Discrete Conformality and Graph Embedding

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ACM Symposium on Computational Geometry

Connections between Analysis and Computational Geometry

University of North Carolina

June 2012

Outline

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- *Combinatorics begets Geometry*

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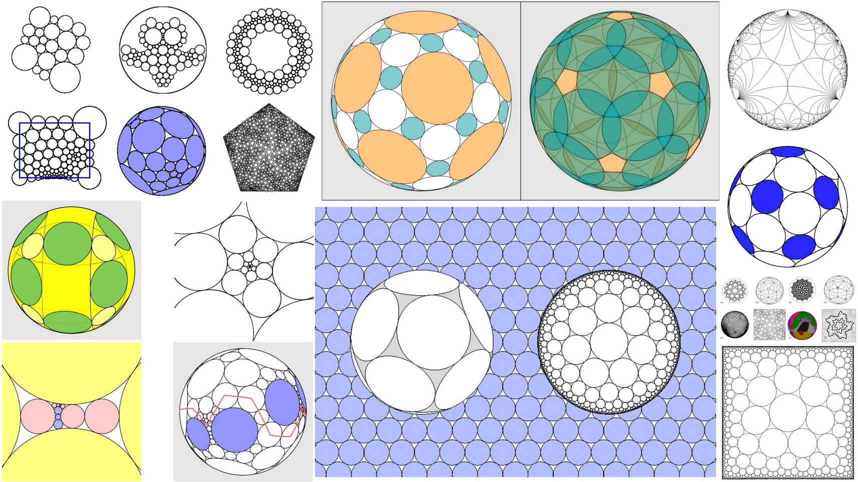
- *Theory pays Dividends*
- Technical/Summary



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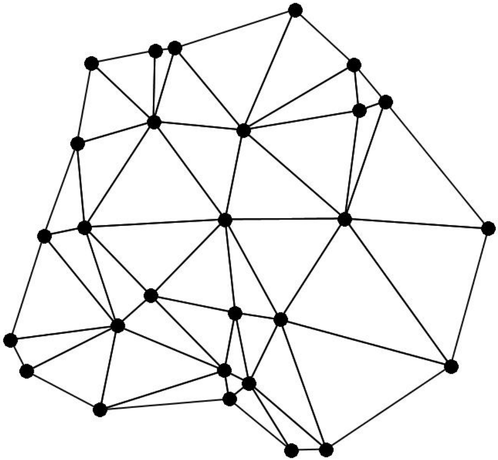

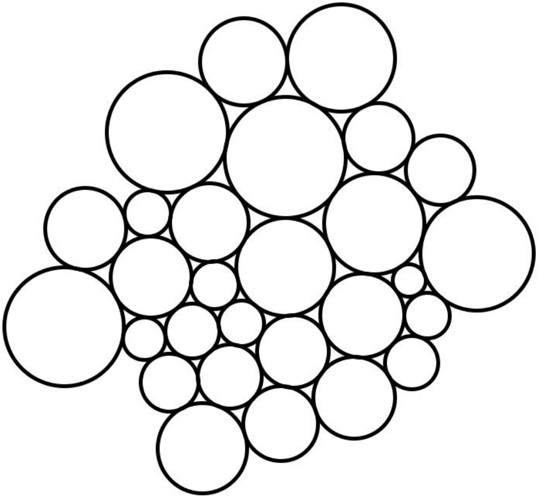

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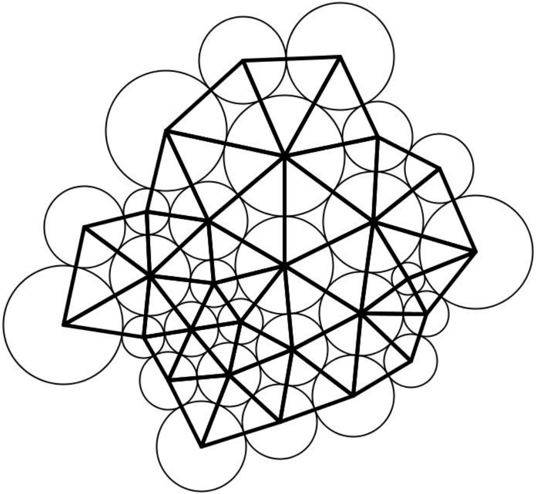

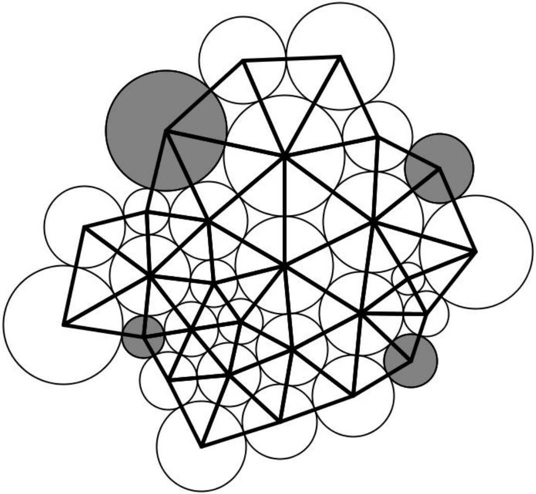

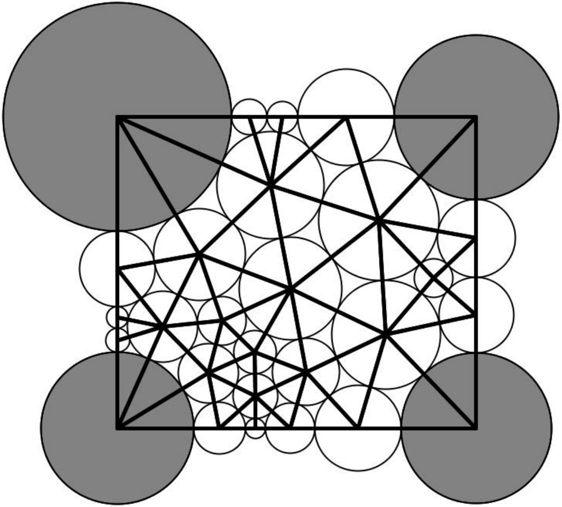

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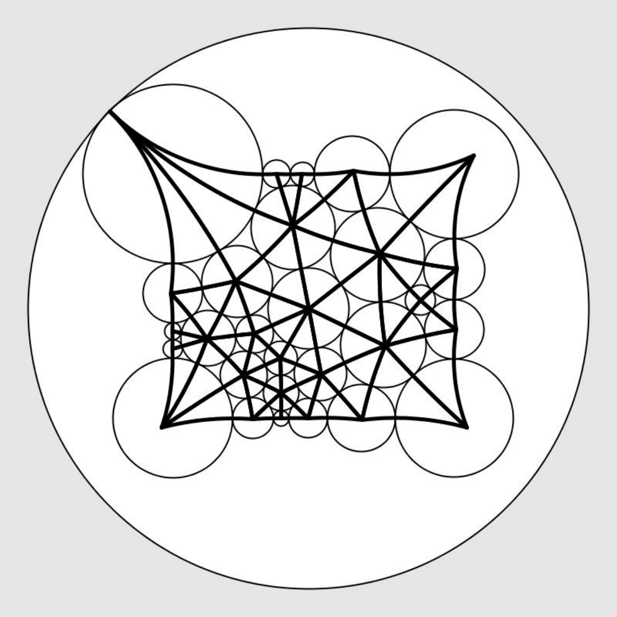

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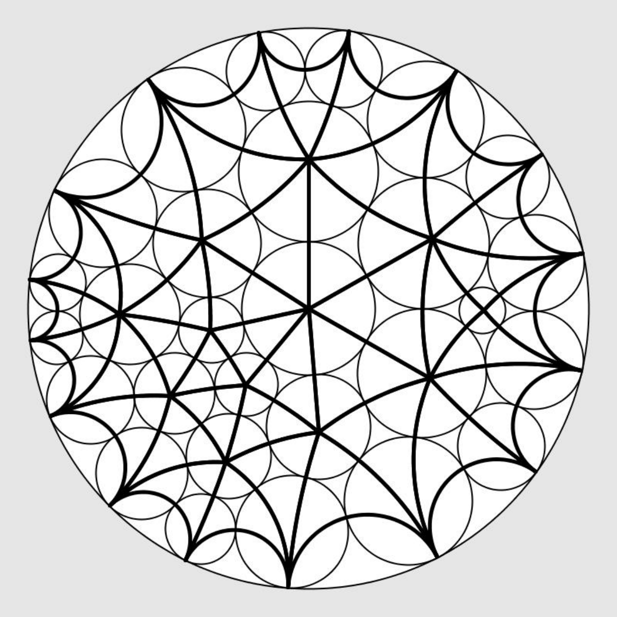

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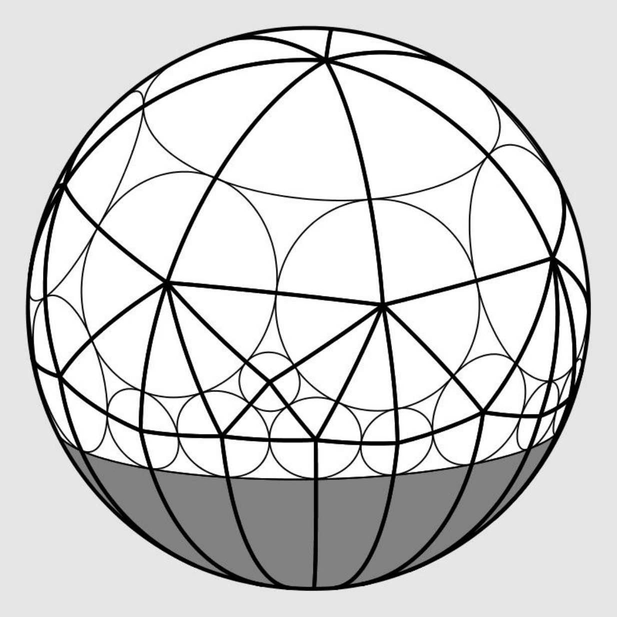

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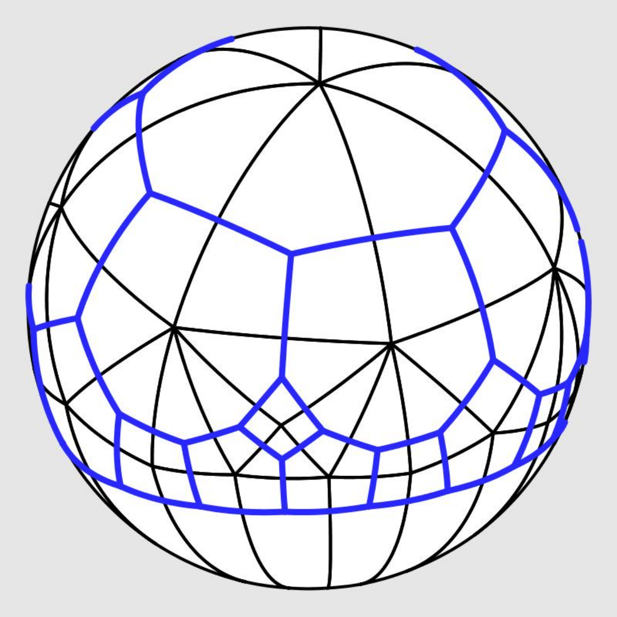

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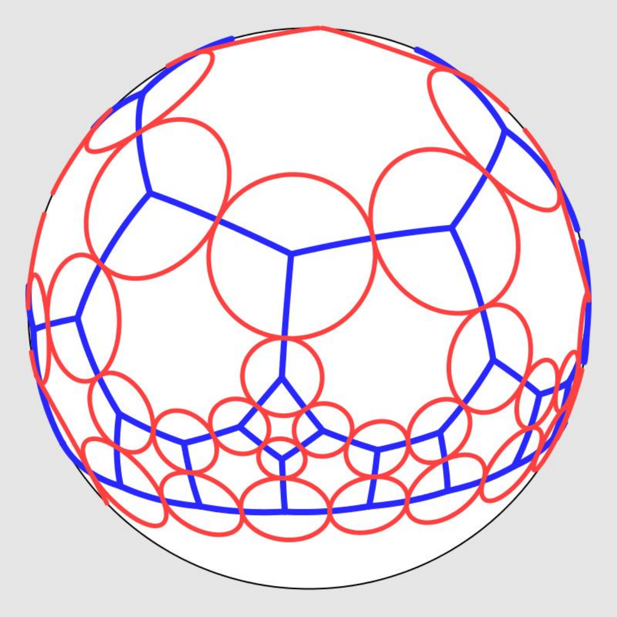

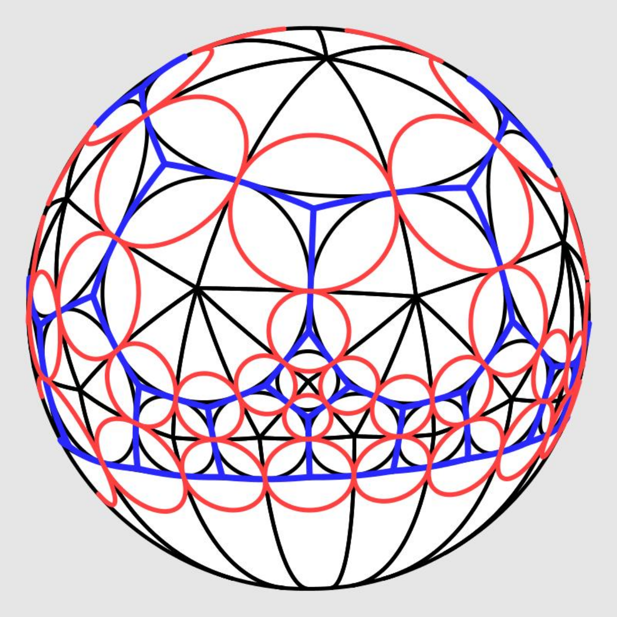

Circle Packing

Definition: *A* **circle packing** *is a configuration of circles with a specified pattern of tangencies.*

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Key Theorem (Koebe-Andreev-Thurston): *For any triangulation K of a sphere, there exists an associated univalent circle packing P_K of the Riemann sphere, unique up to Möbius transformations.*

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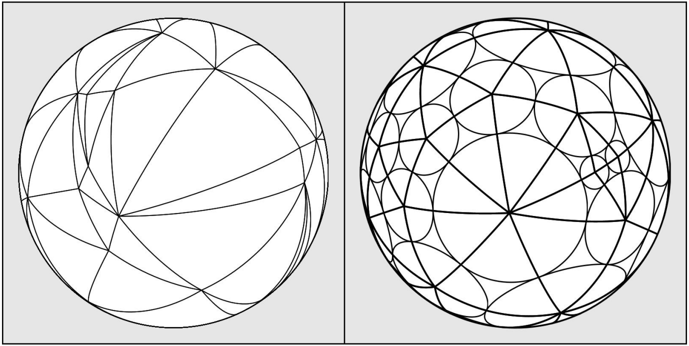

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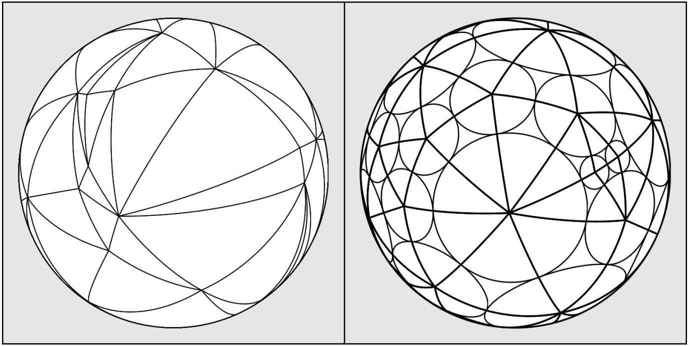

Cautionary Note: Circle packing is NOT 2D sphere packing

Theorem: *Given any triangulation K of an oriented topological surface S, there is an essentially unique conformal structure on S supporting a circle packing P having the combinatorics of K and 'filling' S.*

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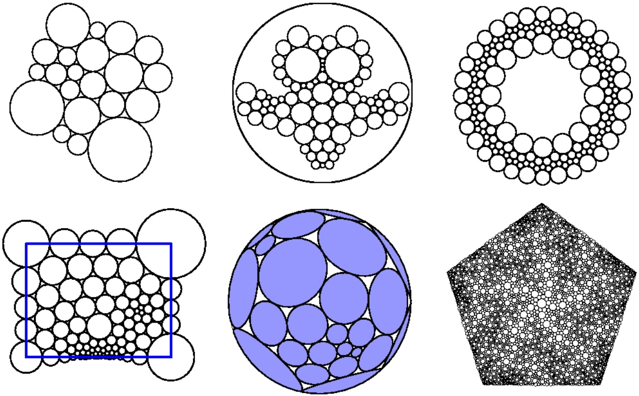
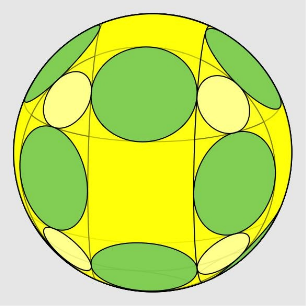
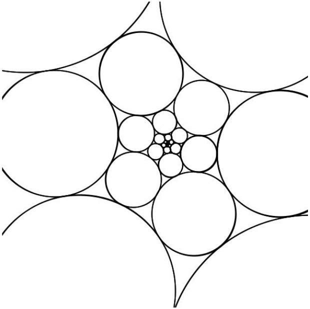
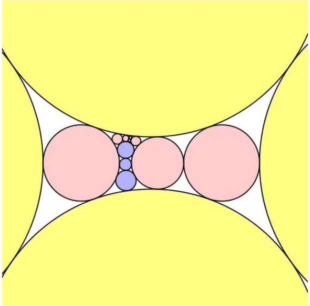
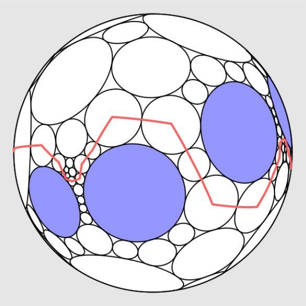


Wonderful Geometry

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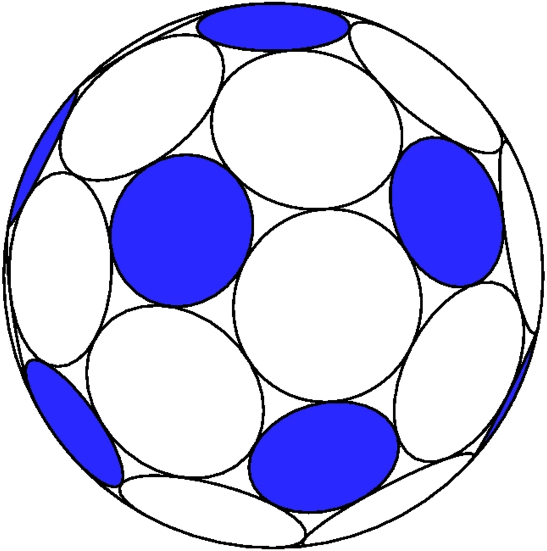

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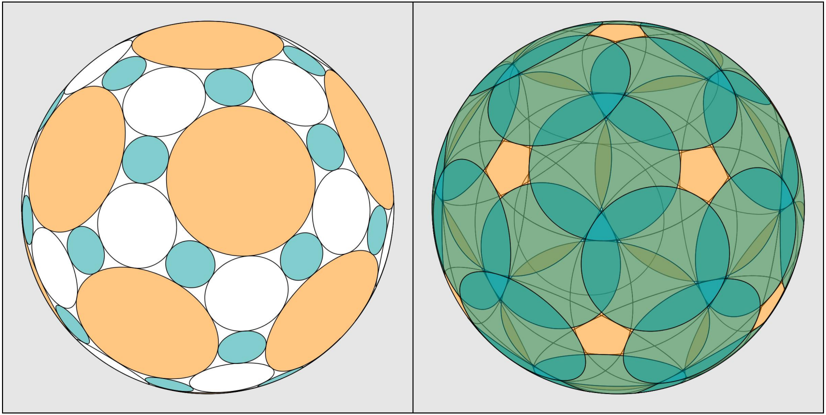

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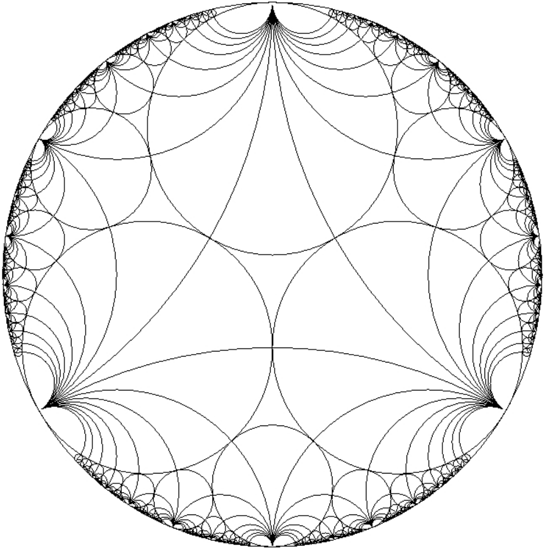

Embedding

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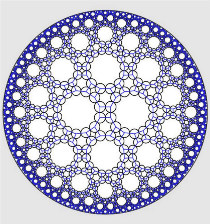
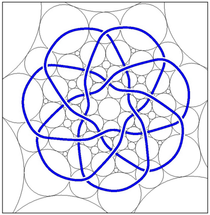
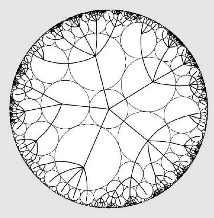
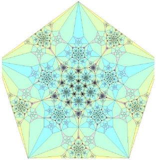
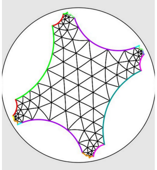
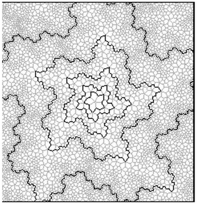
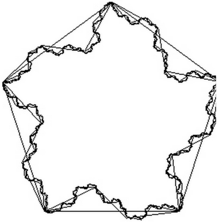
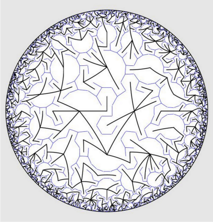
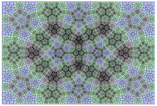
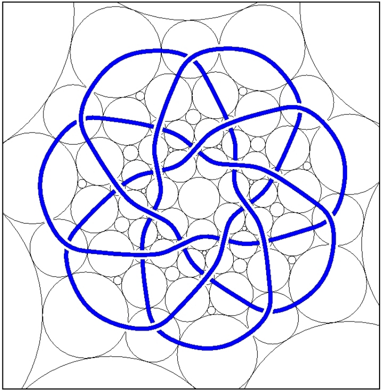
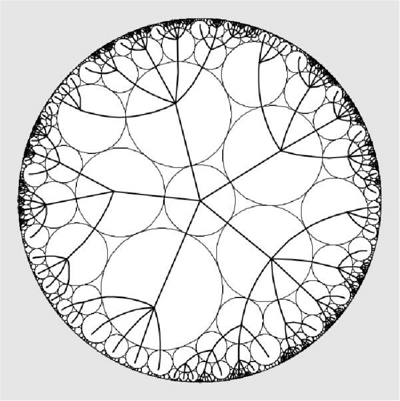
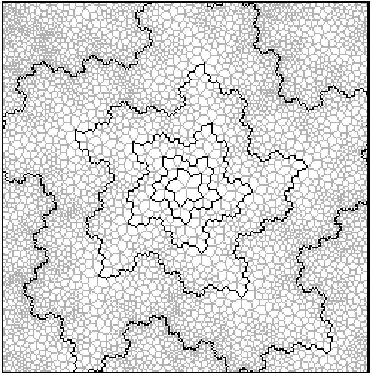










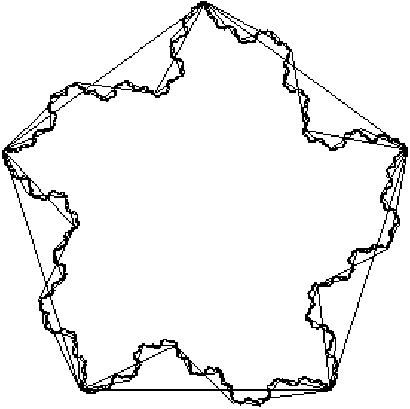
Clean

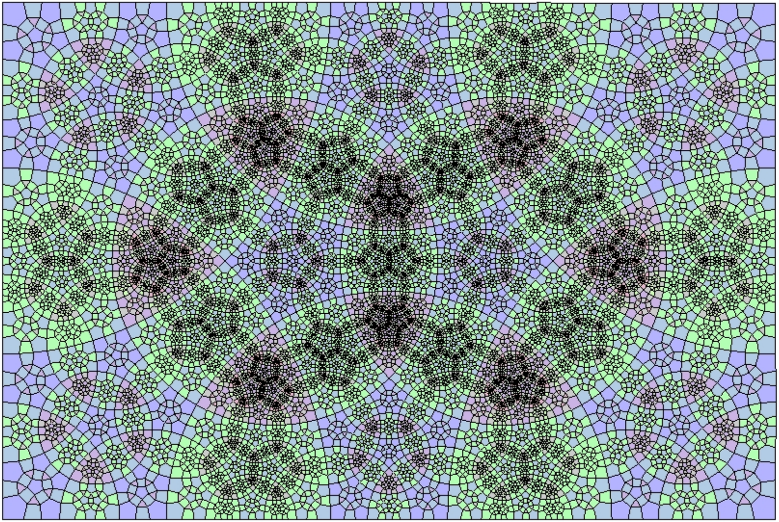


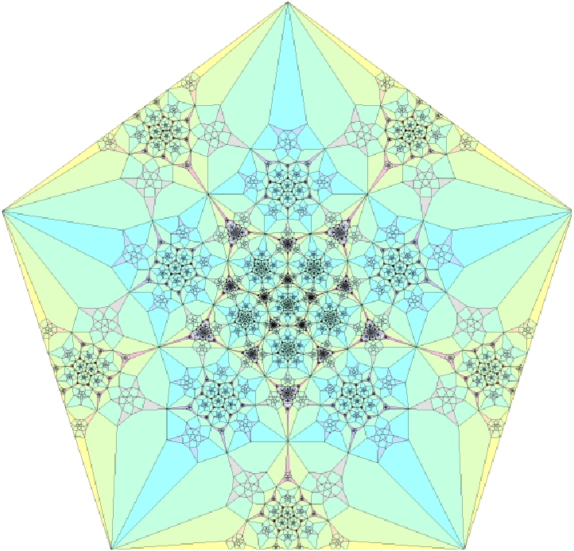


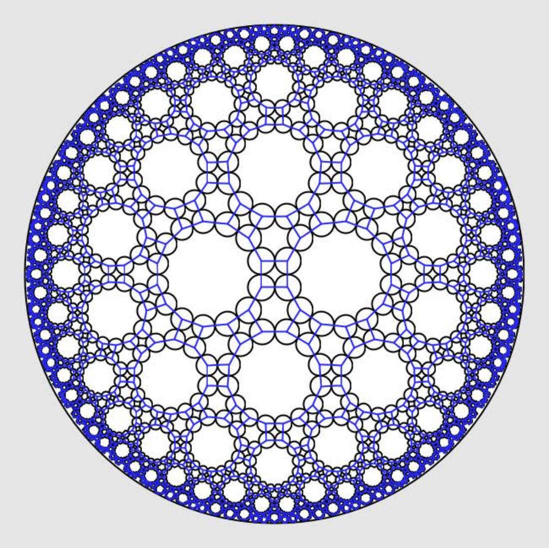
Complicated

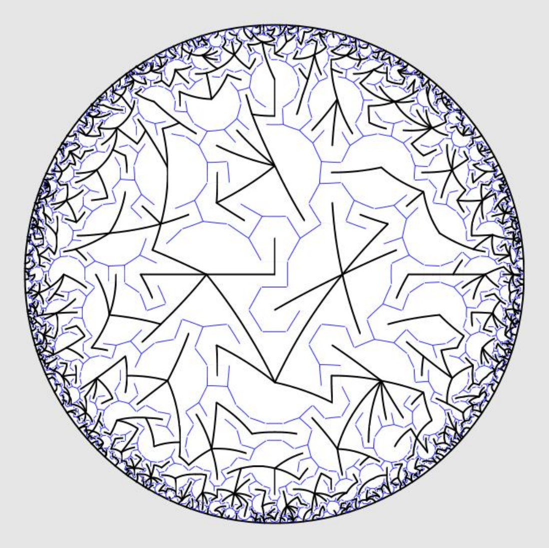


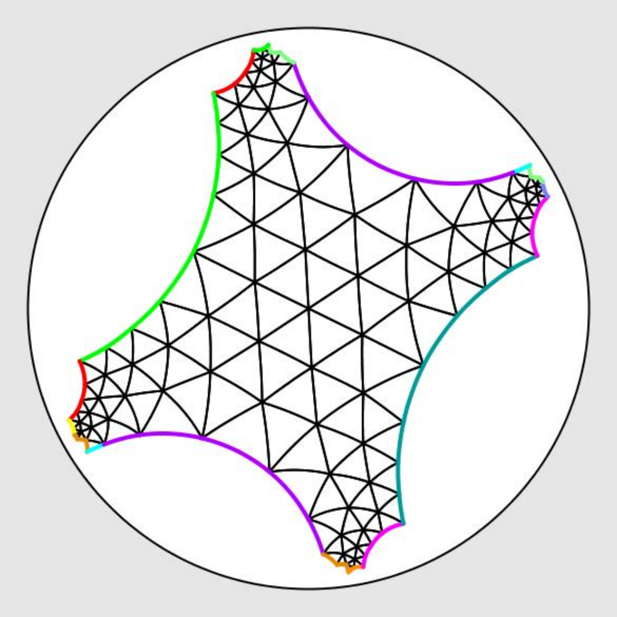












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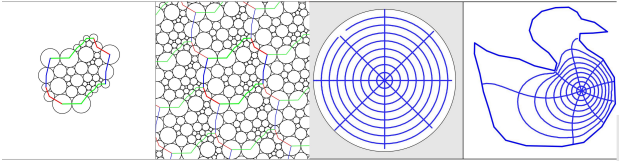
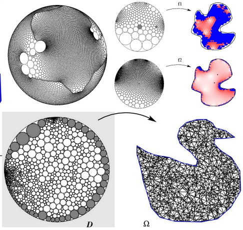
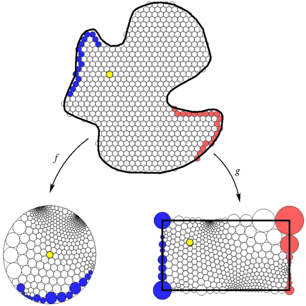
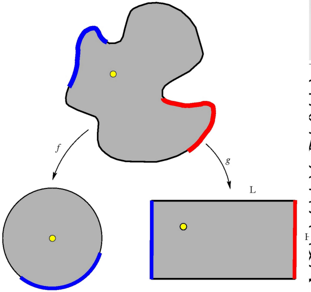


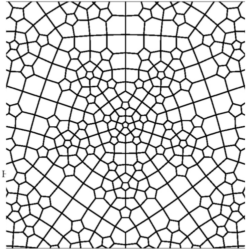




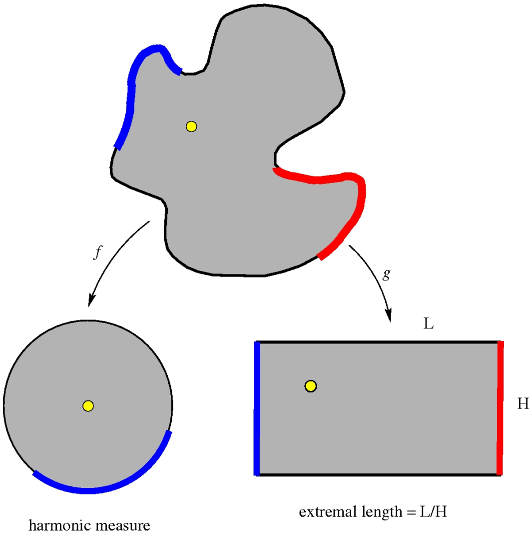


harmonic measure

extremal length = L/H



Discrete Conformal Mapping



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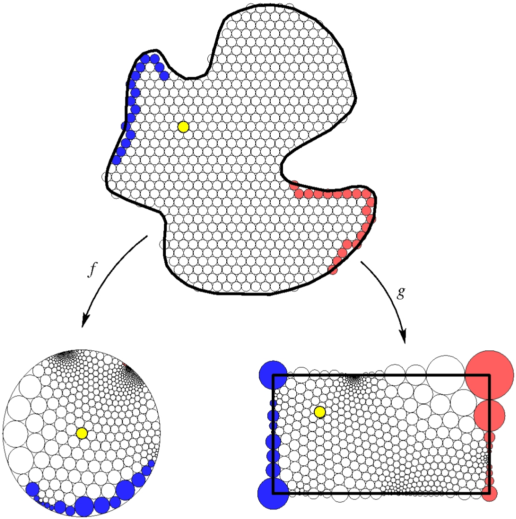
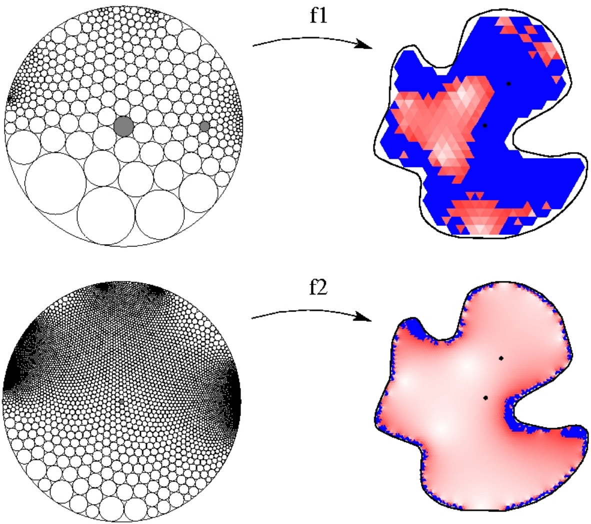
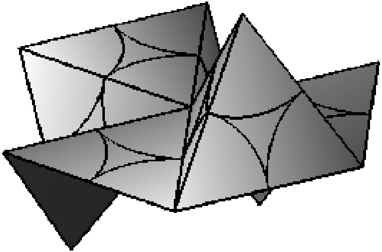
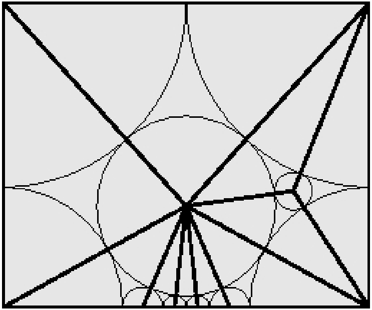


Convergence



Conformal Flow





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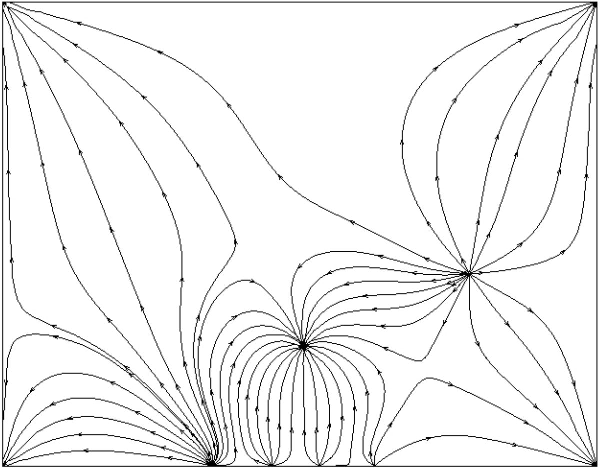

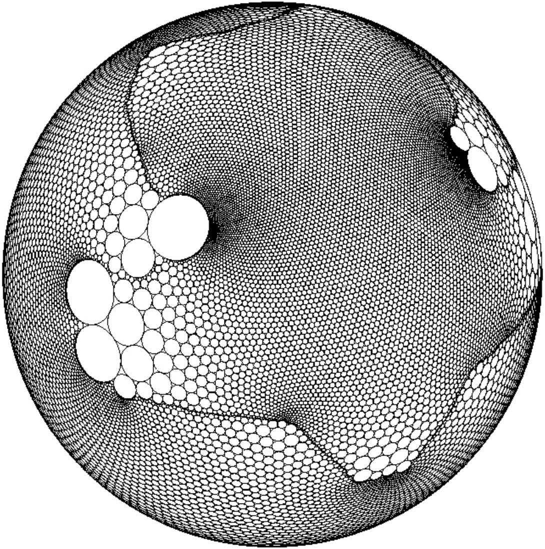

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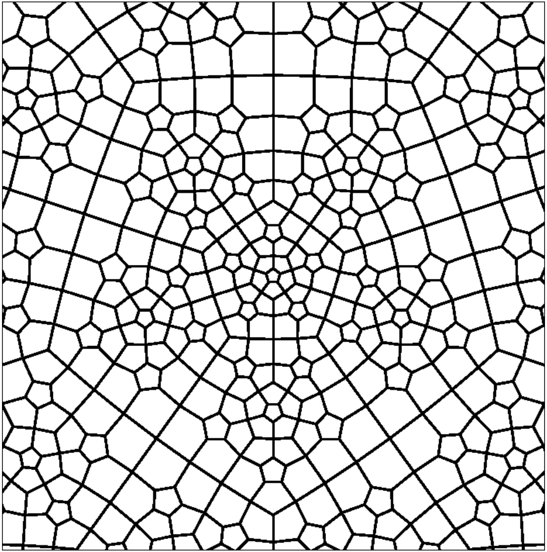
- Conformal Moduli

Synergies

Pentagonal Tiling of the Plane



Synergies — e.g. Conformal Chair Tiling

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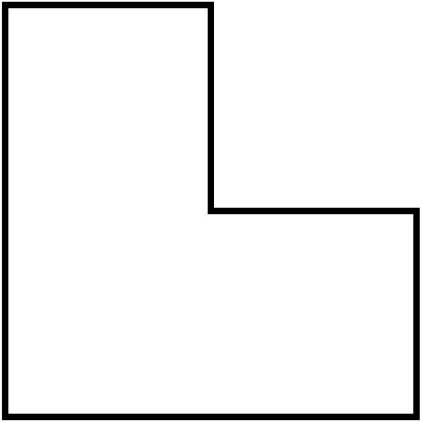
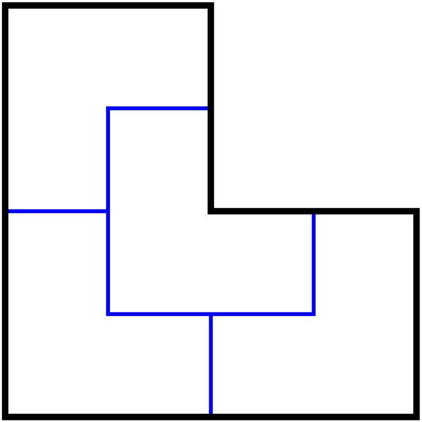
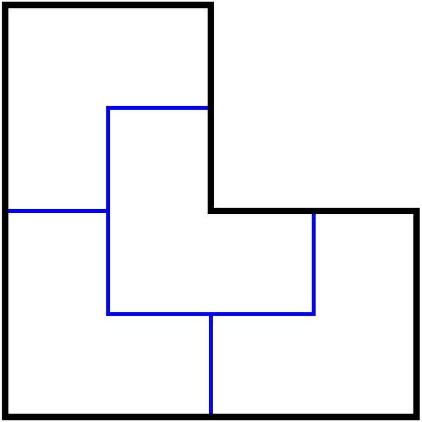


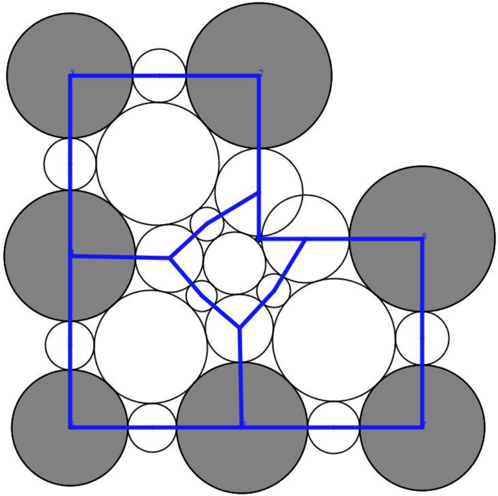
Chair substitution





Chair substitution tiling

Conformal Chairs



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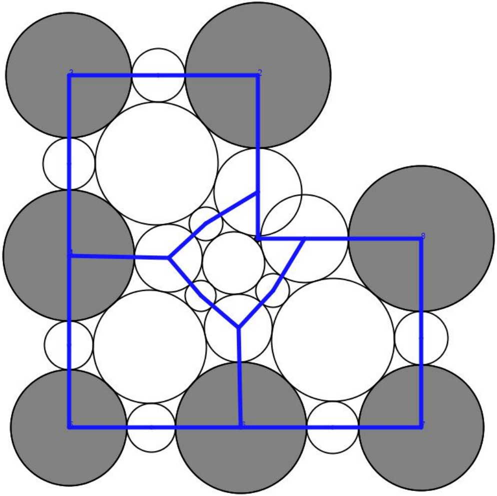
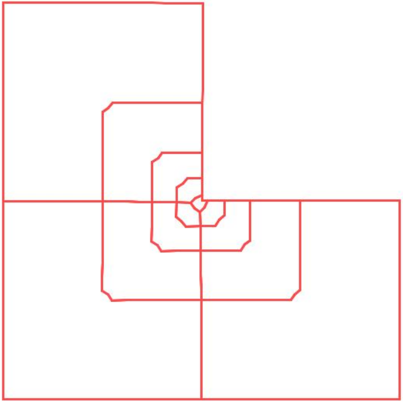


Make each tile into a combinatorial octagon

Conformal Chairs



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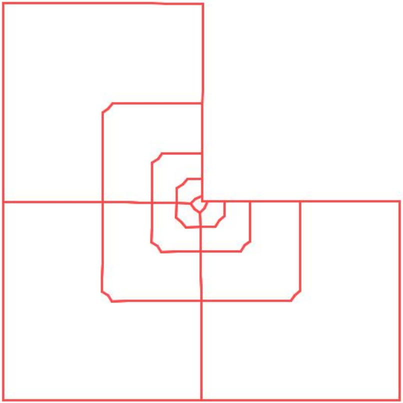

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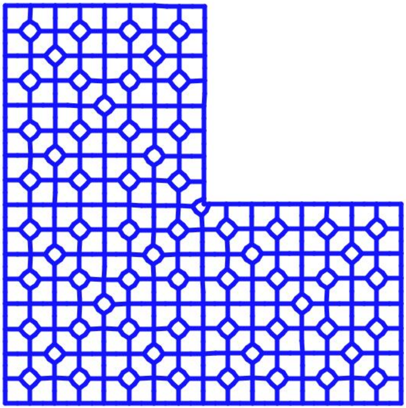

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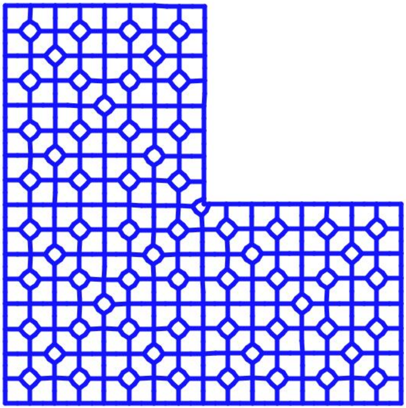

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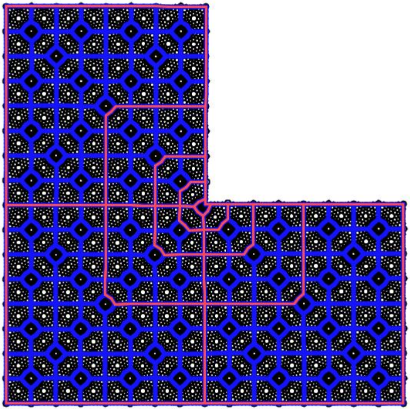

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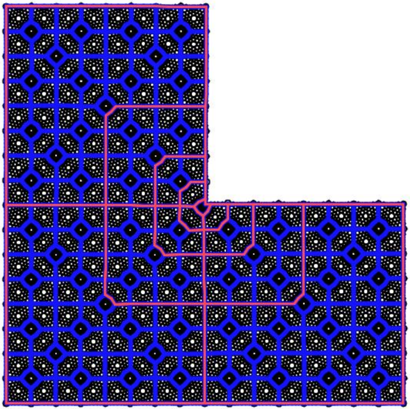
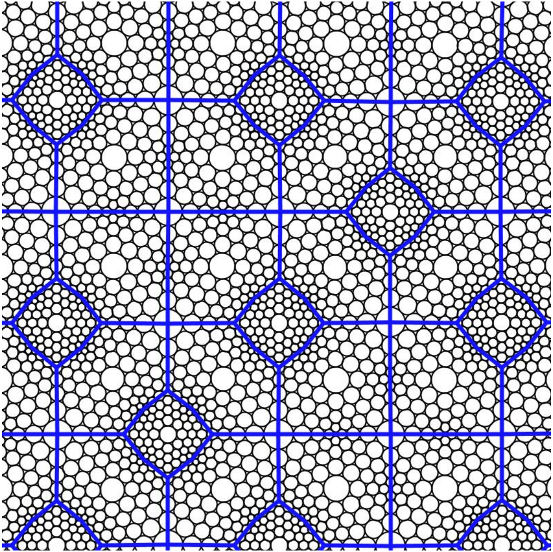
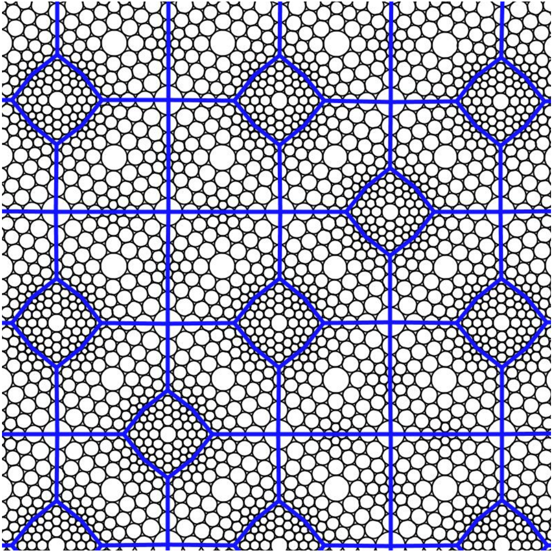


Conformal Chairs





Detail: converging to a conformally regular tiling of octagons

Summary: Synergies

- Grid generation

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- Grid generation
- Conformal tiling

- Grid generation
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- Emergent conformality

Summary: Synergies

- Grid generation
- Conformal tiling
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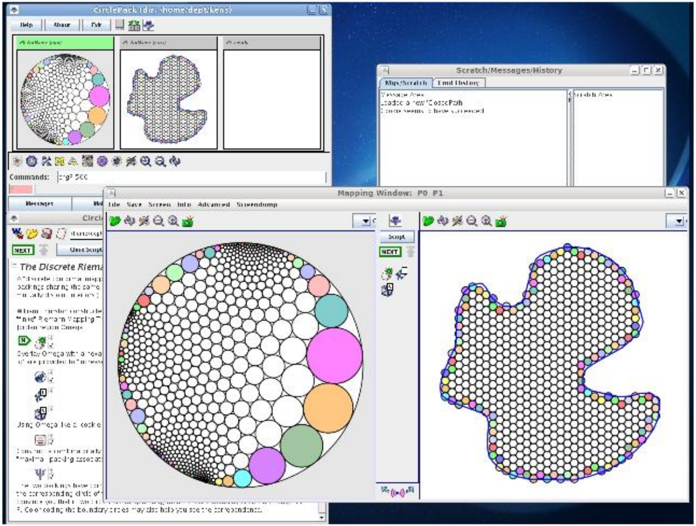
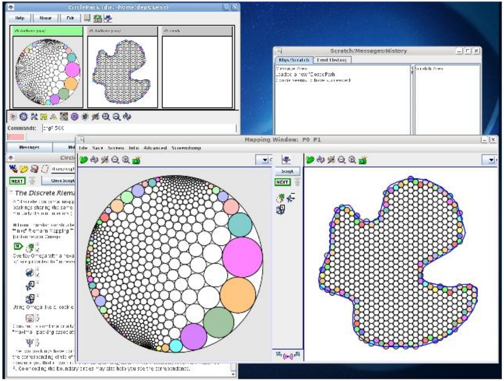
- Spontaneity

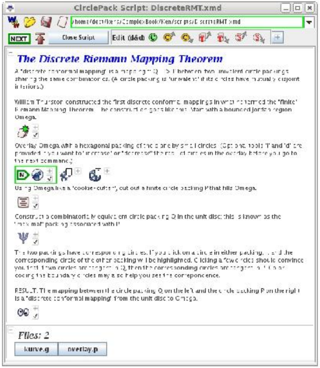
Summary: Synergies

- Grid generation
- Conformal tiling
- Emergent conformality
- Spontaneity — surprises in nearly every experiment

Resources and Acknowledgements

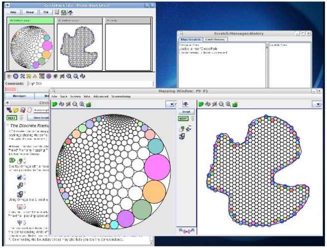


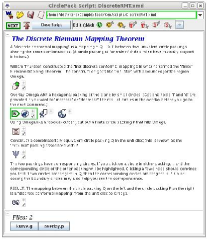


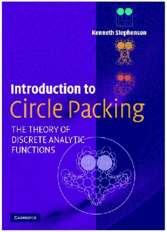


CirclePack, Java, cross-platform, open source

Resources







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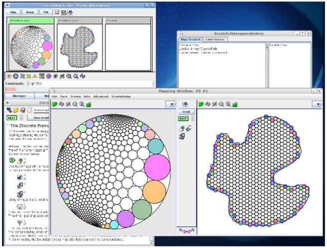

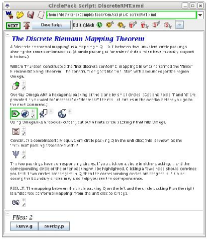

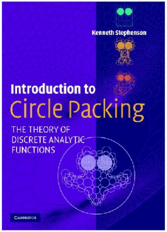

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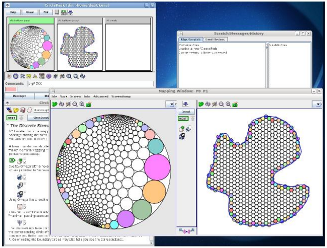

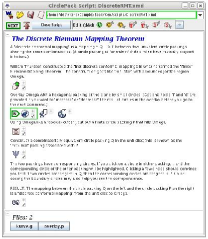

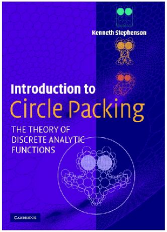

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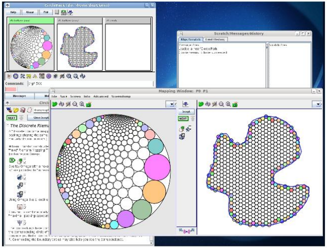

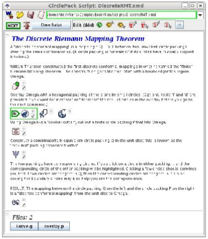

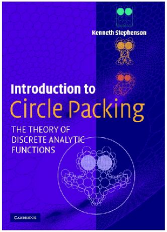

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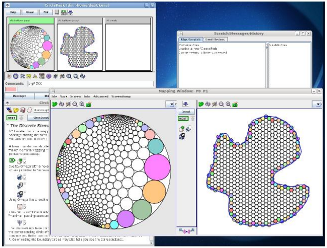

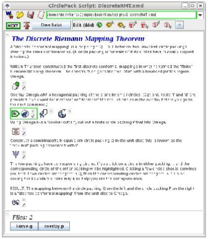

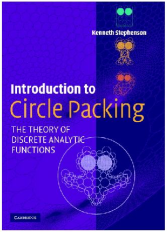

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