Yamabe Invariants,

Weyl Curvature,

and the

Differential Topology of 4-Manifolds

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Stony Brook University

"Not Only Scalar Curvature" Seminar
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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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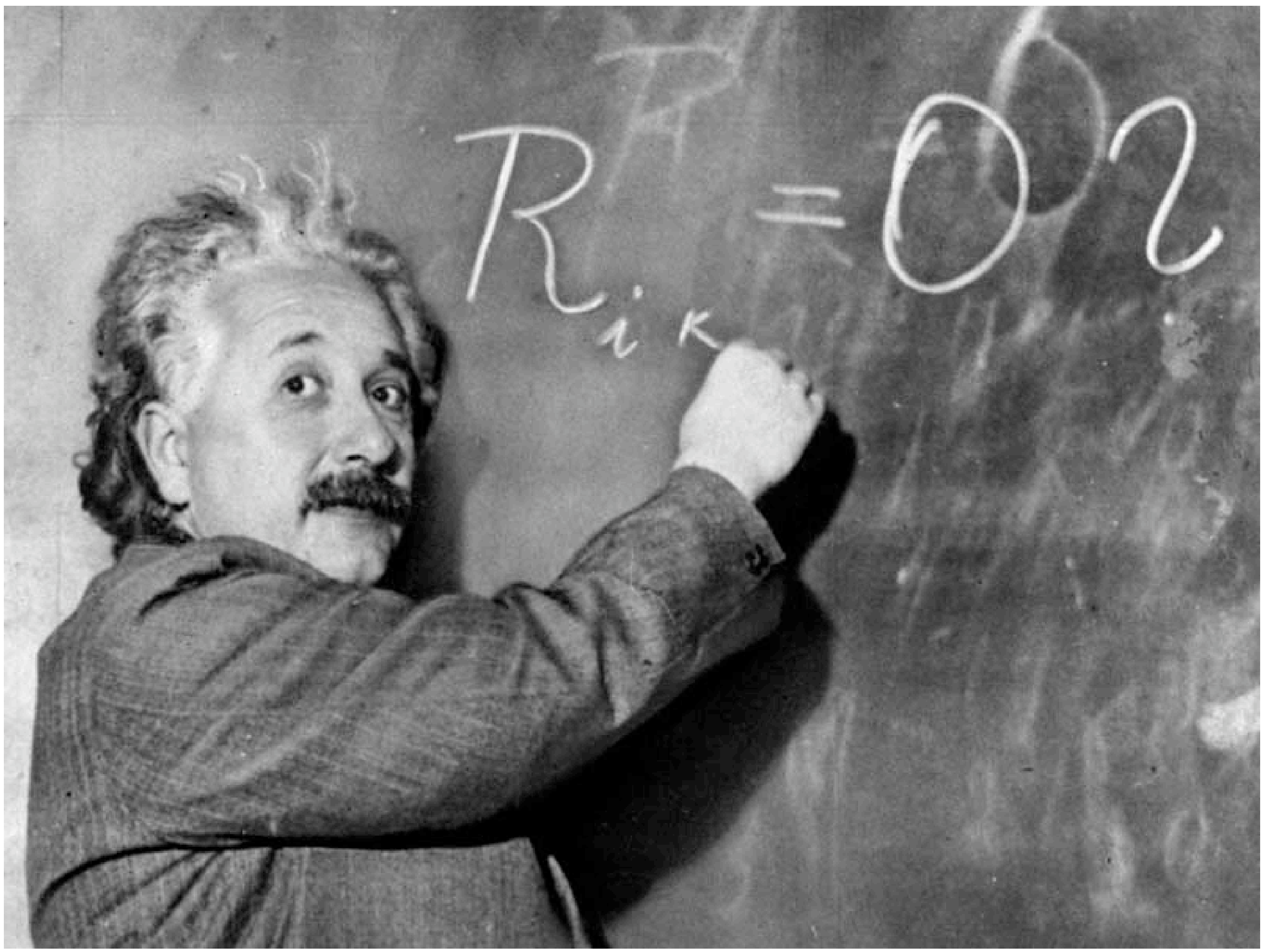

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where $V = \mathrm{Vol}(M, g)$ inserted to make scale-invariant.

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Then restriction $\mathscr{E}|_\gamma$ is bounded below.

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Difficulty: $L^2_1 \hookrightarrow L^p$ bounded, but not compact.

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Has s = constant.

Unique up to scale when $s \leq 0$.

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$$= \quad \text{only for round sphere.}$$

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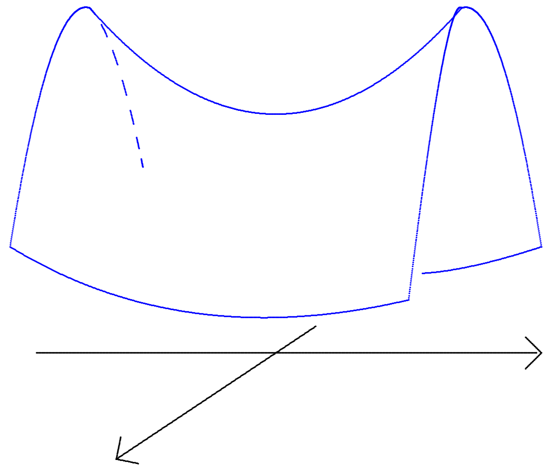

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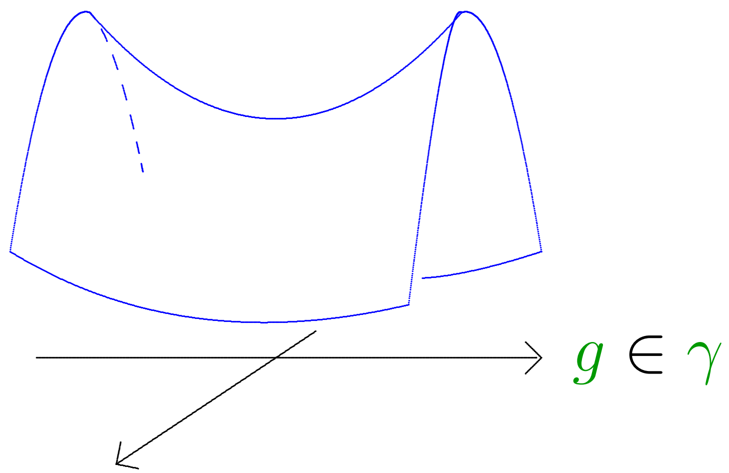

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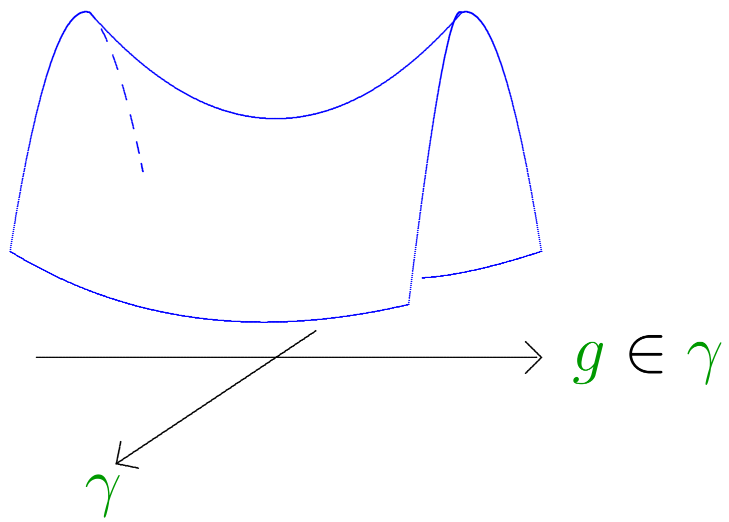

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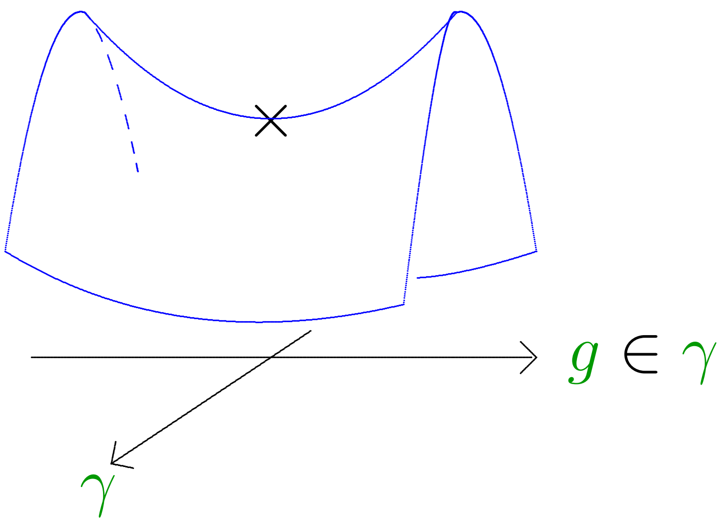

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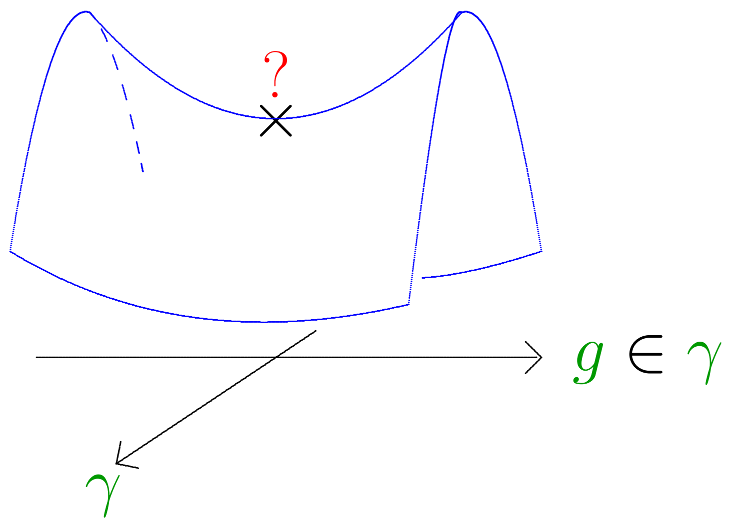

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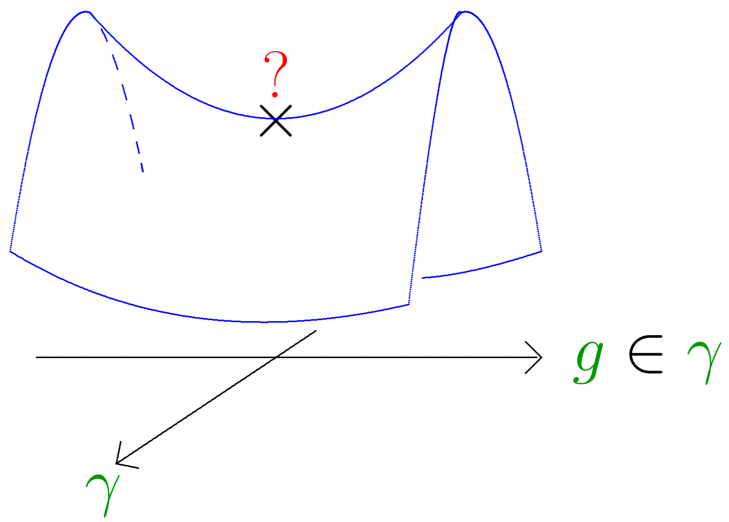

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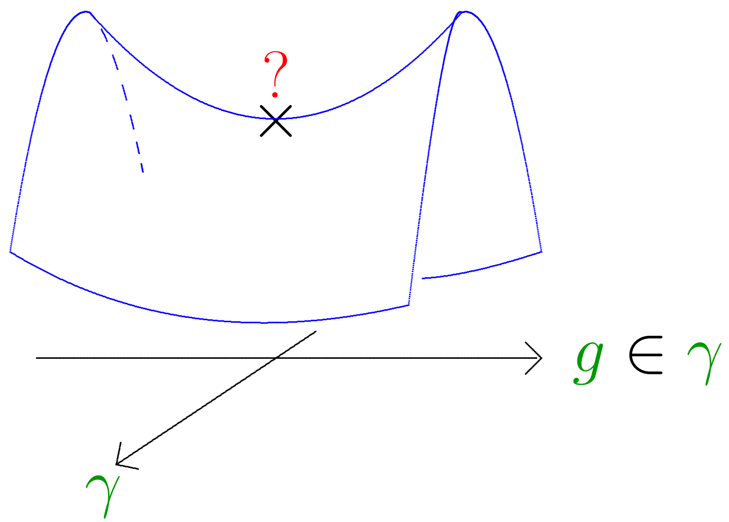

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But gives rise to a smooth-manifold invariant. . .

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R. Schoen ('87): "sigma constant"

O. Kobayashi ('87): "mu invariant"

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Moreover, can choose M_j such that each $\mathscr{Y}(M_j)$ is realized by an Einstein metric g_j.

This last result follows from…

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Kähler-Einstein means that (M, g) is Einstein, with almost-complex structure J s.t. $\nabla J = 0$ w/r to g.

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"square" ${c_1}^2$ with respect to intersection form

$$\cup : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \to \mathbb{Z}$$

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While $c_1(M, J) \in H^2(M, \mathbb{Z})$ depends on J,

$${c_1}^2(M, J) = (2\chi + 3\tau)(M)$$

is an oriented homotopy invariant of M.

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Method of proof: Seiberg-Witten theory.

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Corollary.

$$\mathscr{Y}(\mathbb{CP}_2) = 12\pi\sqrt{2} < 8\pi\sqrt{6} = \mathscr{Y}(S^4).$$

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$\Longrightarrow \nexists$ Einstein metric achieving $\mathscr{Y}(M)$.

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Original proof used perturbed SW equations.

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Shows certain other 4-mfds have $\mathscr{Y}(M) < \mathscr{Y}(S^4)$,

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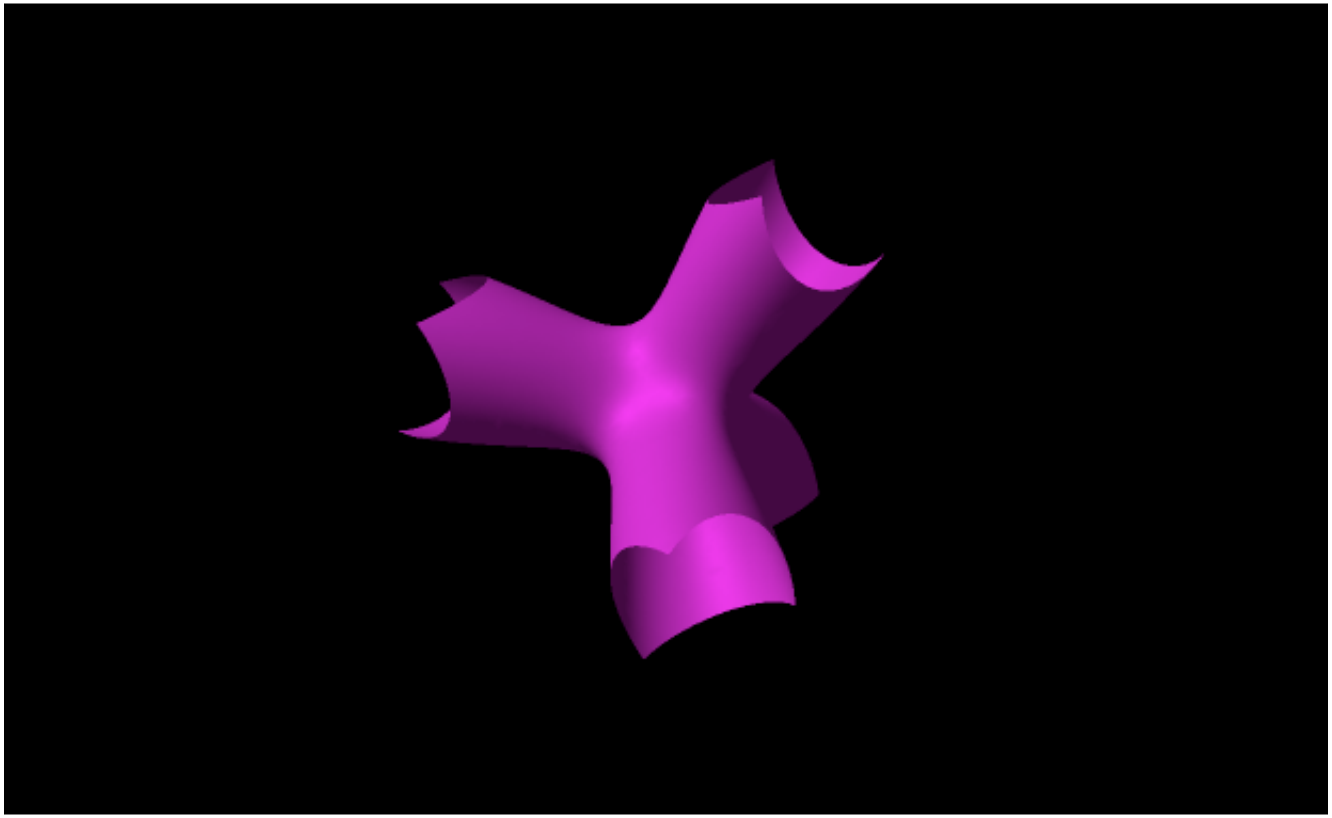

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Yau, Aubin, Siu, et al.

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Theorem (L '96). *There exist compact simply connected* 4*-manifolds* M_j *with* $\mathscr{Y}(M_j) \to -\infty$*.*

Moreover, can choose M_j *such that each* $\mathscr{Y}(M_j)$ *is realized by an Einstein metric* g_j*.*

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These examples also show that the diffeomorphism invariant $\mathscr{Y}(M)$ is not simply a homeomorphism invariant — can detect "exotic" smooth structures.

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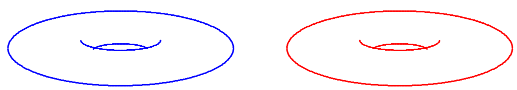

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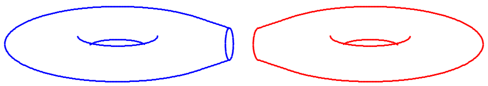

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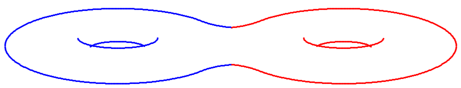

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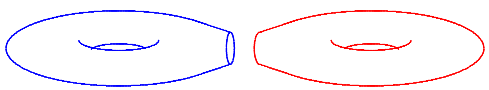

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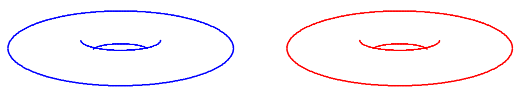

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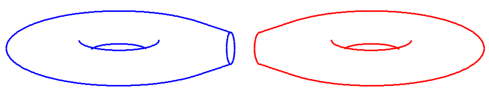

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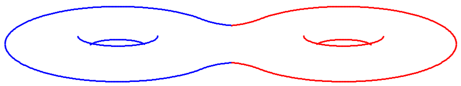

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Also notice $\mathscr{Y}[k(K3)\#\ell(S^2 \times S^2)]$ unachievable!

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Λ^+ self-dual 2-forms.
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Integrals give four scale-invariant functionals.

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Simplifies computation of $\mathscr{Y}(M)$ in negative case!

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But only two of these are genuinely independent!

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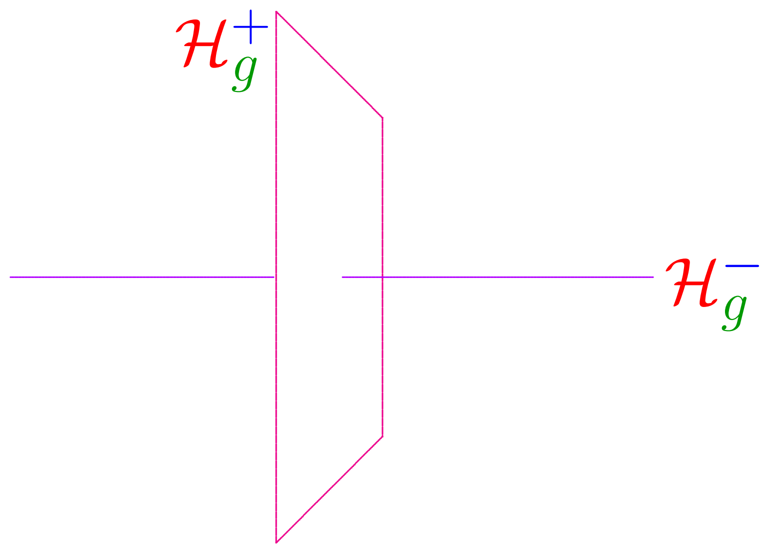

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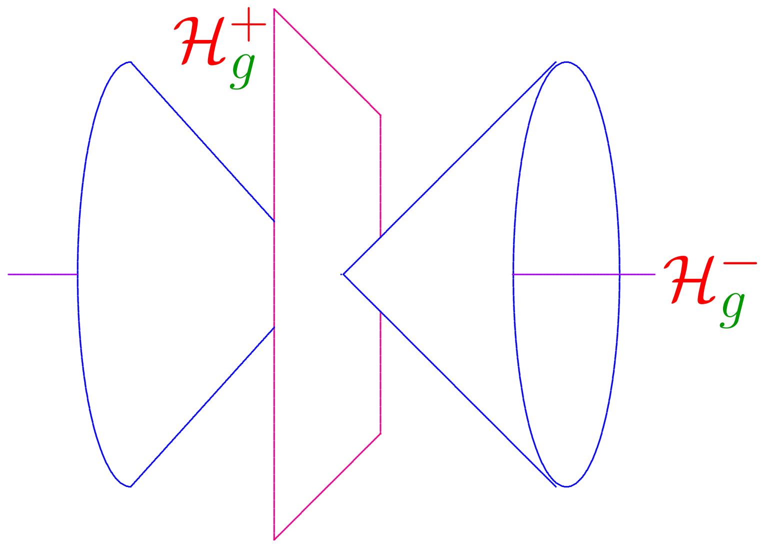

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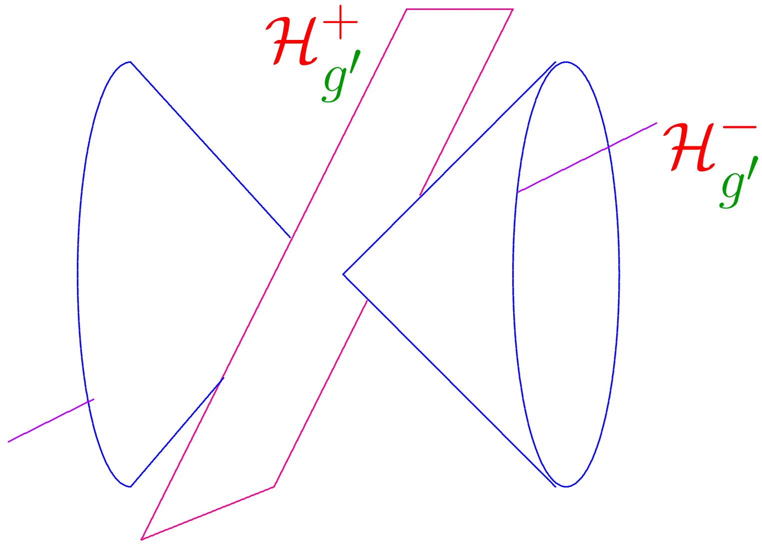

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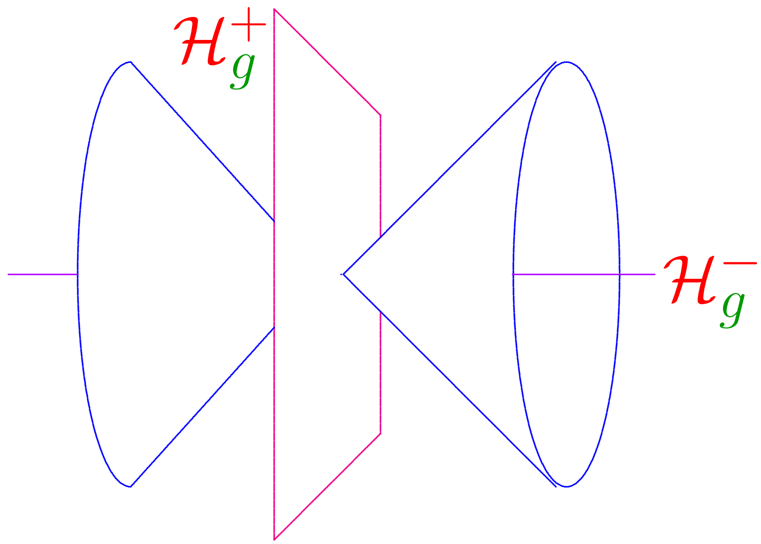

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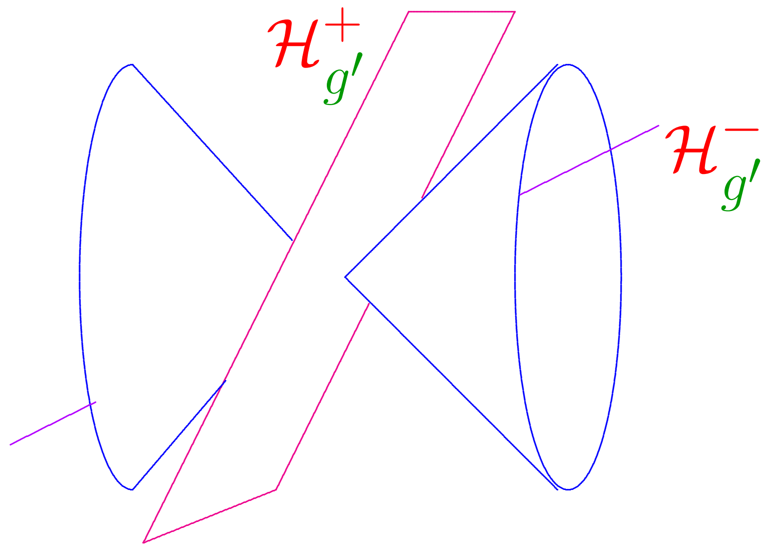

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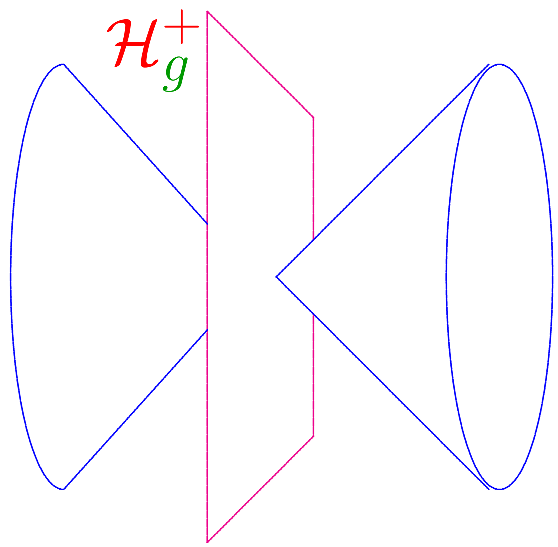

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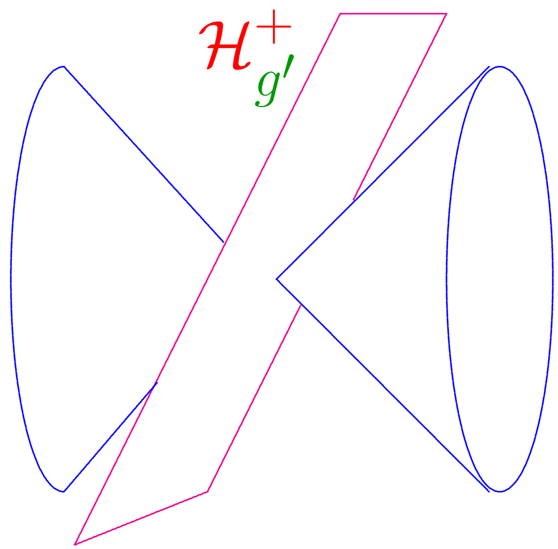

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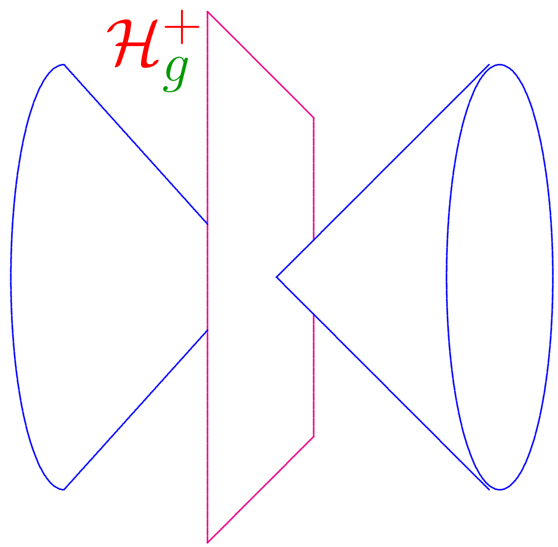

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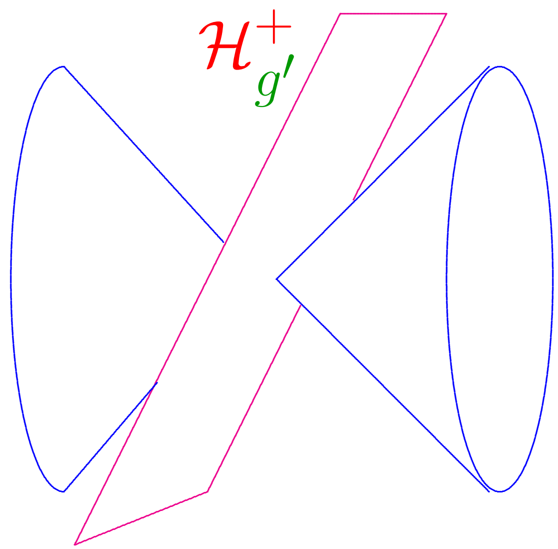

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Bootstrapping with gauge-fixed equations, one gets L^p_k bounds for (Φ, θ) for all k, p.

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Basic strategy becomes: play several spin^c structures off against one another.

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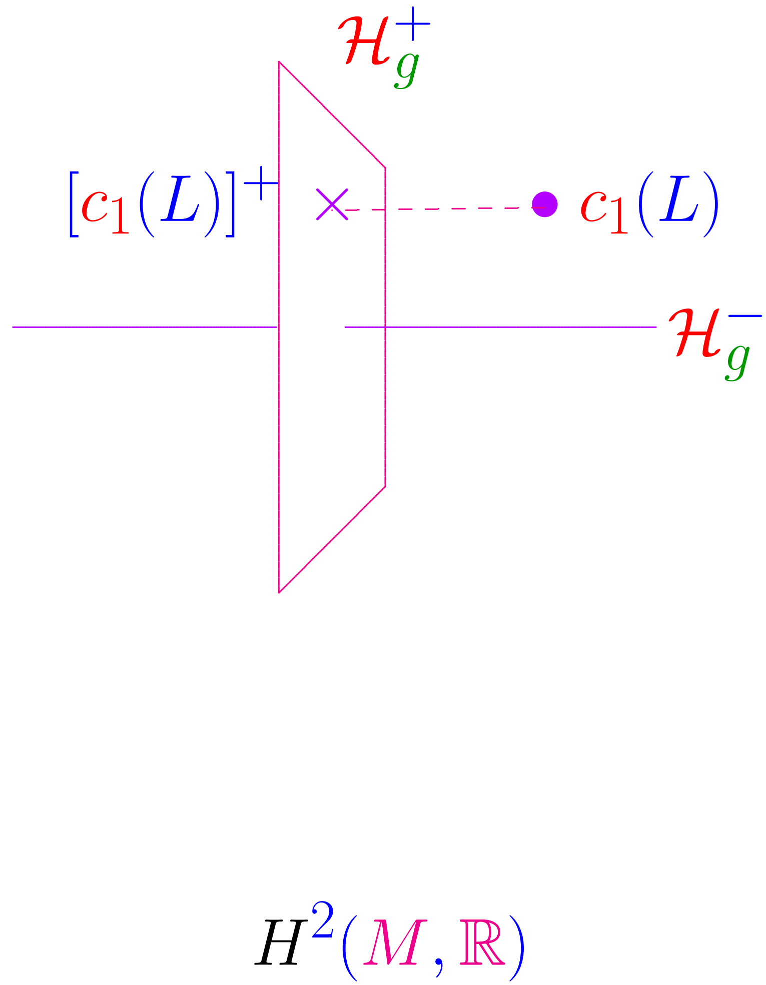

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This played an important role in the original proof, but is used only mildly in what follows.

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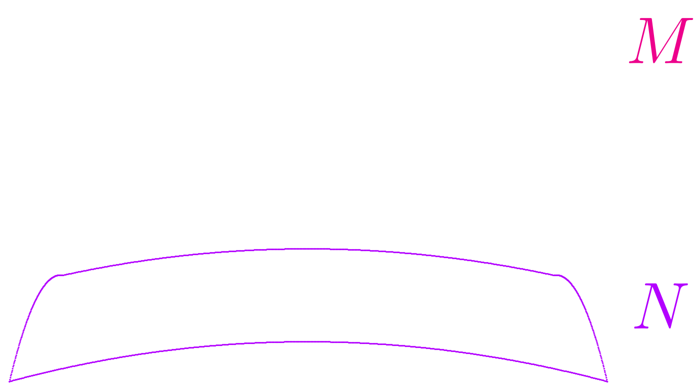

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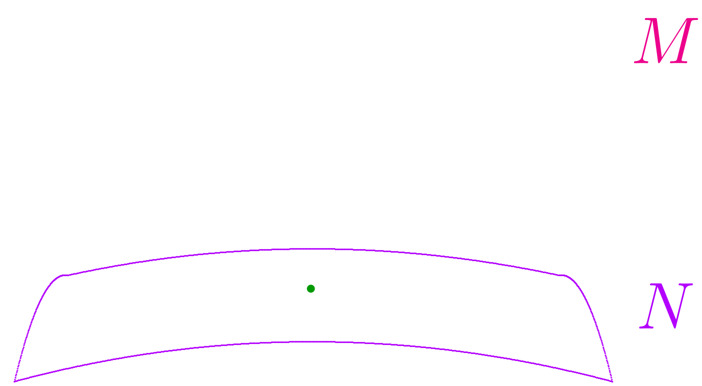

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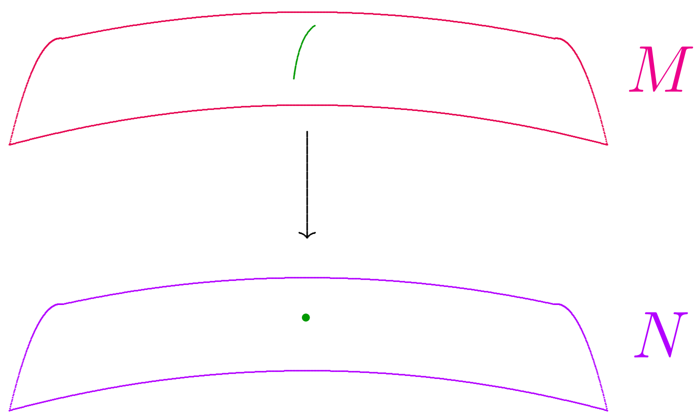

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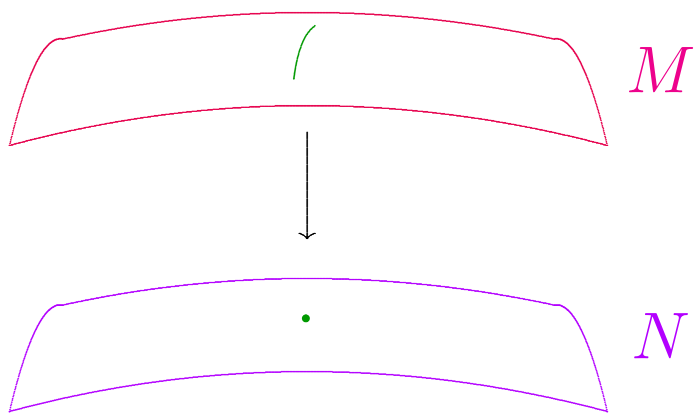

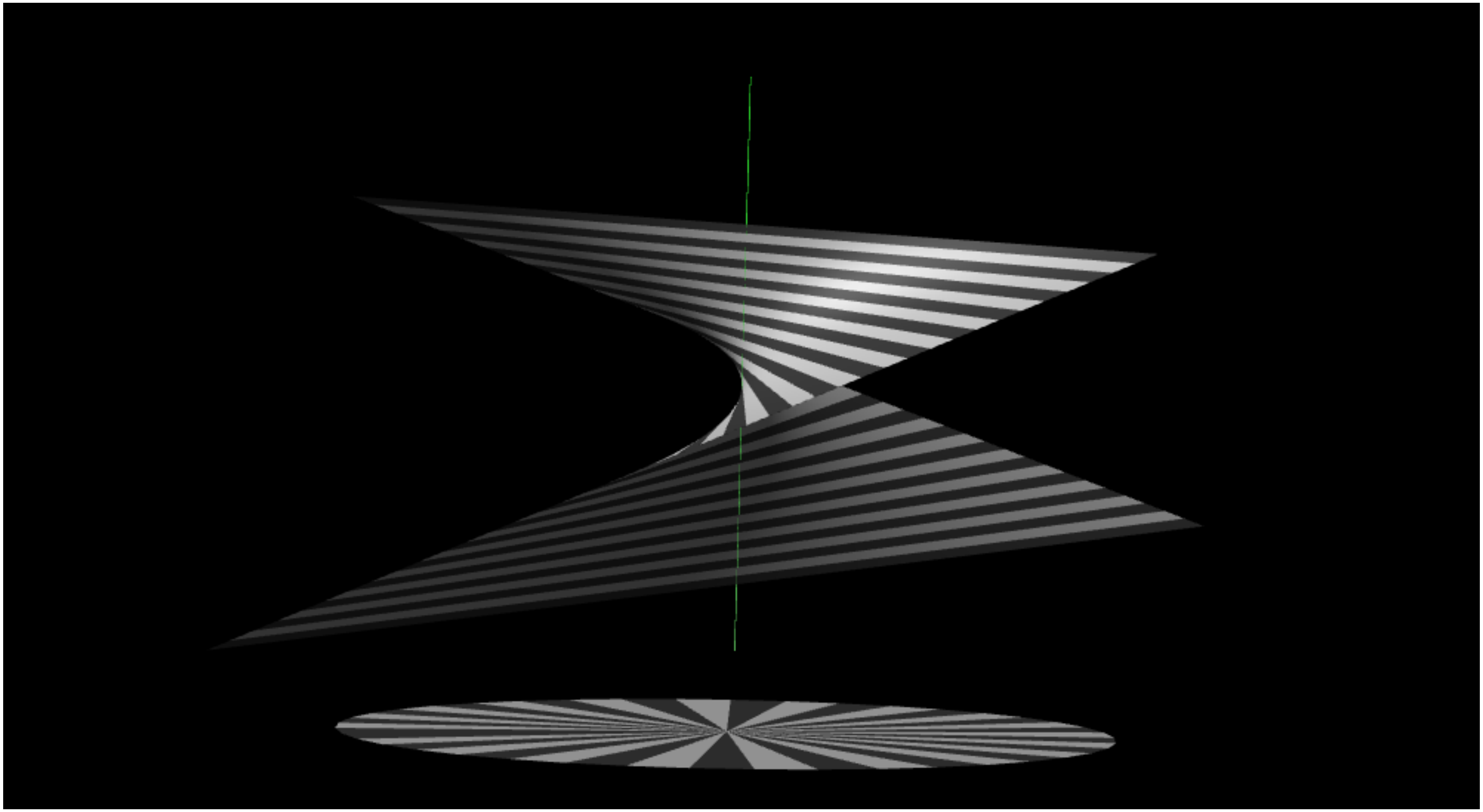

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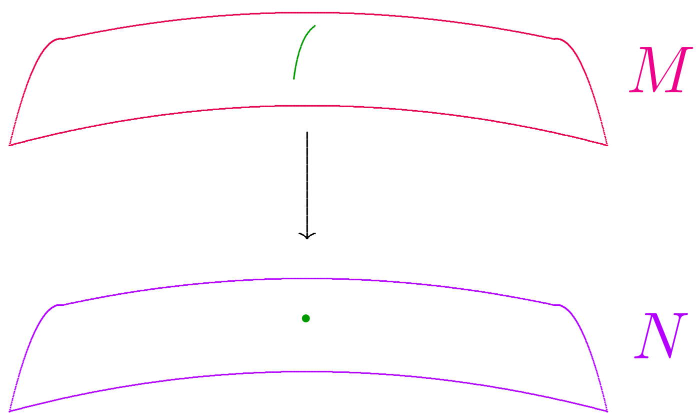

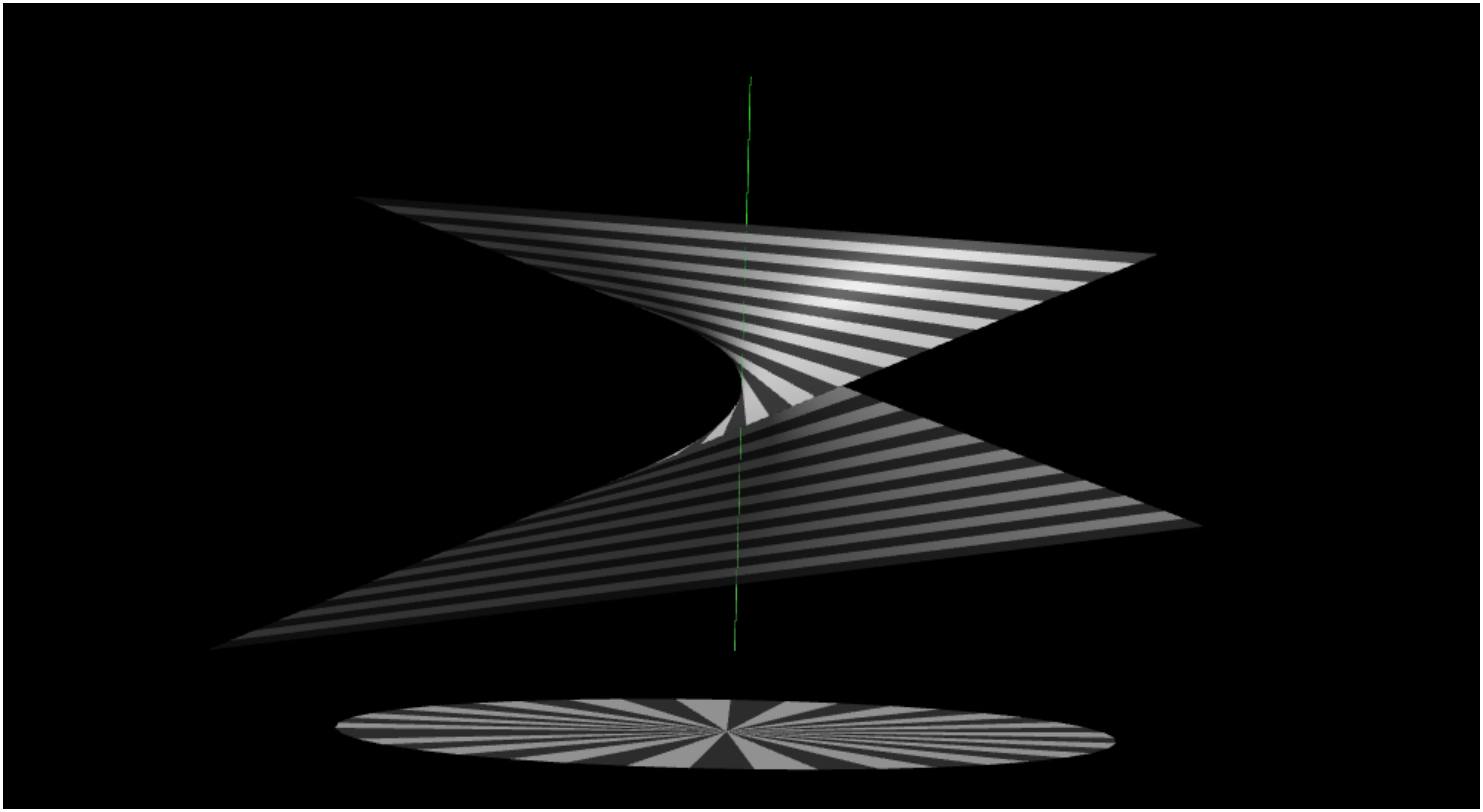

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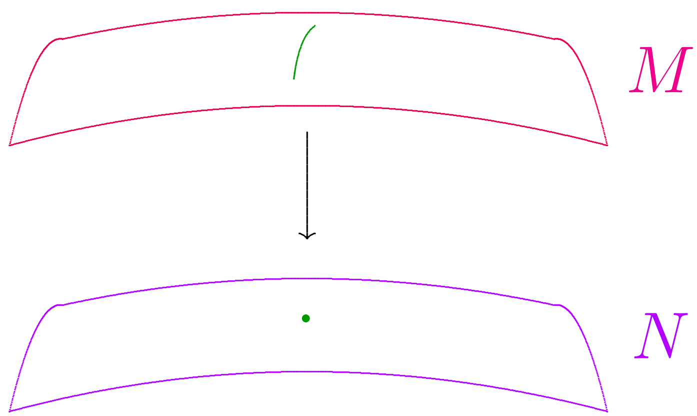

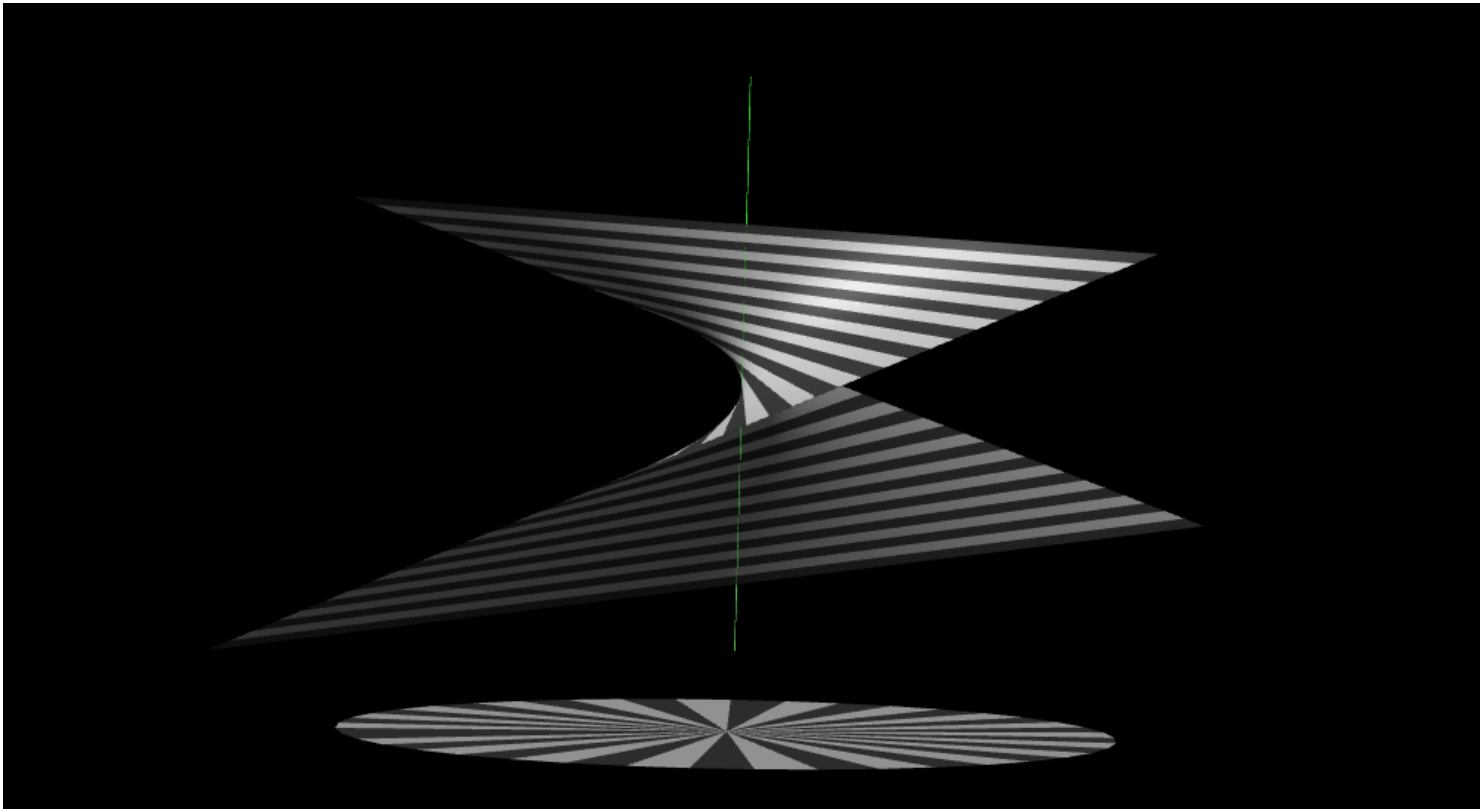

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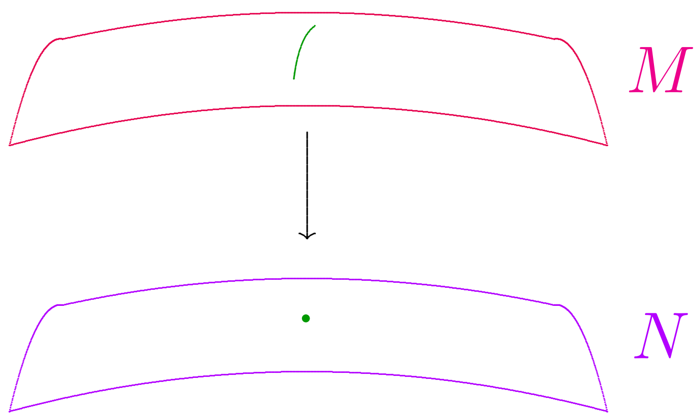

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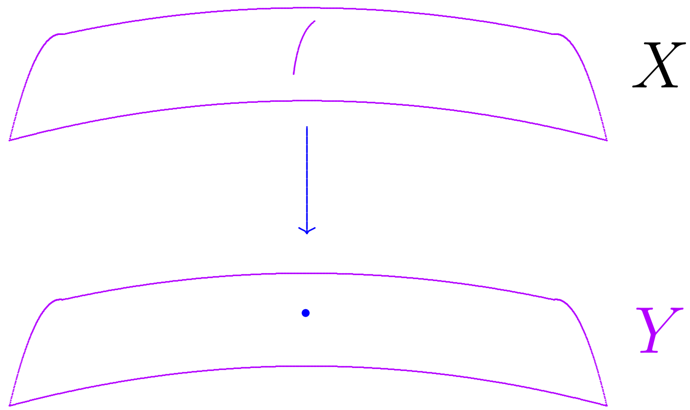

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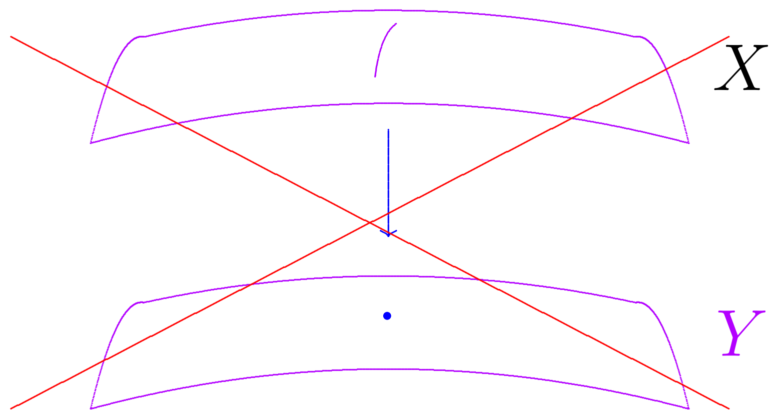

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In this setting, minimal model X is unique.

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Key ingredient: First Curvature estimate.

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Key ingredient: First Curvature estimate.

Next: how to use Second Curvature estimate.

First observe:

$$\frac{s^2}{24}+2|W_+|^2 = \frac{1}{27}\left[\left(s-\sqrt{6}|W_+|\right)^2+\frac{1}{8}\left(s+8\sqrt{6}|W_+|\right)^2\right]$$

First observe:

$$\frac{s^2}{24} + 2|W_+|^2 \quad \geq \quad \frac{1}{27}\left(s - \sqrt{6}|W_+|\right)^2$$

Hence:

$$\int_M \left(\frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g \geq \frac{1}{27} \int_M \left(s - \sqrt{6}|W_+| \right)^2 d\mu_g$$

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$\therefore$ Second curvature estimate implies

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Here one first shows generalized scalar curvature

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would have to be constant if equality held.

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Einstein $\Longrightarrow$ $\quad (2\chi + 3\tau)(M) > \frac{2}{3} {c_1}^2(X)$

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$$\int_M \left(\frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g \geq \frac{1}{27} \int_M \left(s - \sqrt{6}|W_+| \right)^2 d\mu_g$$

$\therefore$ Second curvature estimate implies

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$$\text{Einstein} \Longrightarrow \quad {c_1}^2(M) > \frac{2}{3} {c_1}^2(X)$$

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Einstein $\Longrightarrow \quad {c_1}^2(X) - k > \frac{2}{3}{c_1}^2(X)$

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Einstein $\Longrightarrow \quad \frac{1}{3} {c_1}^2(X) > k$

Theorem. *Let X be a minimal surface of general type,*

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$$M = X \# k\overline{\mathbb{CP}_2}.$$

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So being "very" non-minimal is an obstruction.

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Theorem (Aubin/Yau). *Compact complex manifold* (M^{2m}, J) *admits compatible Kähler-Einstein metric with* $s < 0 \Longleftrightarrow c_1 < 0$.

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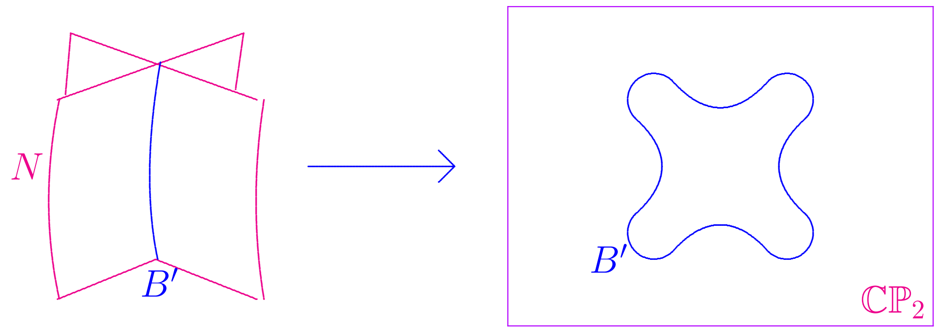

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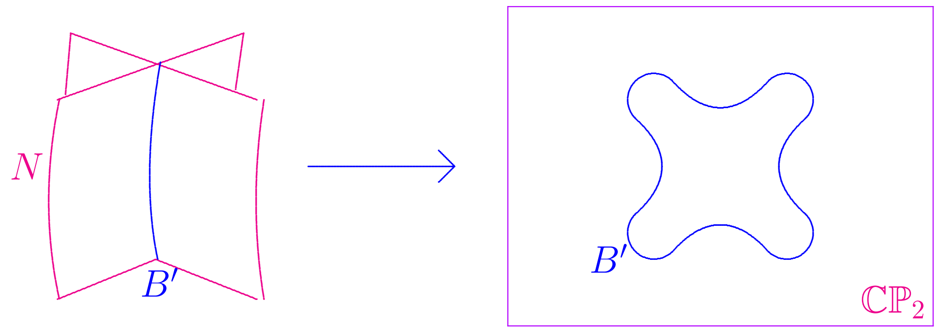
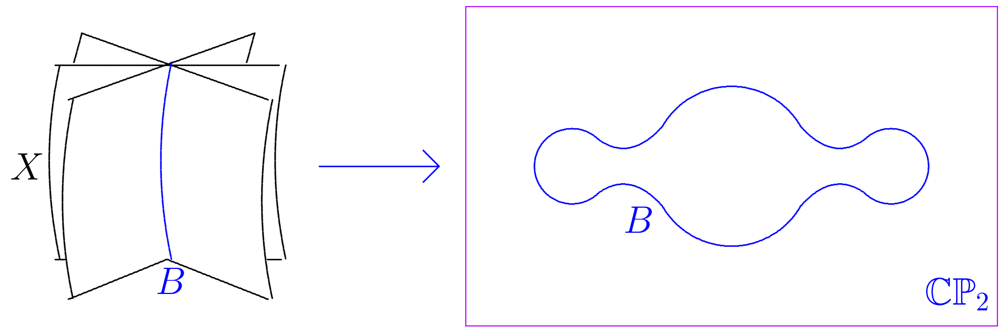


Aubin/Yau $\Longrightarrow$ N carries Einstein metric.

Now let X be a triple cyclic cover $\mathbb{CP}_2$, ramified at a smooth sextic



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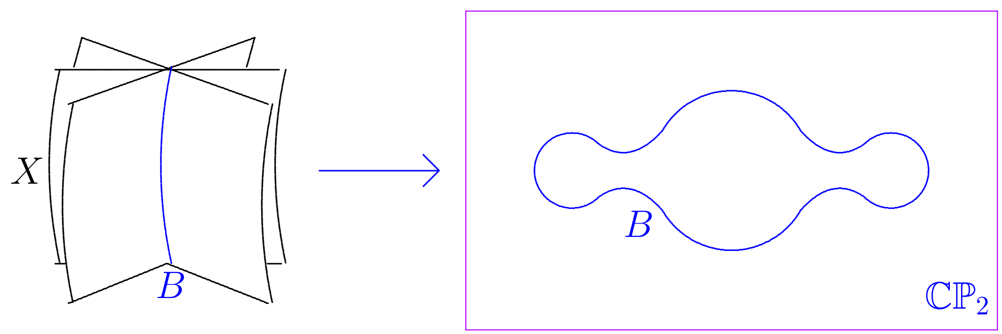

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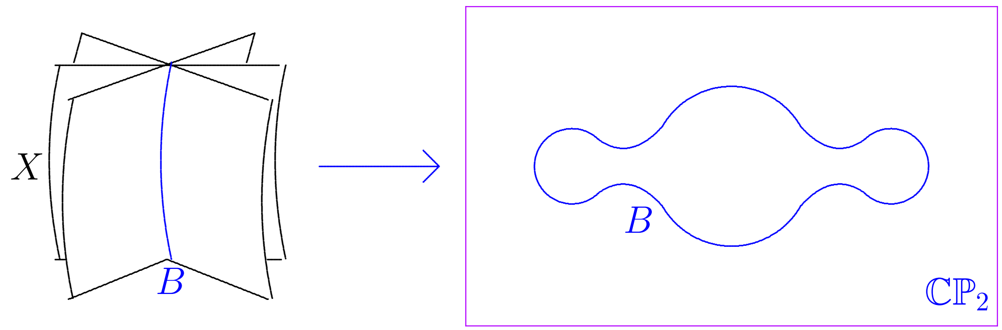

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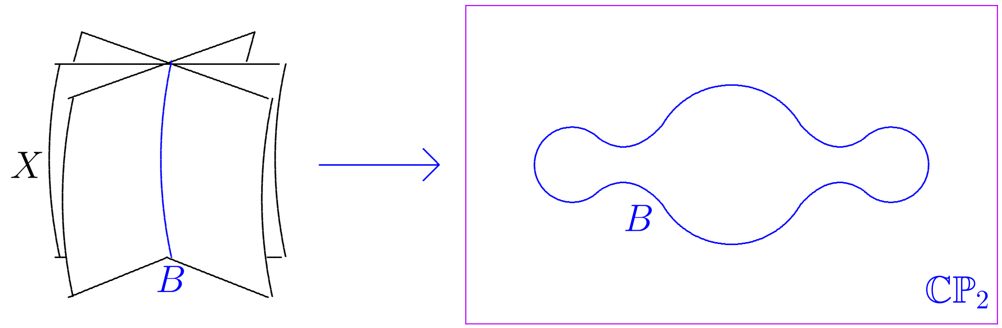
In example:

$${c_1}^2(X) = 3$$
$$k = 1 = {c_1}^2(X)/3$$

X is triple cover $\mathbb{CP}_2$ ramified at sextic



$$M = X \# \overline{\mathbb{CP}}_2.$$

Theorem $\Longrightarrow$ *no* Einstein metric on M.

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Moral: Existence depends on diffeotype!

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