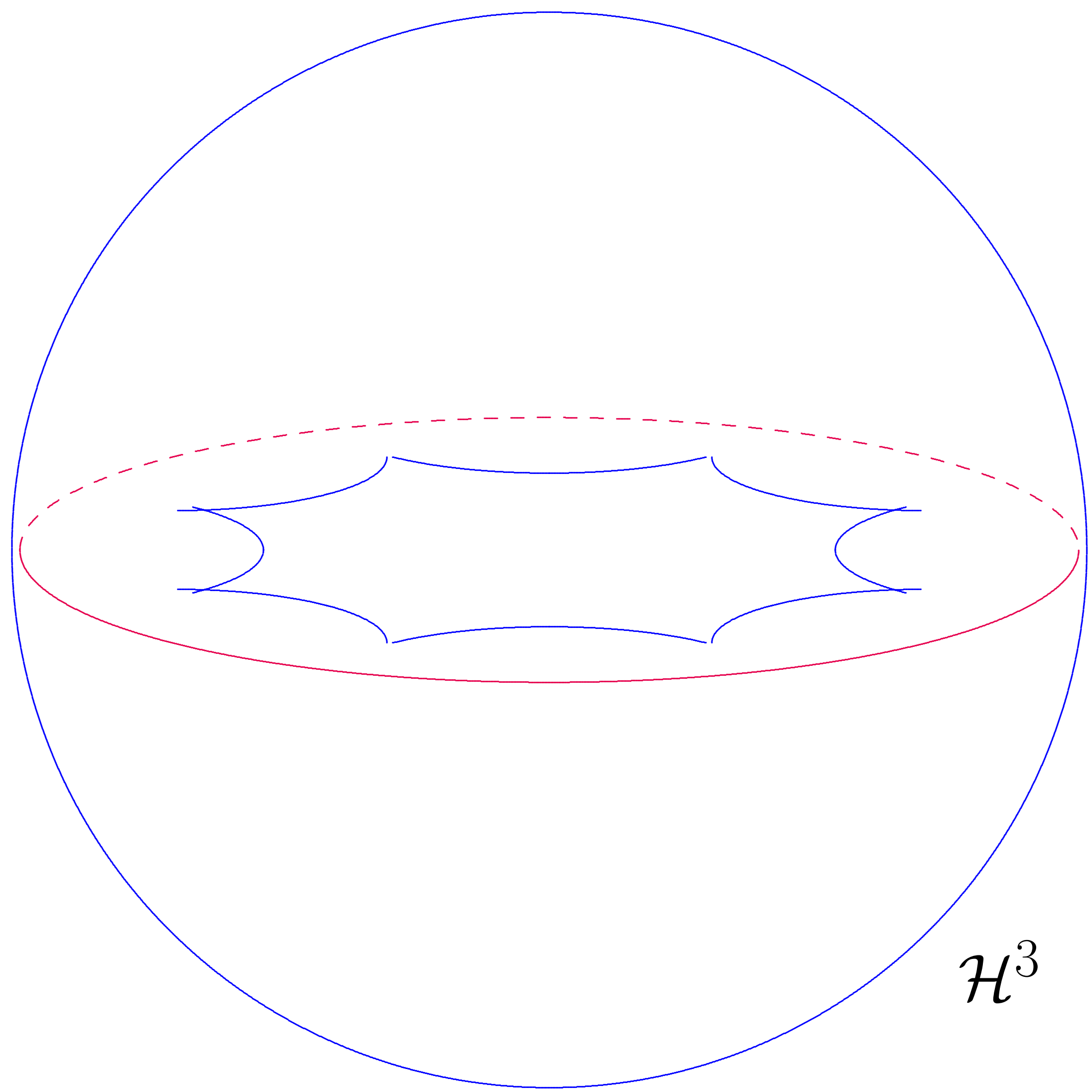
Quasi-conformally conjugate to Fuchsian.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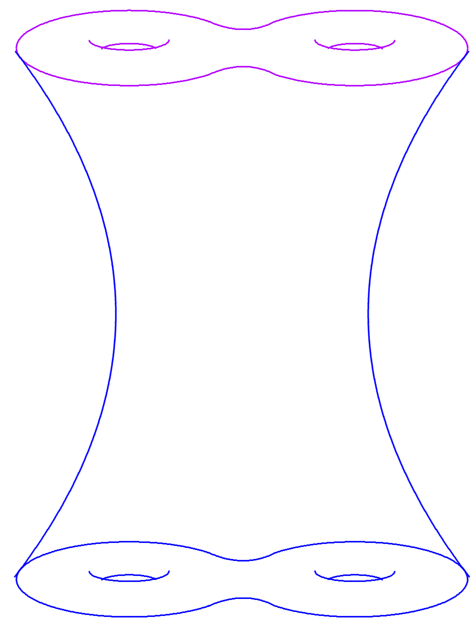
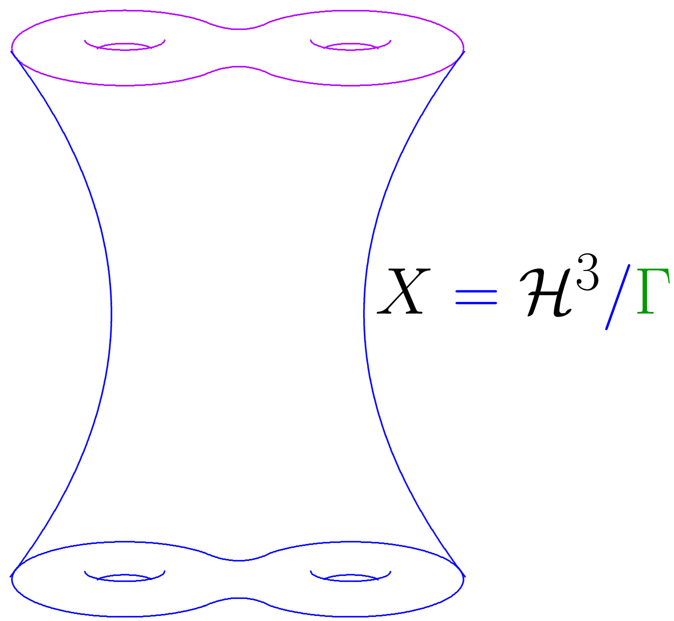
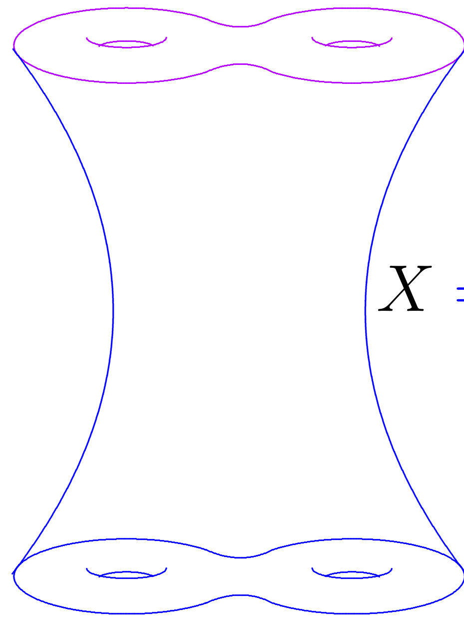



$X = \mathcal{H}^3/\Gamma$



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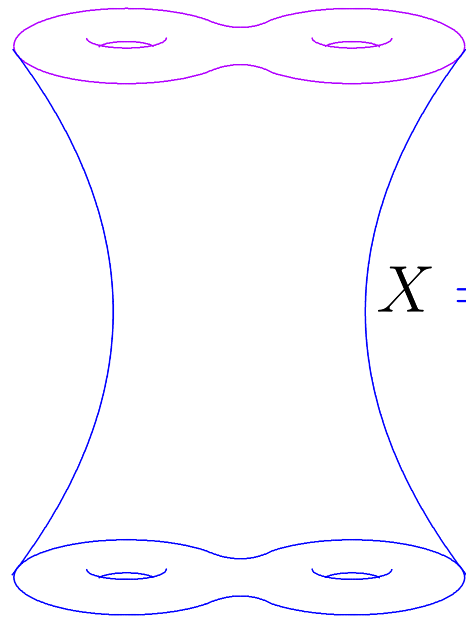
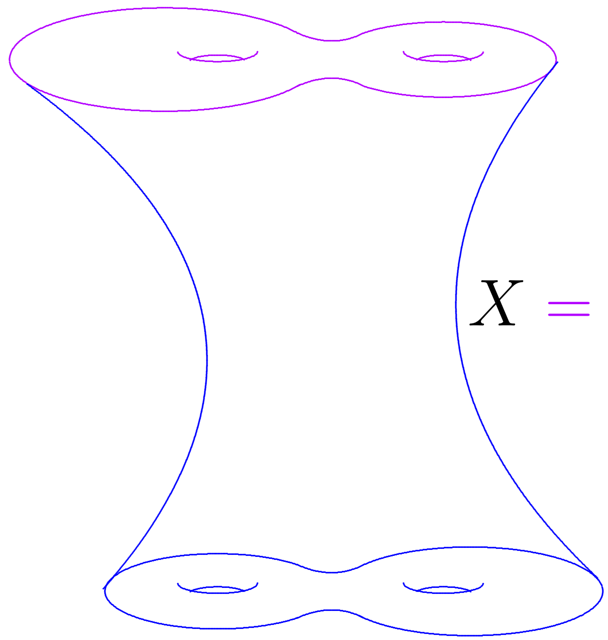


$$X = \mathcal{H}^3/\Gamma$$

Γ Fuchsian

$$X \approx \Sigma \times \mathbb{R}$$



$X = \mathcal{H}^3/\Gamma$

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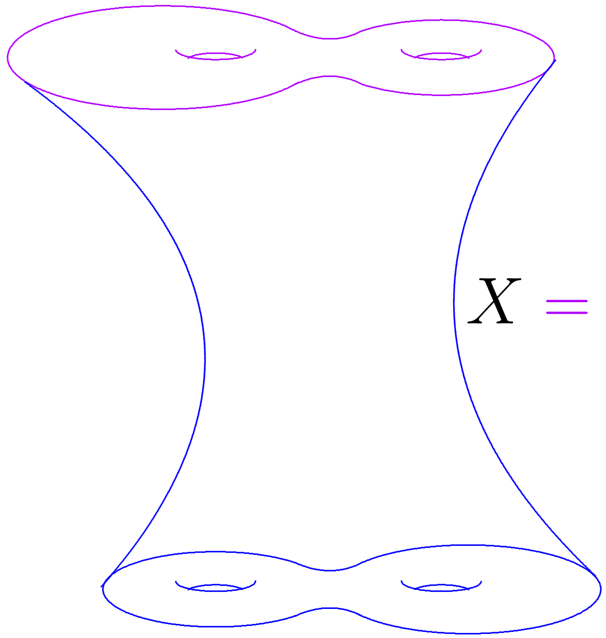

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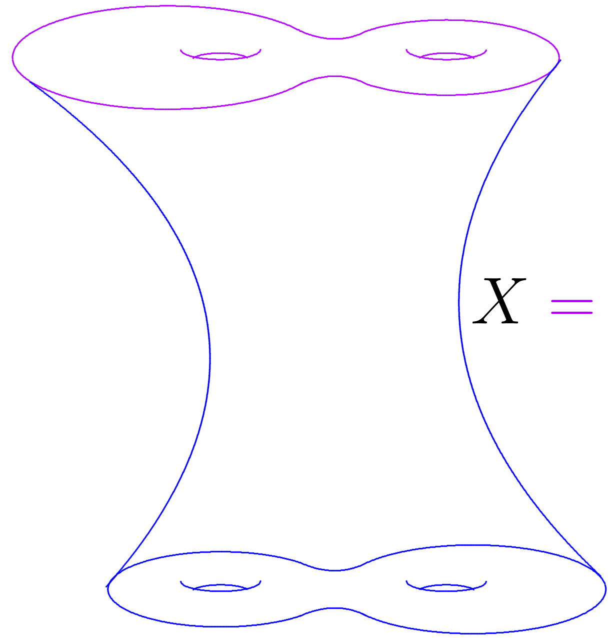
Freedom: two points in Teichmüller space.



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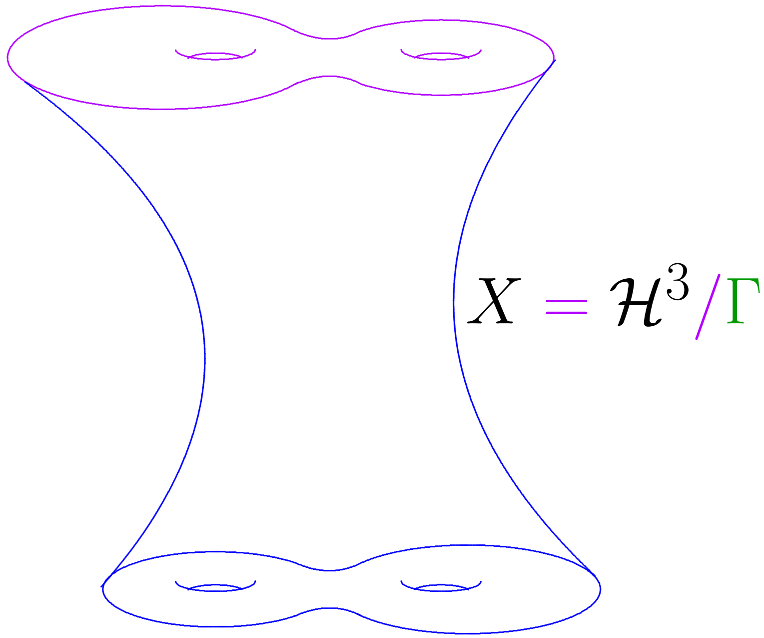

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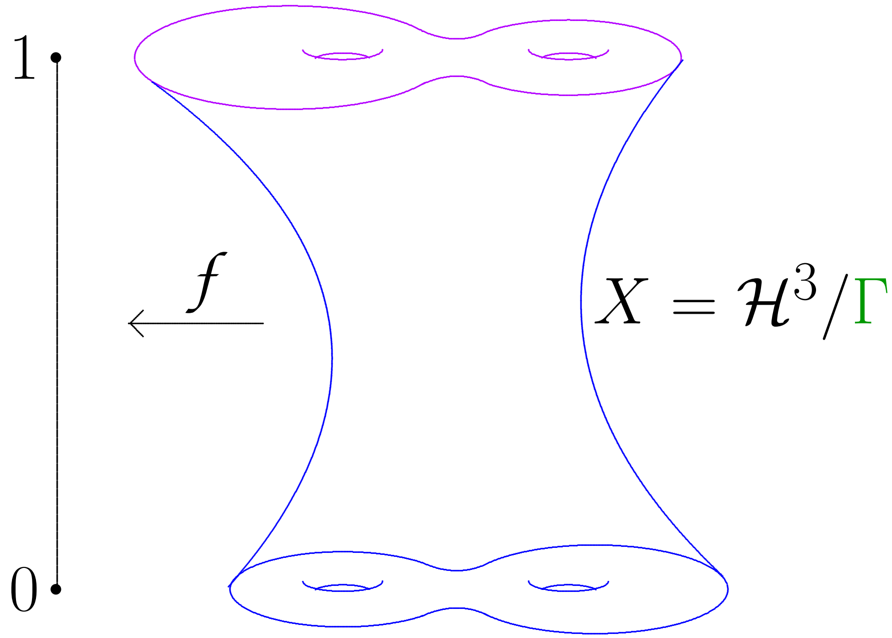

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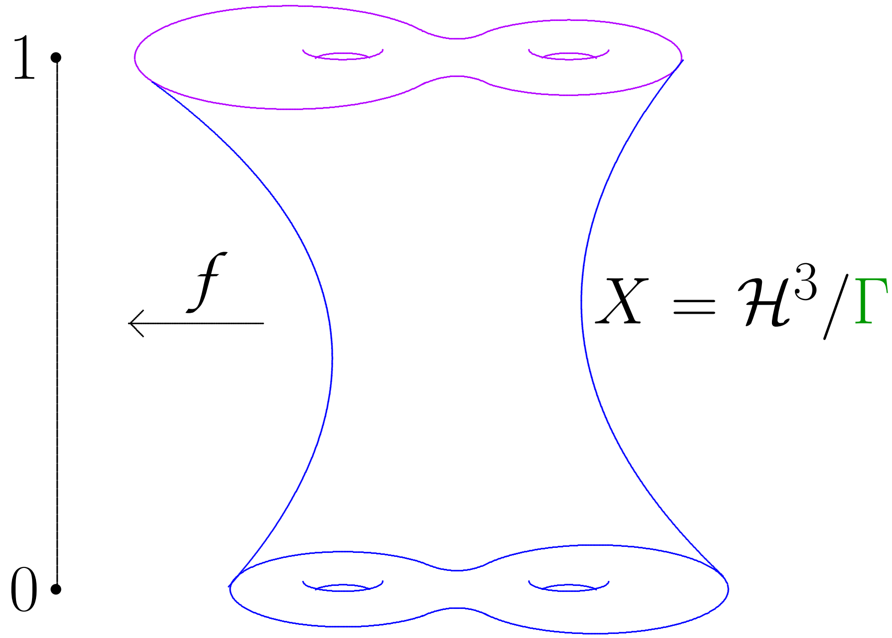

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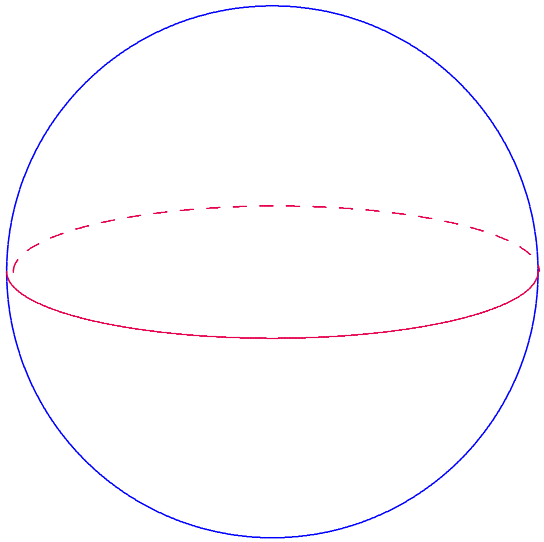

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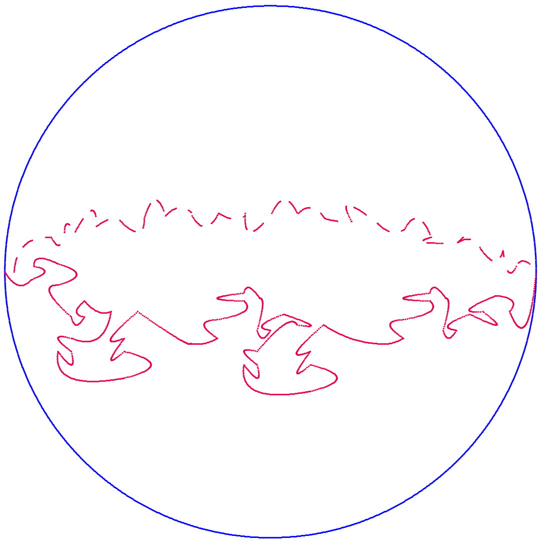

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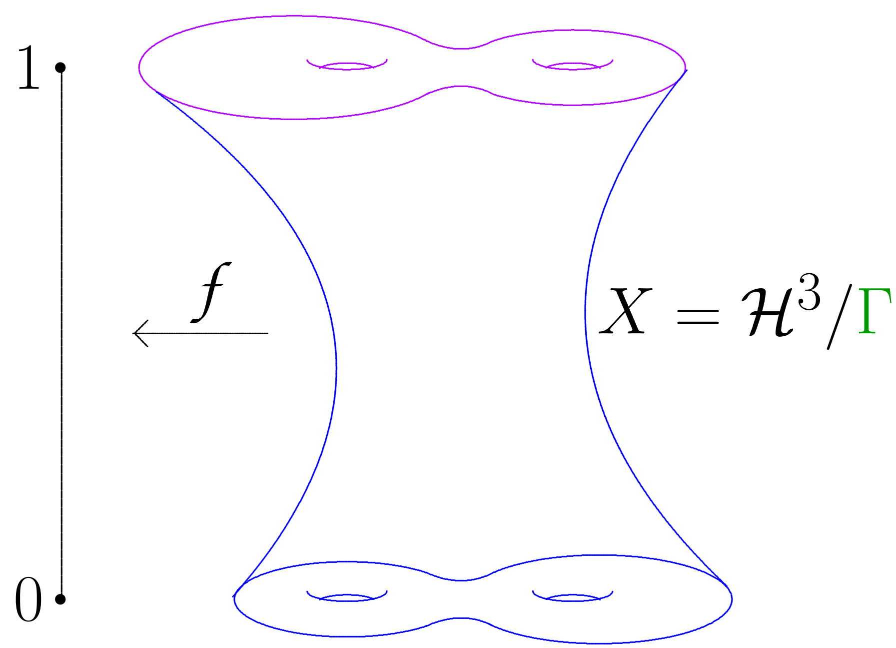

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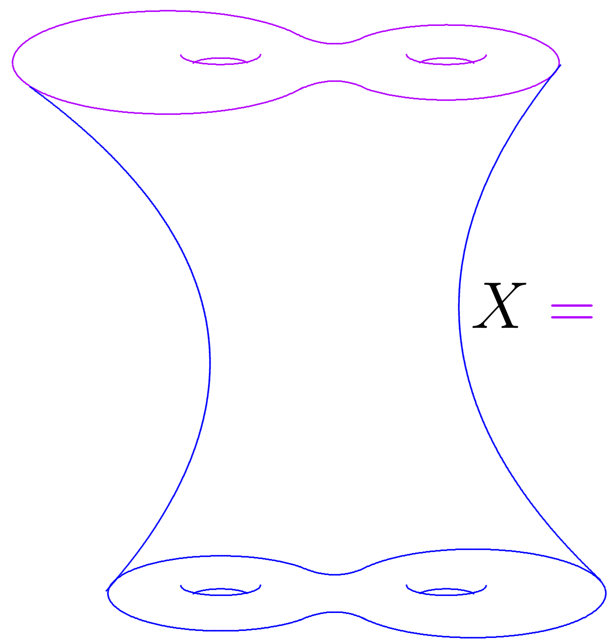

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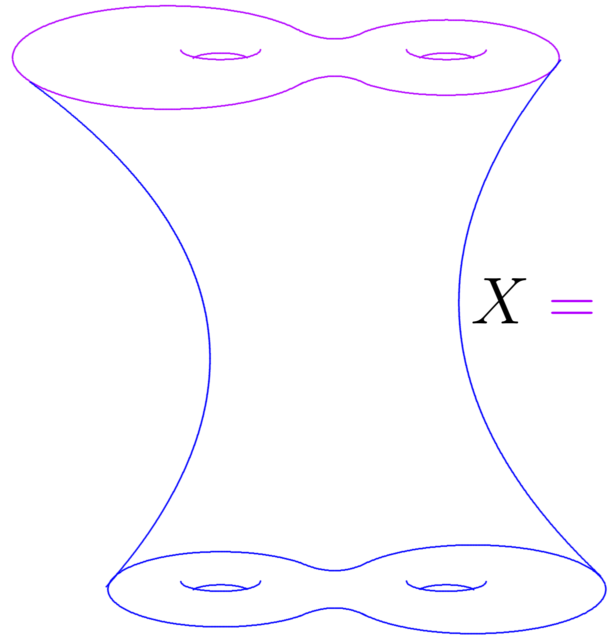

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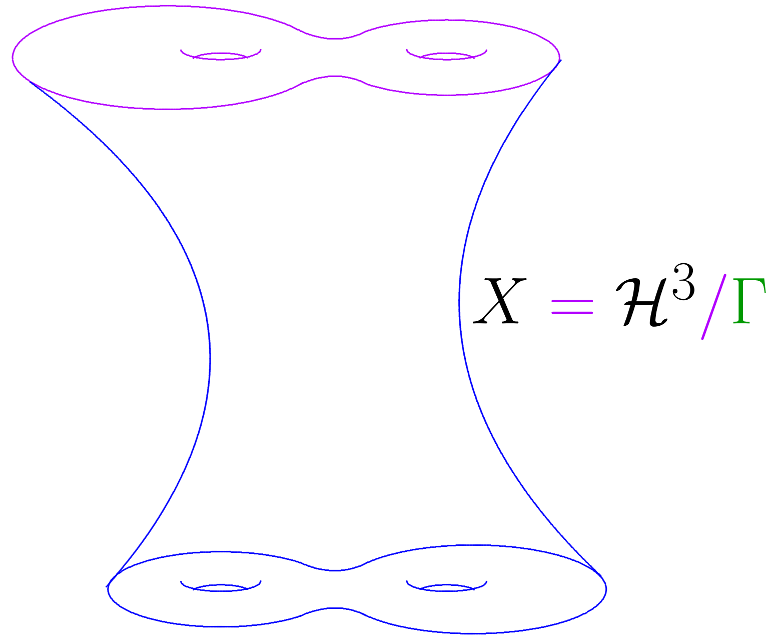
Construction of conformally flat 4-manifolds:



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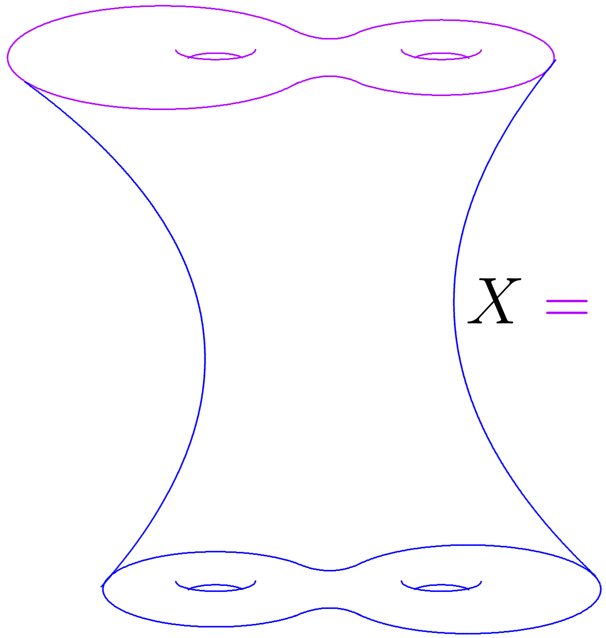

Construction of conformally flat 4-manifolds:

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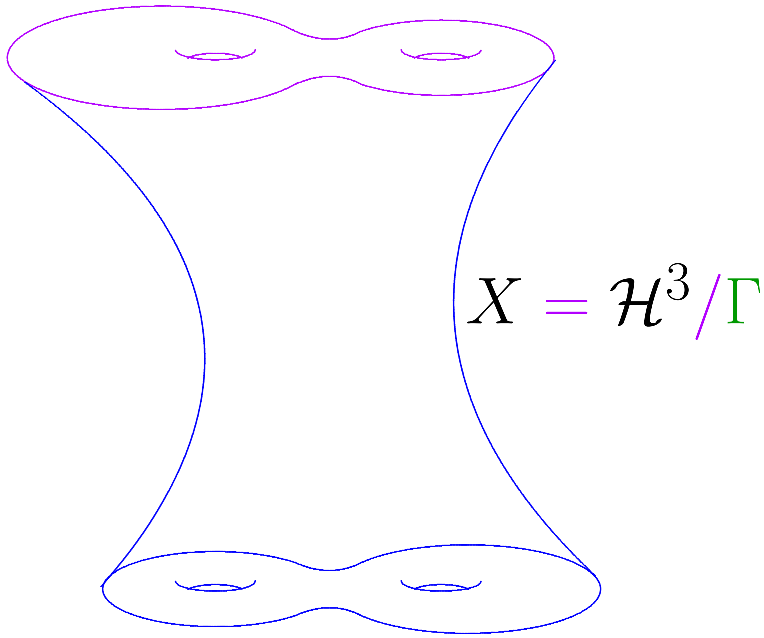
$\sim$: crush $\partial\overline{X} \times S^1$ to $\partial\overline{X}$.



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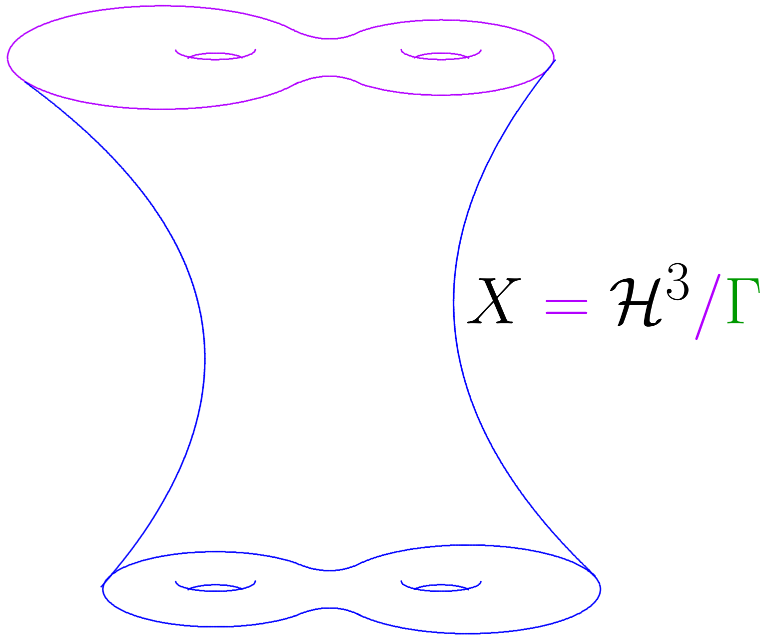

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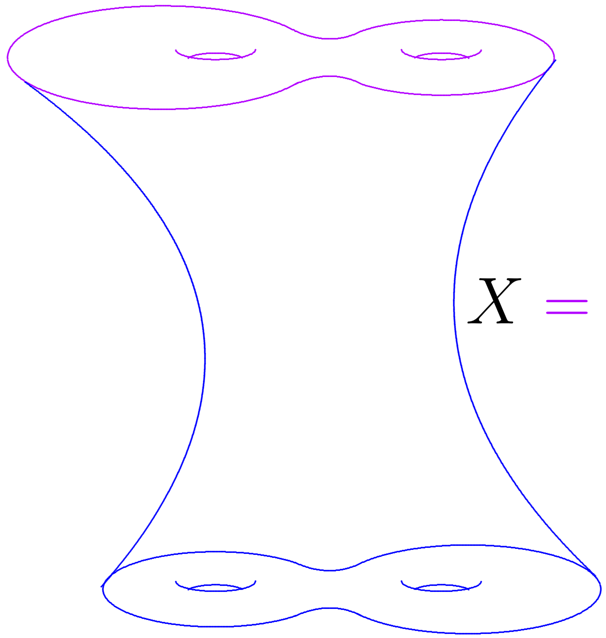

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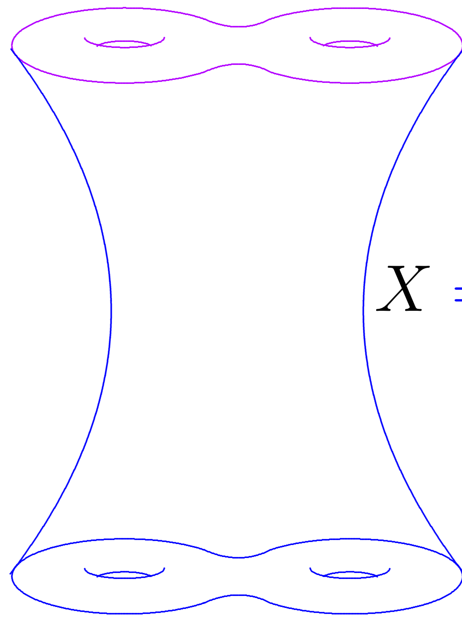

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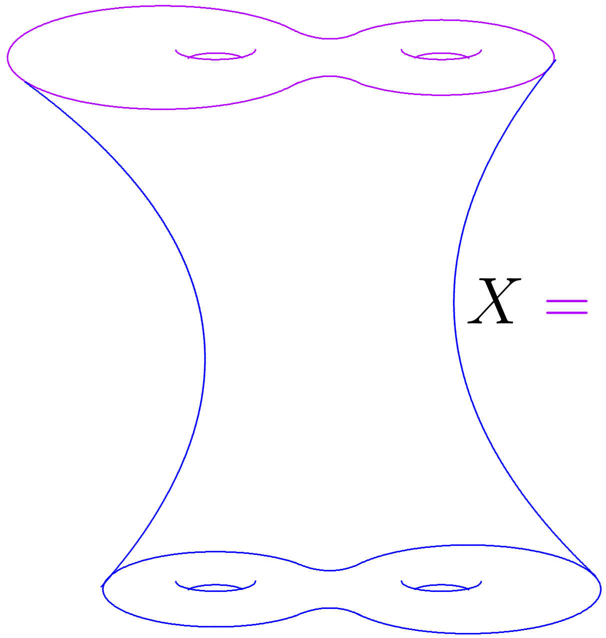

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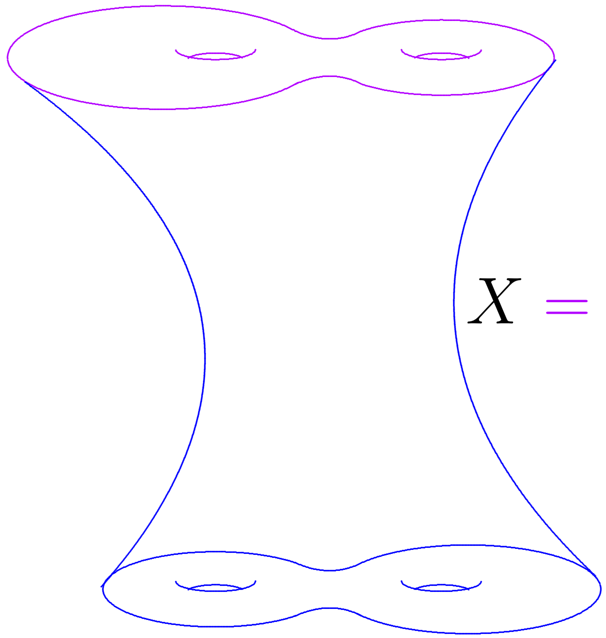

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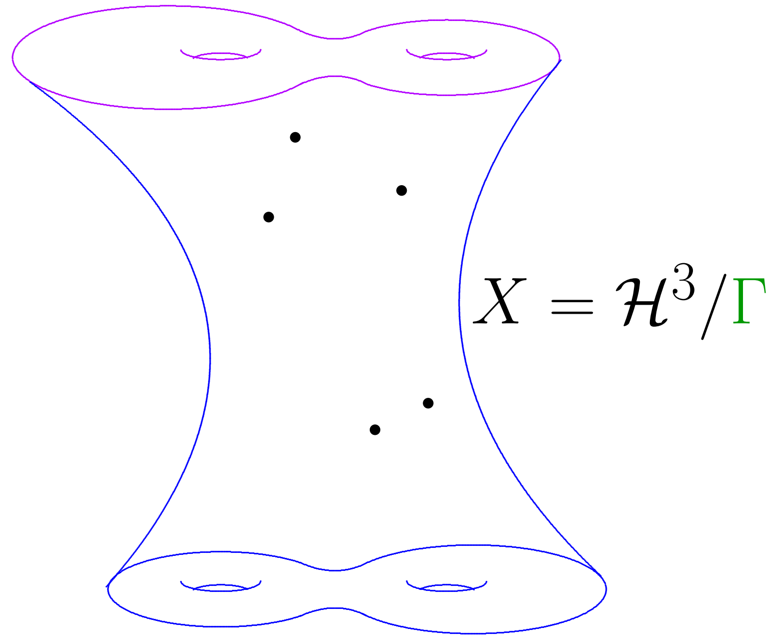
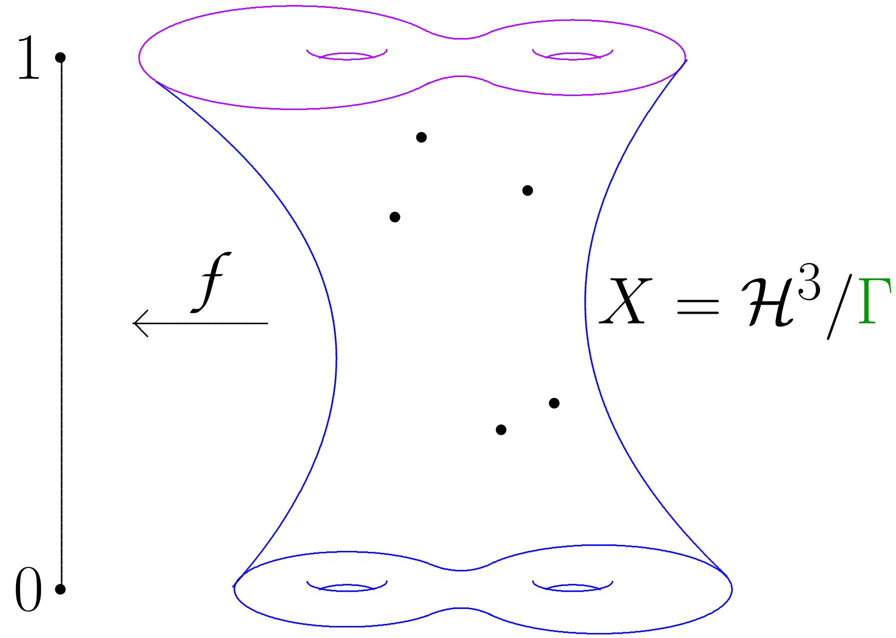


Construction of ASD 4-manifolds:

Choose k points $p_1, \ldots, p_k \in X$



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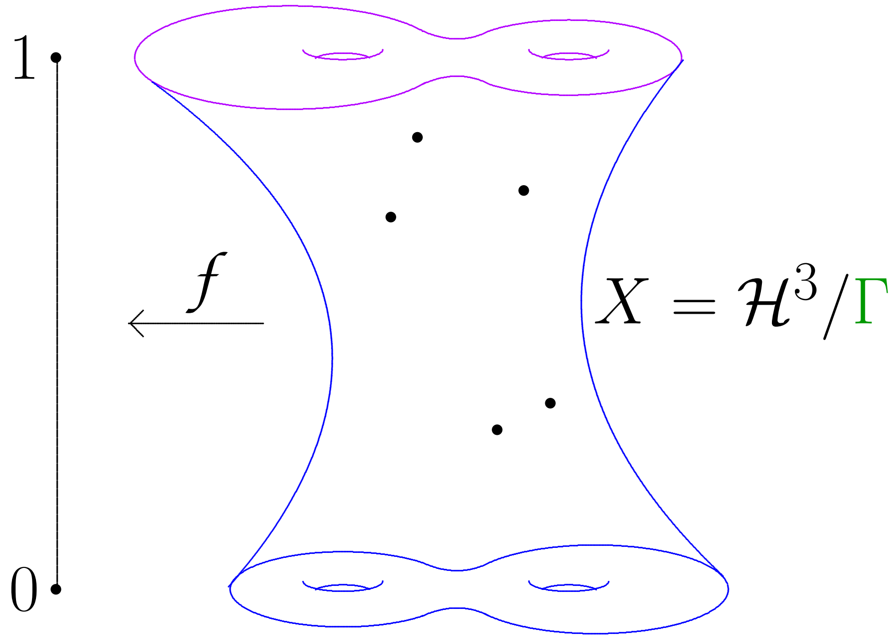

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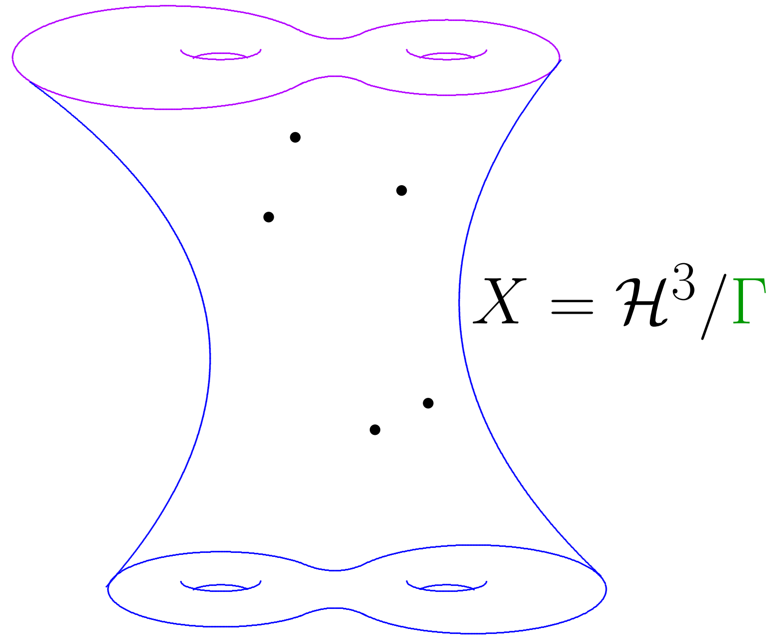

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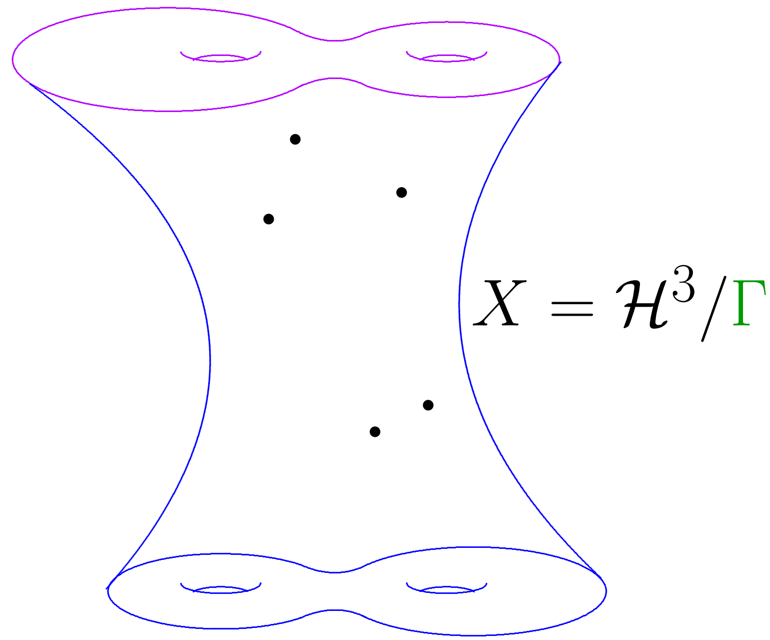

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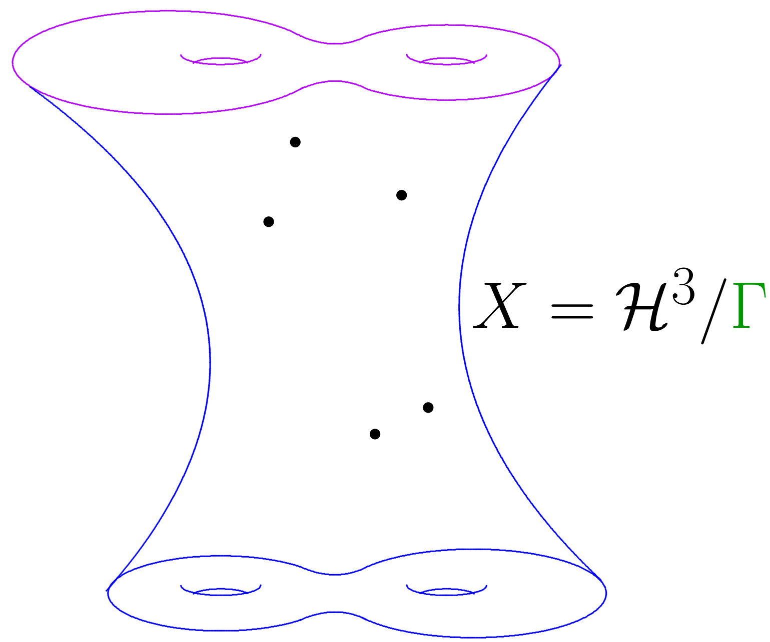
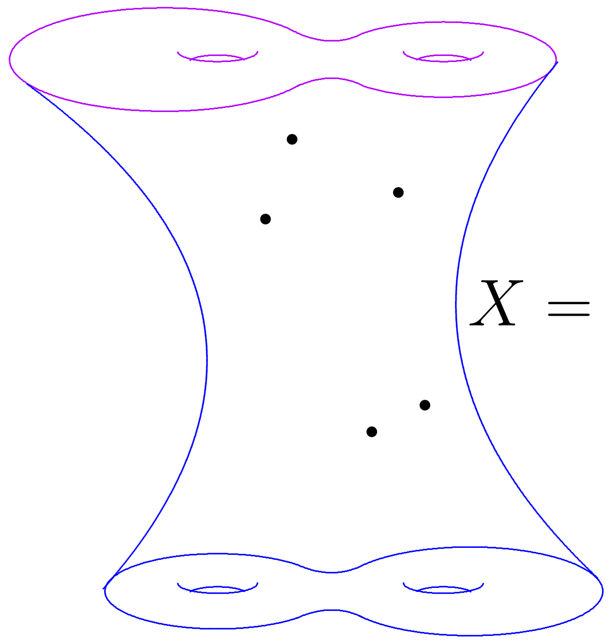
$$\Delta G_j = 2\pi\delta_{p_j}, \qquad G_j \to 0 \text{ at } \partial\overline{X}$$



Construction of ASD 4-manifolds:

Let G_j be the Green's function of p_j, and set

$$V = 1 + \sum_{j=1}^{k} G_j.$$



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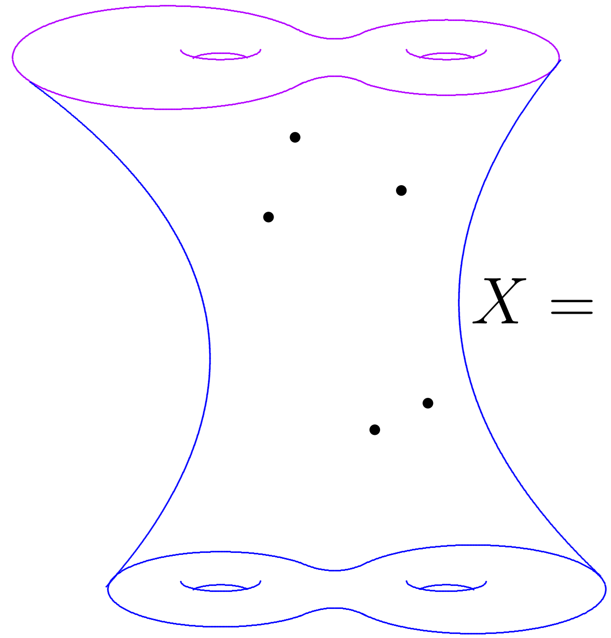

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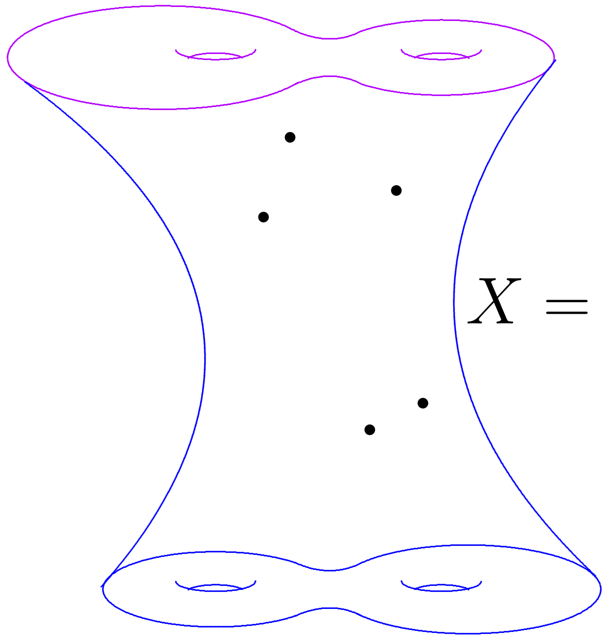
Choose $P \to (X - \{p_1, \ldots, p_k\})$ circle bundle with connection form θ such that

$$d\theta = \star dV .$$



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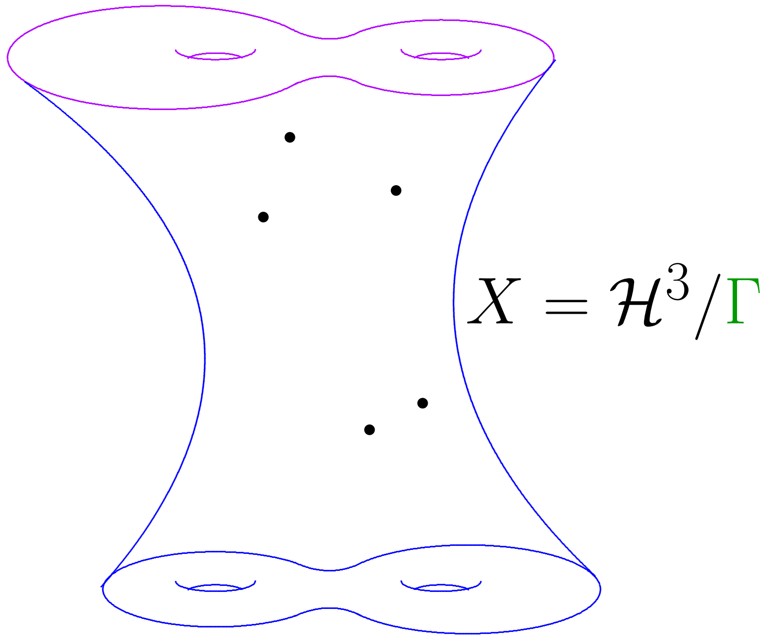

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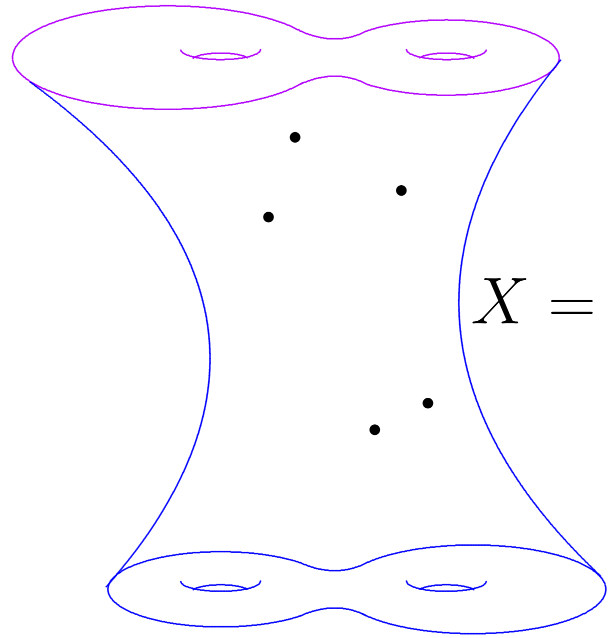

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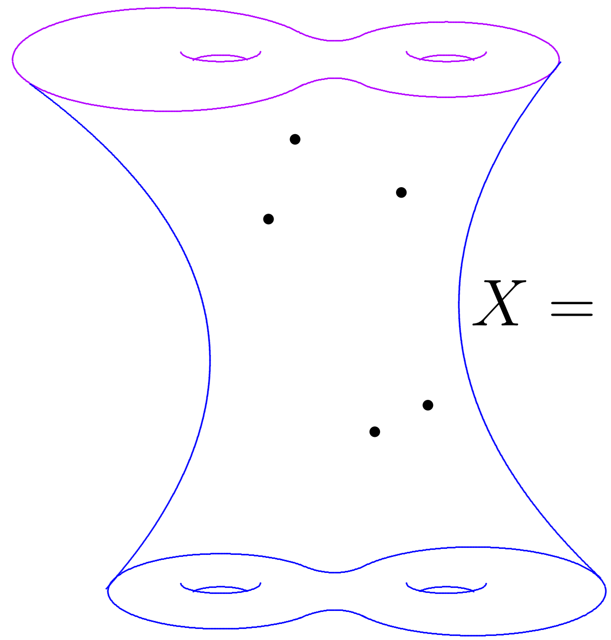

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\downarrow & & \downarrow & & \downarrow & & \downarrow \\
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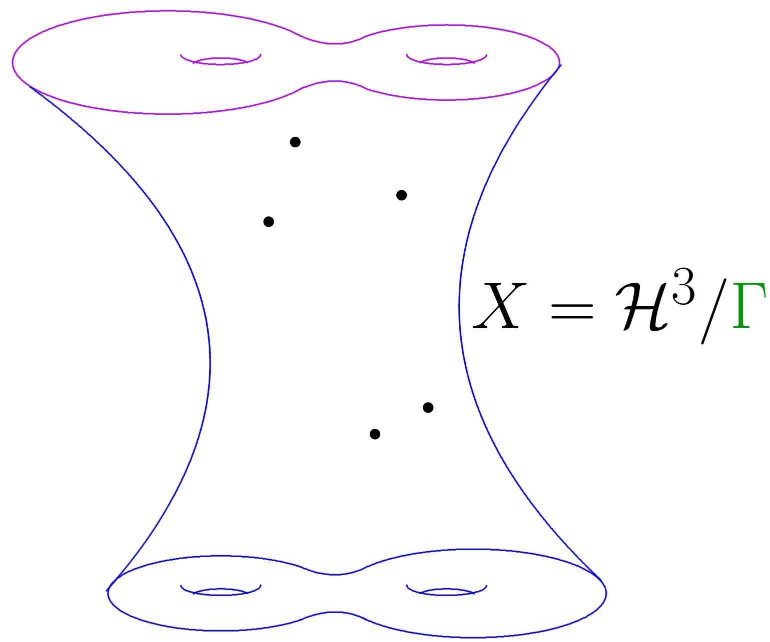
\end{array}$$



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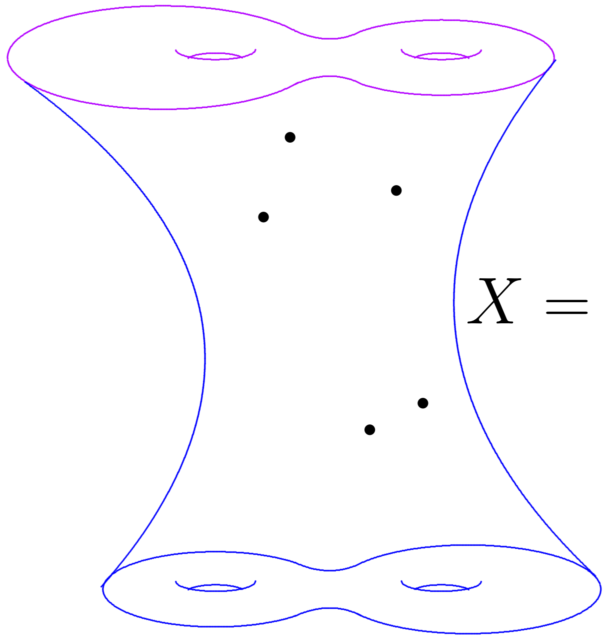

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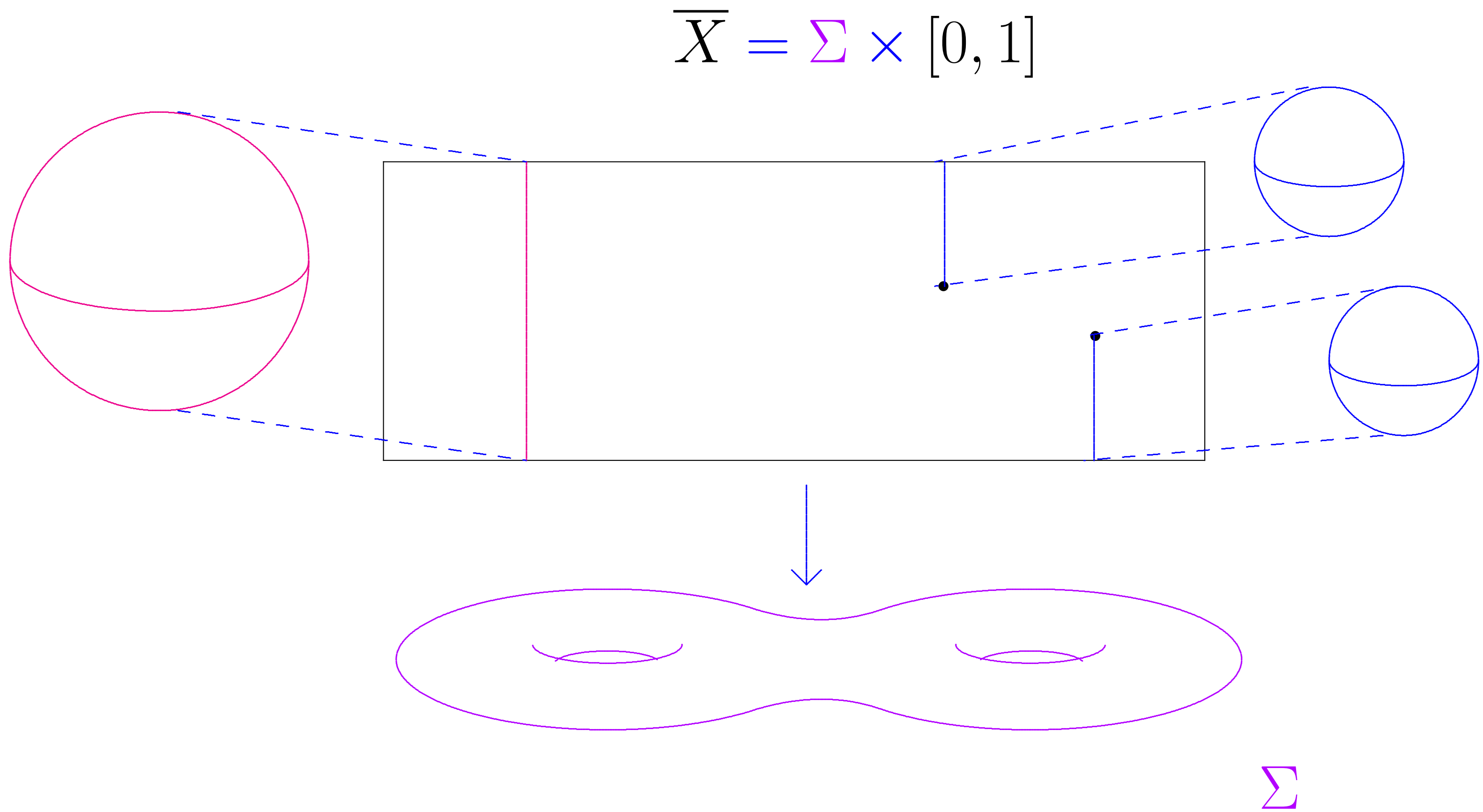

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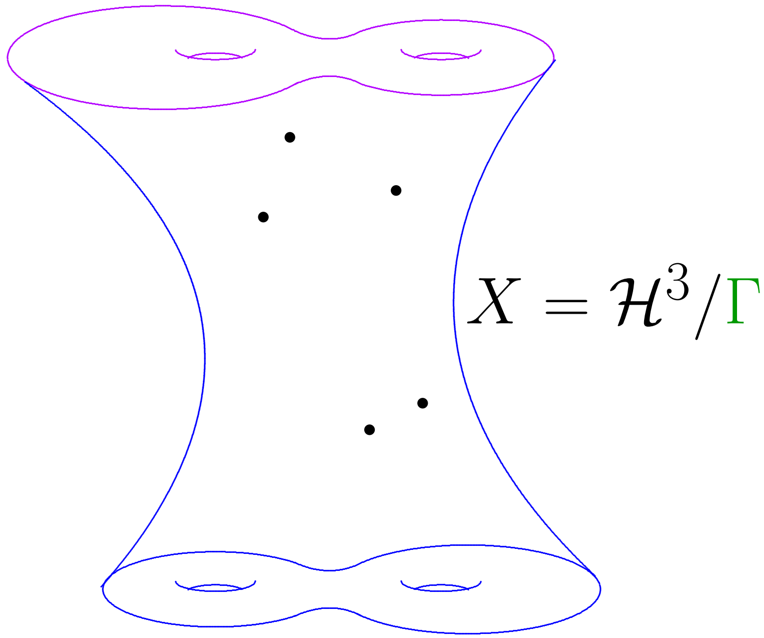

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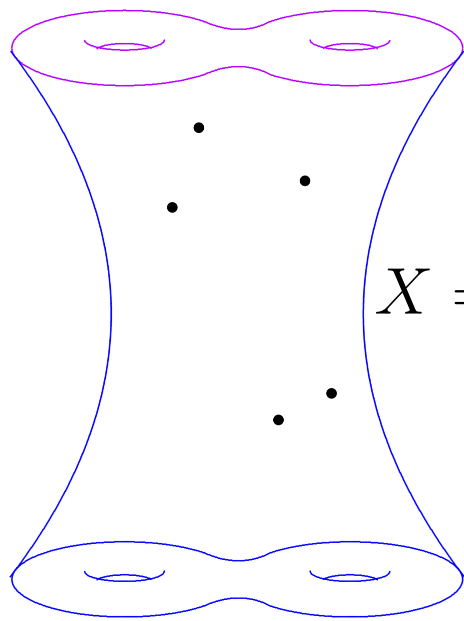

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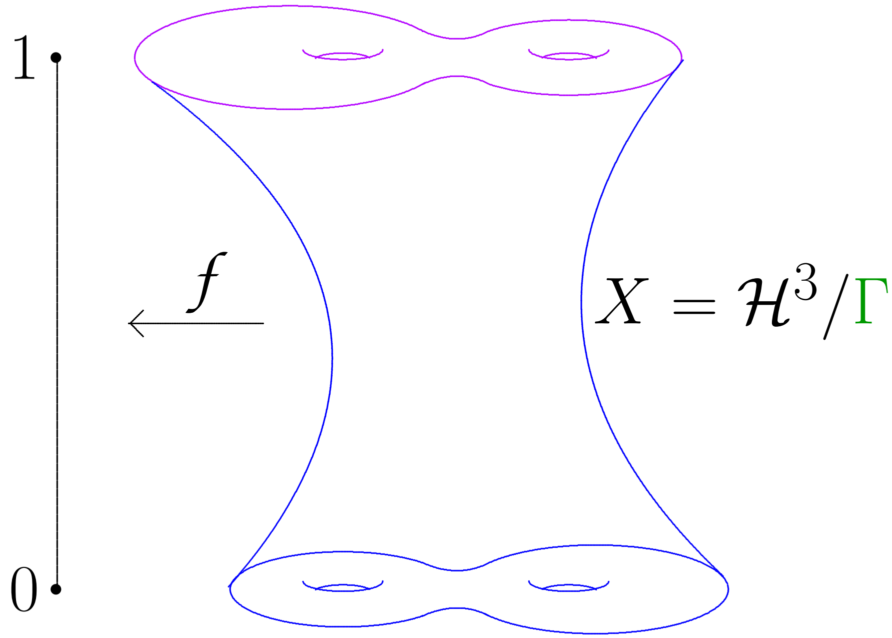

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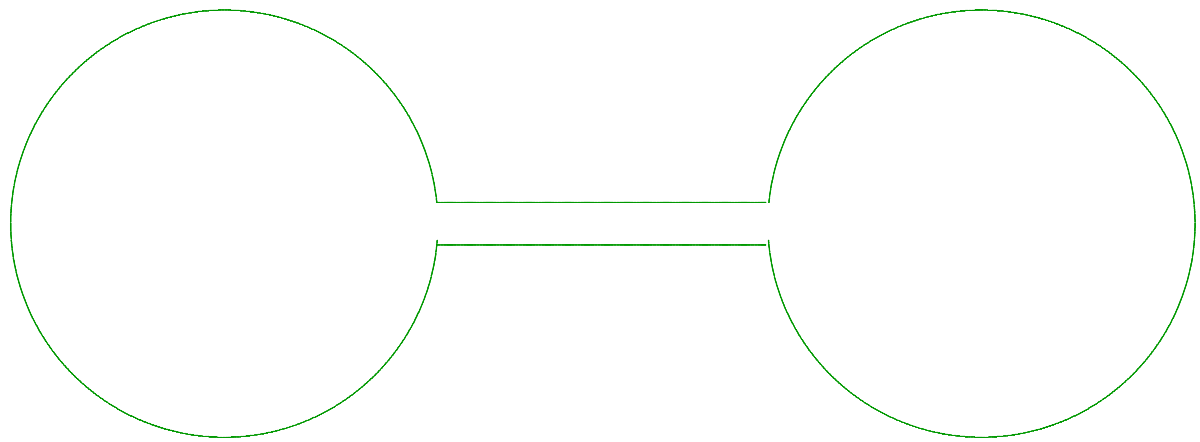
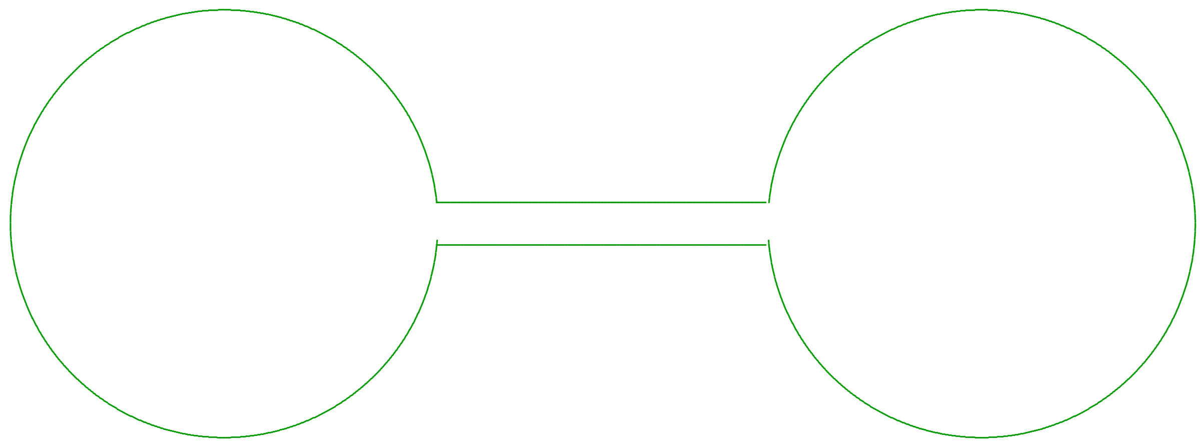
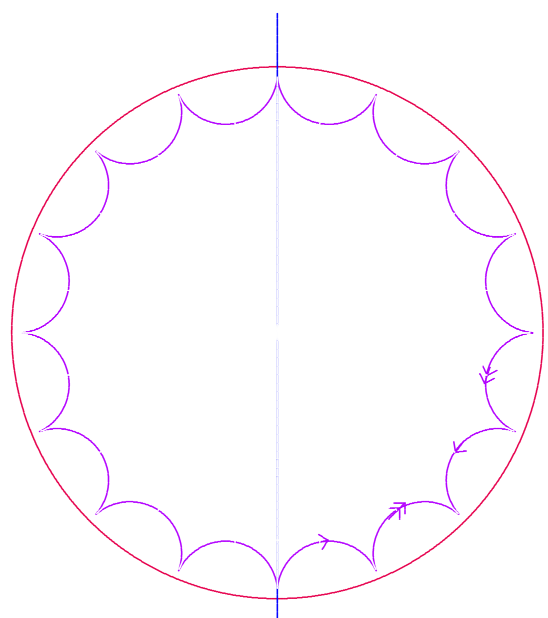
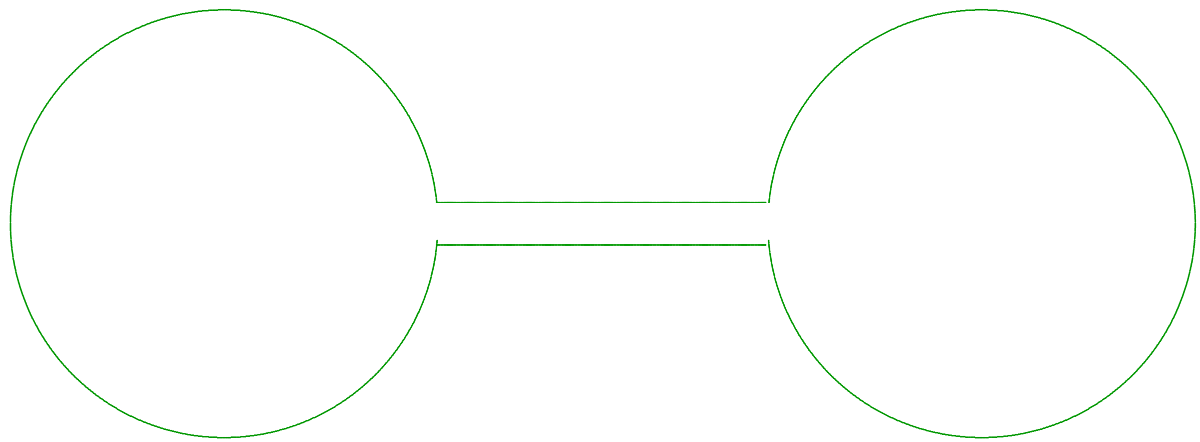
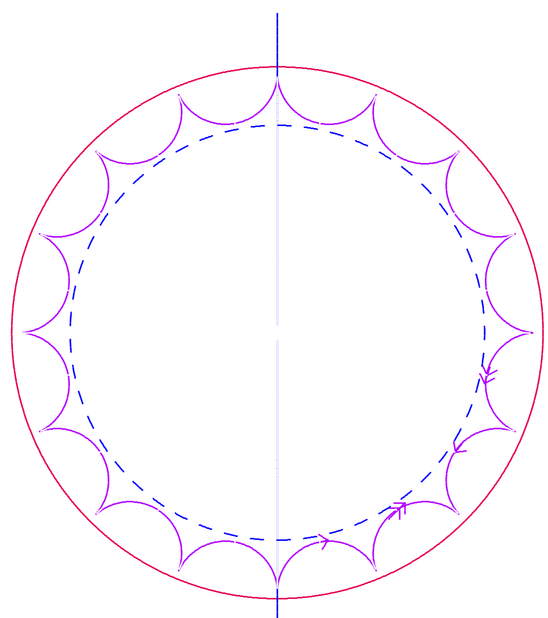
Ahlfors-Bers: Quasi-conformal mappings

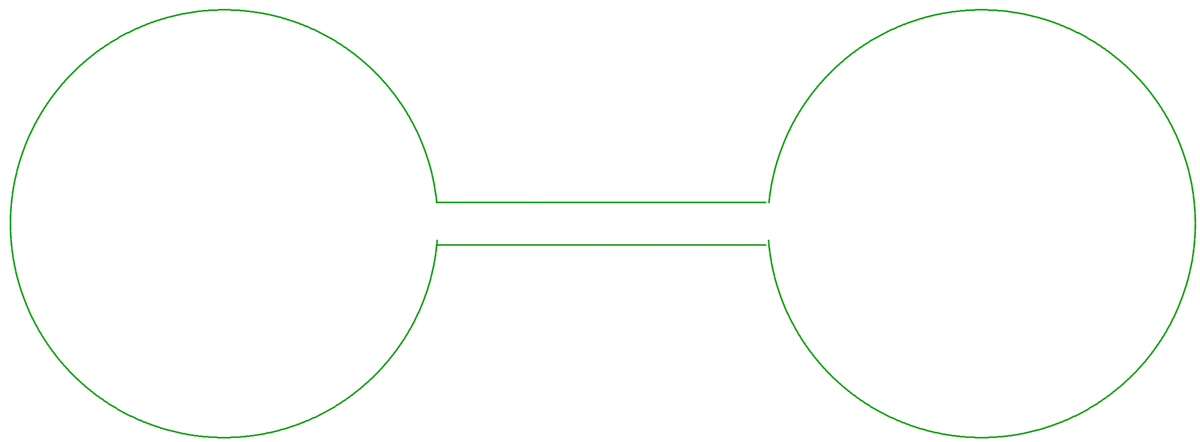


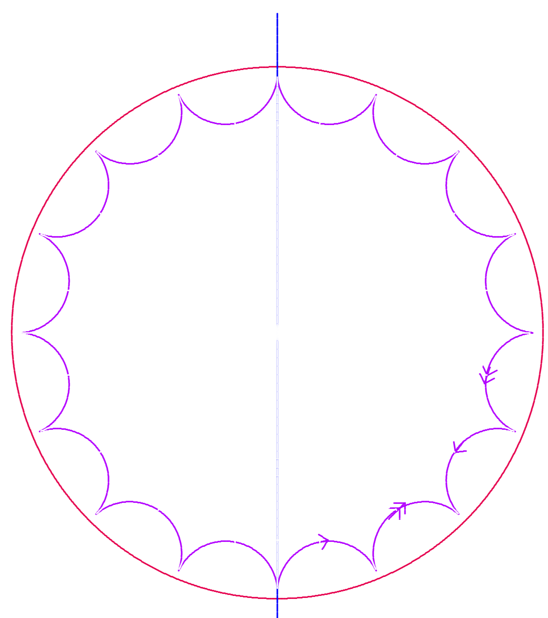


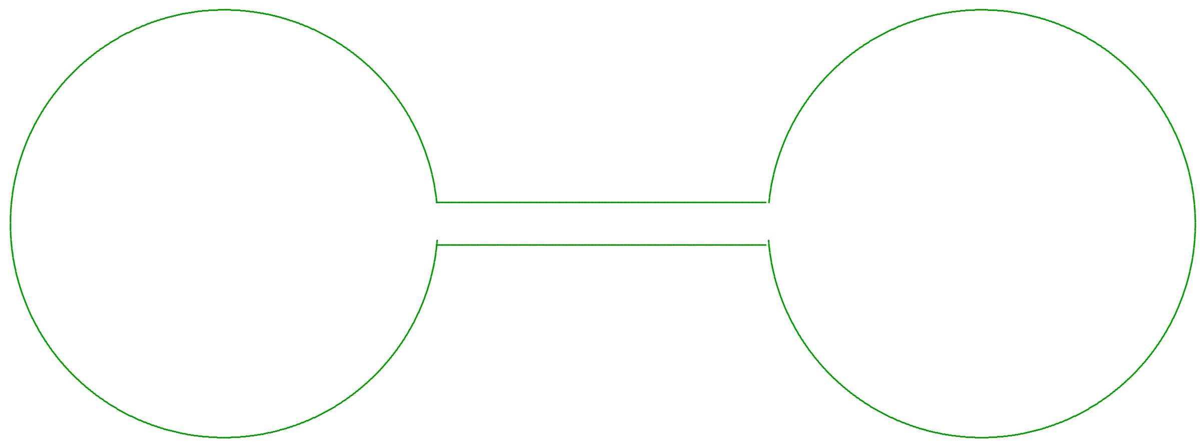


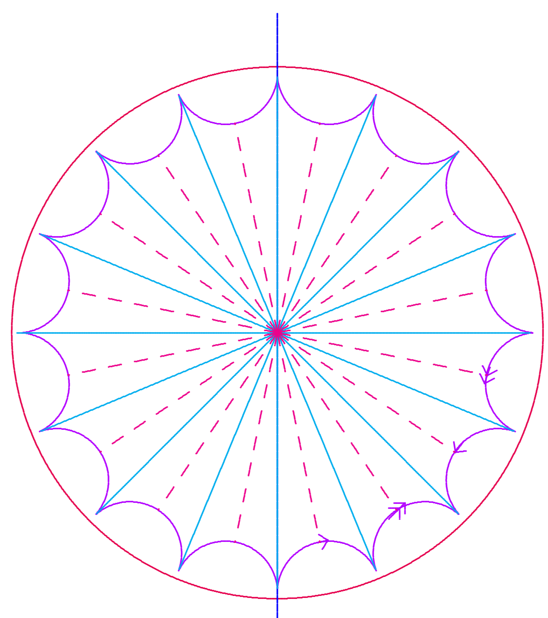


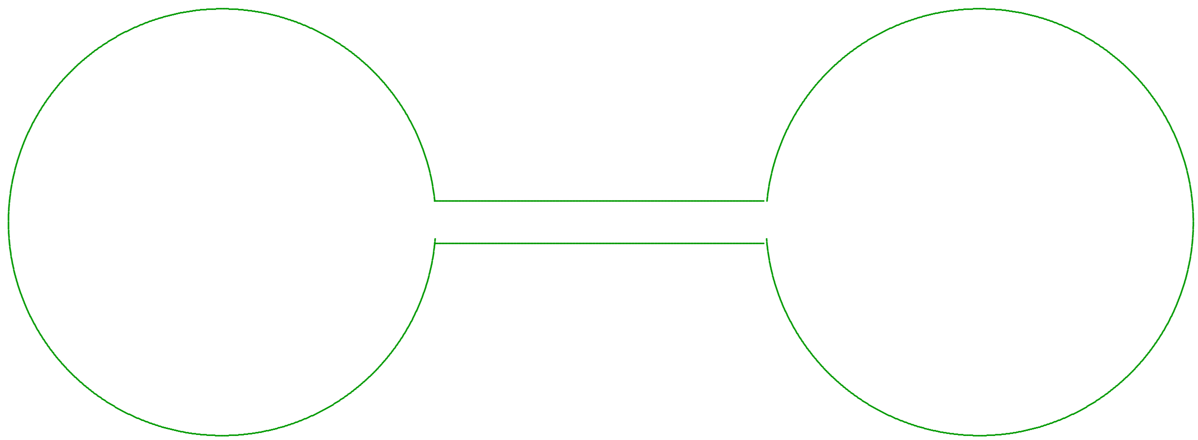












Theorem A. *Consider* 4*-manifolds* $M = \Sigma \times S^2$, *where* Σ *compact Riemann surface of genus* $\mathbf{g}$.

Then $\forall$ *even* $\mathbf{g} \gg 0$, $\exists$ *family* $[g_t]$, $t \in [0,1]$, *of locally-conformally-flat classes on* M, *such that*

- $\exists$ *scalar-flat Kähler metric* $g_0 \in [g_0]$; *but*
- $\nexists$ *almost-Kähler metric* $g \in [g_1]$.

Theorem B. *Fix an integer* $k \geq 2$*, and then consider the 4-manifolds* $M = (\Sigma \times S^2)\#k\overline{\mathbb{CP}}_2$*, where* Σ *compact Riemann surface of genus* g*.*

Then $\forall$ *even* $\mathsf{g} \gg 0$*,* $\exists$ *family* $[g_t]$*,* $t \in [0,1]$*, of anti-self-dual conformal classes on* M*, such that*

- $\exists$ *scalar-flat Kähler metric* $g_0 \in [g_0]$*; but*
- $\nexists$ *almost-Kähler metric* $g \in [g_1]$*.*

Kia Ora!