Einstein Manifolds,

Self-Dual Weyl Curvature, &

Conformally Kähler Geometry

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Stony Brook University

Geometry/Topology Seminar
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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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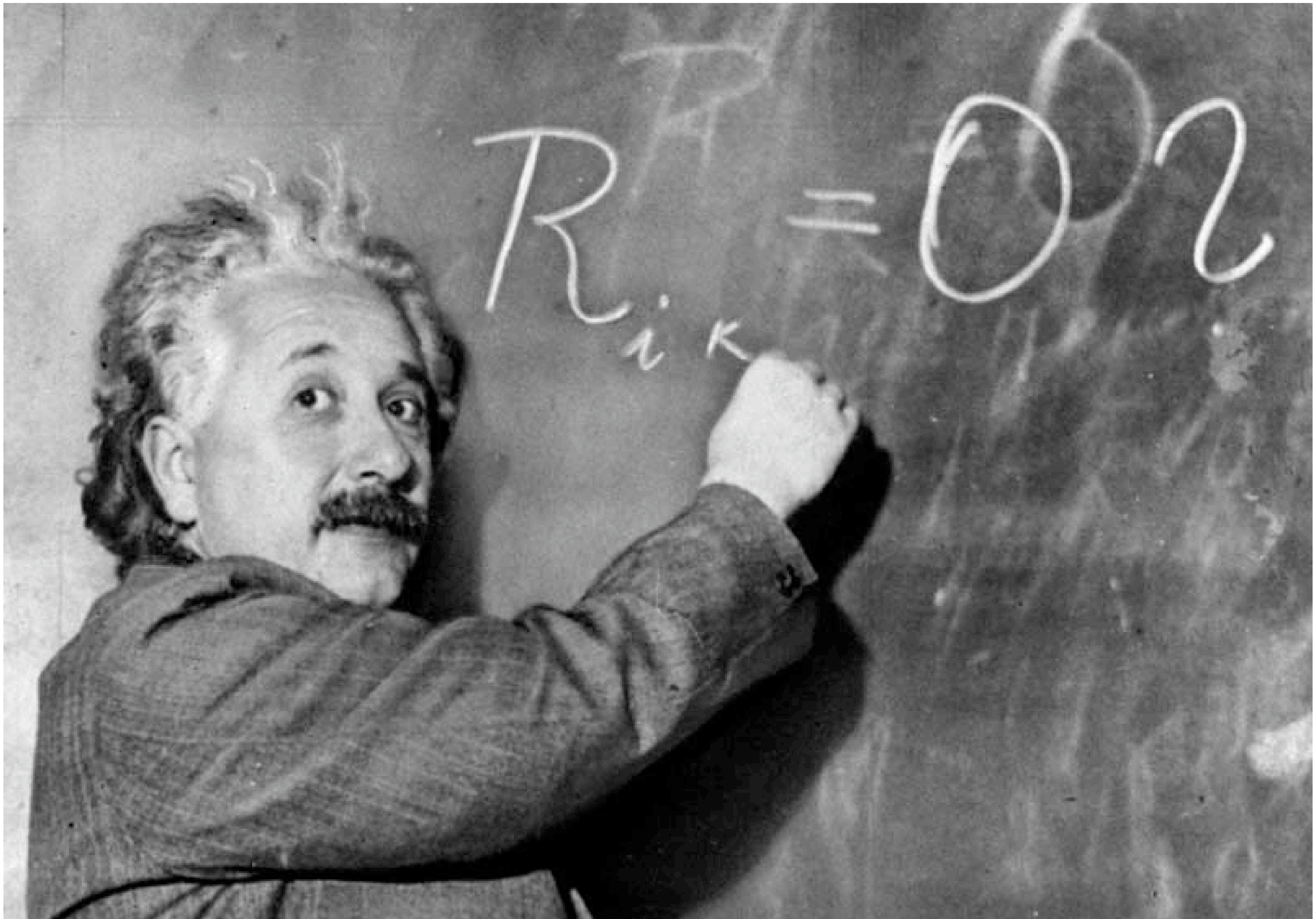

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Has same sign as the *scalar curvature*

$$s = r_j{}^j = \mathcal{R}^{ij}{}_{ij}.$$

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When $n = 4$, Einstein metrics satisfy a remarkable conformally-invariant condition.

On Riemannian n-manifold (M, g),

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$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2} \mathring{r}^{[a}{}_{[c} \delta^{b]}{}_{d]} + \frac{2}{n(n-1)} \, s \, \delta^{a}_{[c} \delta^{b}_{d]}$$

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$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

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Proposition. *Assume* $n \geq 4$. *Then* (M^n, g) *locally conformally flat* $\iff W \equiv 0$.

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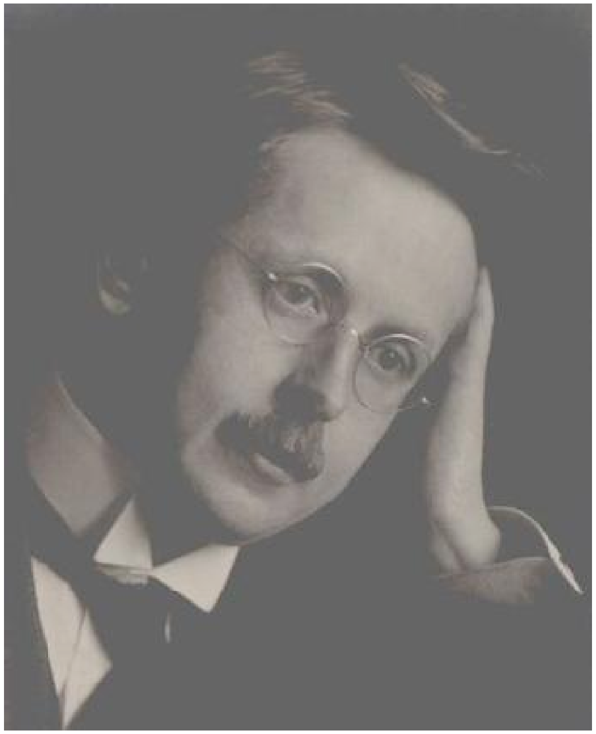

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Measures deviation $[g]$ from conformal flatness.

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When $n = 4$, conf. Einstein $\Rightarrow$ critical for $\mathscr{W}$.

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$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array}\right)$$

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	Λ^{+*}	Λ^{-*}
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$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

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Hence

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A case of special interest:

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(M^4, g, J) Kähler.

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$$|W_{+}|^{2} = \frac{s^{2}}{24}$$

Restriction of $\mathscr{W}_+$ to Kähler metrics?

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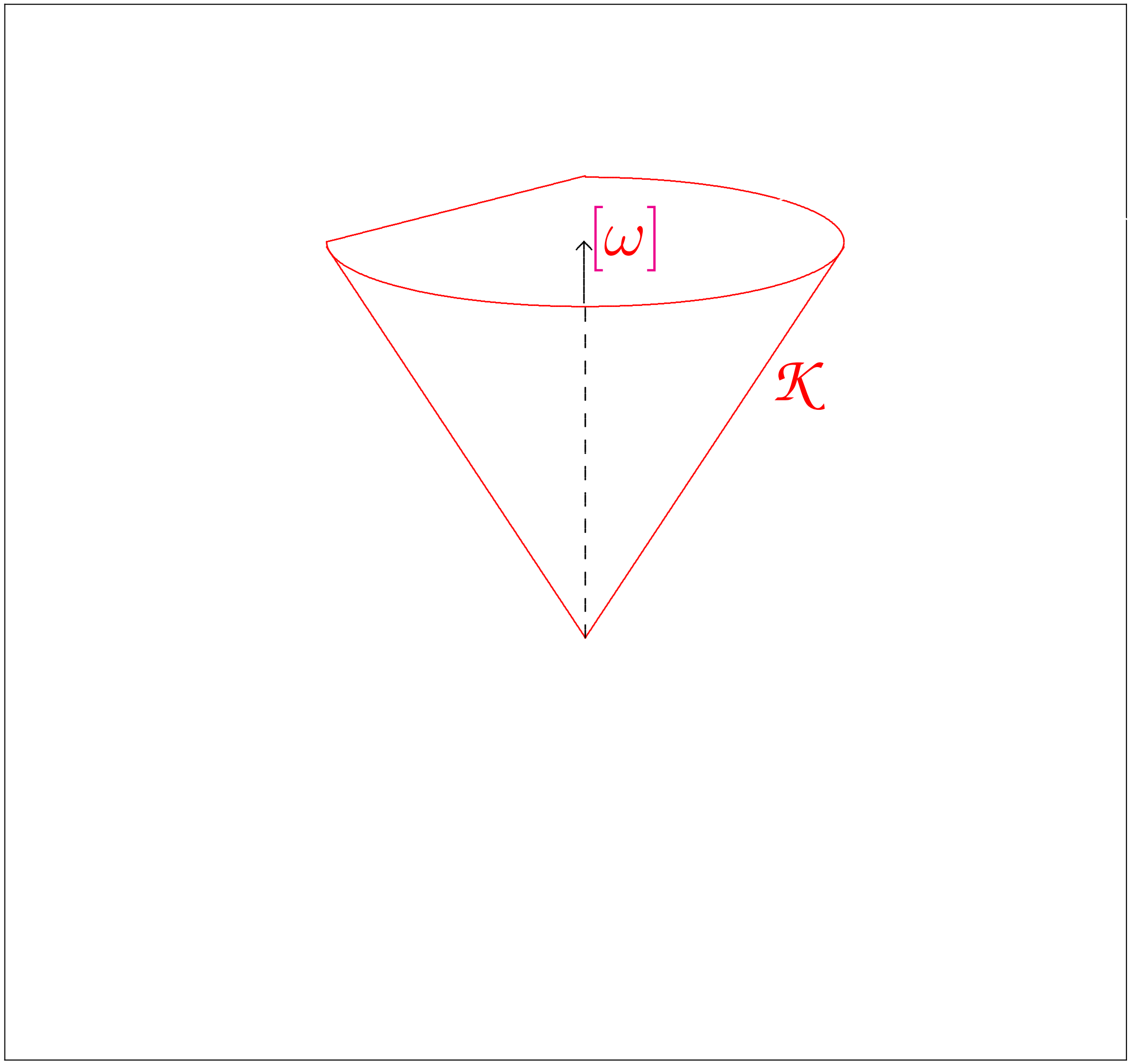

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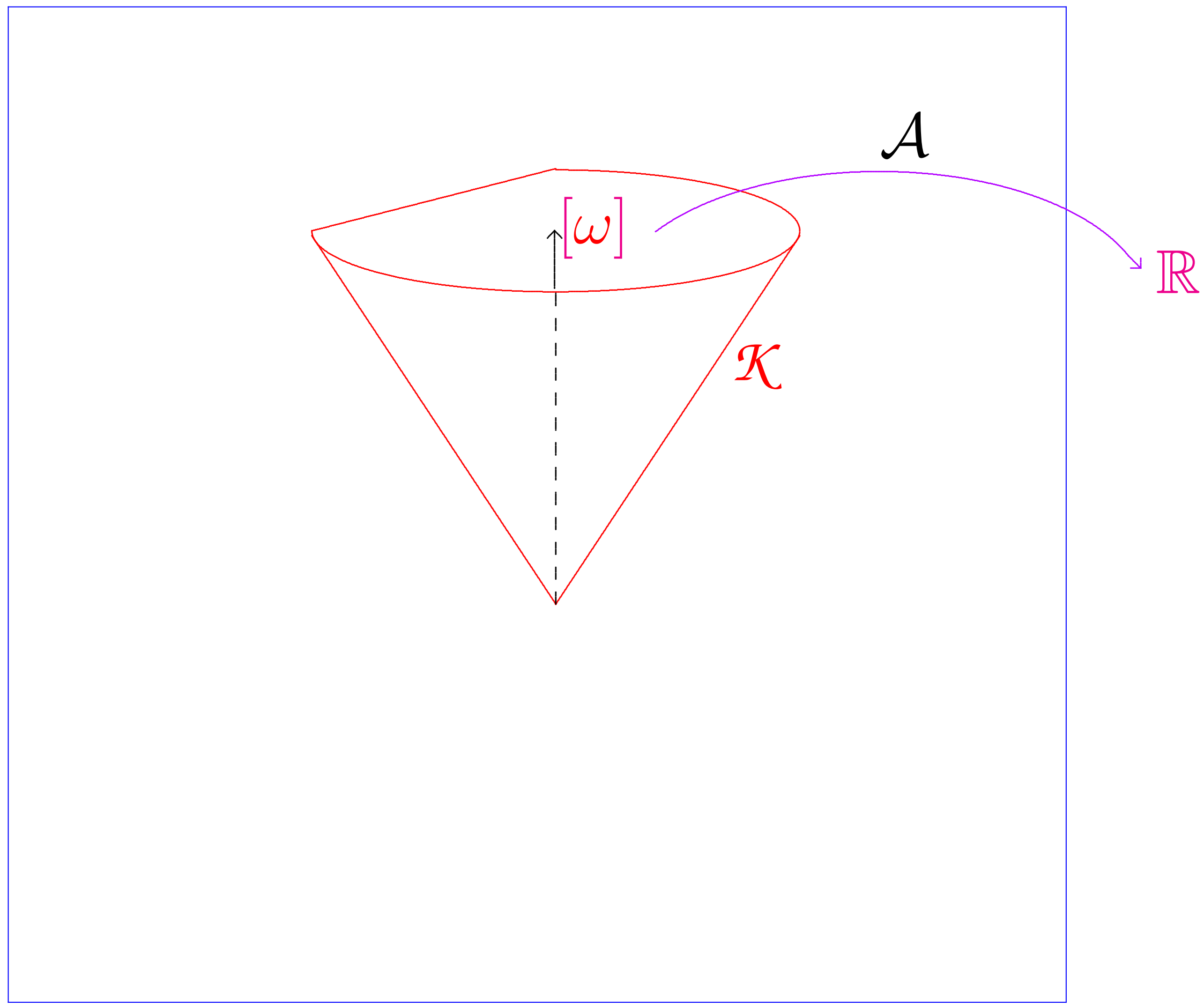

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- *g is an extremal Kähler metric; and*

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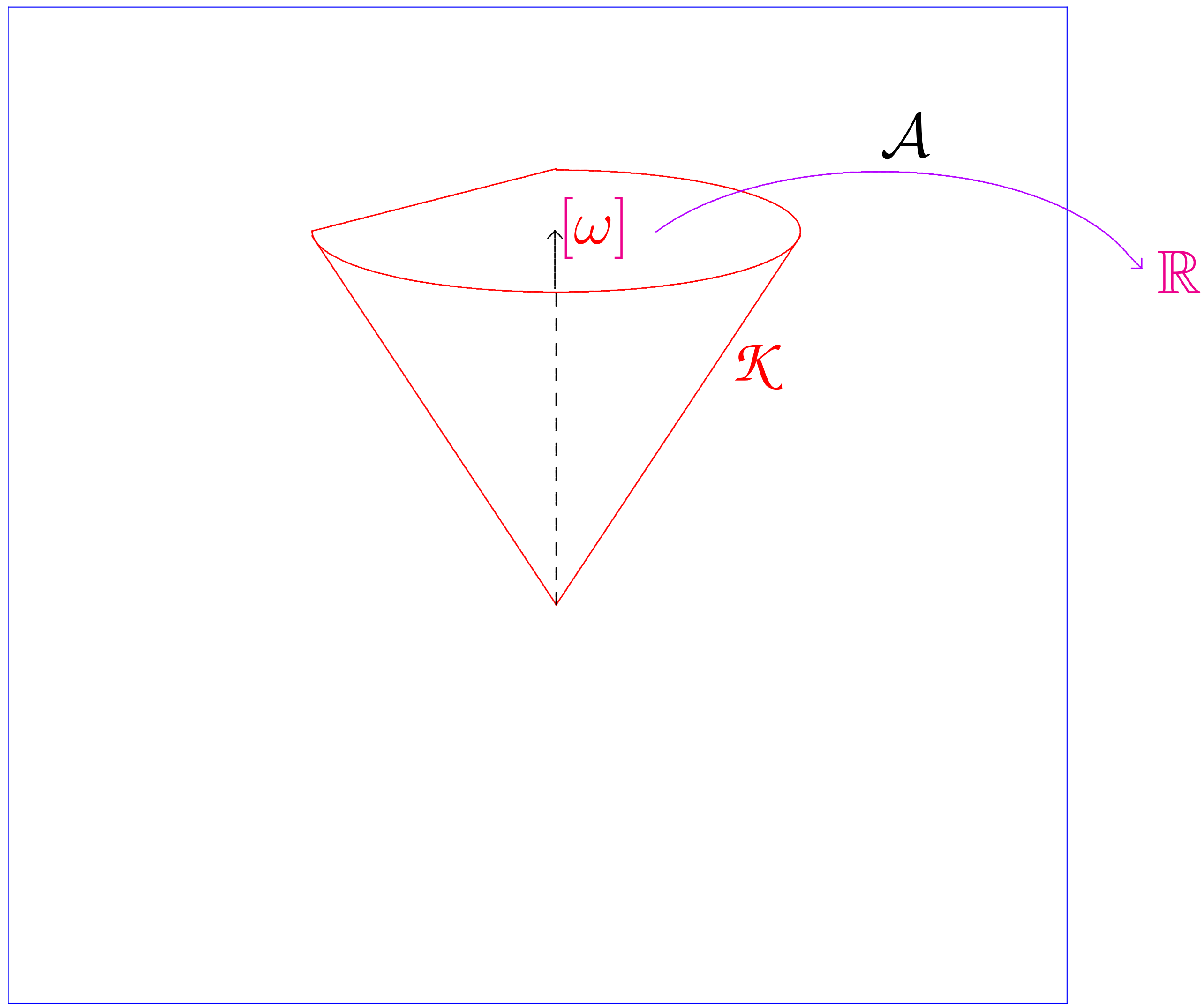
Proposition. *If g is a Kähler metric on a compact complex surface (M^4, J), with Kähler class $[\omega]$, then g satisfies $B = 0 \iff$*

- *g is an extremal Kähler metric; and*
- *$[\omega]$ is a critical point of $\mathcal{A} : \mathcal{K} \to \mathbb{R}$.*



$$\mathcal{K} \subset H^{1,1}(M,\mathbb{R}) \subset H^2(M,\mathbb{R})$$

Restriction of $\mathscr{W}_+$ to Kähler metrics?

On Kähler metrics,

$$\int |W_+|^2 d\mu = \int \frac{s^2}{24} d\mu$$

so any critical point of restriction must be extremal in sense of Calabi.

Andrzej Derdziński : For Kähler metrics g,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\Big]$$

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Global implications?

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Moreover, each case actually occurs.

Main interest today:

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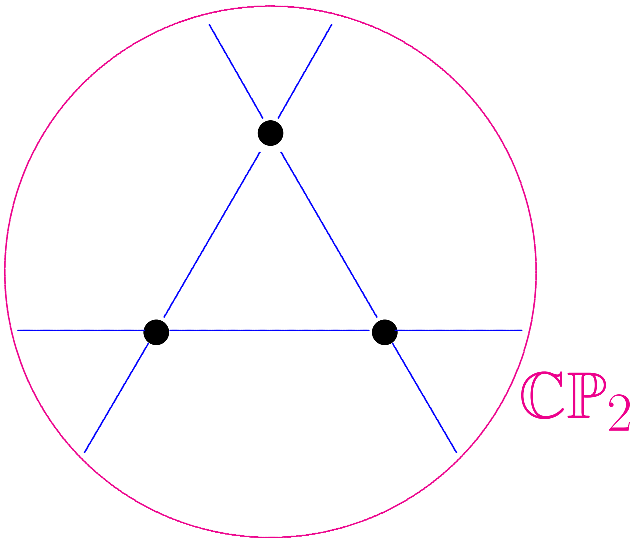

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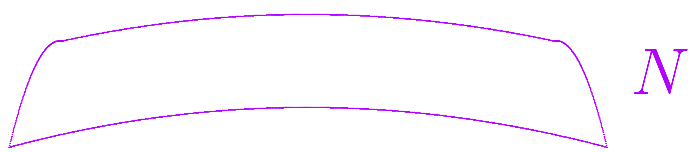

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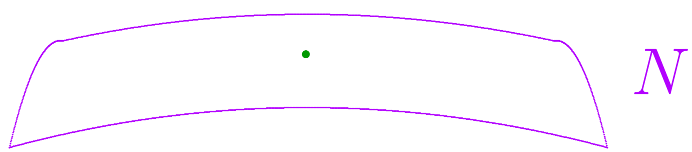

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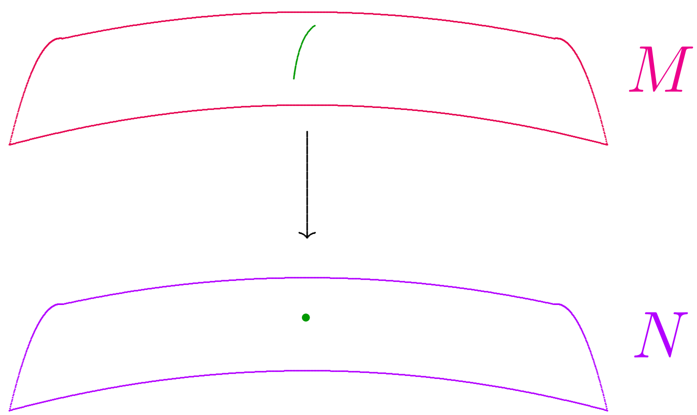

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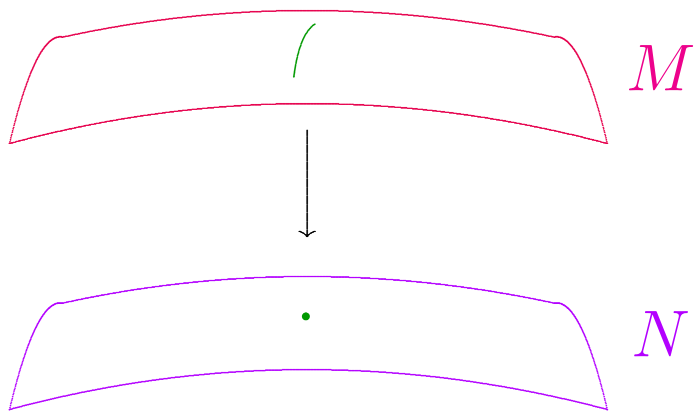

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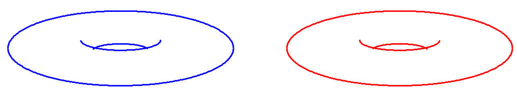

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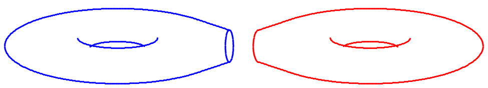

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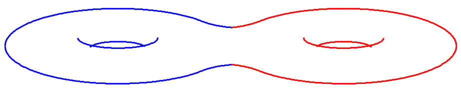

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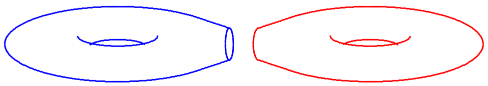

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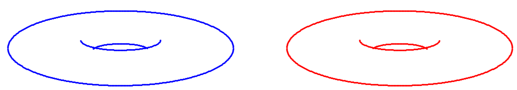

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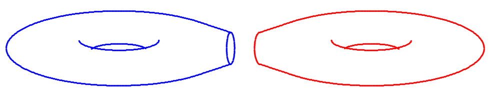

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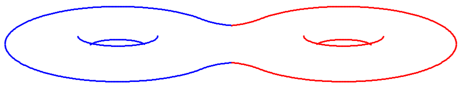

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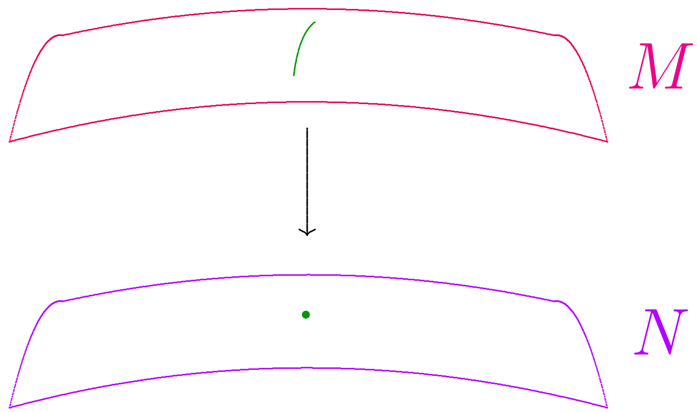

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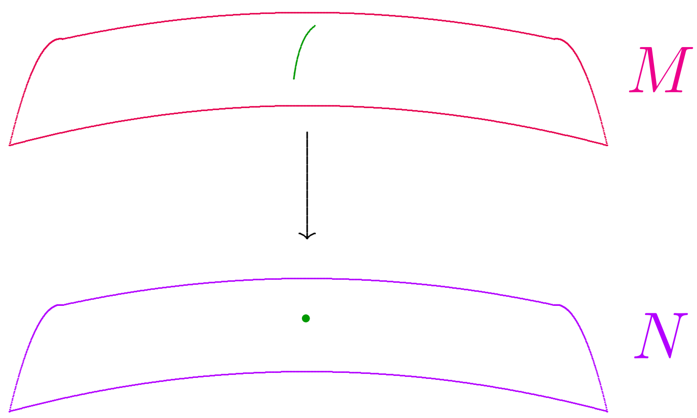

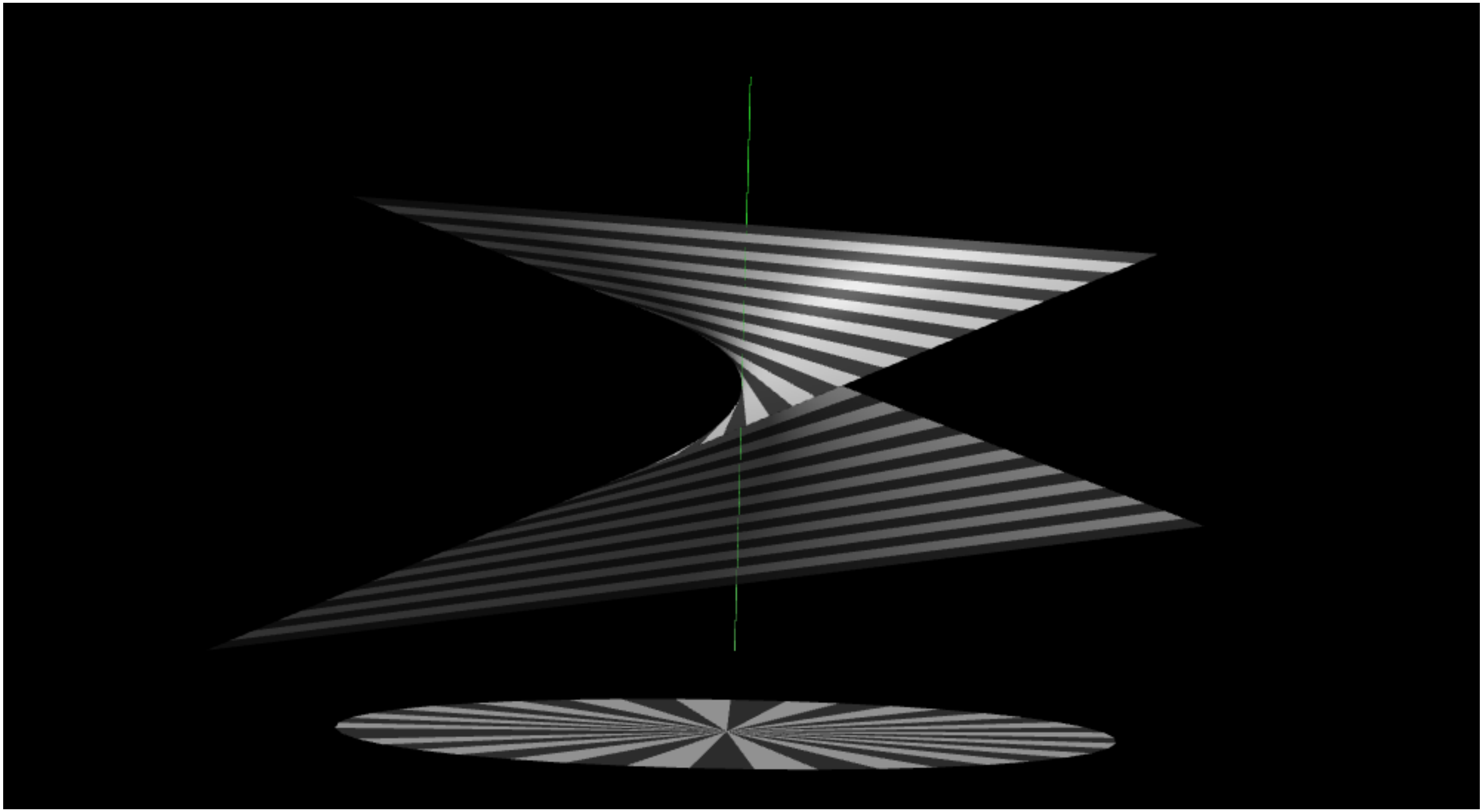

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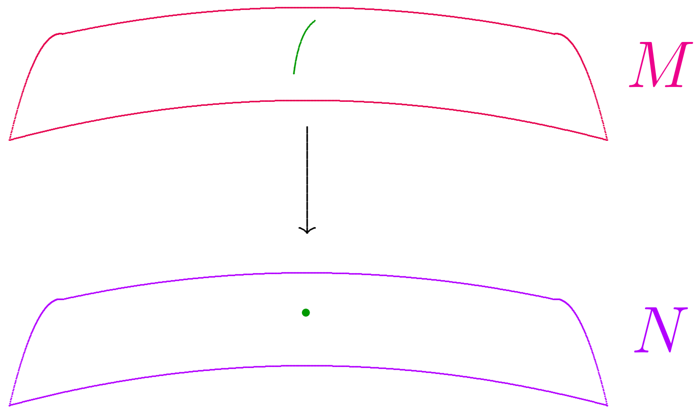

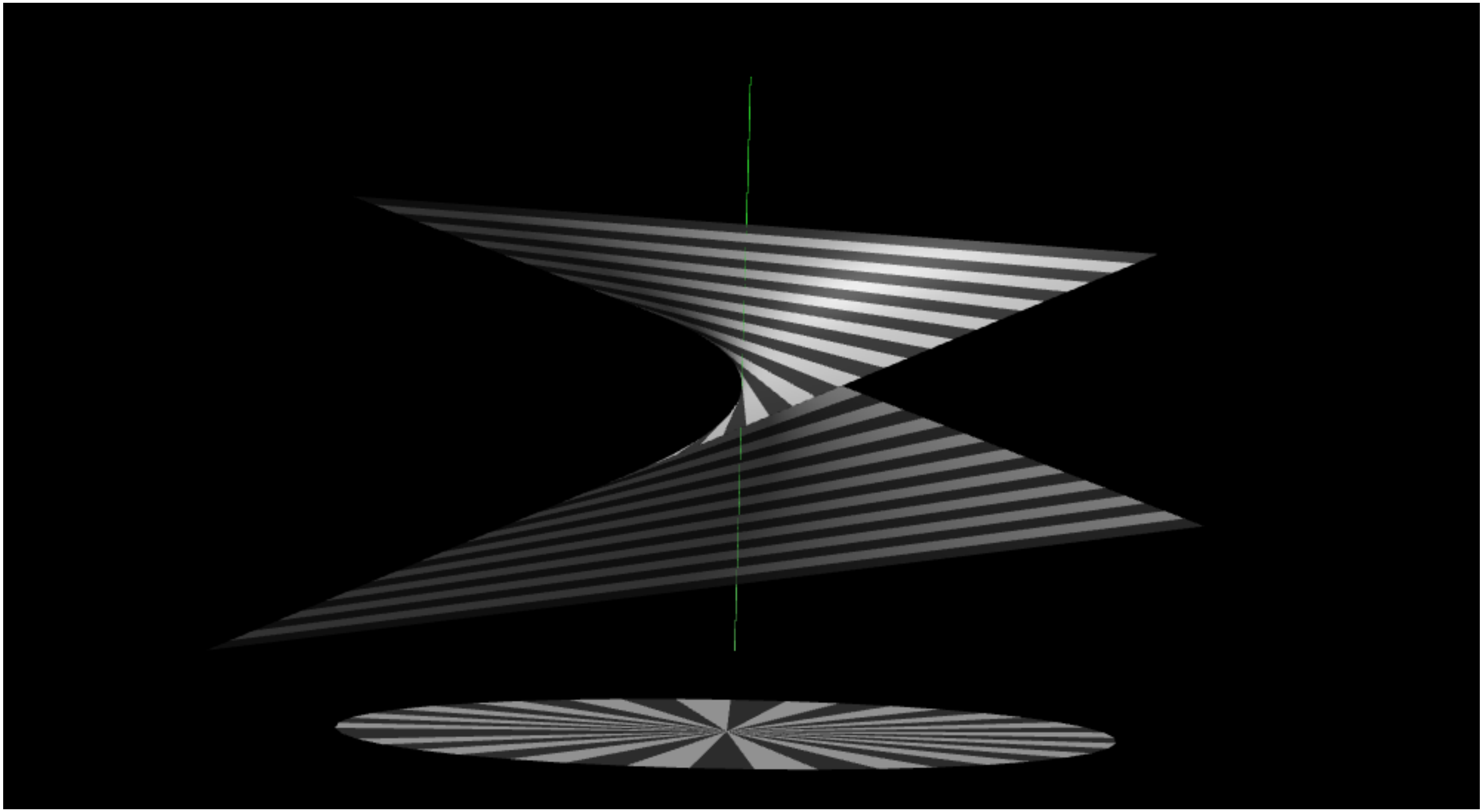

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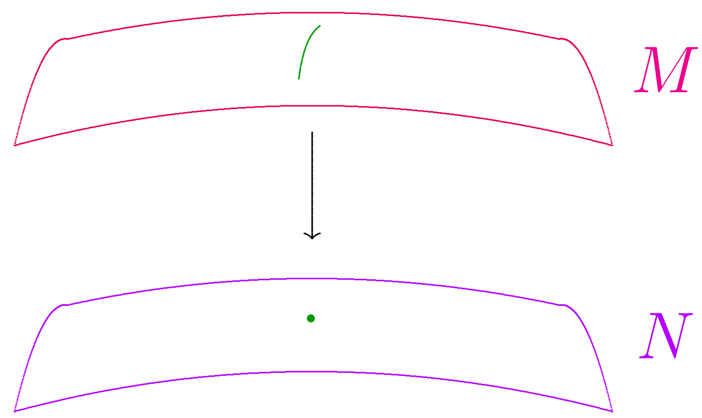

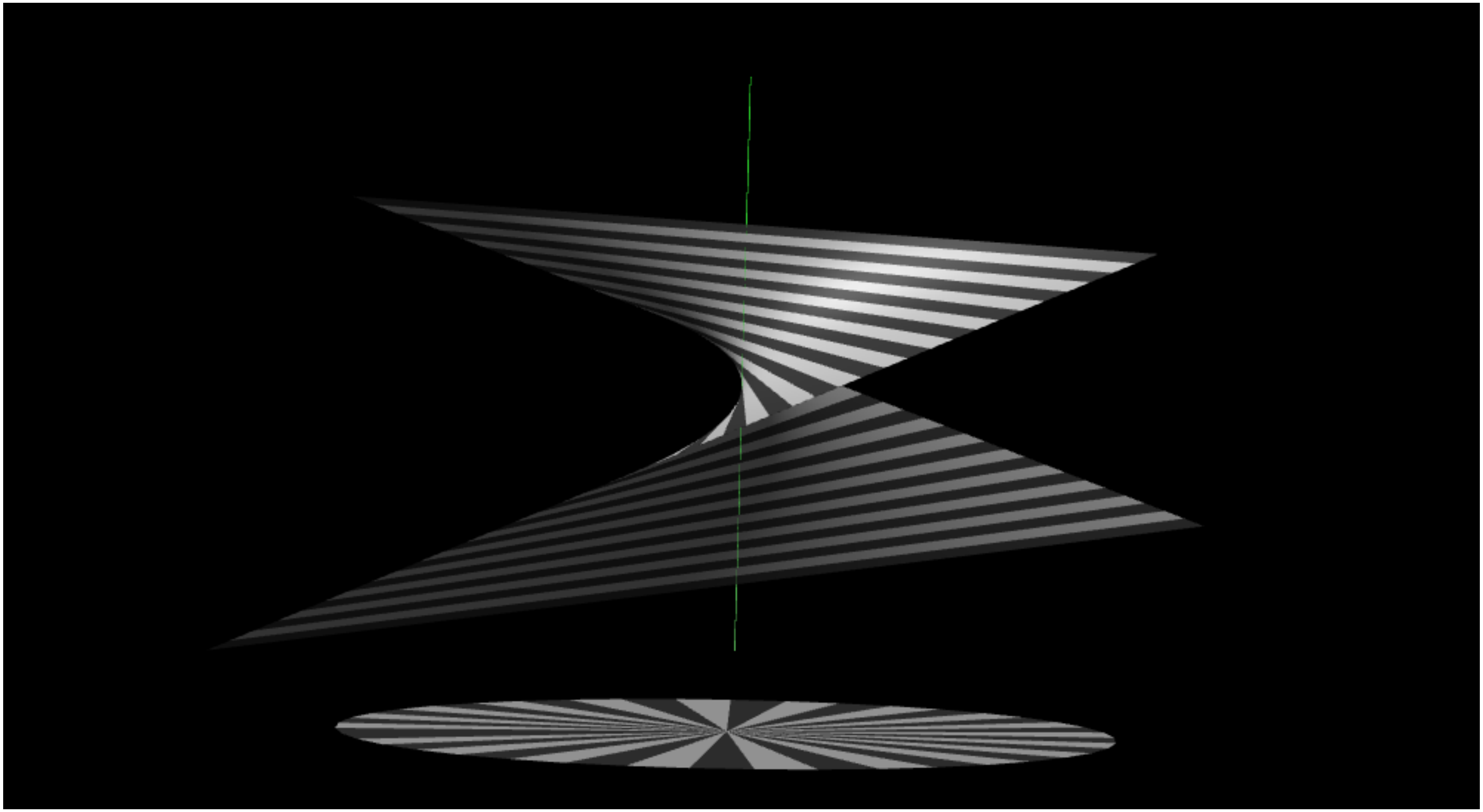

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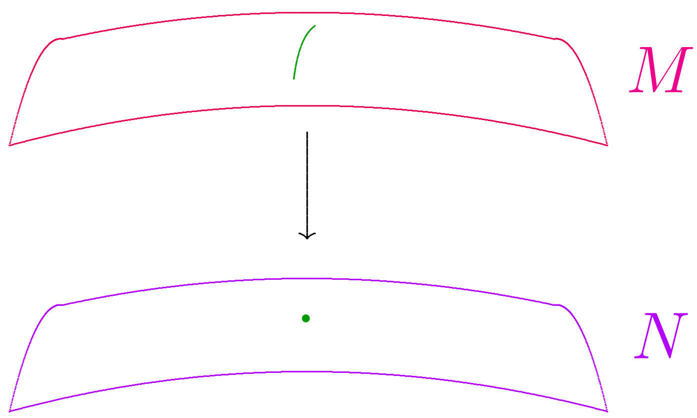
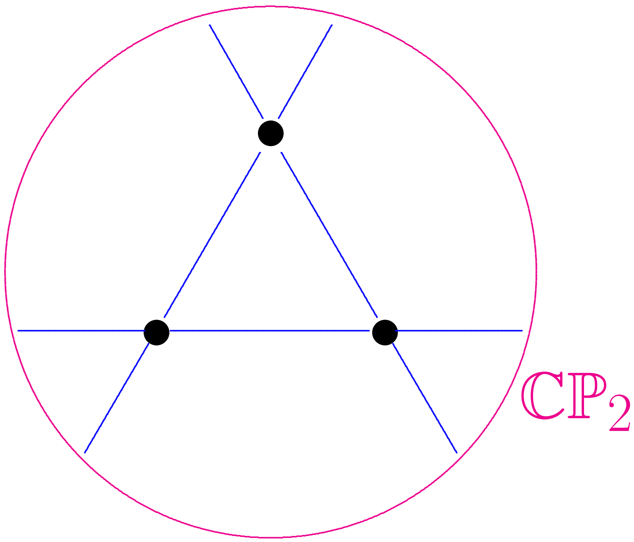


Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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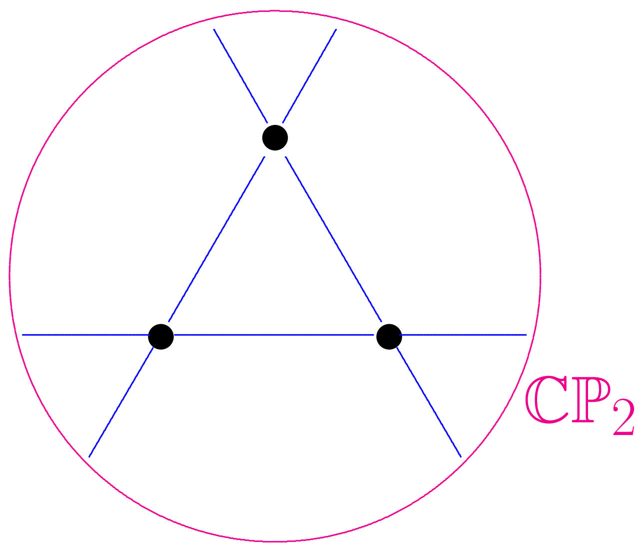

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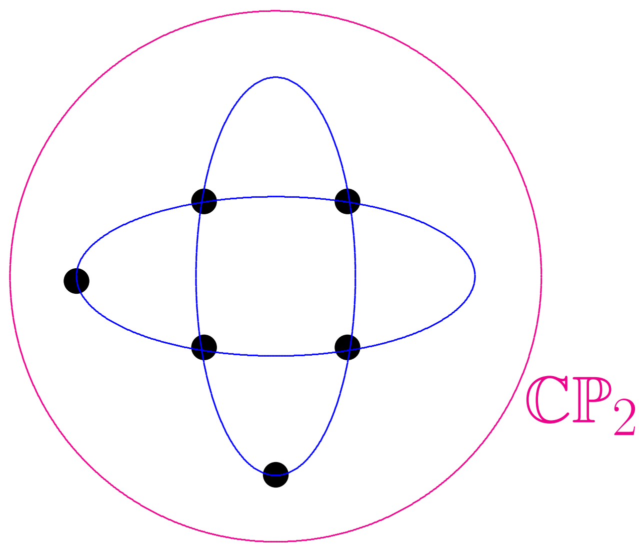

No 3 on a line, no 6 on conic,

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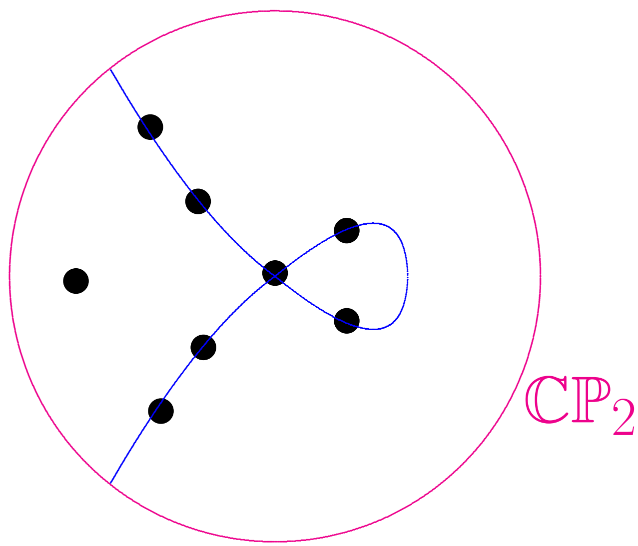

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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Theorem.

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One reason this seems satisfying. . .

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Exactly one connected component of moduli space!

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$$W^+ = \text{trace-free part of } \begin{bmatrix} 0 & & \\ & 0 & \\ & & \frac{s}{4} \end{bmatrix}$$

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for these metrics & conformal rescalings:

$$g \rightsquigarrow h = f^2 g \quad \Longrightarrow \quad \det(W^+) \rightsquigarrow f^{-6} \det(W^+).$$

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either on M or double cover $\widetilde{M}$.

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Claim: (M, h) compact Einstein $\Longrightarrow$ J integrable.

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Theorem B. *Let* (M, h) *be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

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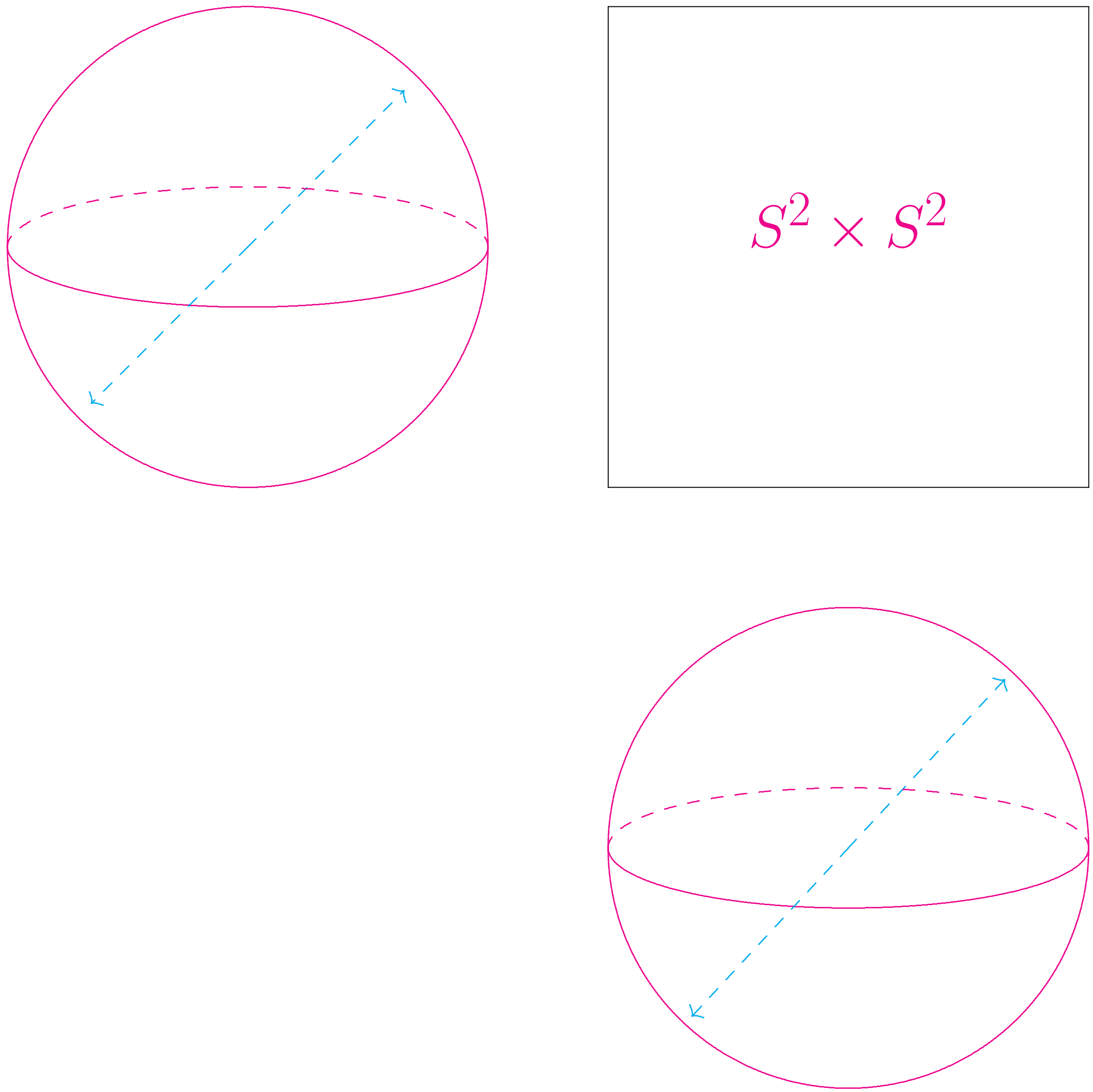
Excludes 5 types with $\pi_1 = \mathbb{Z}_2$ and $b_+(M) = 0$.


$S^2 \times S^2$

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Correctly understand equation $\delta W^+ = 0$.

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$$0 = \nabla^*\nabla W^+ + \frac{s}{2}W^+ - 6W^+ \circ W^+ + 2|W^+|^2 I$$

for $W^+ \in \mathrm{End}(\Lambda^+)$, with respect to h.

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holds whenever $h = f^2 g$ satisfies $\delta W^+ = 0$.

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This identity has many applications.

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thus showing that g must actually be Kähler.

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Now choose $\omega \in \Gamma\Lambda^+$ so that

$$W_g^+(\omega) = \alpha\, \omega, \quad |\omega|_g \equiv \sqrt{2},$$

after at worst passing to double cover $\hat{M} \to M$.

$$0 = \int_{\hat{M}} \Big[\langle W^+, \nabla^* \nabla (\omega \otimes \omega) \rangle + \frac{s}{2} W^+(\omega, \omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2 \Big] f \, d\mu$$

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$$0 = \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) - 2W^+(\omega, \nabla^e \nabla_e \omega) + \frac{s}{2} W^+(\omega,\omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2 \Big] f \, d\mu$$

$$0 = \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) - 2\alpha\langle \omega, \nabla^e \nabla_e \omega\rangle$$

$$+\frac{s}{2}\alpha|\omega|^2 - 6\alpha^2|\omega|^2 + 2|W^+|^2|\omega|^2\Big] f \, d\mu$$

because

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because

$$|W_g{}^+|^2 \geq \frac{3}{2}\alpha^2$$

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$$|\omega|_g^2 = 2 \implies (\nabla_e \omega) \perp \omega$$

$$\det(W^+) > 0 \implies W^+(\nabla_e \omega, \nabla^e \omega) \leq 0$$

$$0 \geq \int_M \Big[2\alpha\langle \omega, \nabla^*\nabla\omega\rangle + \frac{s}{2}\alpha|\omega|^2 - 3\alpha^2|\omega|^2 \Big] f \; d\mu$$

$$|\omega|^2_g = 2 \quad \Longrightarrow \quad (\nabla_e \omega) \perp \omega$$

$$\det(W^+) > 0 \quad \Longrightarrow \quad -W^+(\nabla_e\omega, \nabla^e\omega) \geq 0$$

$$0 \geq \int_M \left[2\alpha \langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2} \alpha |\omega|^2 - 3\alpha^2 |\omega|^2 \right] f \, d\mu$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2}|\omega|^2 - 3\alpha|\omega|^2\right](\alpha f)\ d\mu$$

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But

$$\alpha f \equiv 1$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2}|\omega|^2 - 3|\omega|^2 \alpha \right] d\mu$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle - 3W^+(\omega, \omega) + \frac{s}{2}|\omega|^2 \right] d\mu$$

$$0 \geq \int_M \left[\frac{1}{2}|\nabla\omega|^2 + \frac{3}{2}\langle\omega, \left(\nabla^*\nabla - 2W^+ + \frac{s}{3}\right)\omega\rangle\right] d\mu$$

$$0 \geq \int_M \left[\tfrac{1}{2}|\nabla\omega|^2 + \tfrac{3}{2}\langle\omega, \quad (d+d^*)^2 \quad \omega\rangle\right] d\mu$$

Because

$$(d+d^*)^2 = \nabla^*\nabla - 2W^+ + \frac{s}{3}$$

on $\Gamma\Lambda^+$.

$$0 \geq \frac{1}{2} \int_M |\nabla \omega|^2 \, d\mu + 3 \int_M |d\omega|^2 \, d\mu$$

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So $\nabla\omega \equiv 0$, and g is Kähler!

Theorem B. *Let* (M, h) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M*. Then* h *is conformally Kähler, and* M *is a Del Pezzo surface.*

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Produces harmonic ω with $W^+(\omega, \omega) > 0$.

Now use my earlier result!

Theorem C. *Let* (M, h) *be a compact oriented Riemannian* 4*-manifold with* $\delta W^+ = 0$*. If*

$$W^+ \neq 0 \quad \textit{and} \quad \det(W^+) \geq -\frac{5\sqrt{2}}{21\sqrt{21}}|W^+|^3$$

everywhere on M*, then actually* $\det(W^+) > 0$*. In particular, if* (M, h) *is a simply-connected Einstein manifold, then* h *is conformally Kähler, and* M *is a Del Pezzo surface.*

Thanks for the invitation!

It's a pleasure to be here!