MAPPINGS AND MESHES

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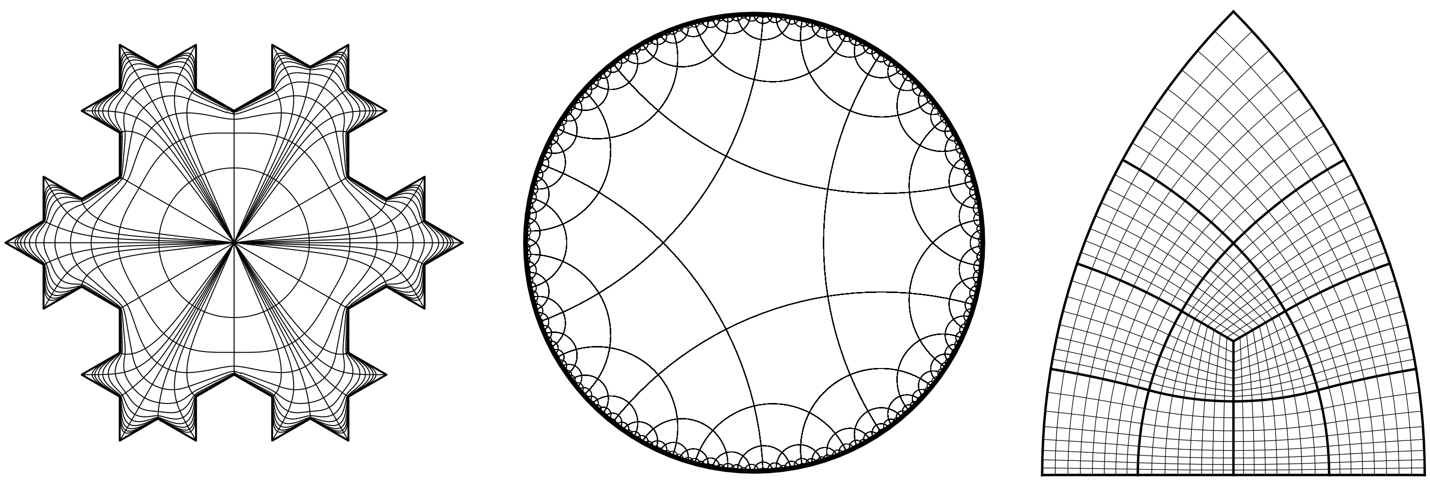
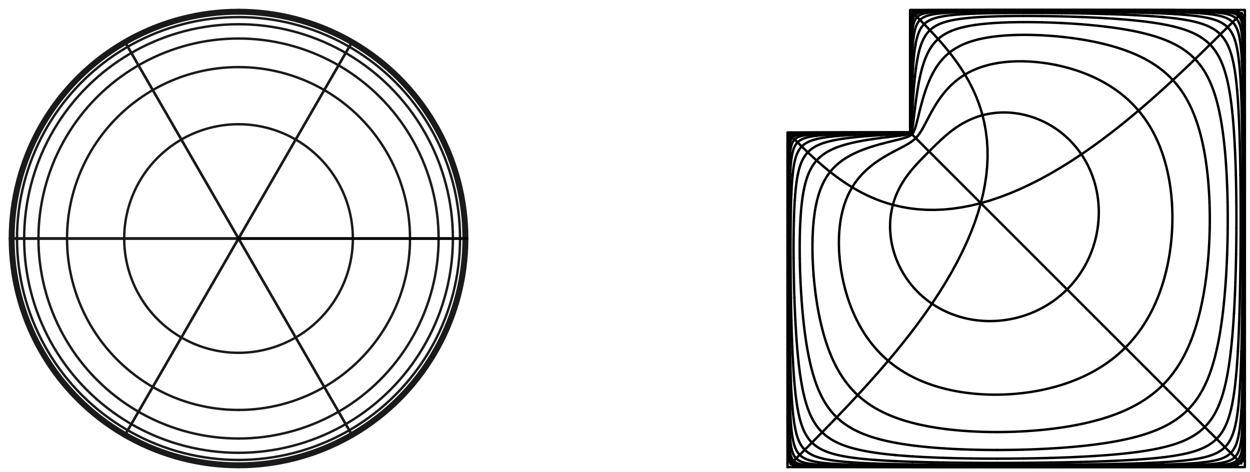
SoCG, June 18, 2012

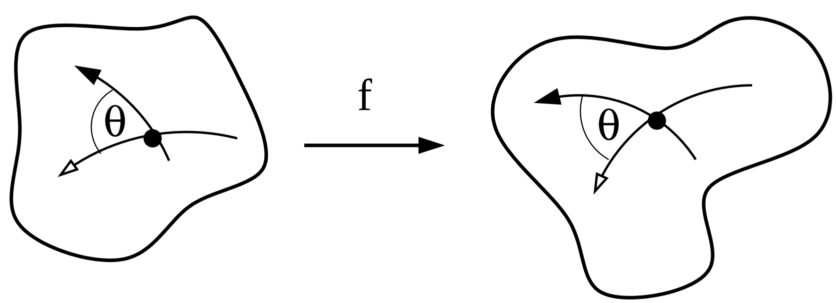


www.math.sunysb.edu/~bishop/SoCG12

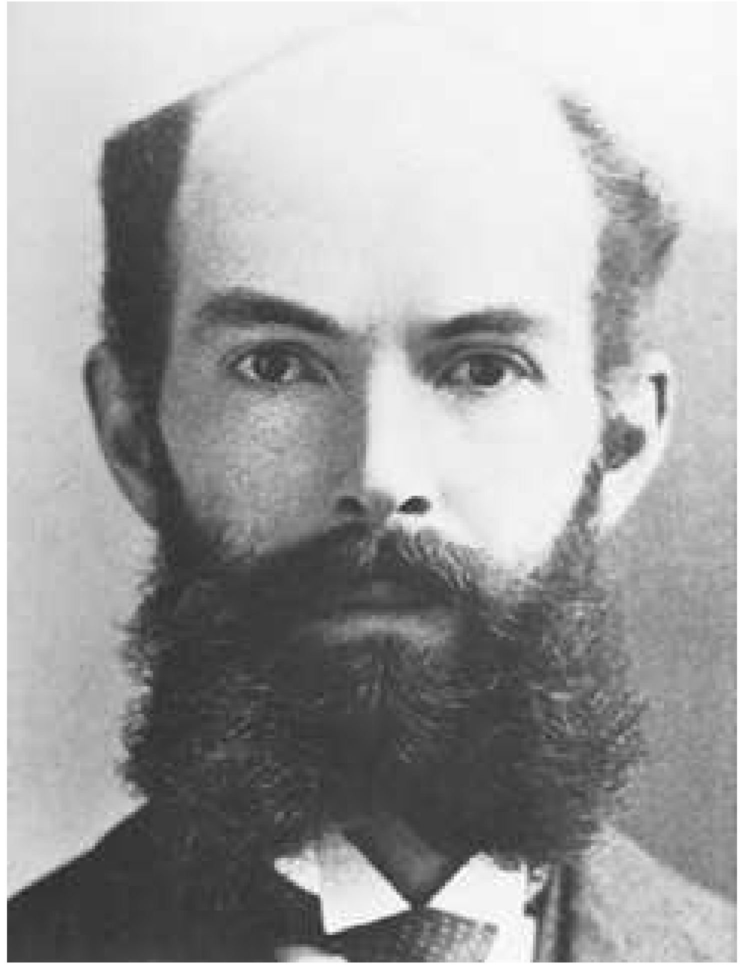
Riemann Mapping Theorem: If $\Omega \subsetneq \mathbb{R}^2$ is simply connected, then there is a conformal map $f : \mathbb{D} \to \Omega$.



Conformal = angle preserving



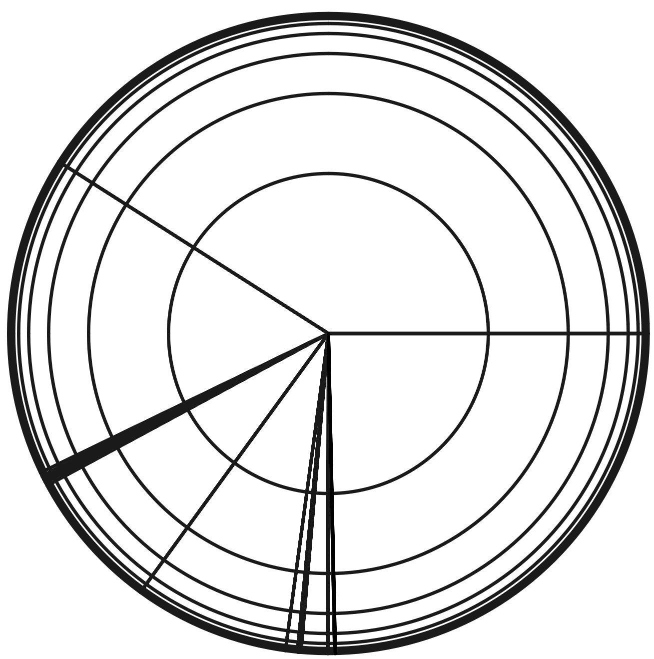
Georg Friedrich Bernhard Riemann
Stated RMT in 1851 thesis
a gloriously fertile originality - Gauss

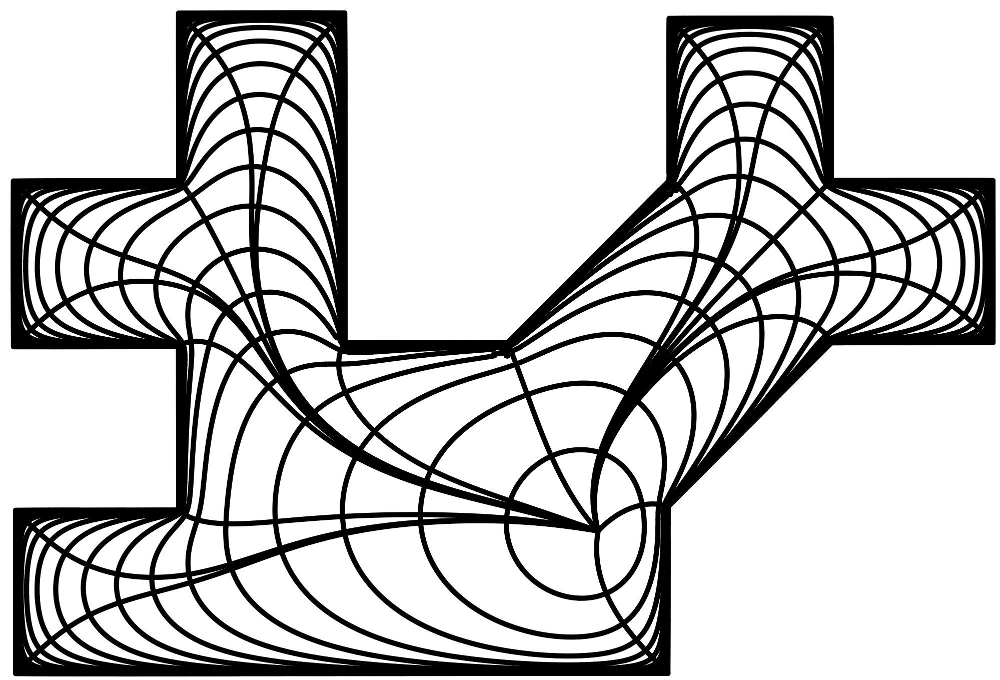


William Fogg Osgood
First proof of RMT, 1900

Fact: Conformal map to a polygon is determined by the points on circle that map to the vertices.

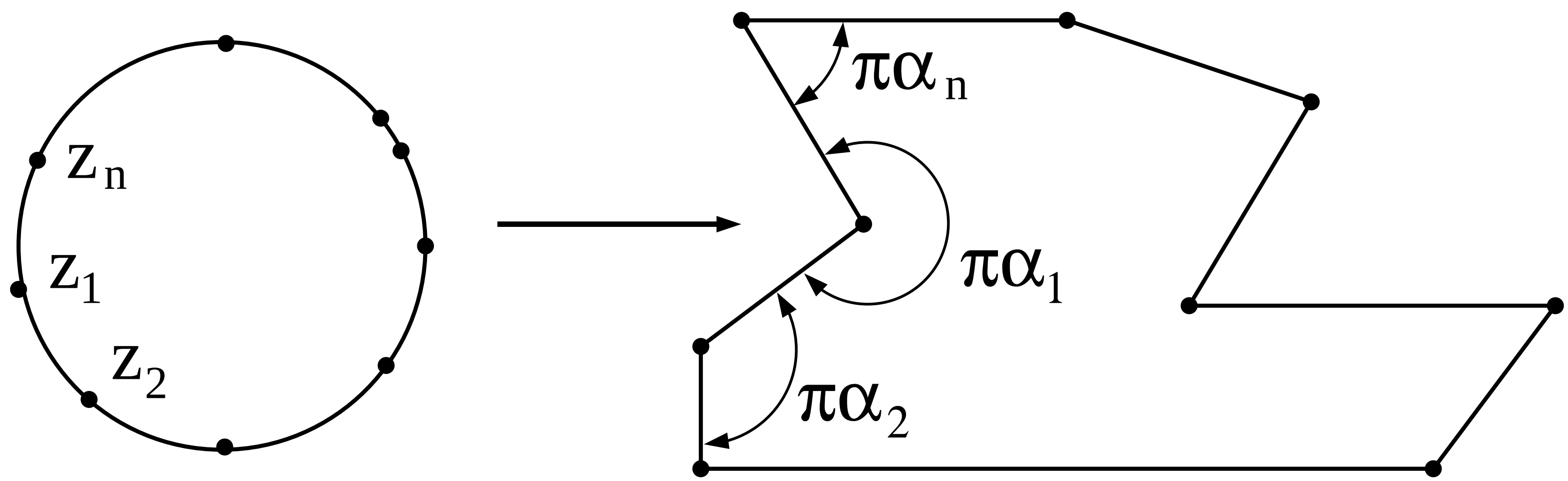
Problem: given n-gon, compute these n points.





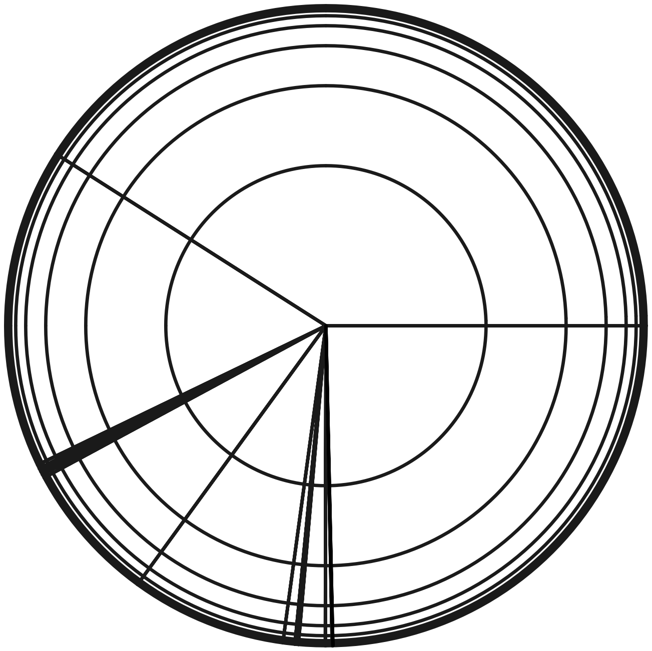
Schwarz-Christoffel formula (1867):

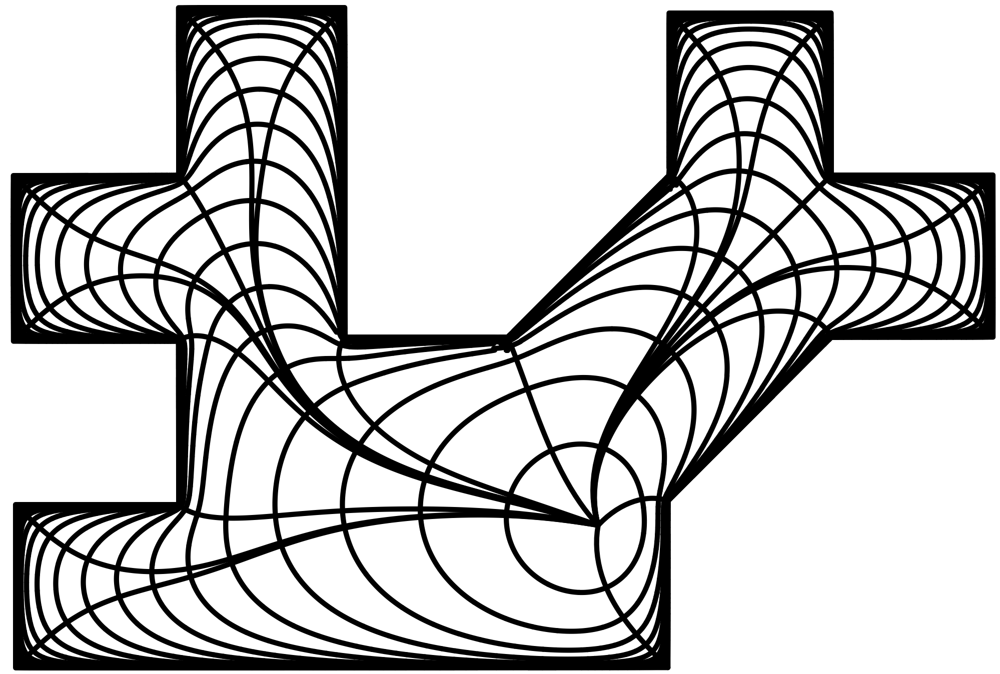
$$f(z) = A + C \int^z \prod_{k=1}^{n} \left(1 - \frac{w}{z_k}\right)^{\alpha_k - 1} dw,$$

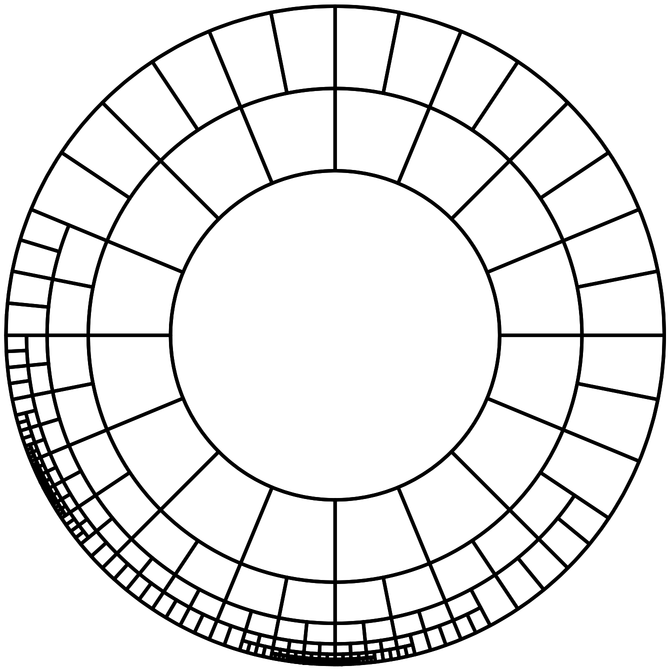


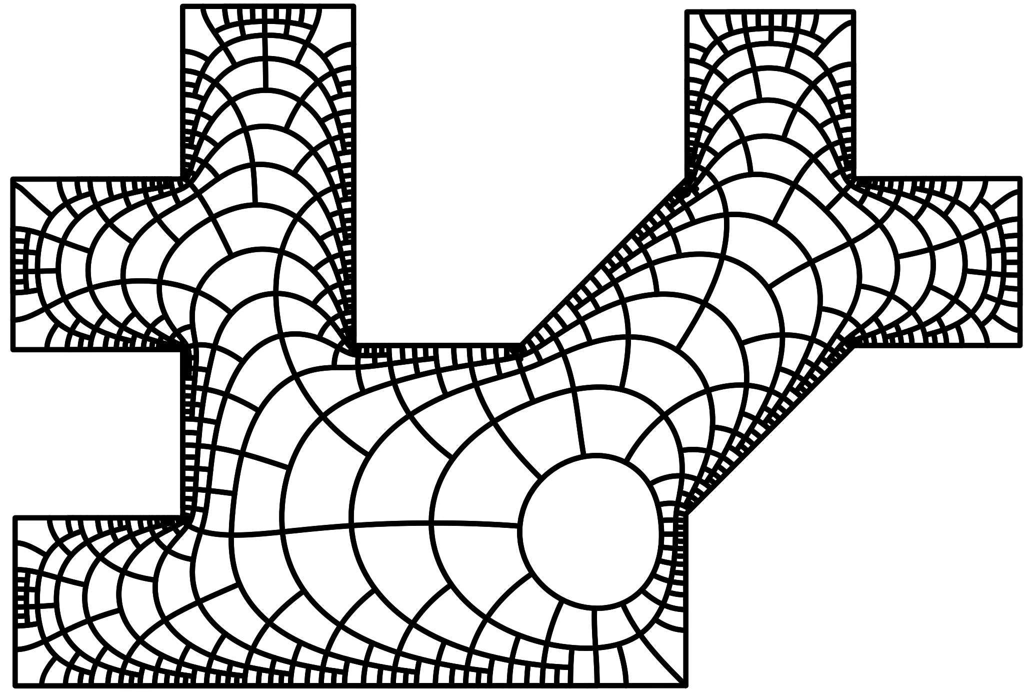
α's known

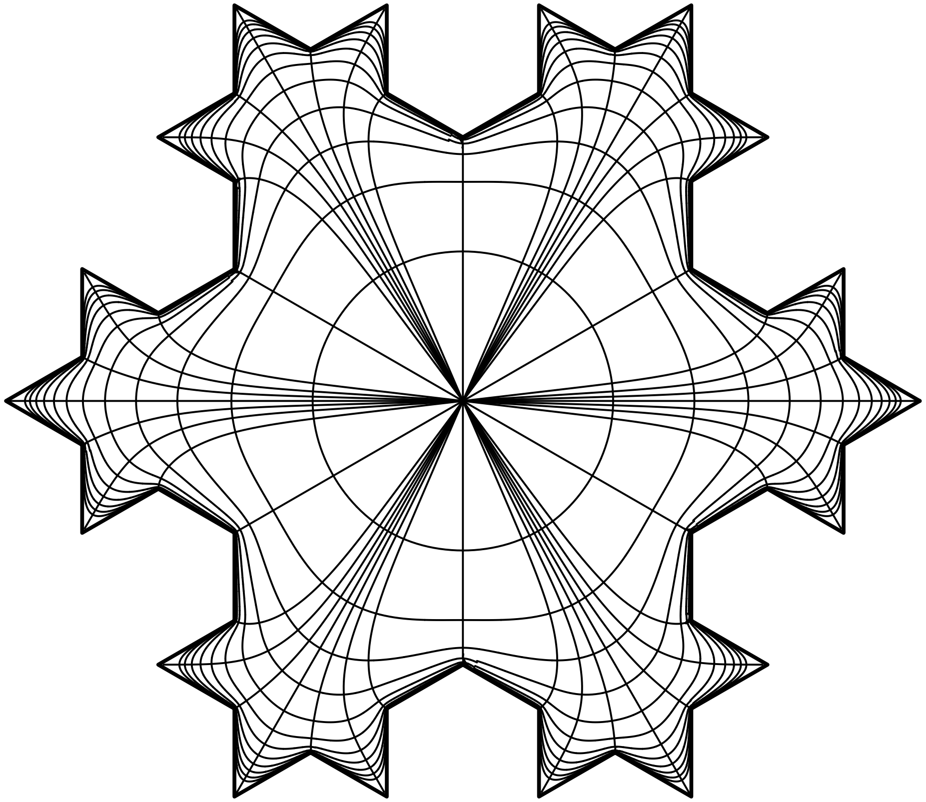
z's unknown (= **SC-parameters** = **pre-vertices**)

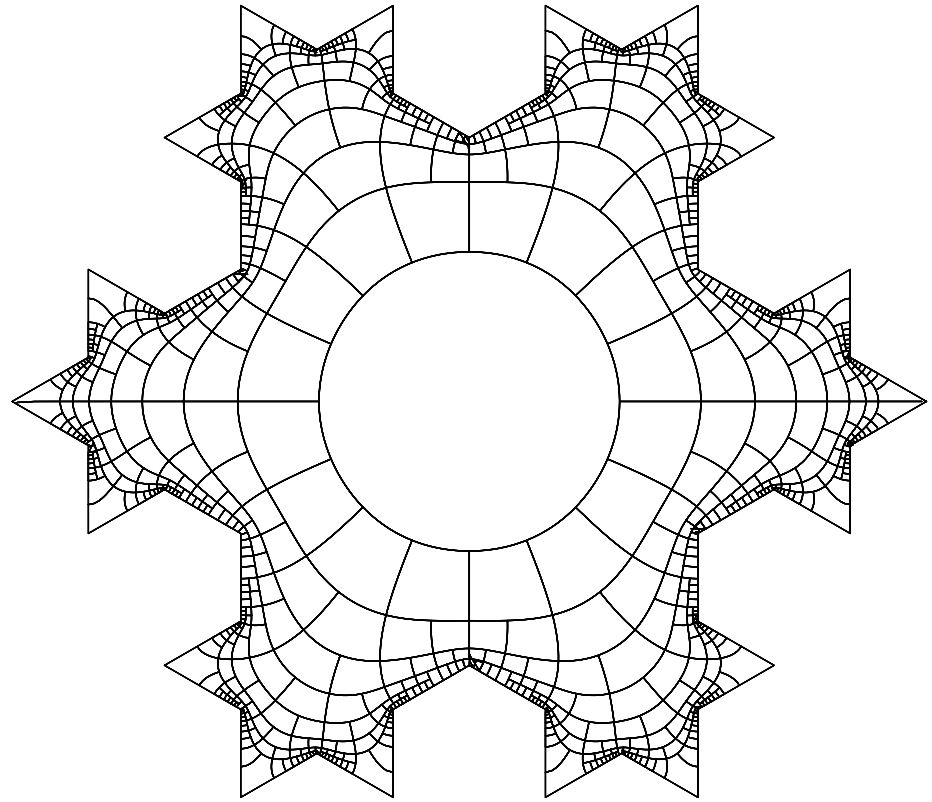


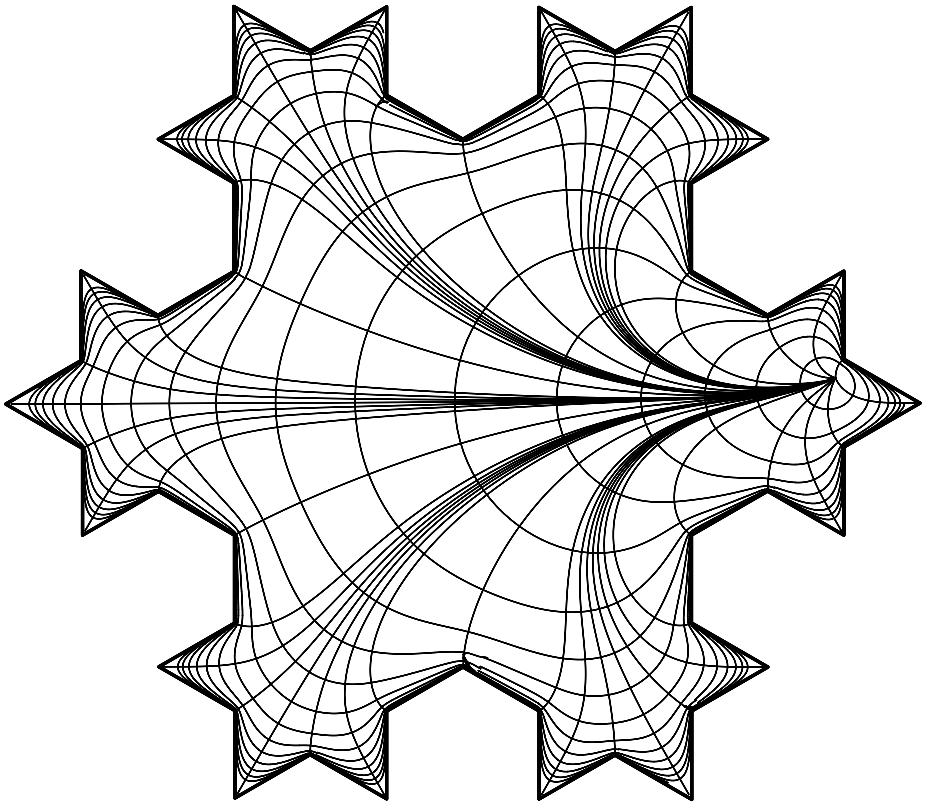


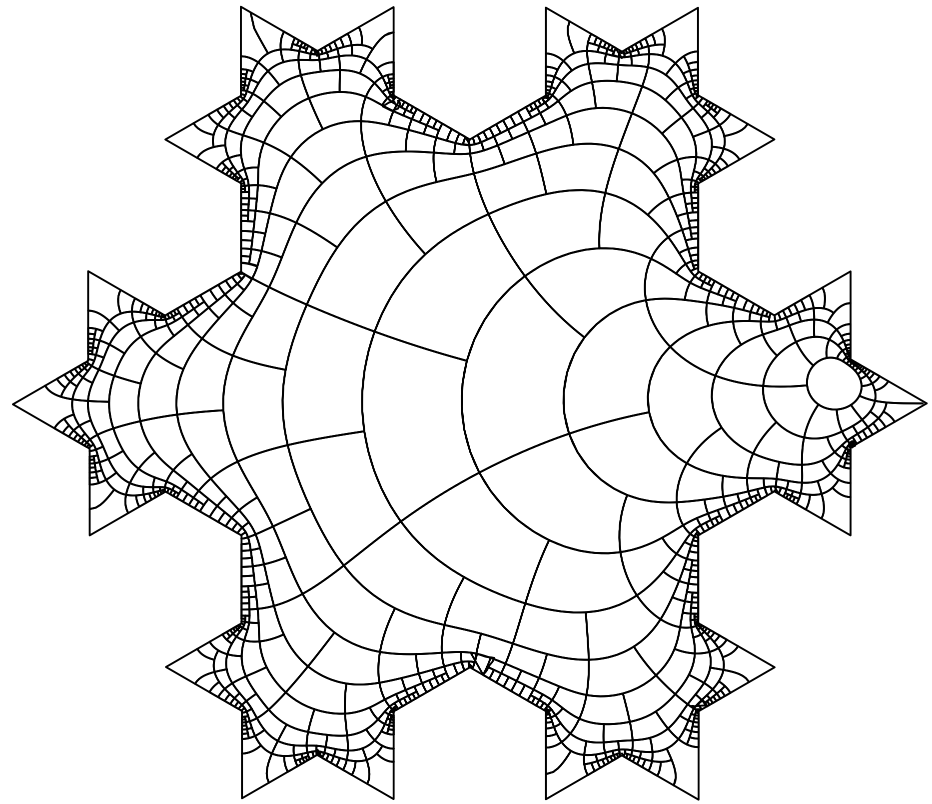












How fast can we compute the pre-vertices?

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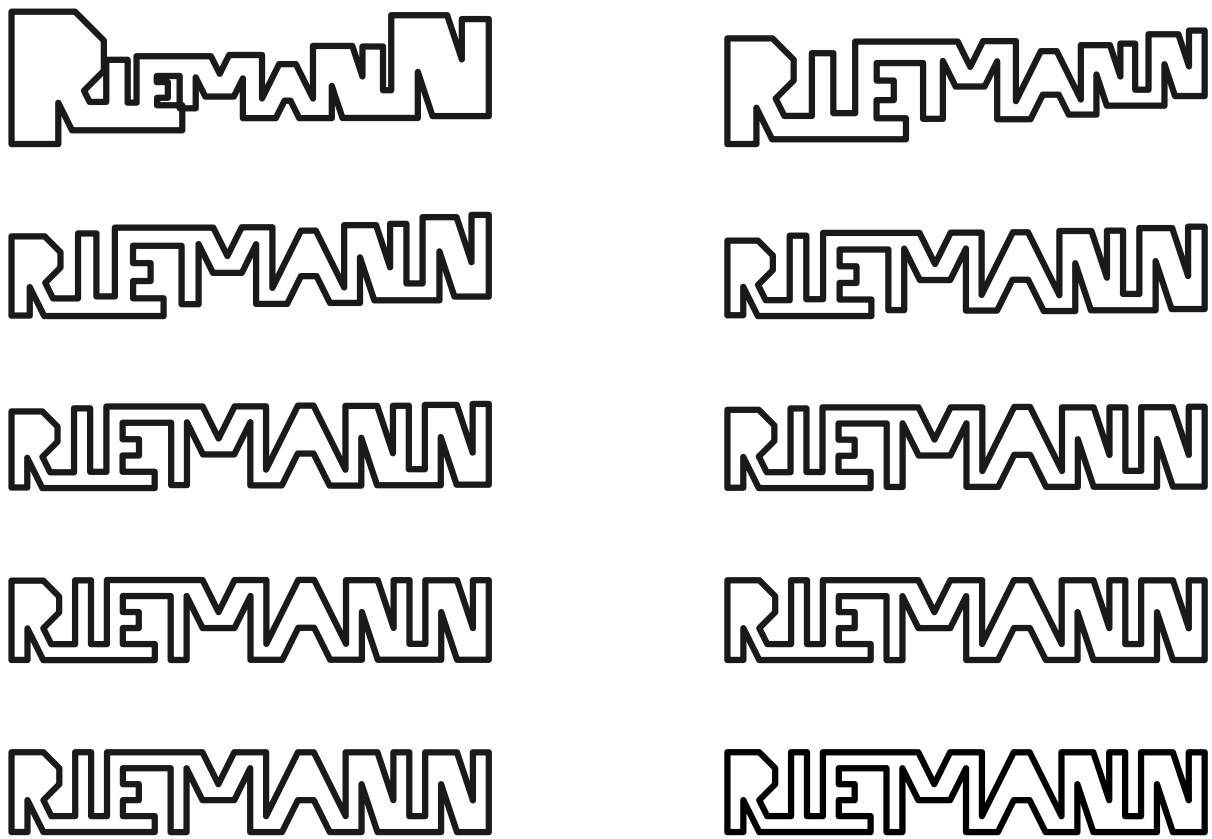
Theorem: Can compute SC-parameters in time $C_\epsilon \cdot n$.

Idea: Newton's method on n-tuples with initial guess that is **close** to correct answer and **fast** to compute.



Idea: Newton's method on n-tuples with initial guess that is **close** to correct answer and **fast** to compute.

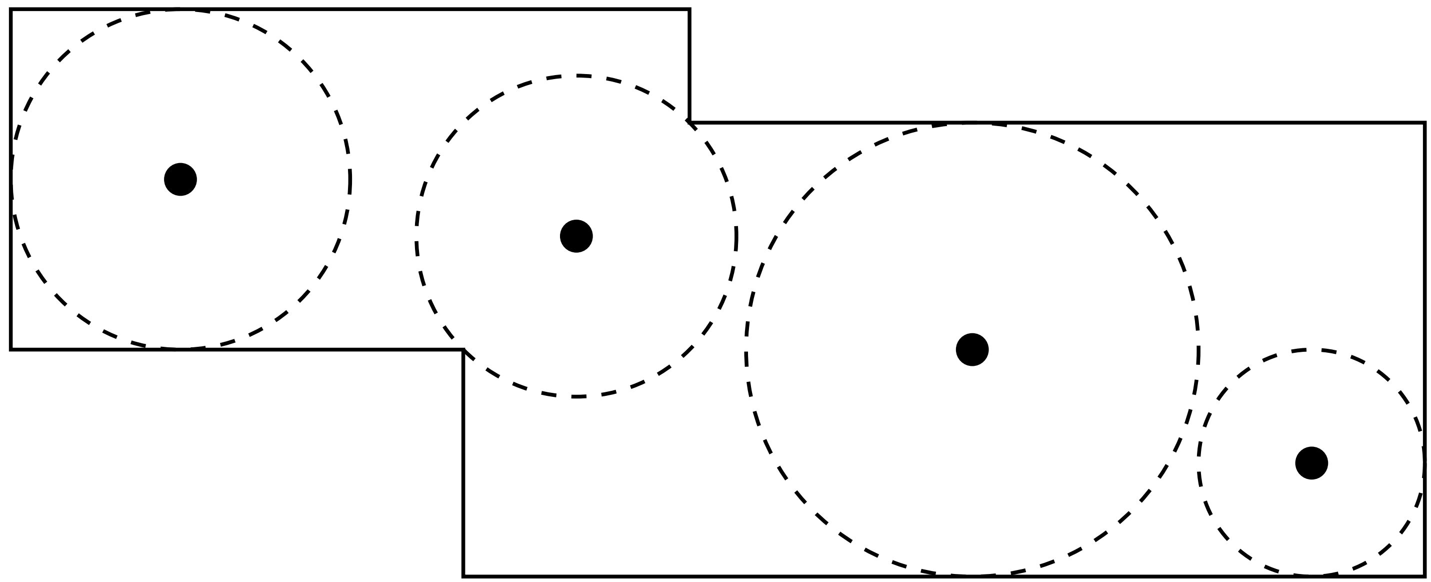
- **close** comes from hyperbolic geometry.

- **fast** comes from computational geometry.

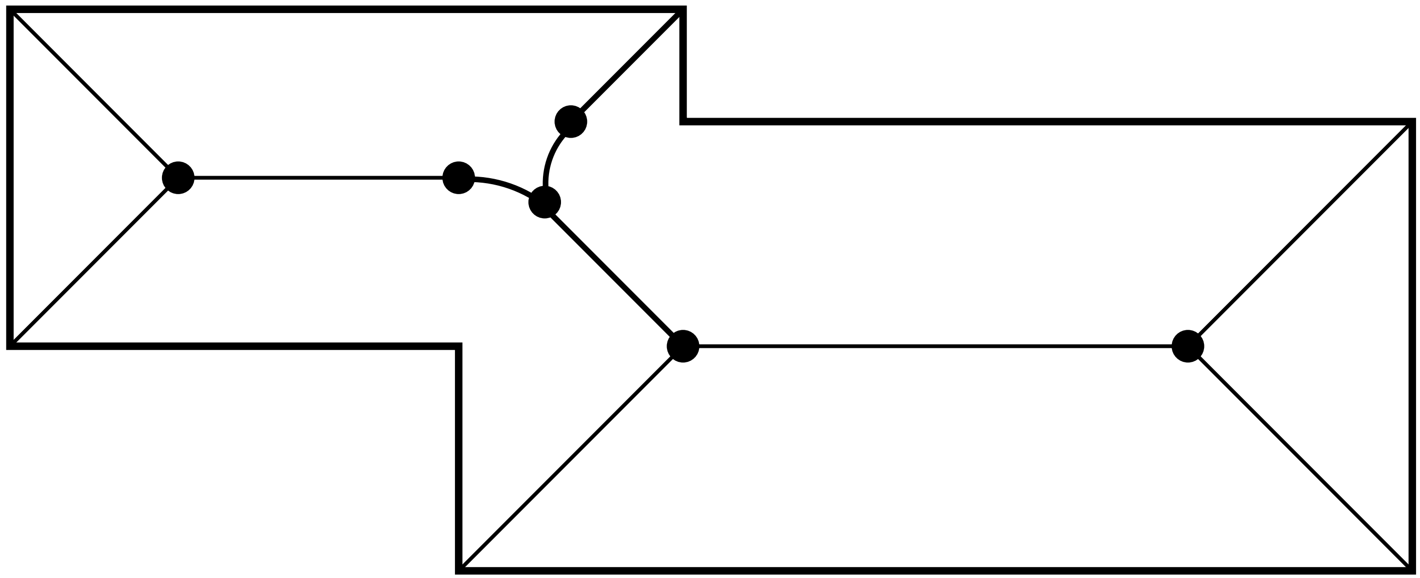
The initial guess is built using the medial axis.

Medial axis:

centers of disks that hit boundary in at least two points.

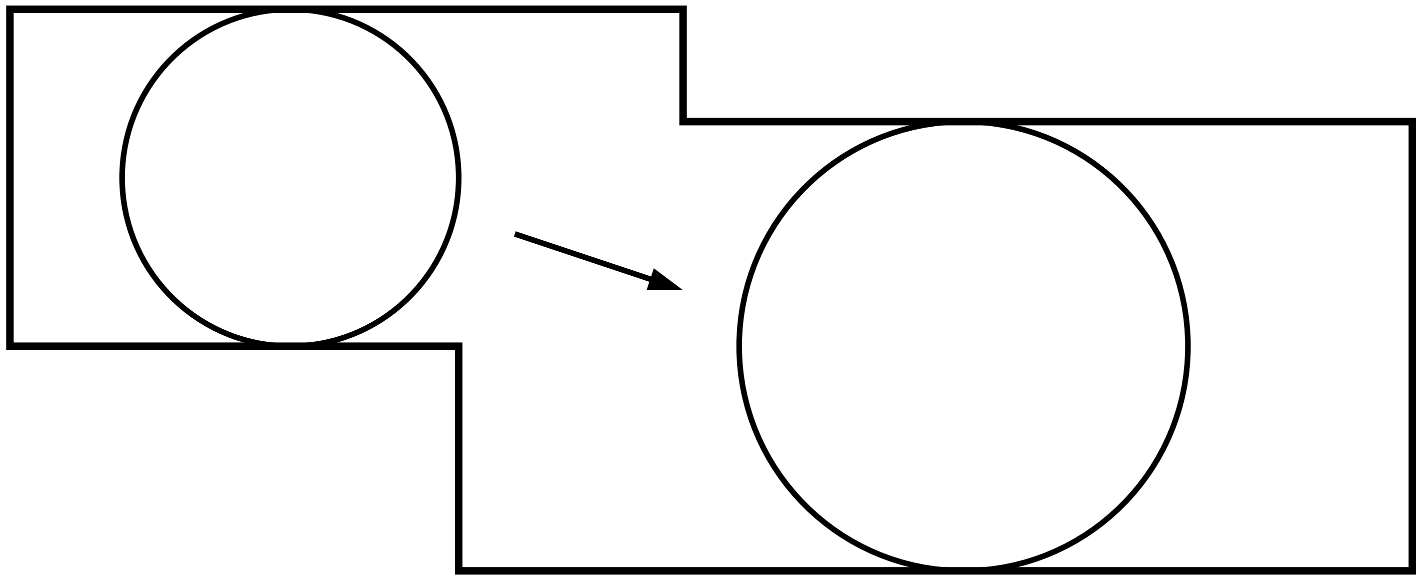


Medial axis:
centers of disks that hit boundary in at least two points.



Medial axis of a polygon is a finite tree.

Medial axis:
centers of disks that hit boundary in at least two points.



There is a “natural” choice of conformal map between any two medial axis disks.

A **Möbius transformation** is a map of the form

$$z \to \frac{az+b}{cz+d}.$$

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Exactly the 1-1, conformal self-maps of $S^2 = \mathbb{R}^2 \cup \{\infty\}$.

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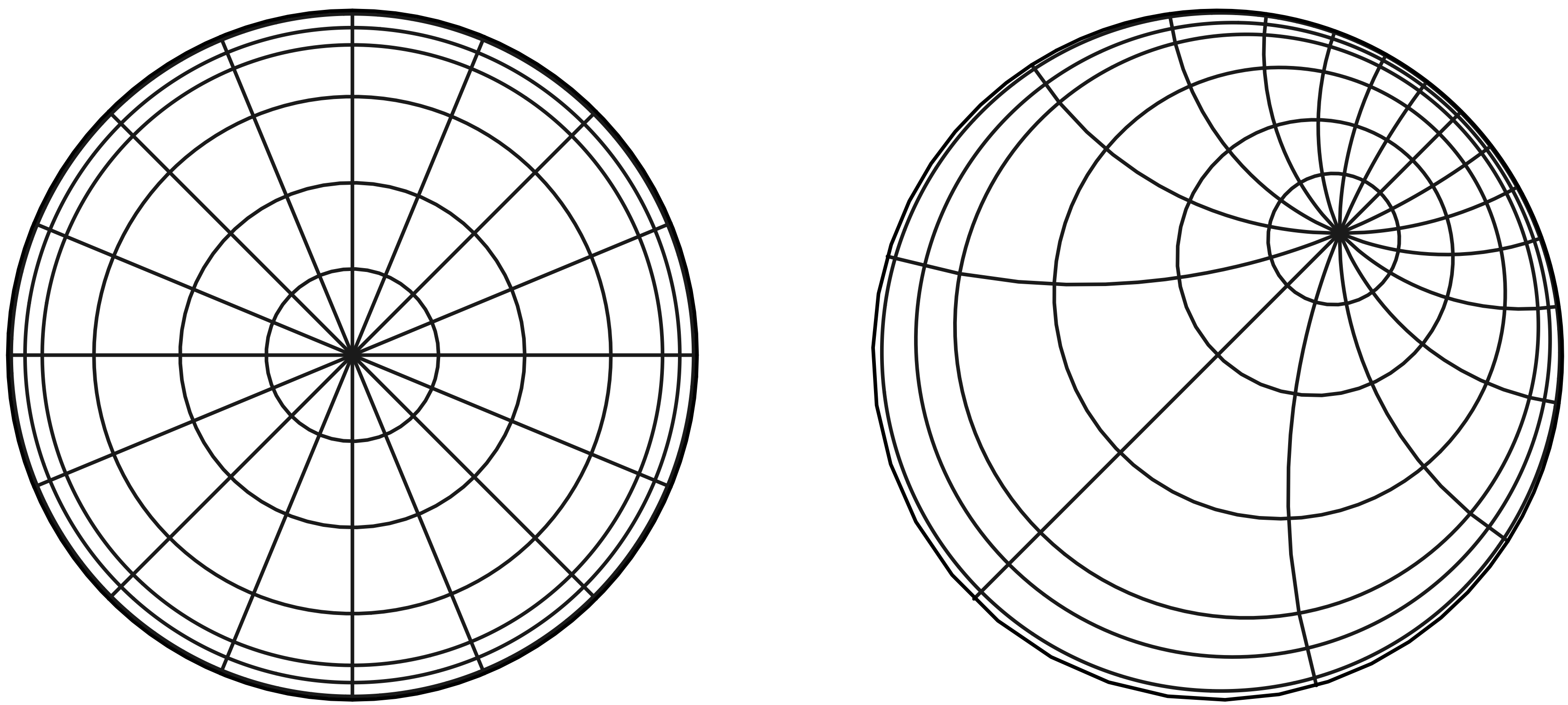
All conformal maps between disks have this form.



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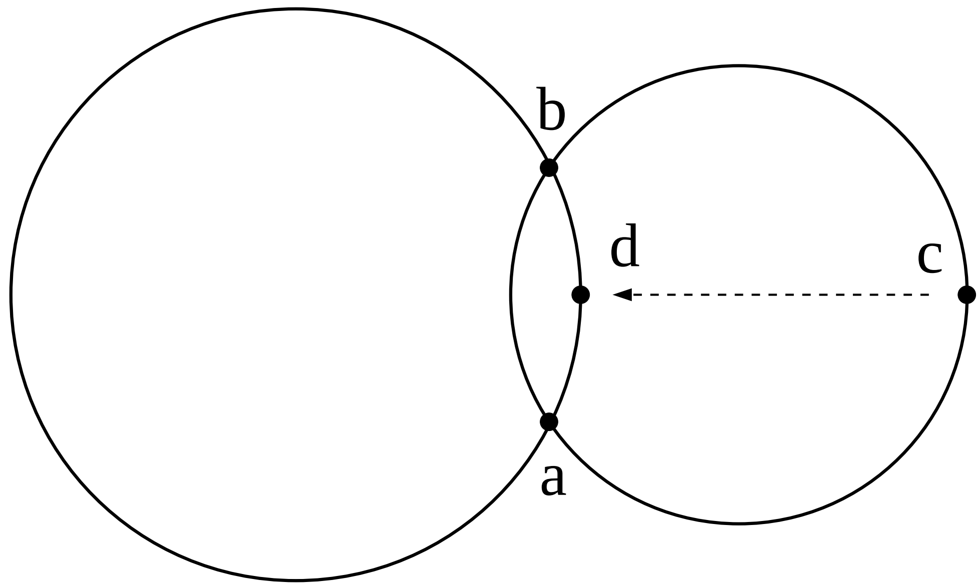
$$z \to \frac{az+b}{cz+d}.$$

Form a group under composition.

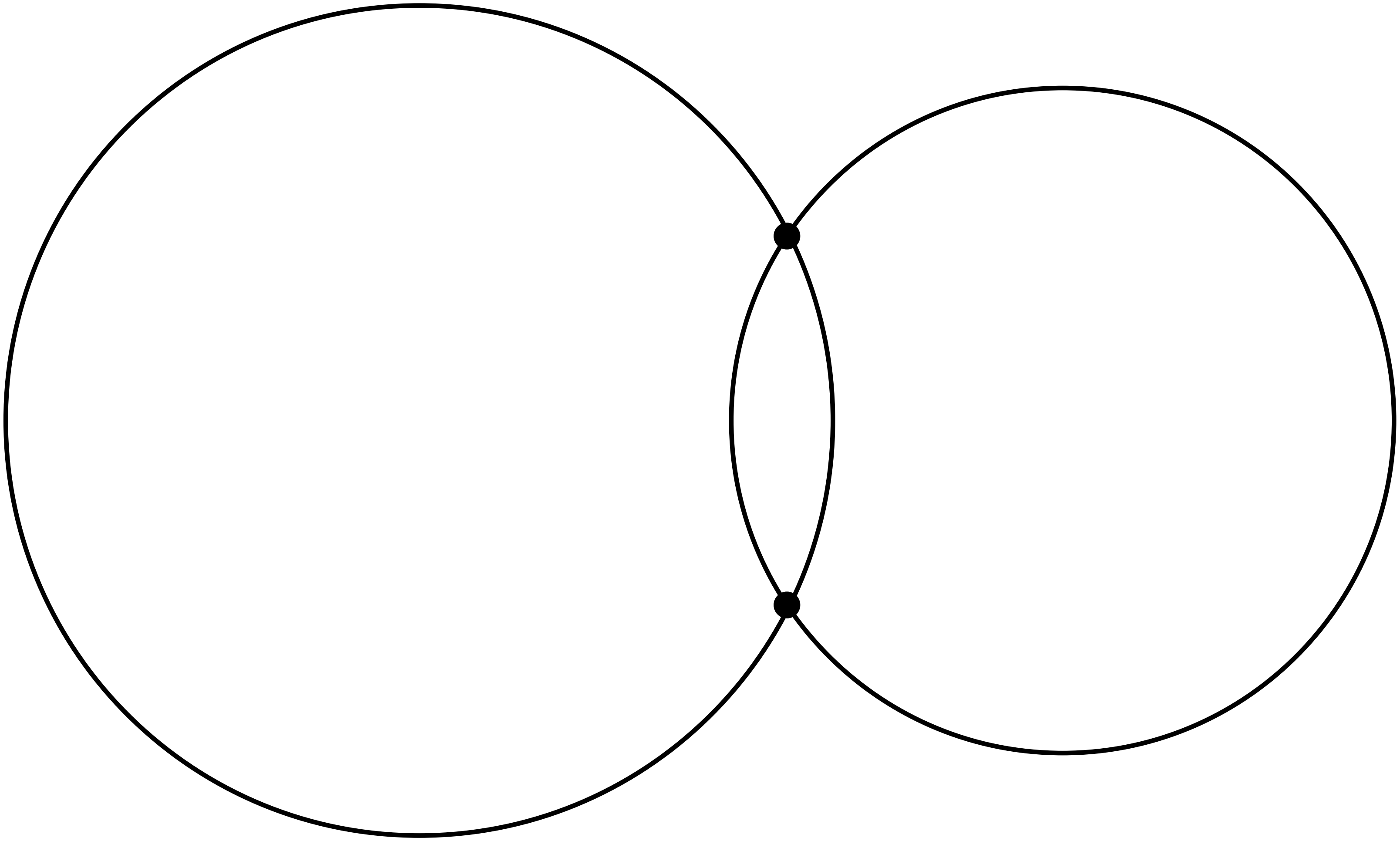
Uniquely determined by images of 3 distinct points.

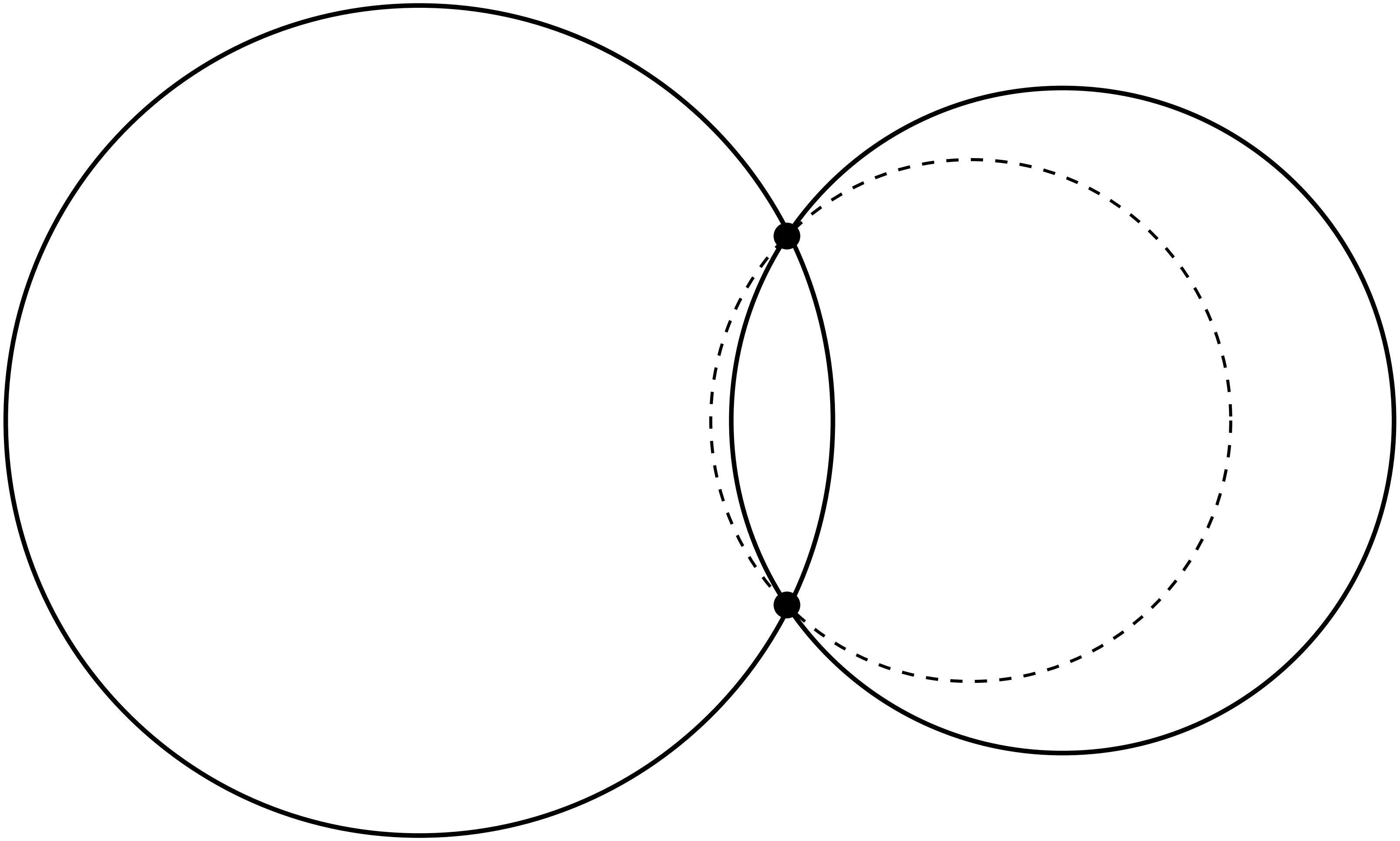


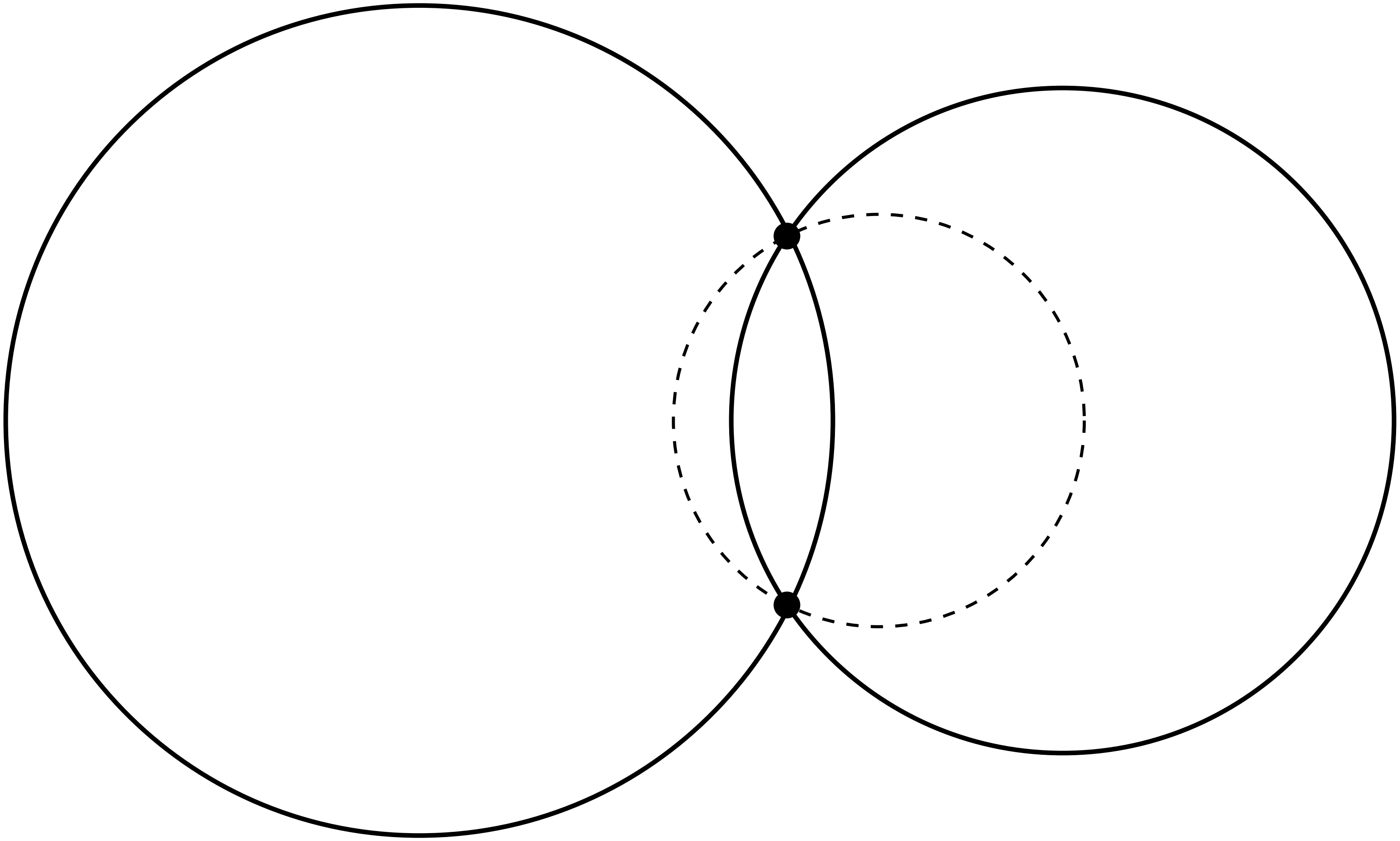
Fix intersection points a, b and map $c \to d$ as shown.

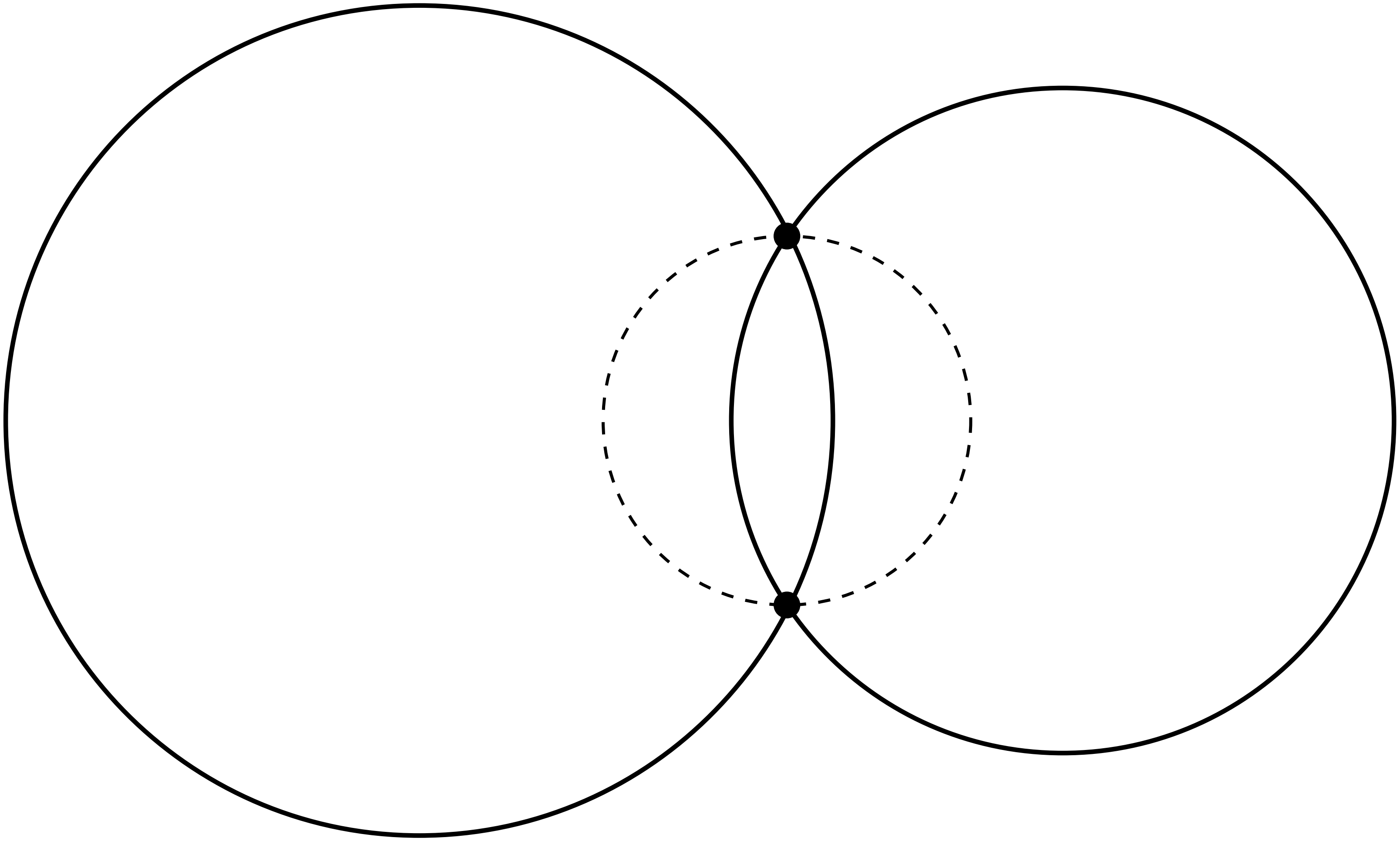
Determines unique Möbius map between disks.

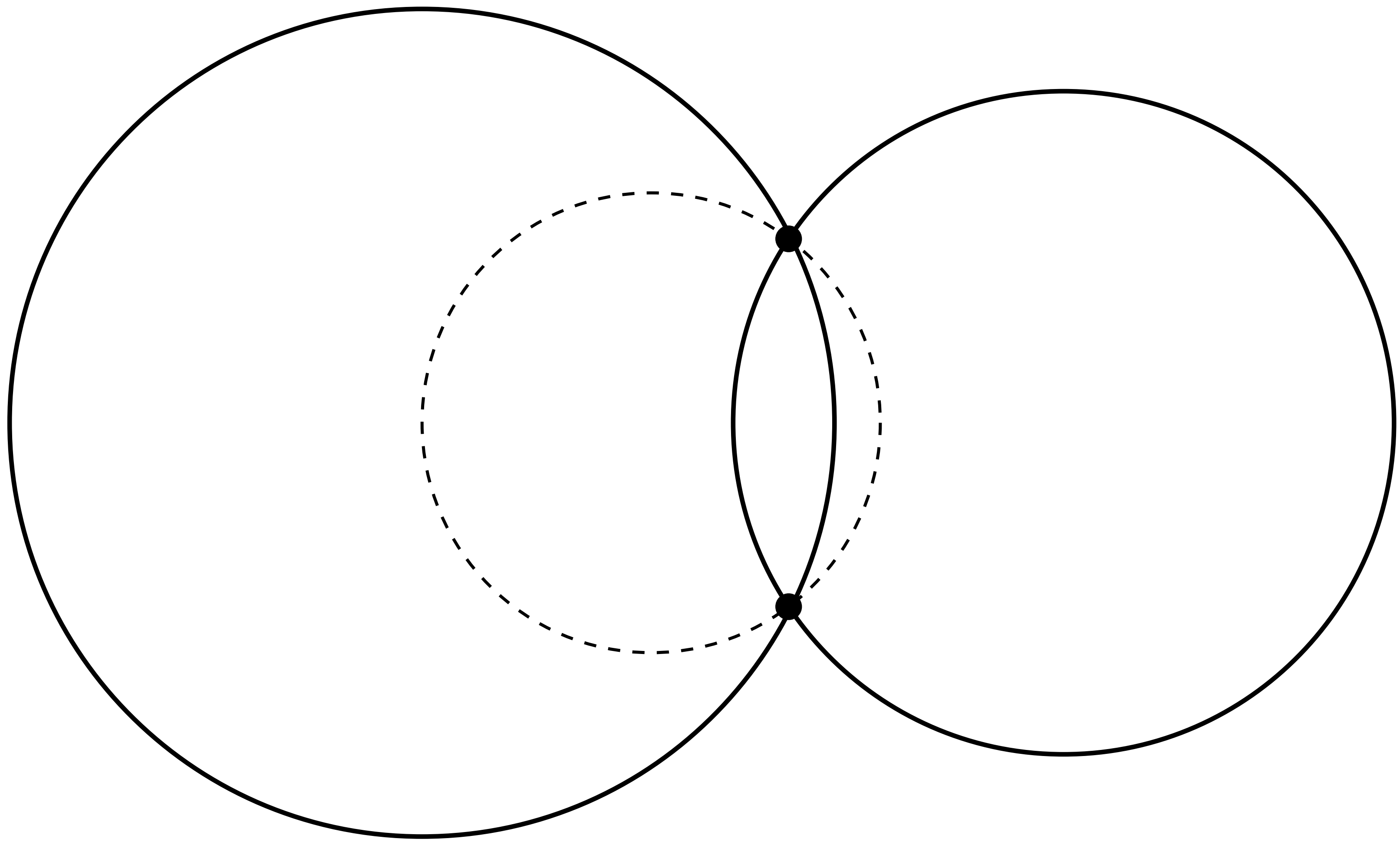
Part of 1-parameter symmetric family fixing a, b.

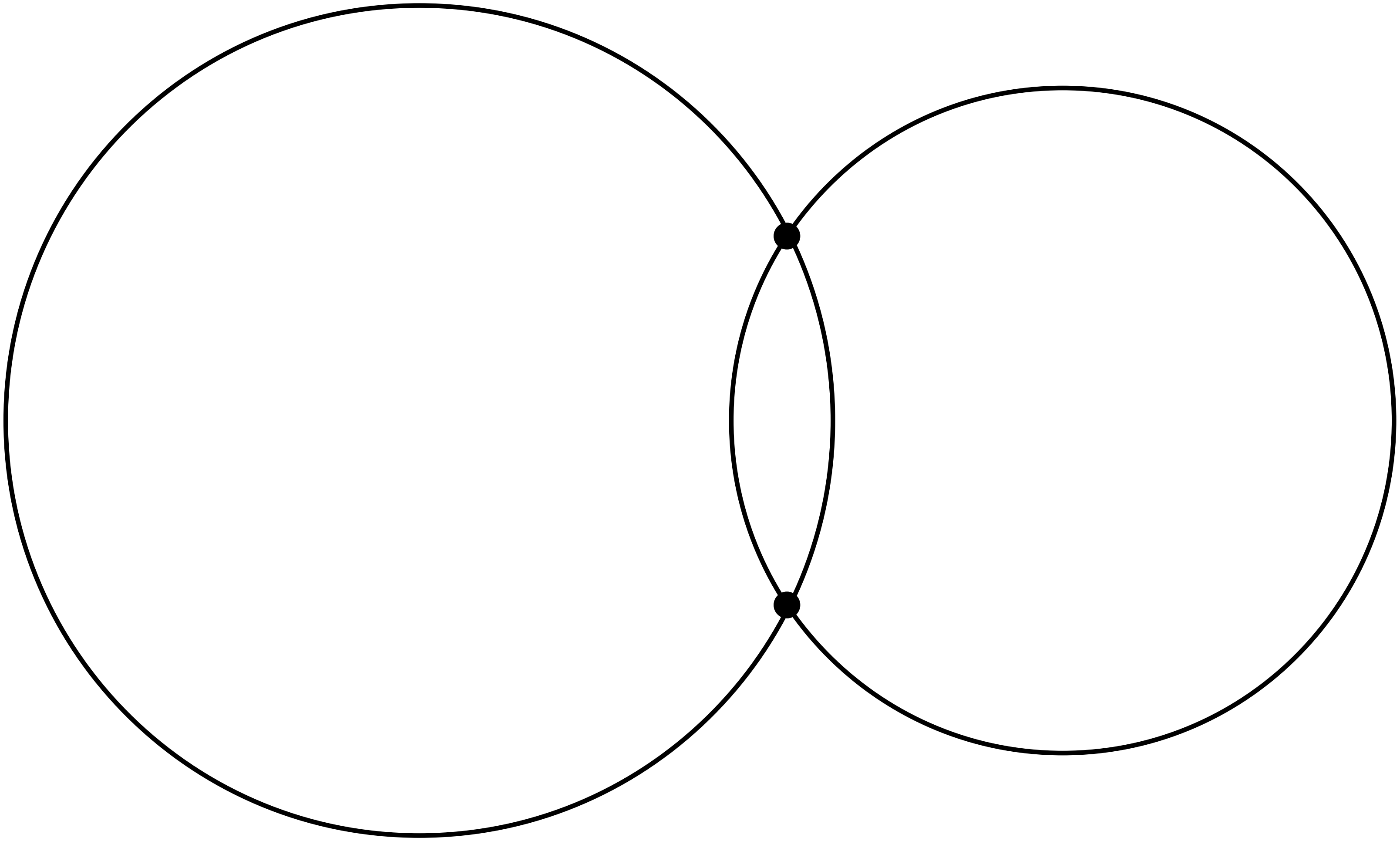


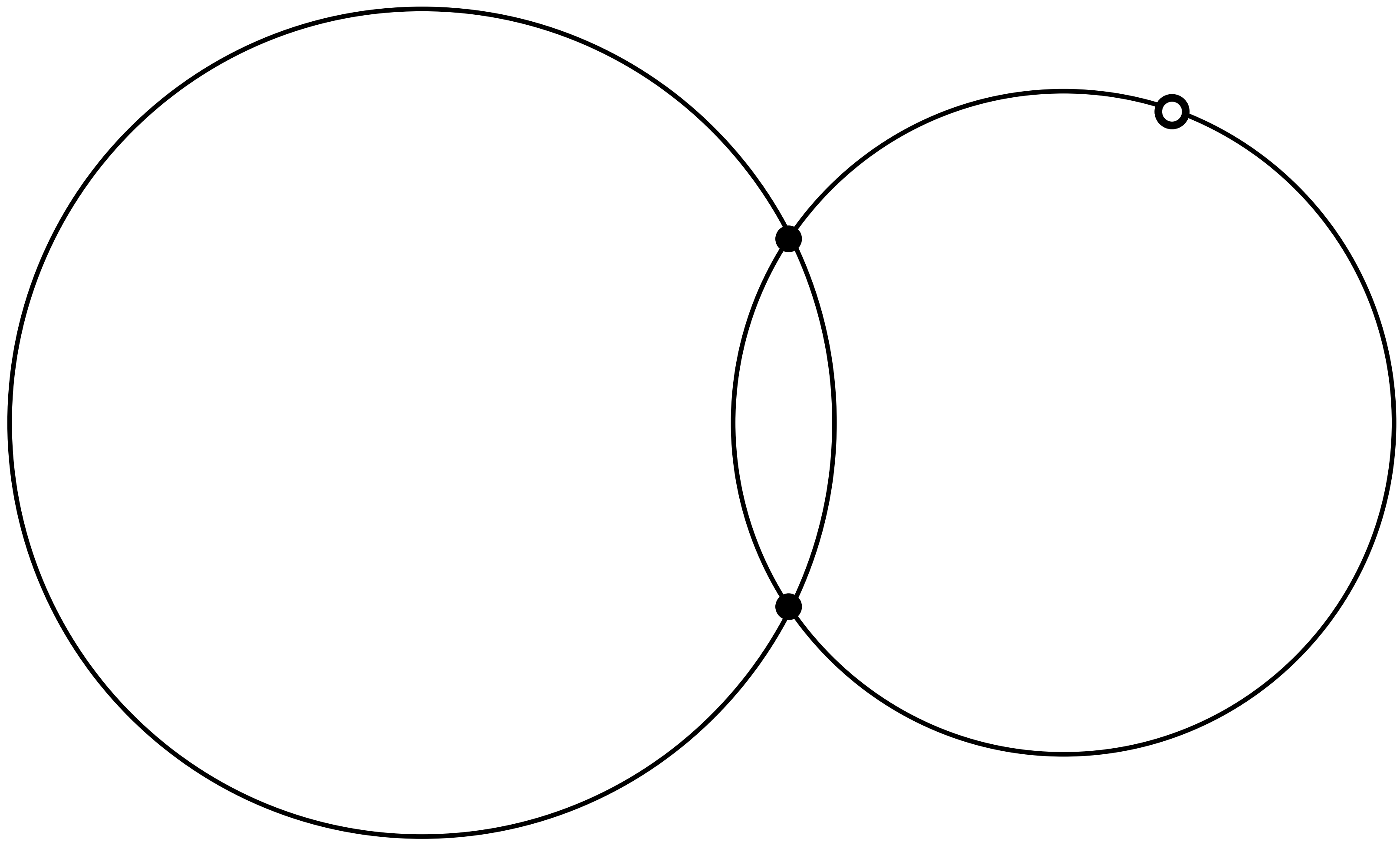


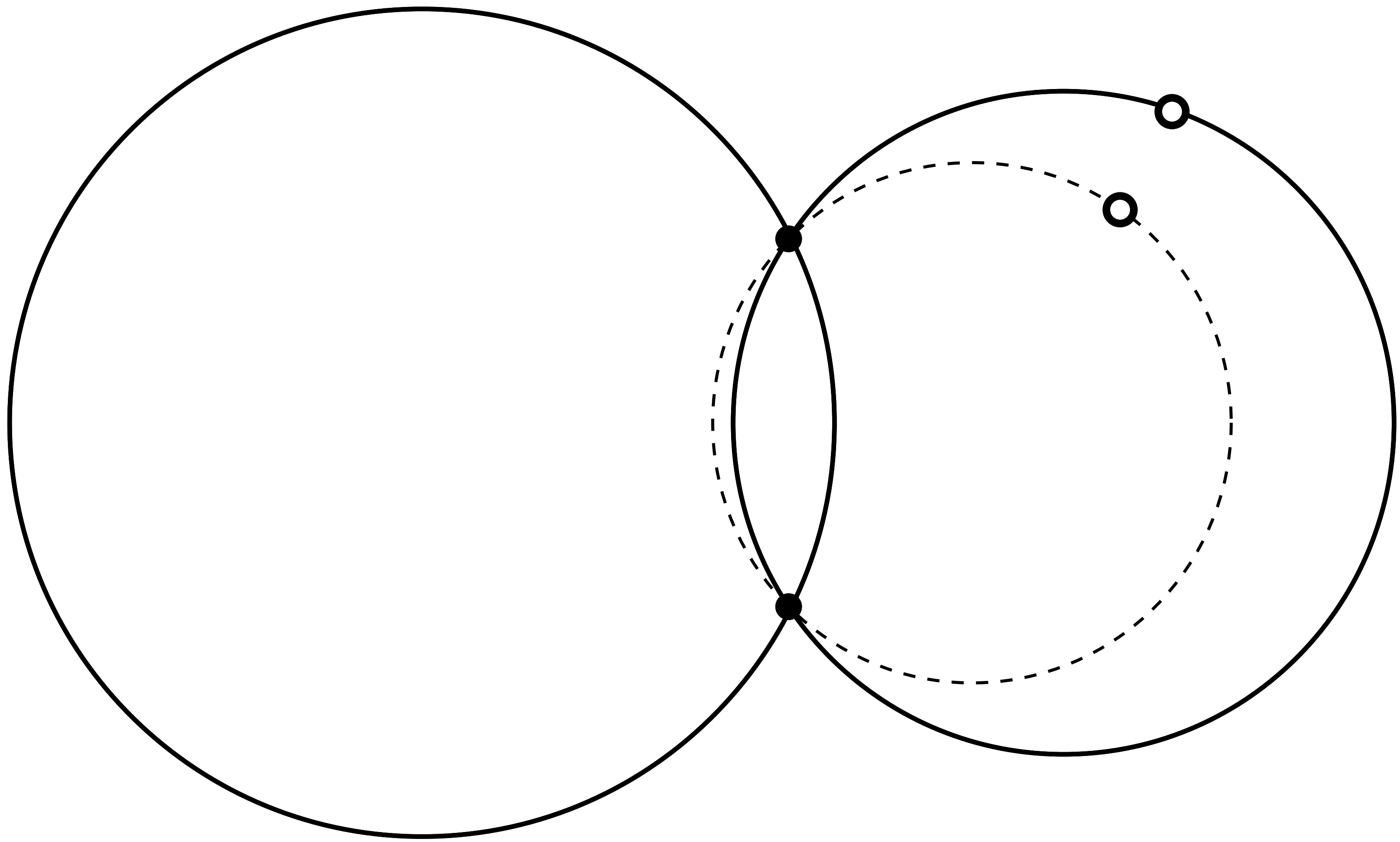


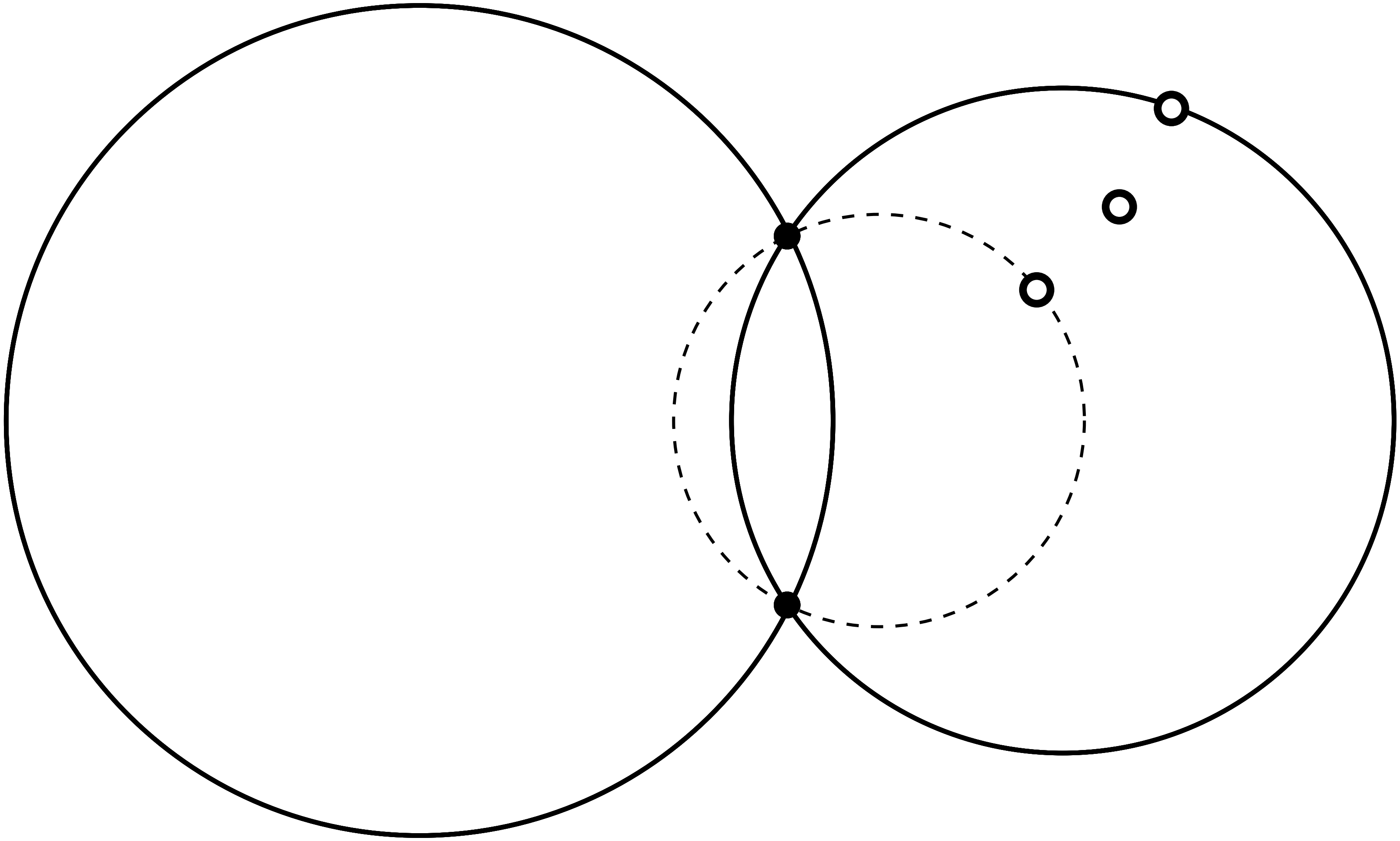


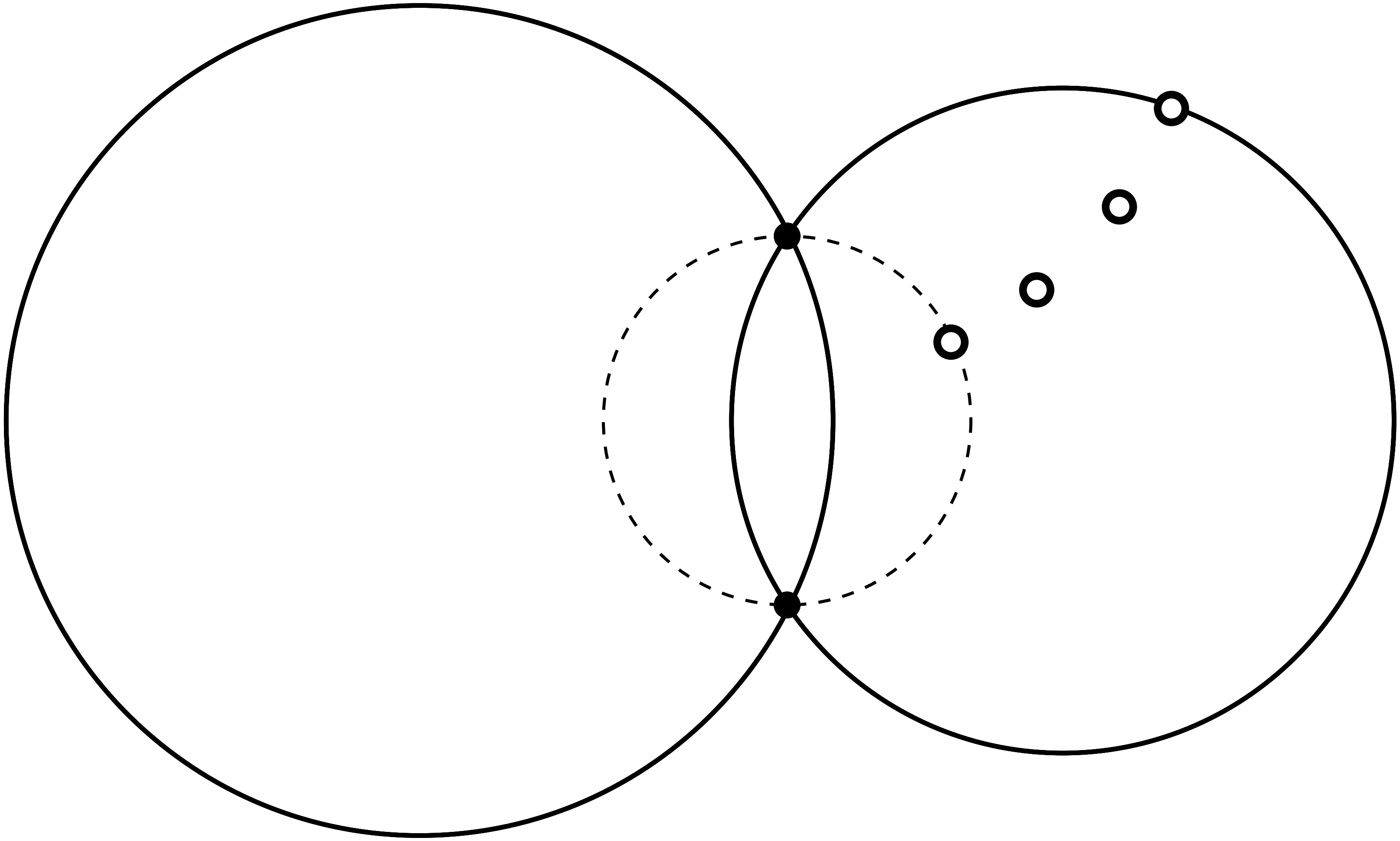


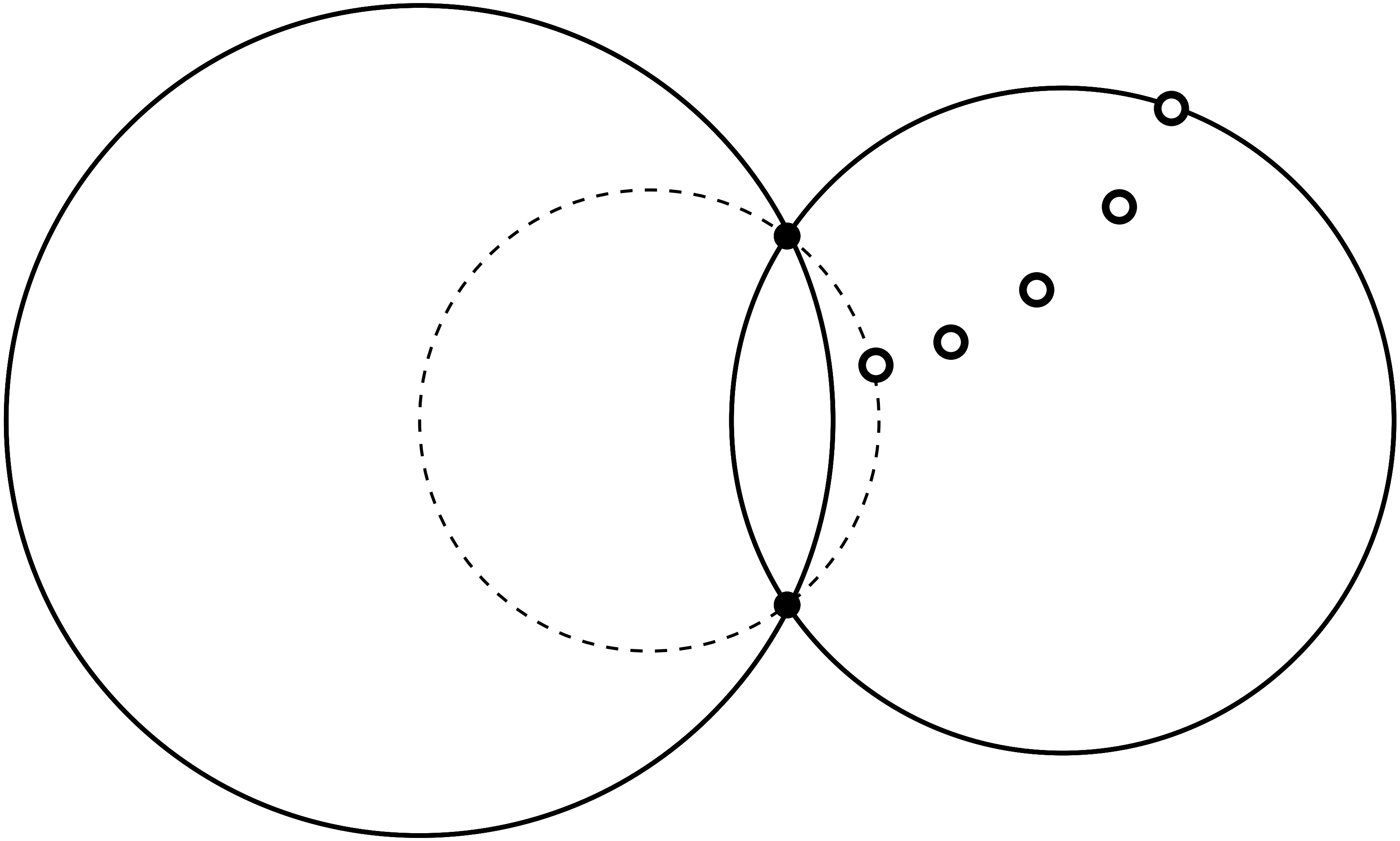


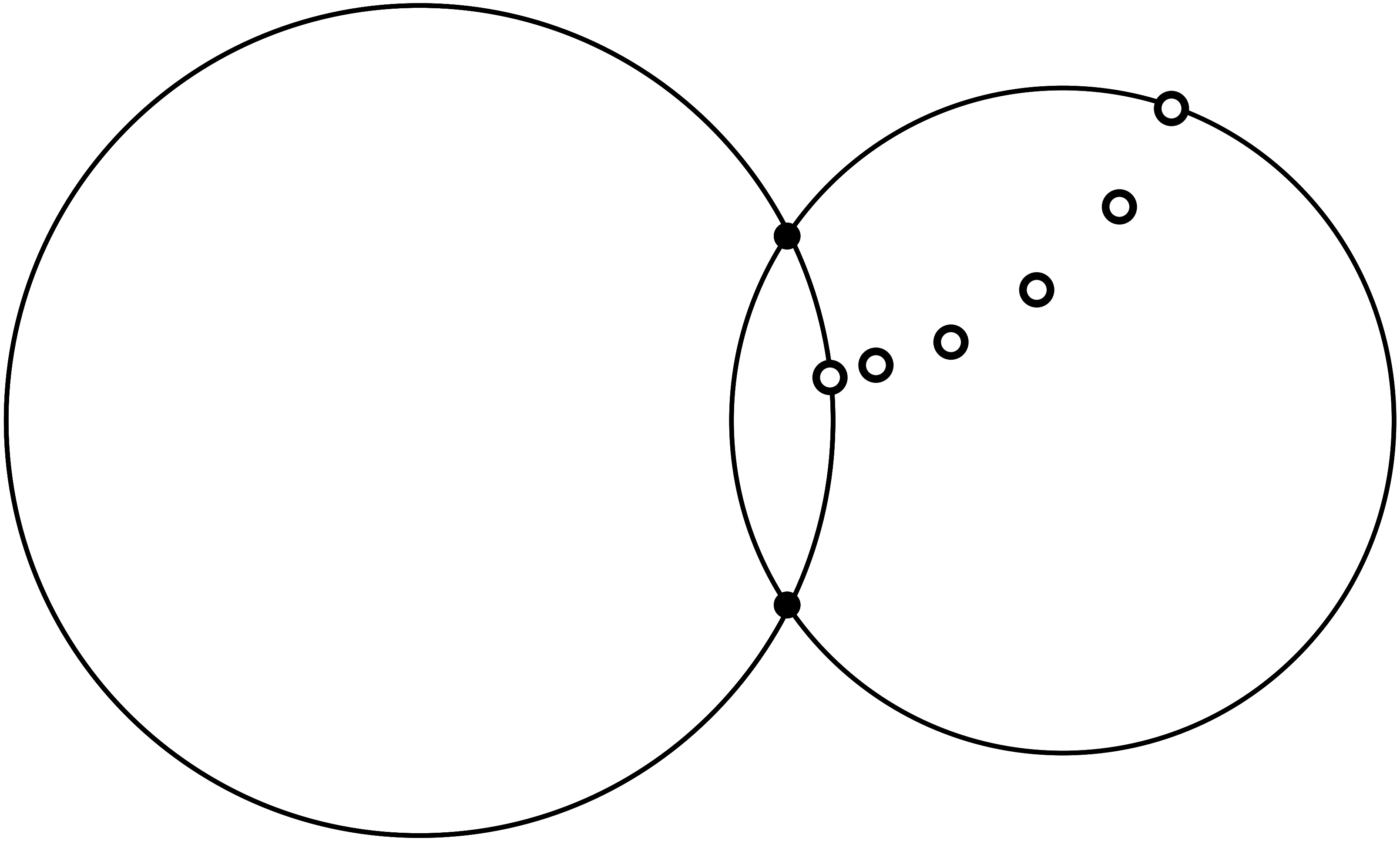


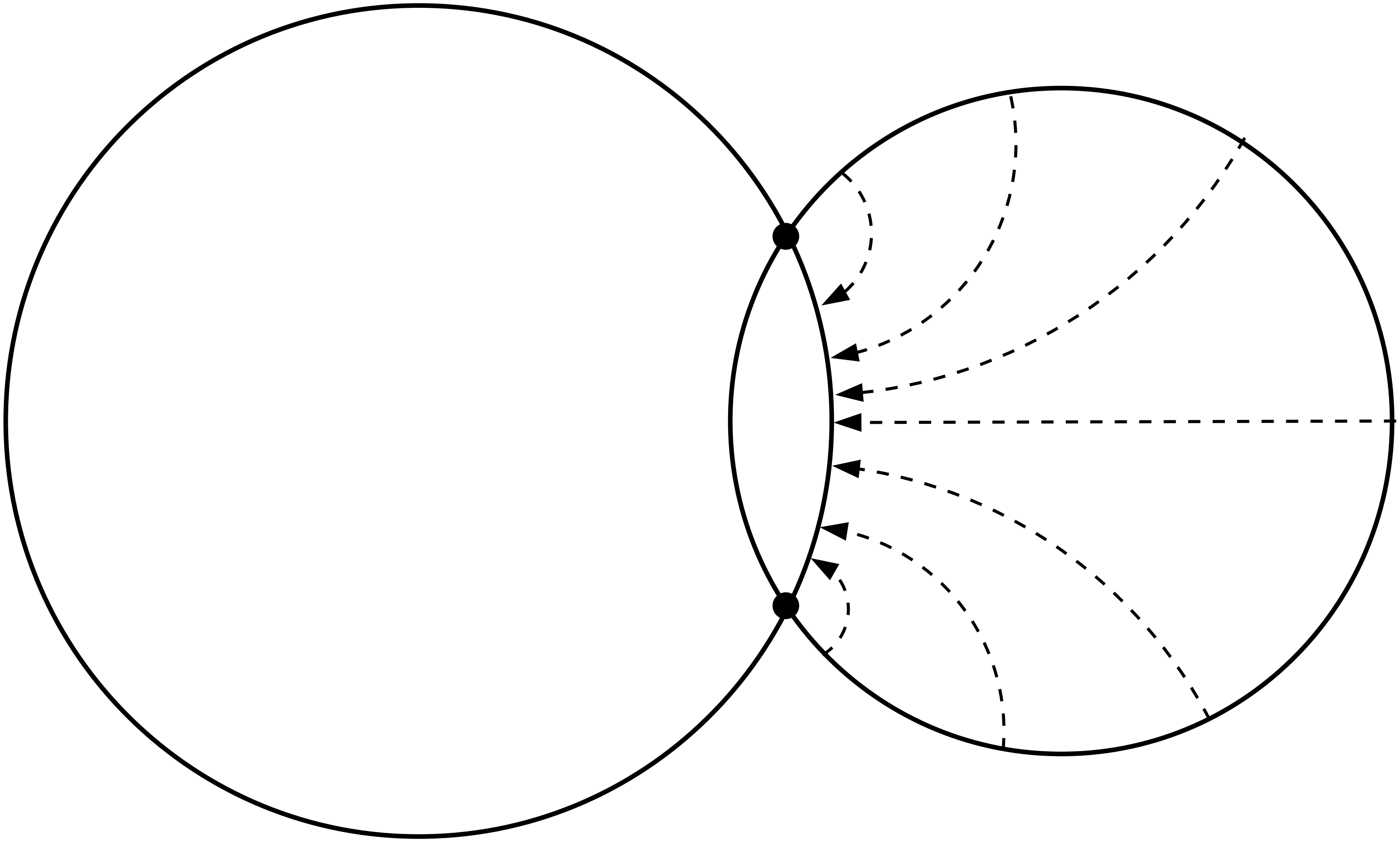






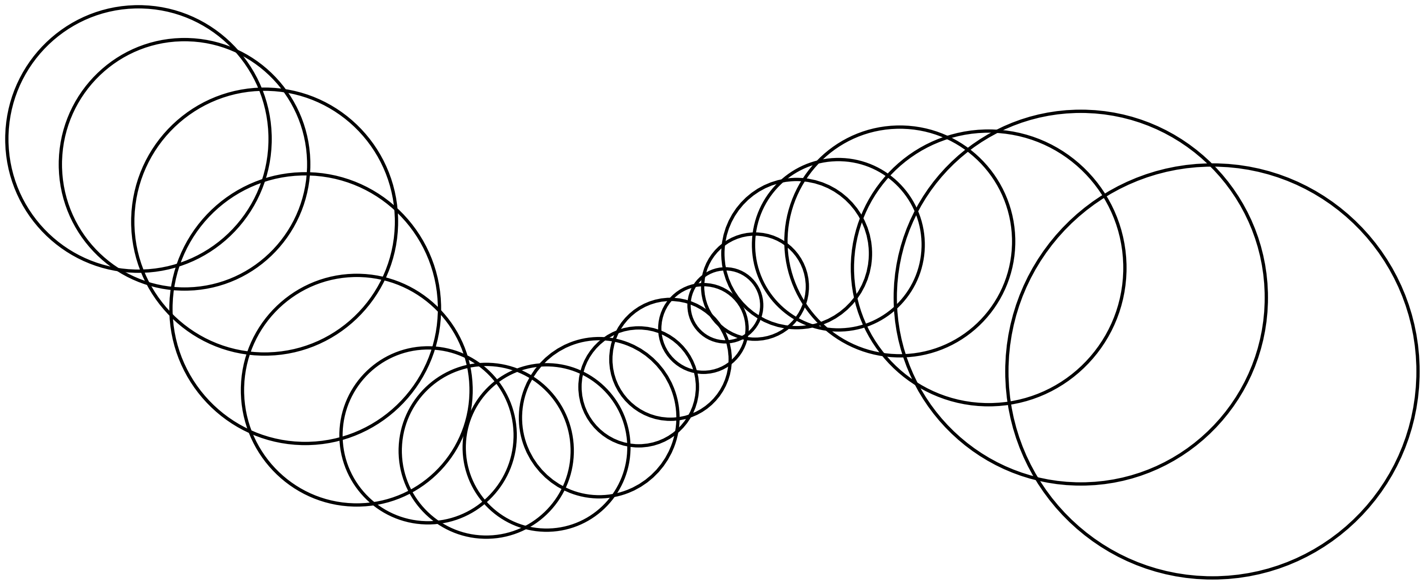






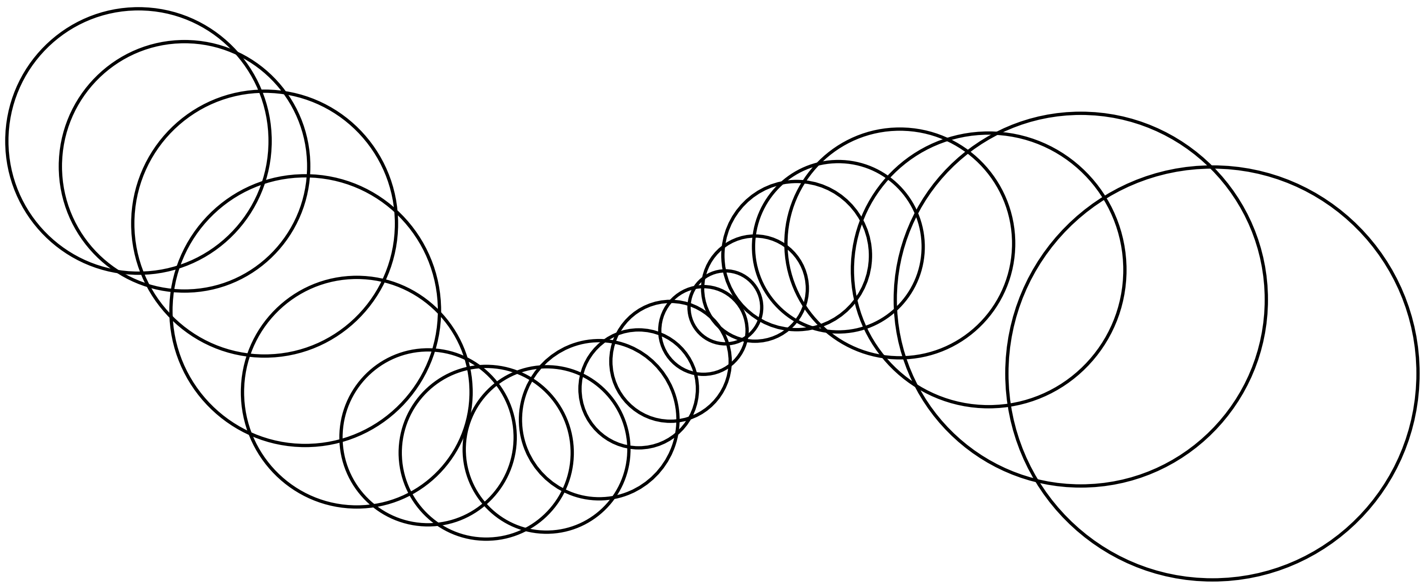
Points follow circular paths, perpendicular to boundary.

Compose along chain to map first disk to last.



Composition is Möbius (maps form group).

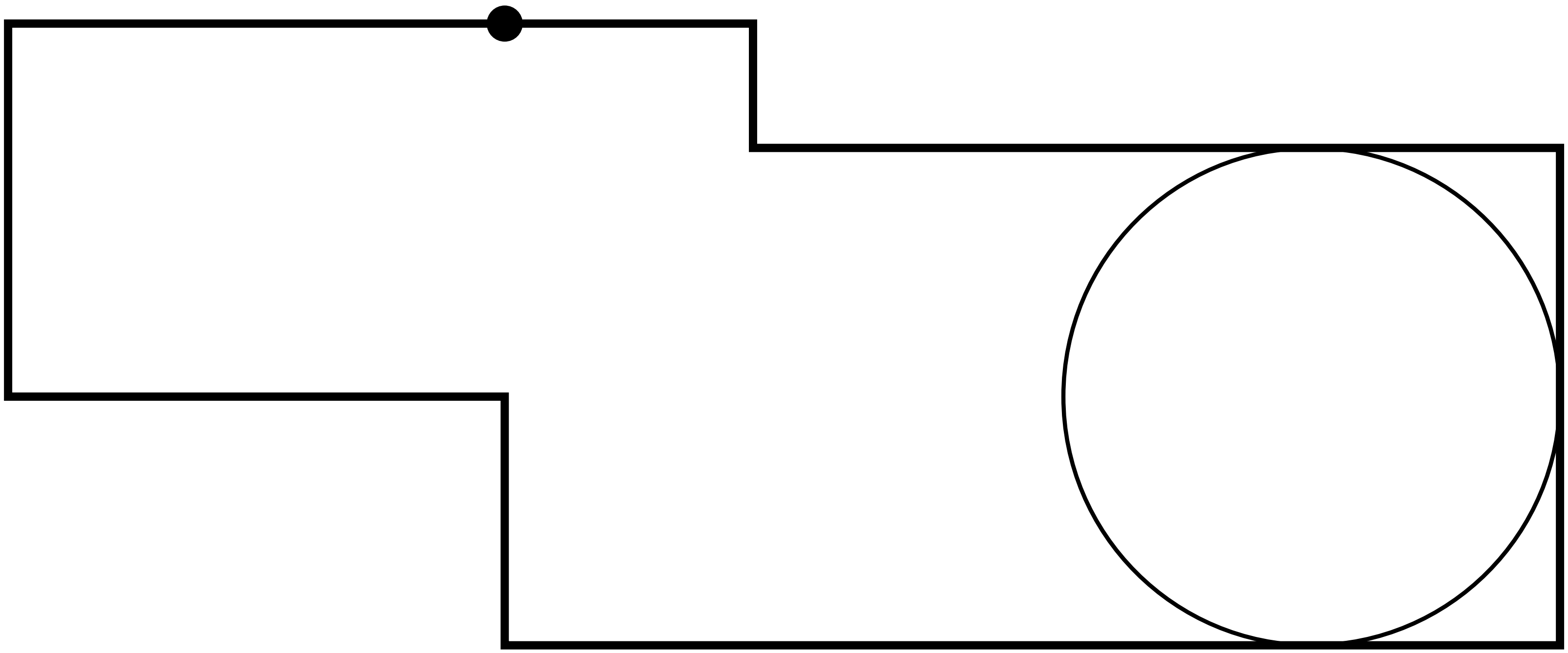
Compose along chain to map first disk to last.



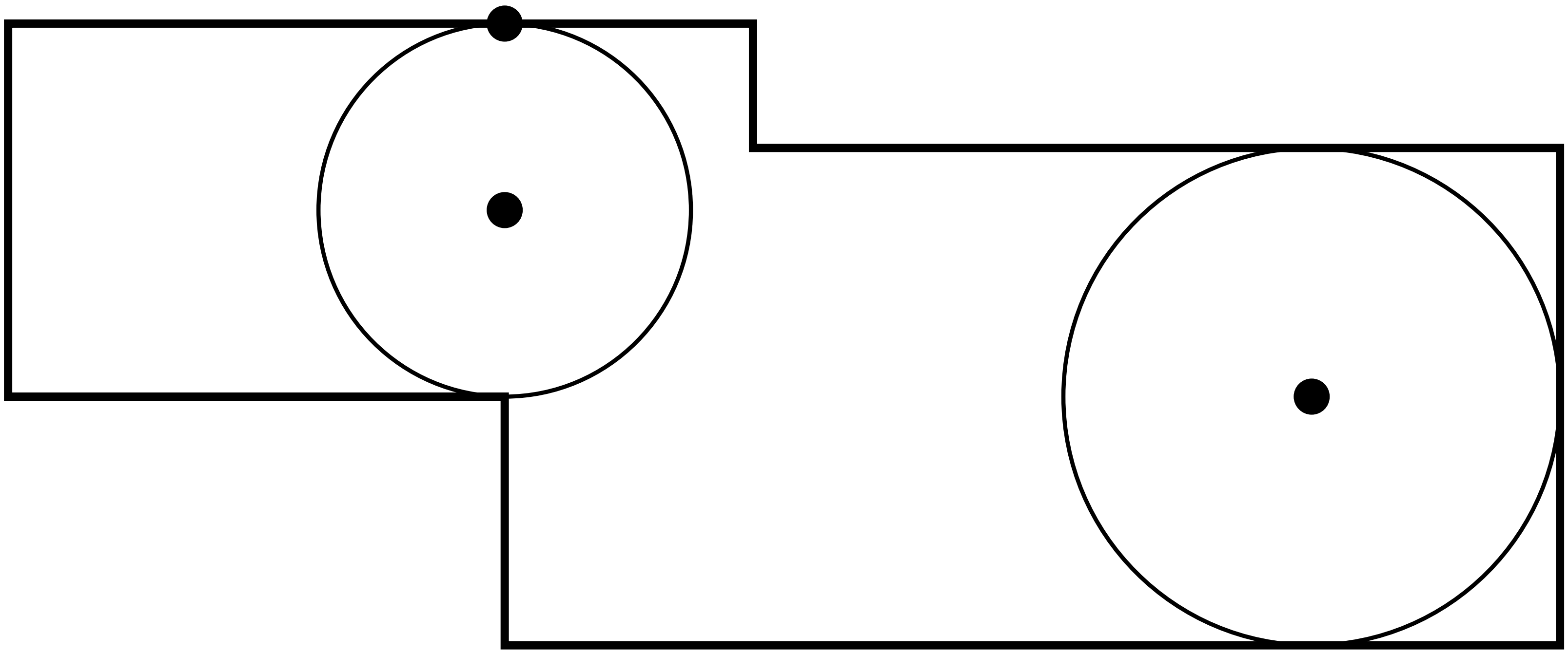
Composition is Möbius (maps form group).

Take limit to associate Möbius map to path of circles.

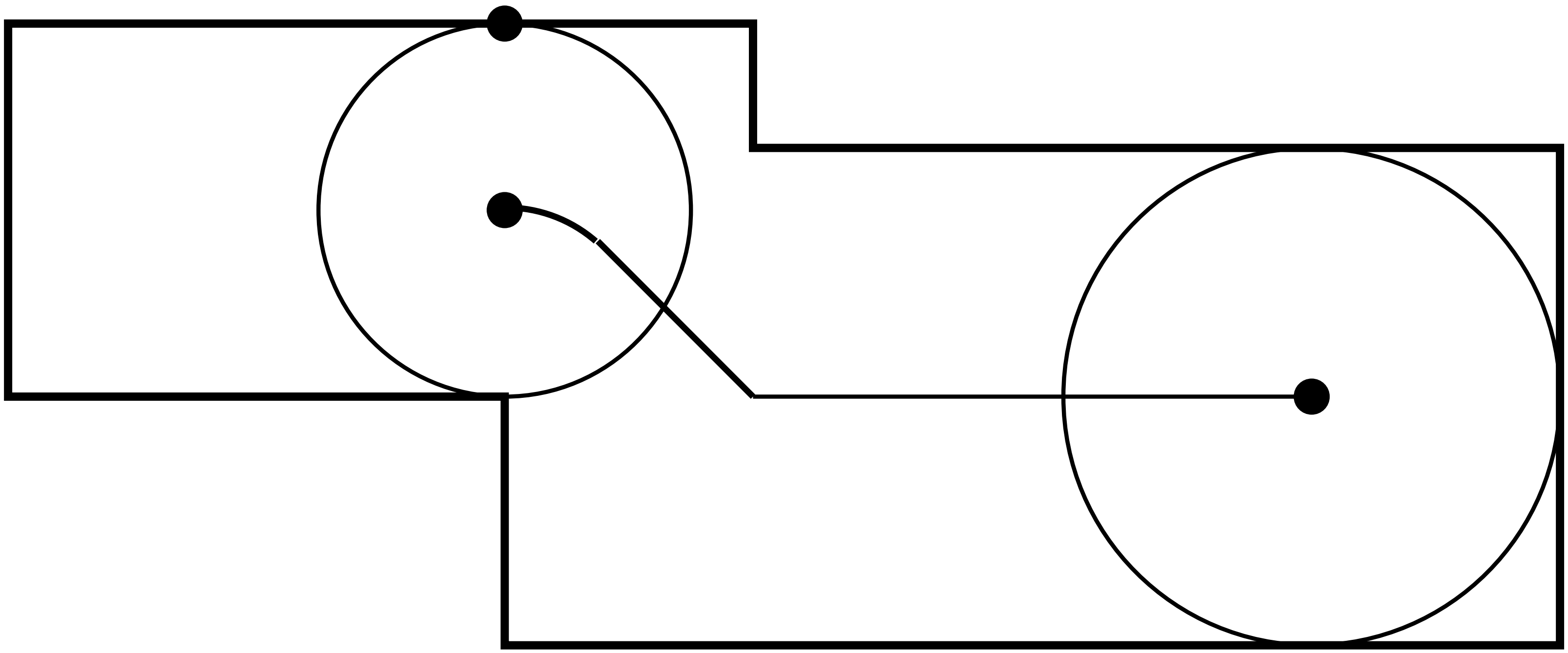
How does this give a map from polygon P to a circle C?



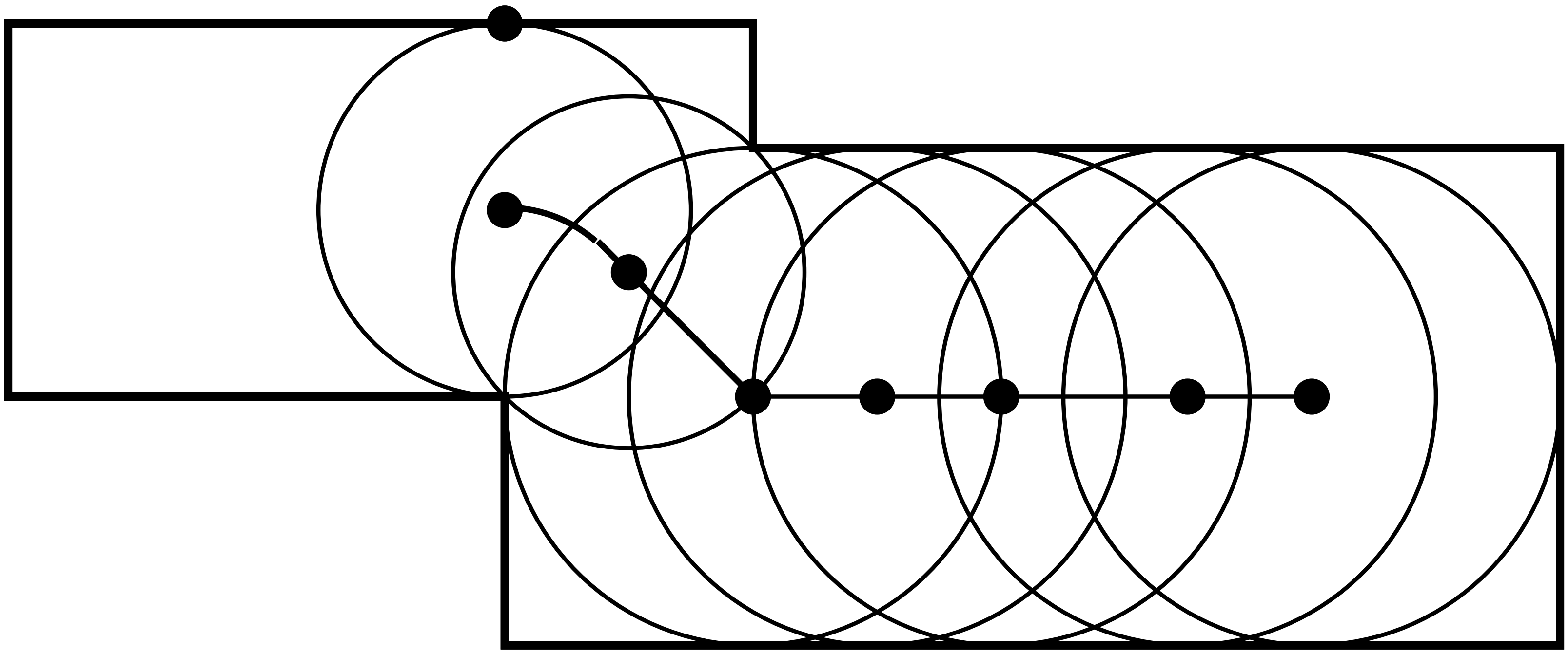
- Fix a “root” MA disk D. C is boundary of D.

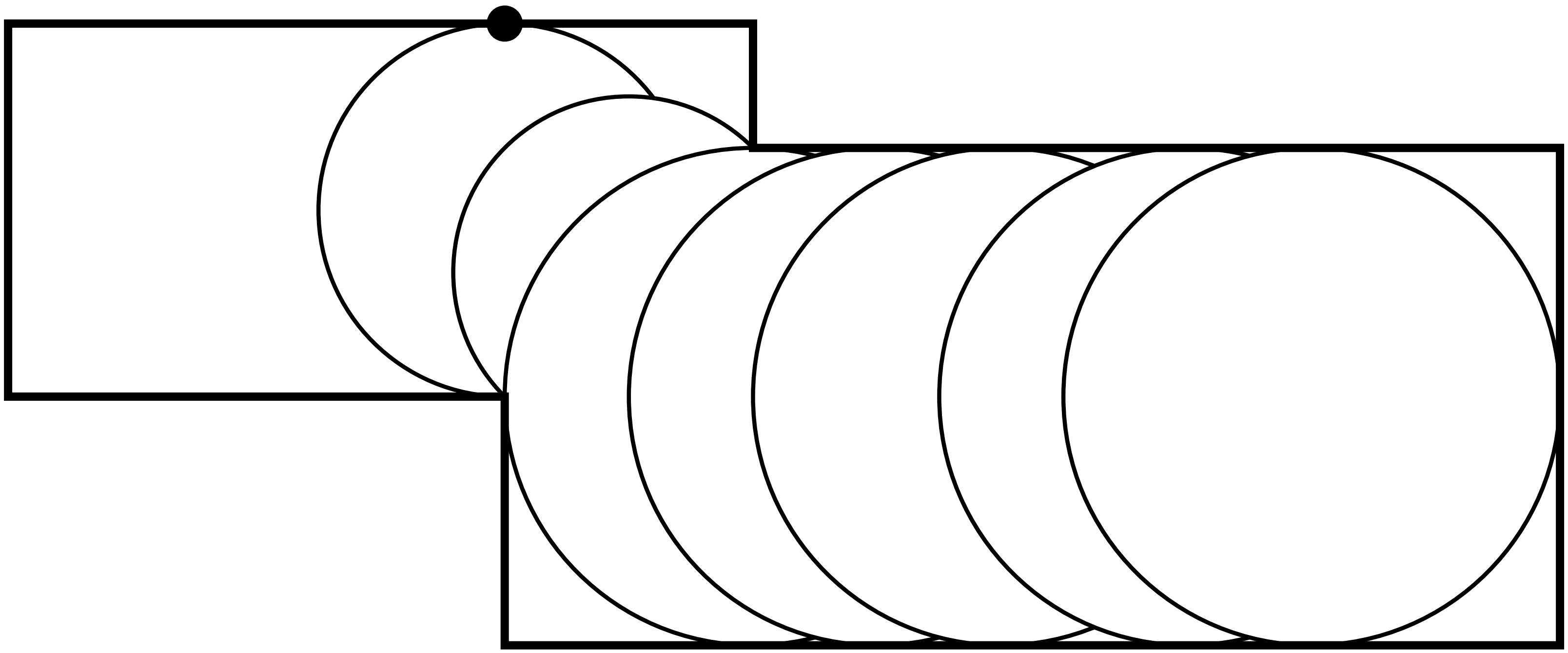


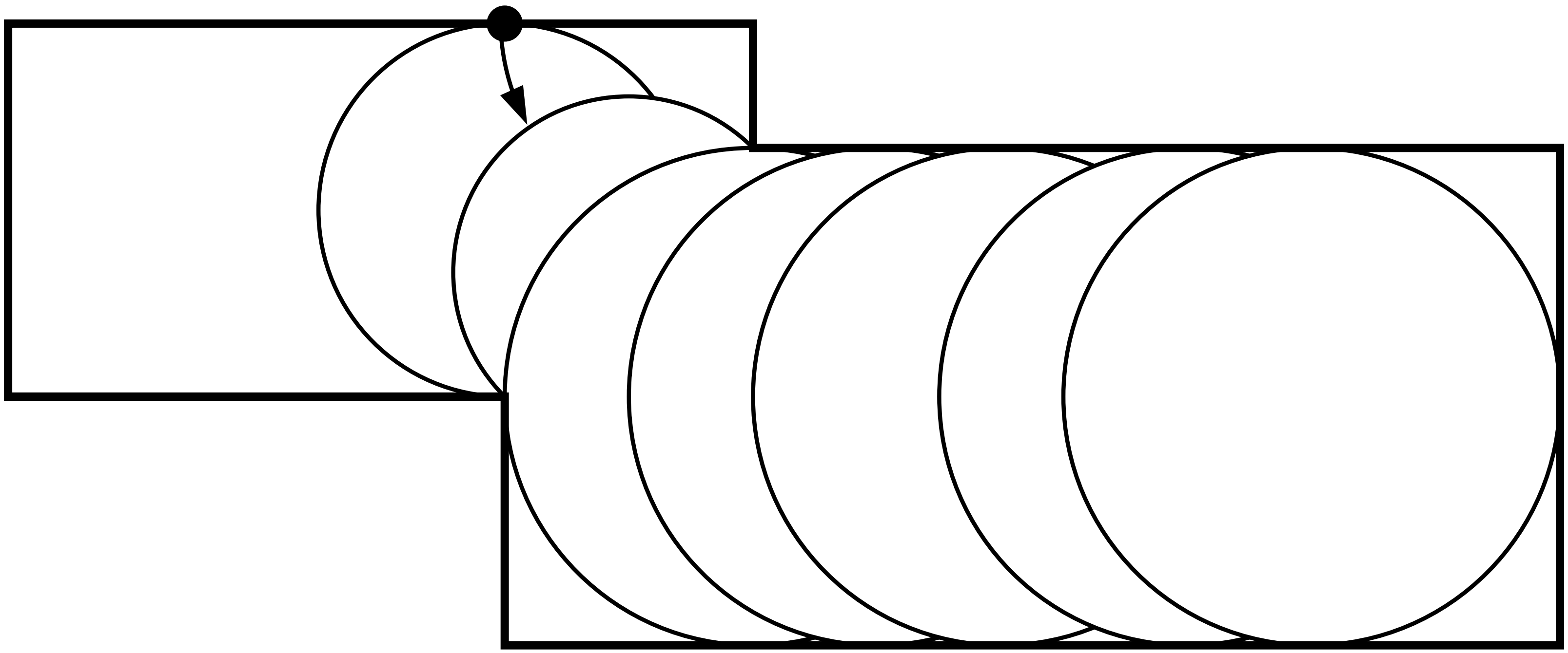
- For any $z \in P$, take MA disk D_z touching z.

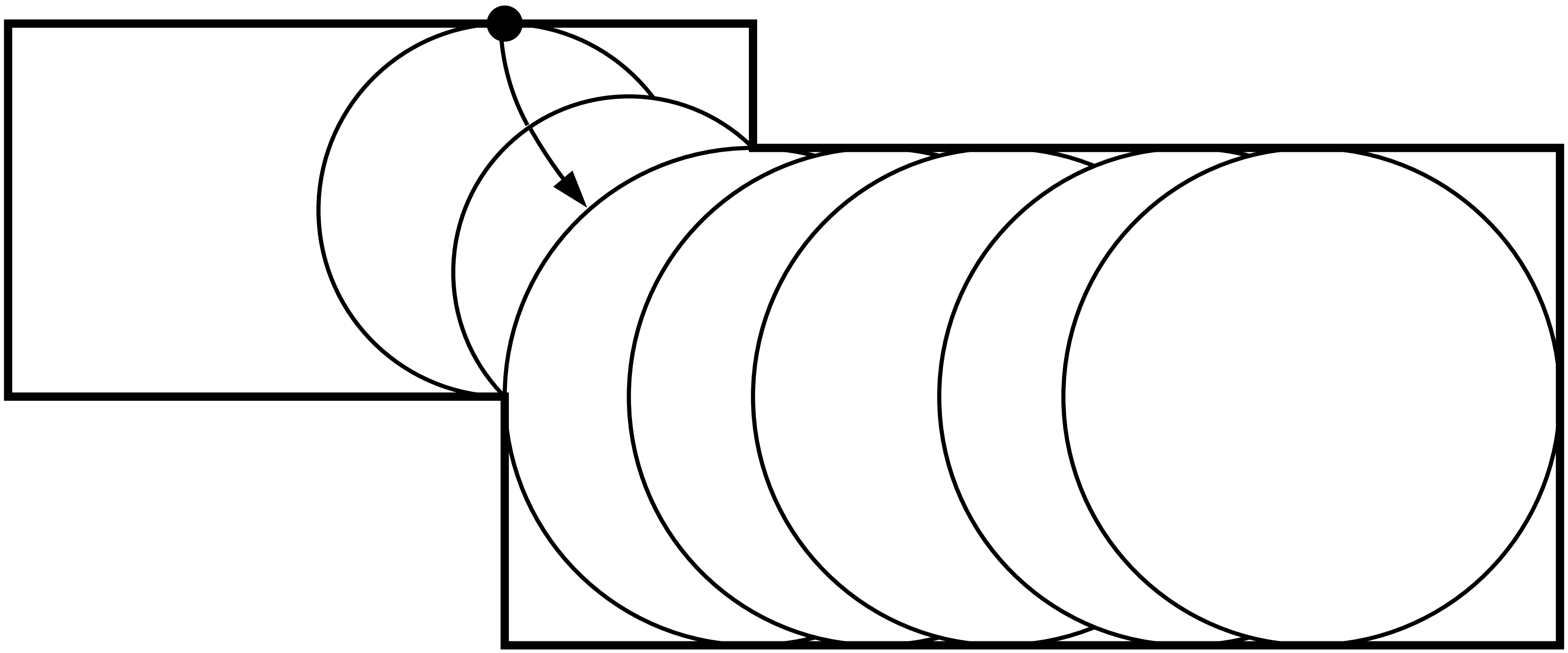


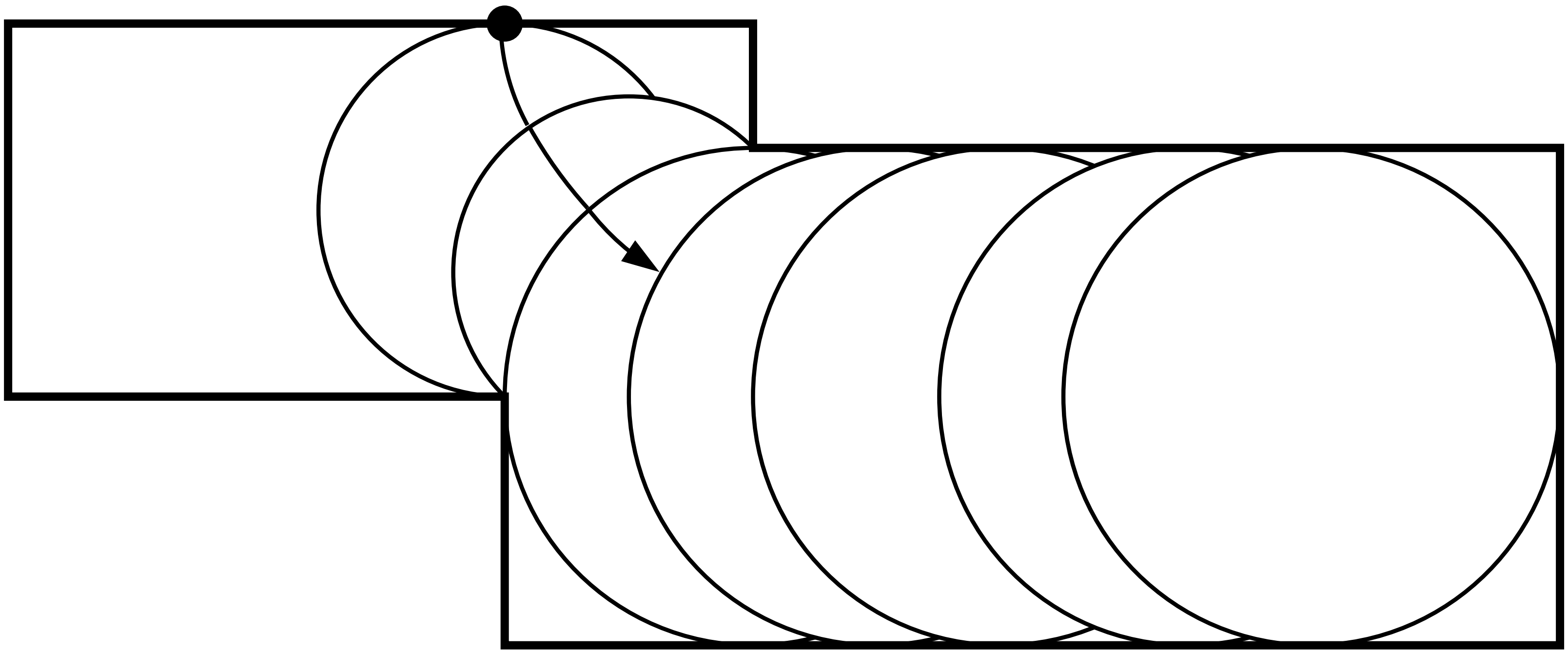
- Connect D_z to D on MA. Path map sends z into C.

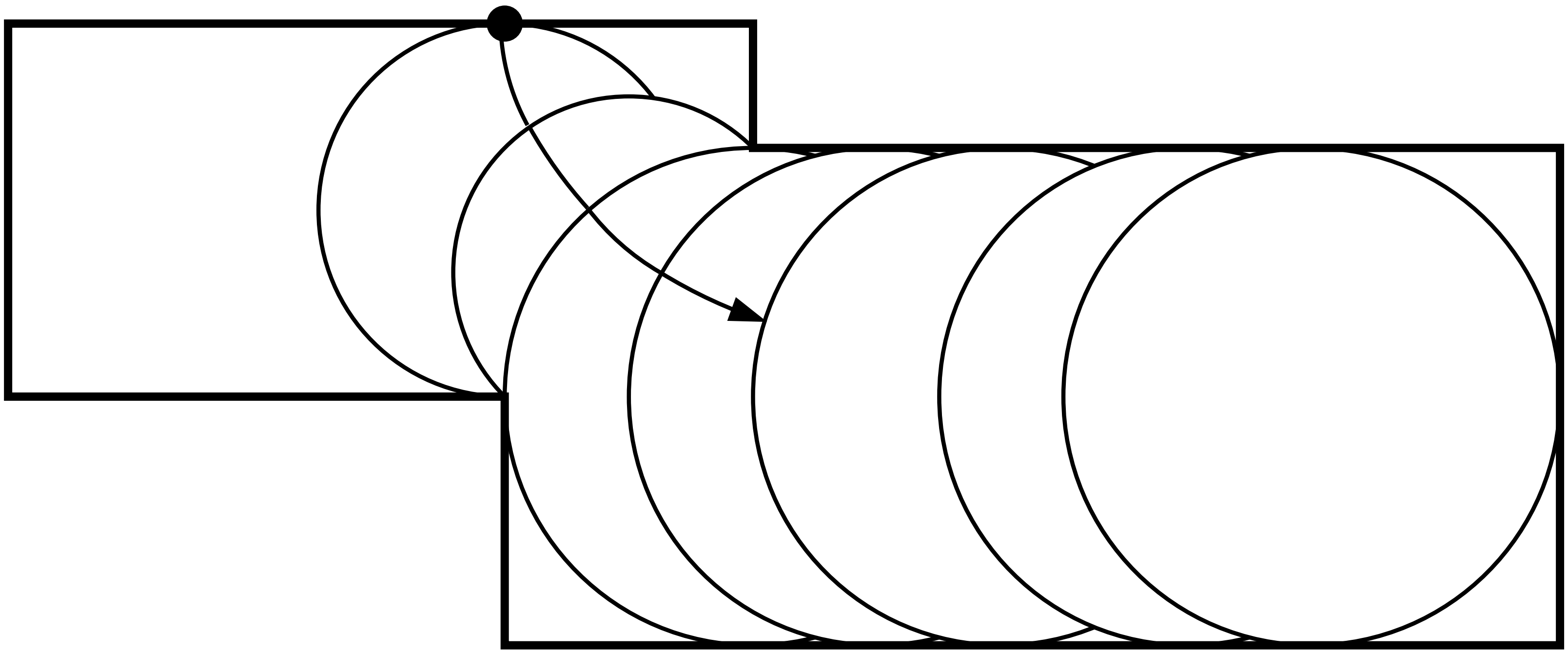


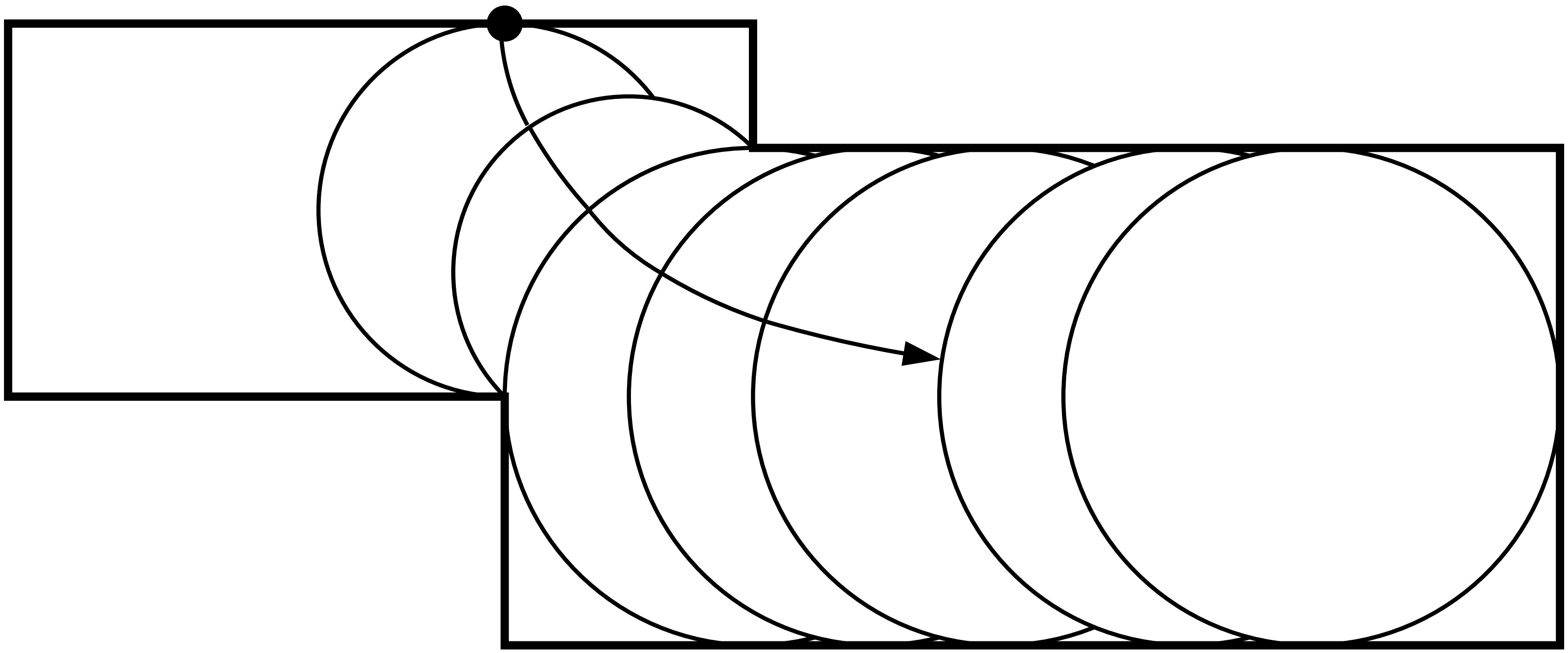


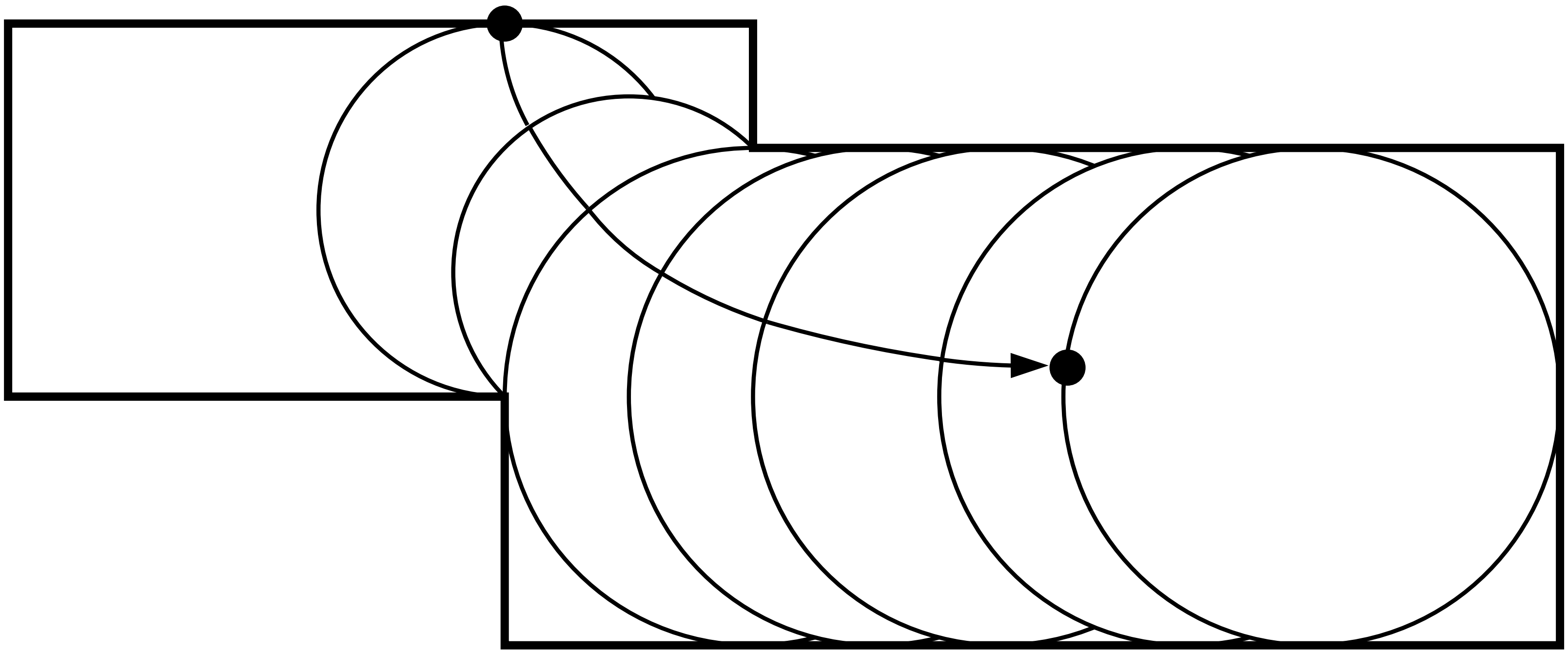


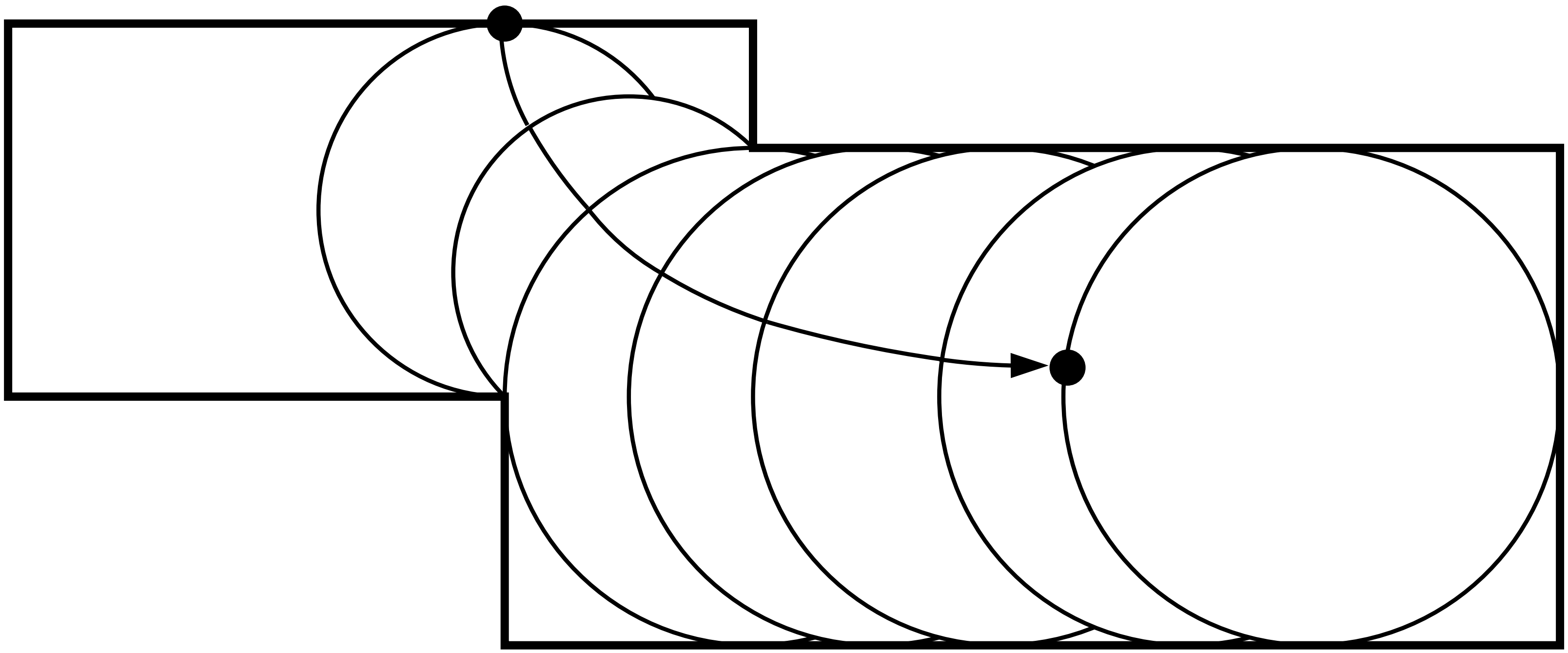






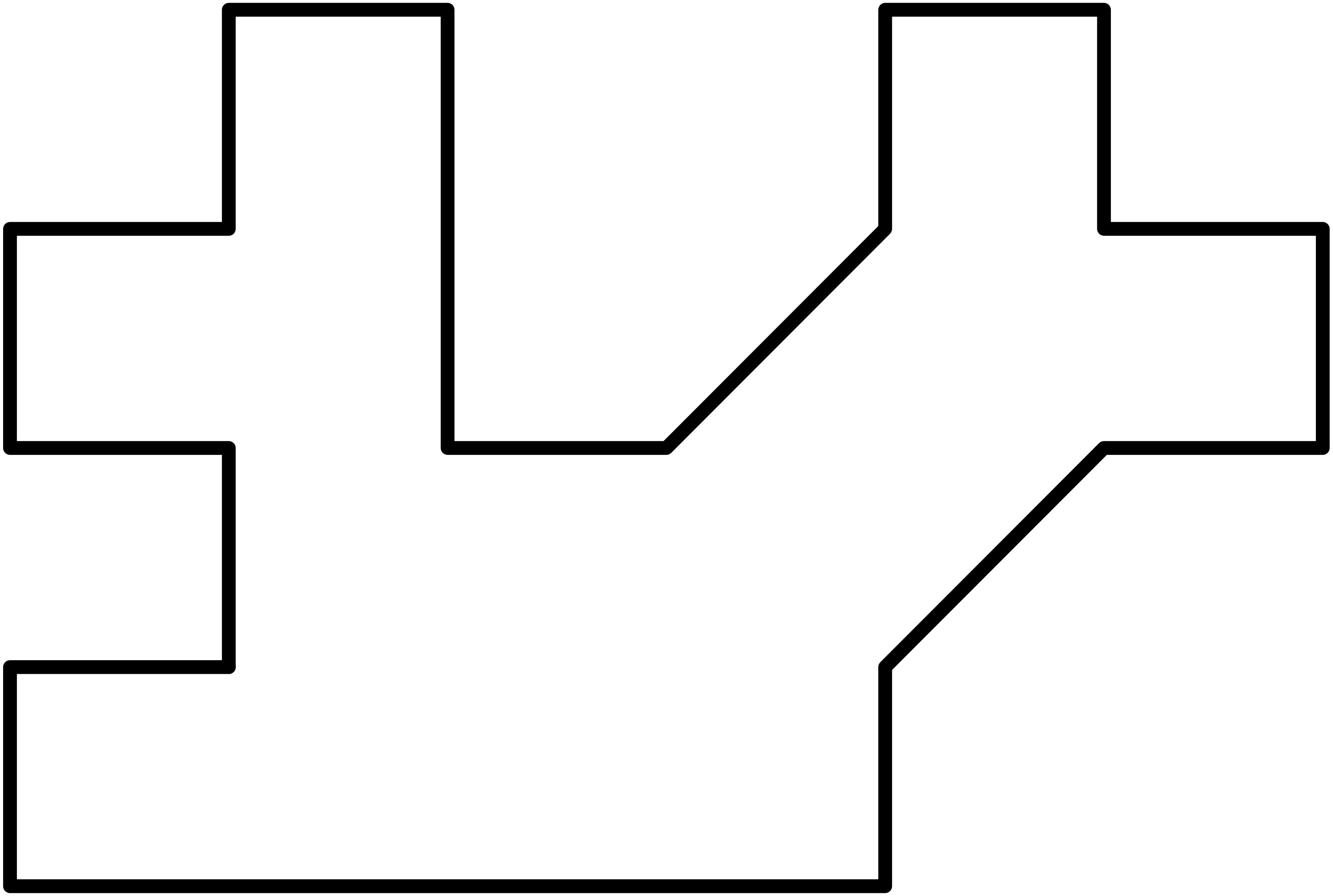


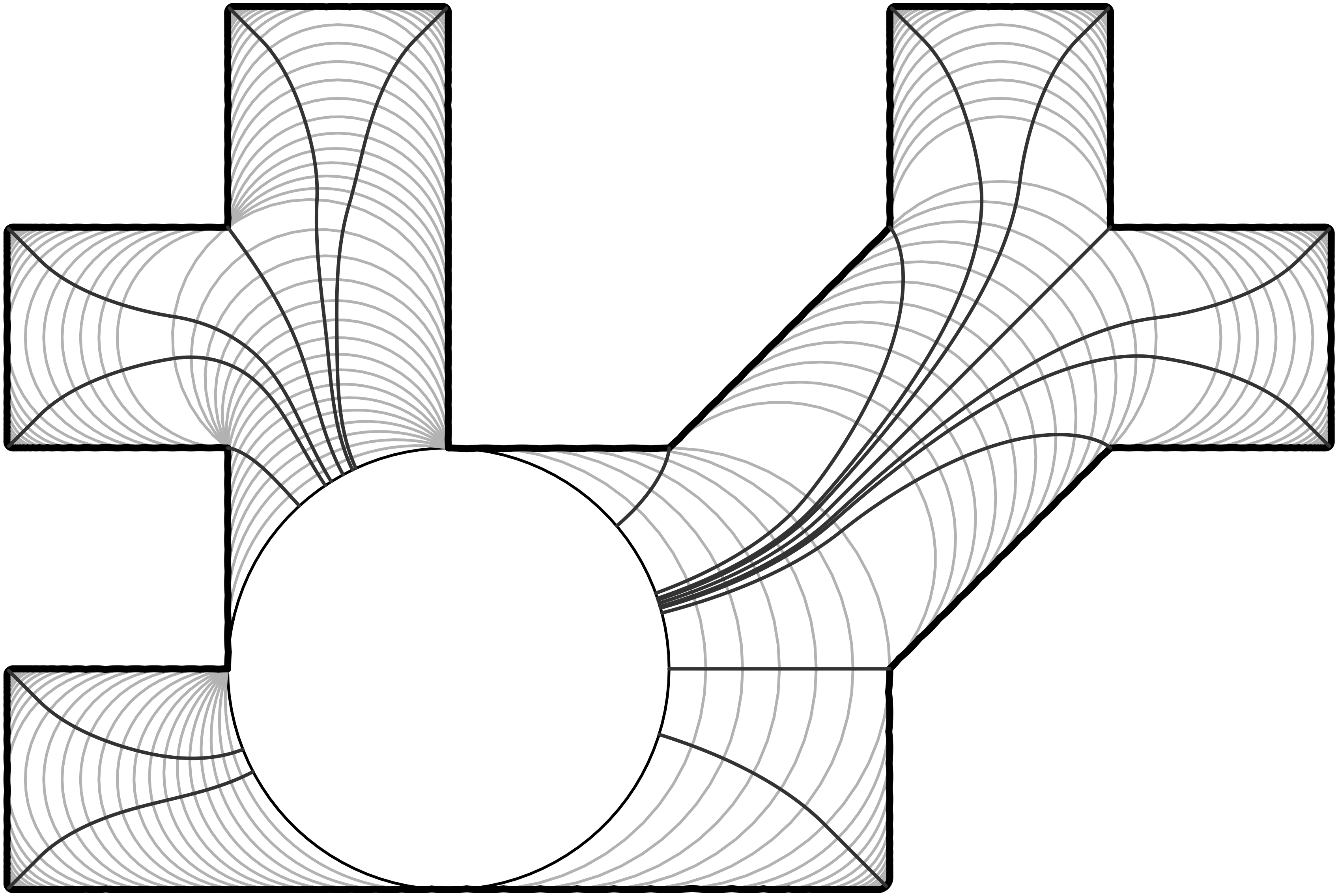


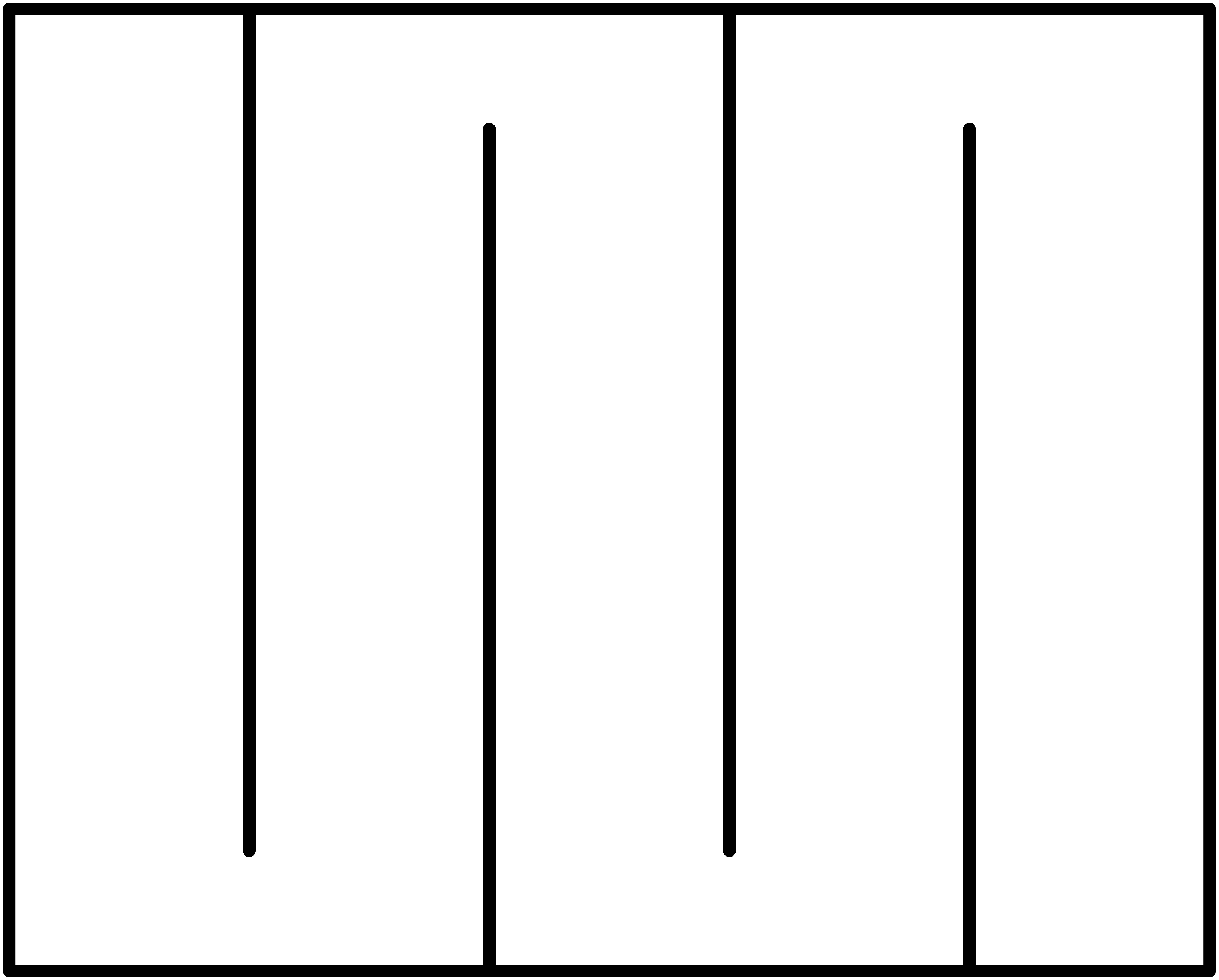


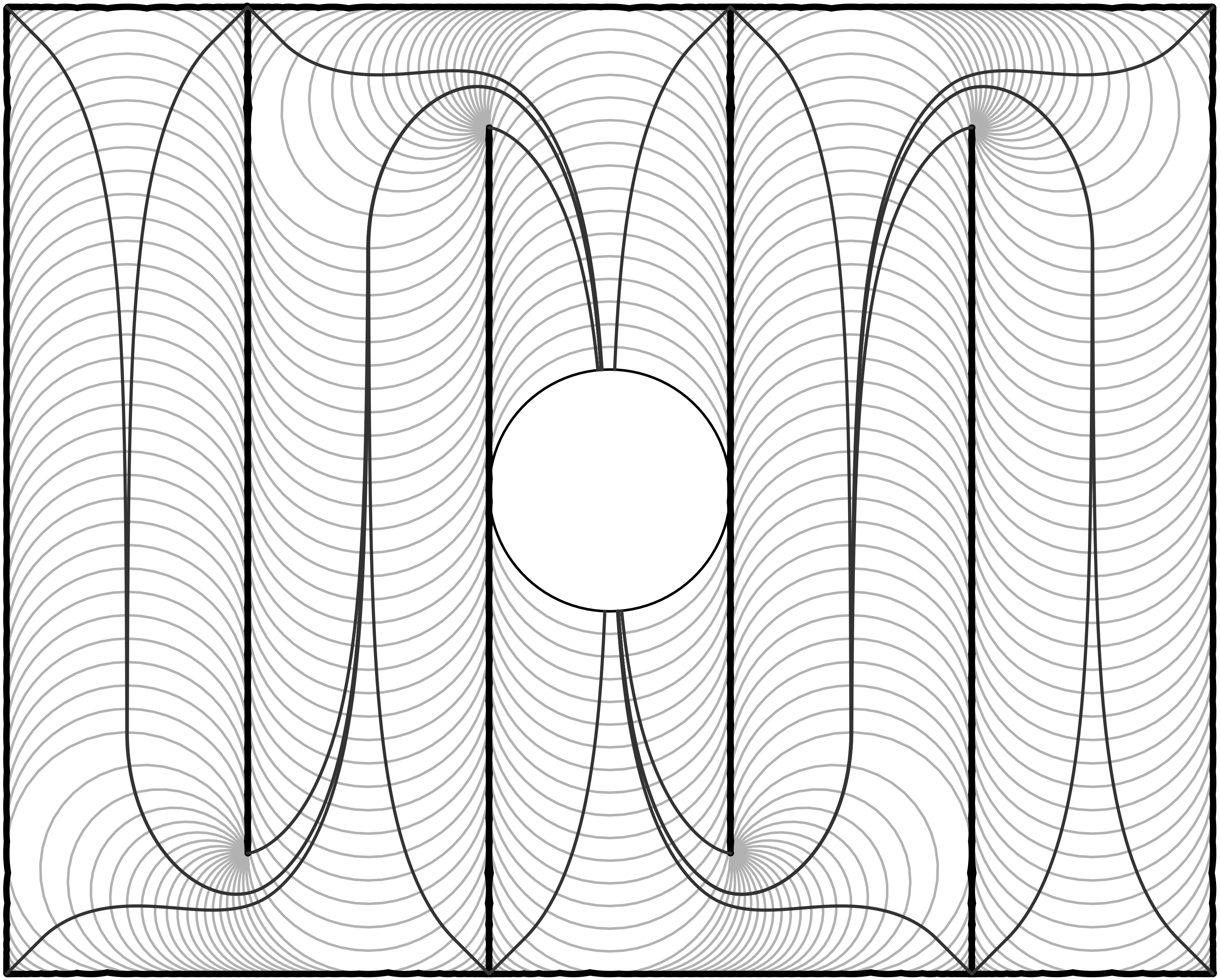
We discretize only to draw picture.

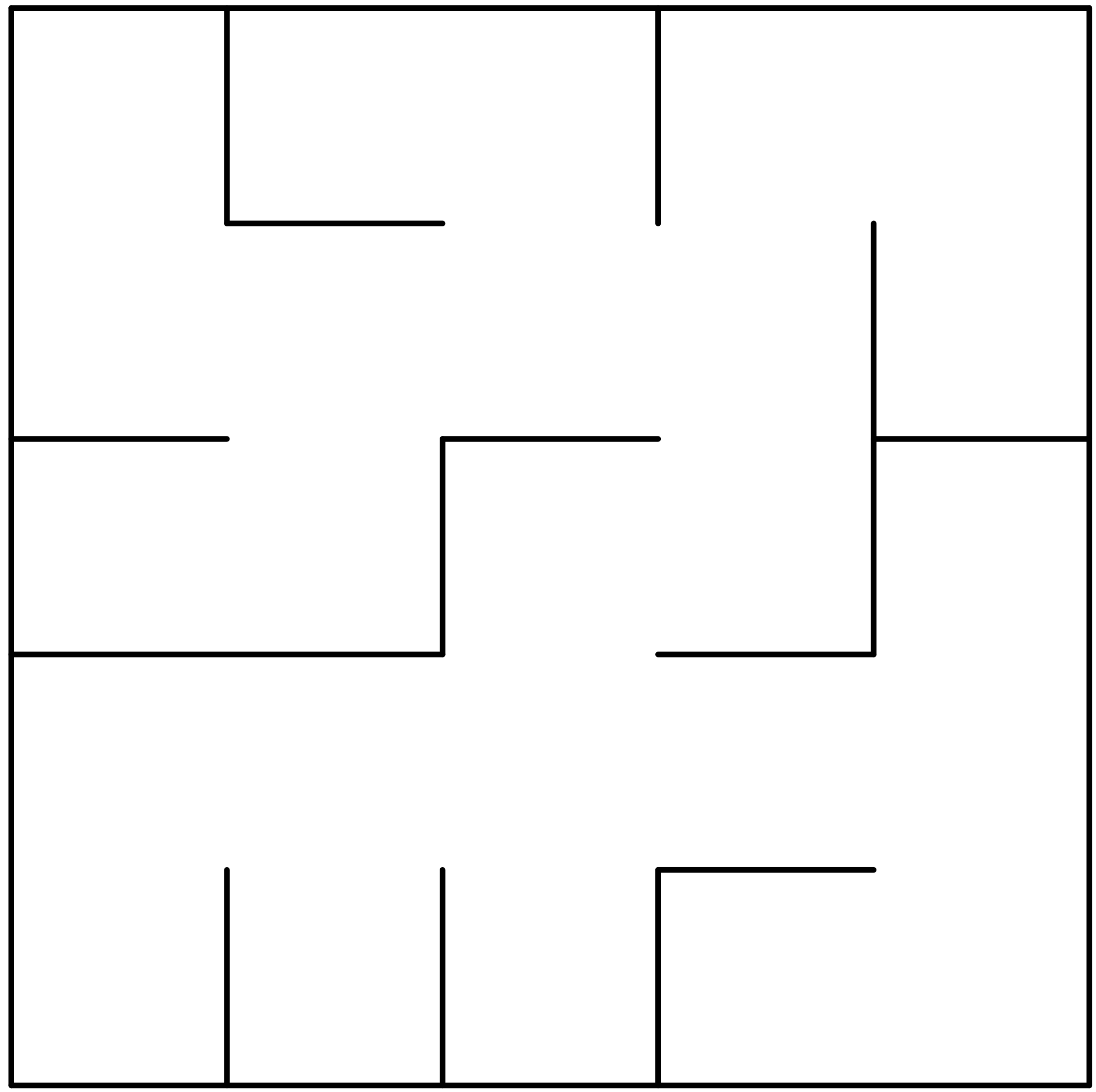
Limiting map has formula in terms of medial axis.

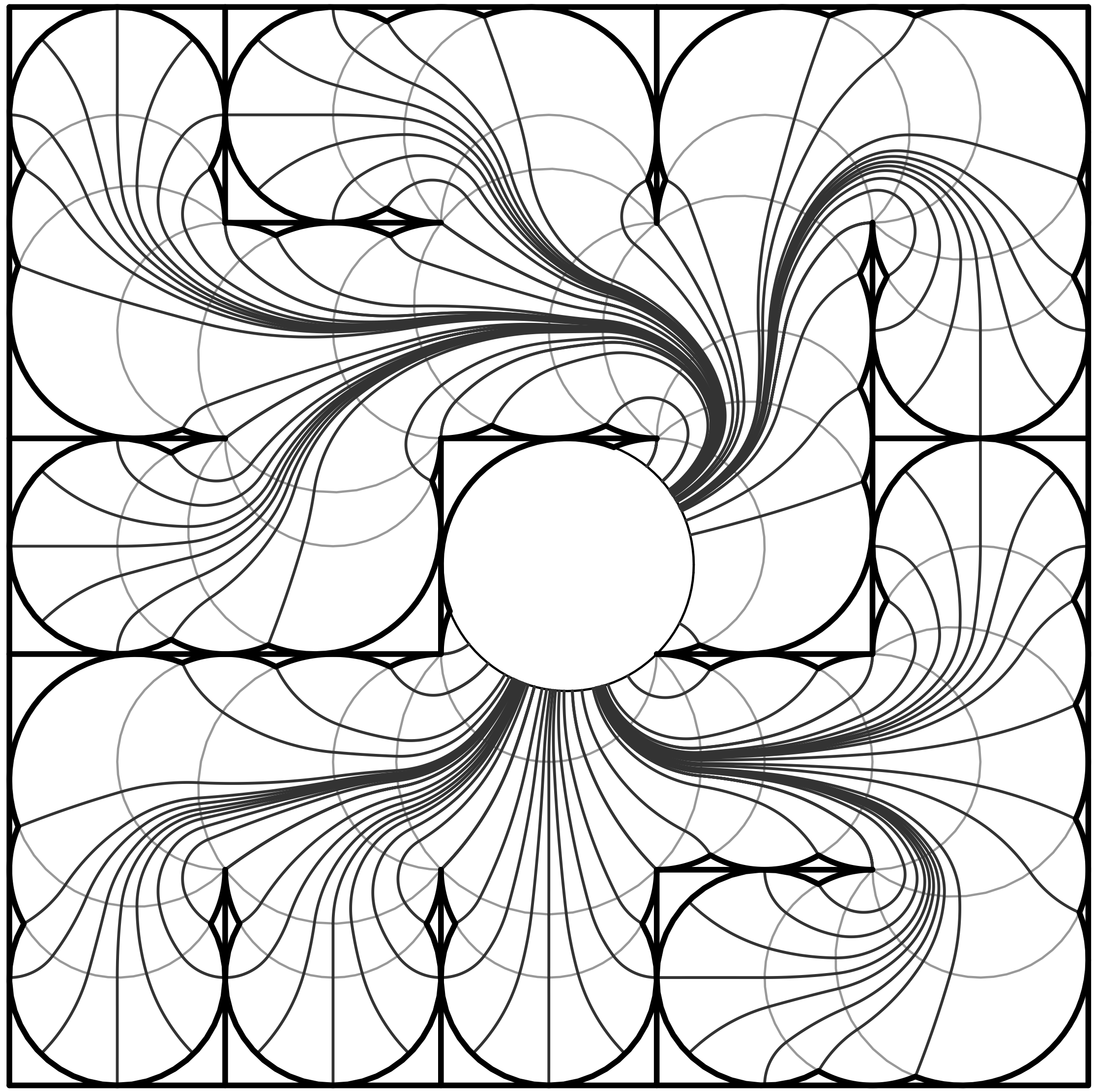






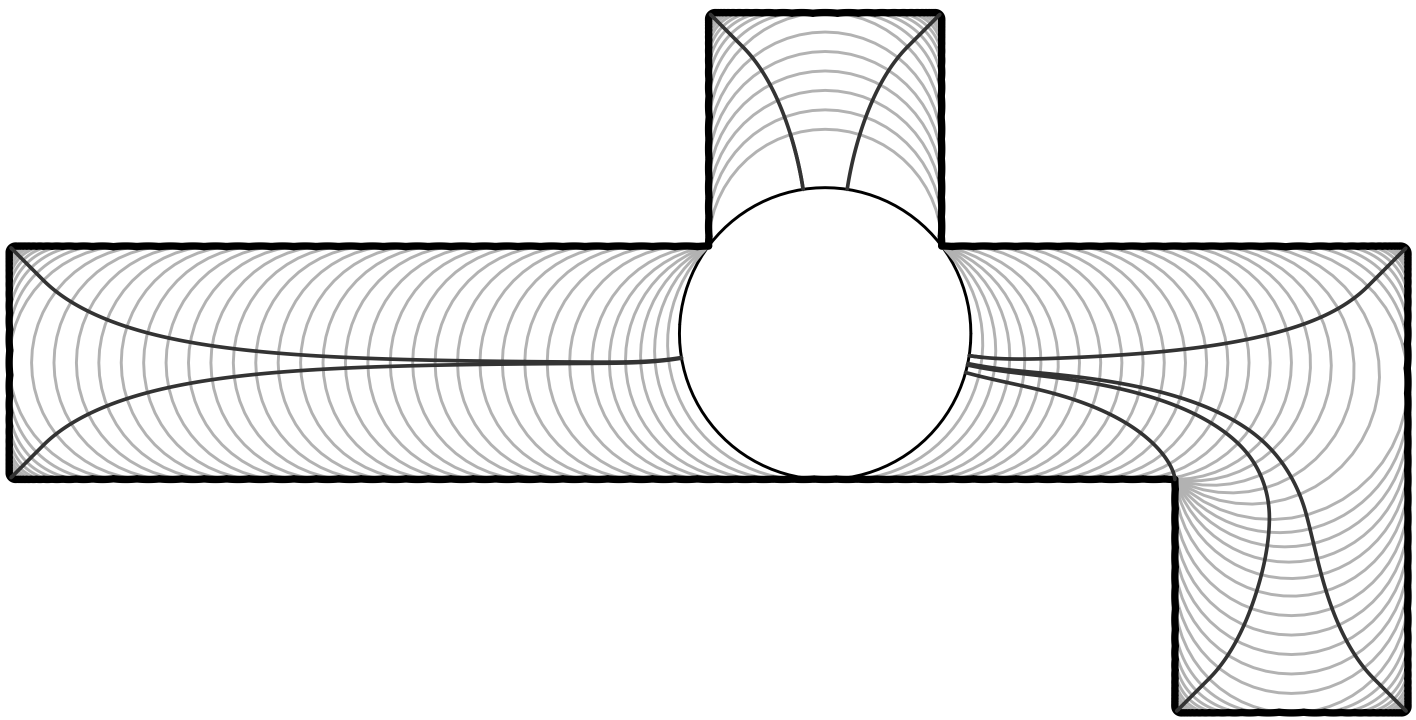






Theorem: Mapping all n vertices takes $O(n)$ time.

Uses linear time computation of MA (Chin-Snoeyink-Wang) and book-keeping with cross ratios.



How close is medial axis map to conformal map?

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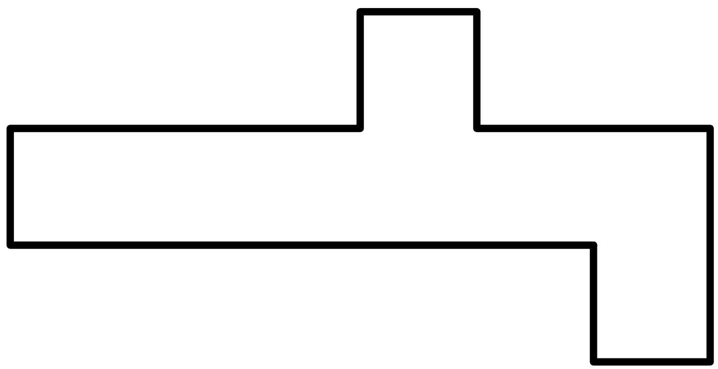
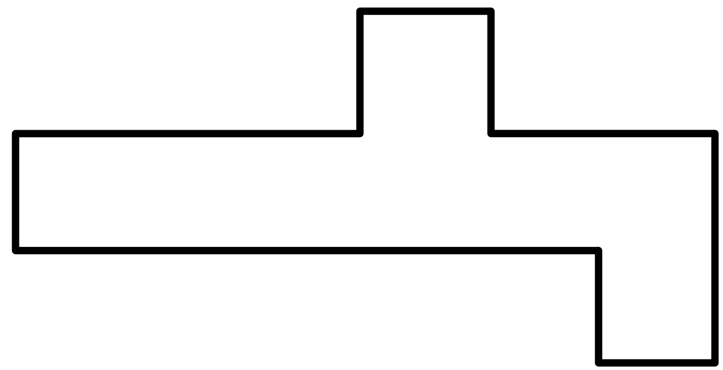
Use "MA-parameters" in Schwarz-Christoffel formula.



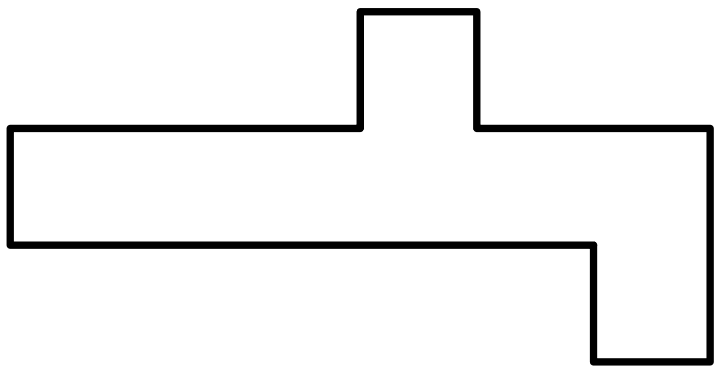
Target Polygon



MA Parameters

How close is medial axis map to conformal map?

Use “MA-parameters” in Schwarz-Christoffel formula.



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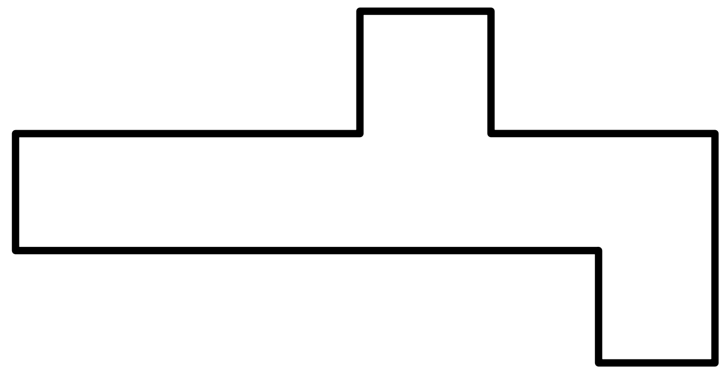


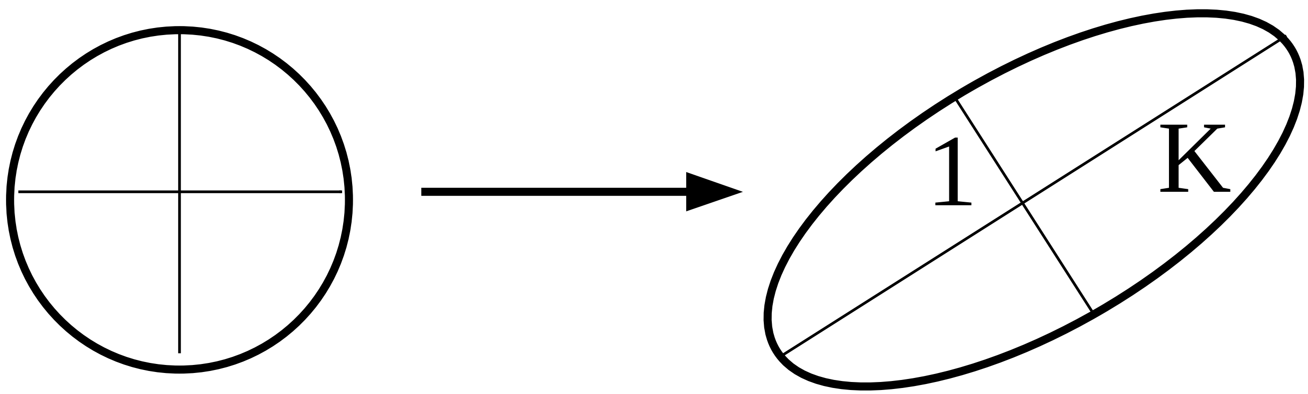
MA Parameters

Looks pretty close. How do we measure error?

Quasiconformal maps: bounded angle distortion

Roughly, a K-QC map multiplies angles by $\leq K$.

More precisely: tangent map sends circles to ellipses a.e.

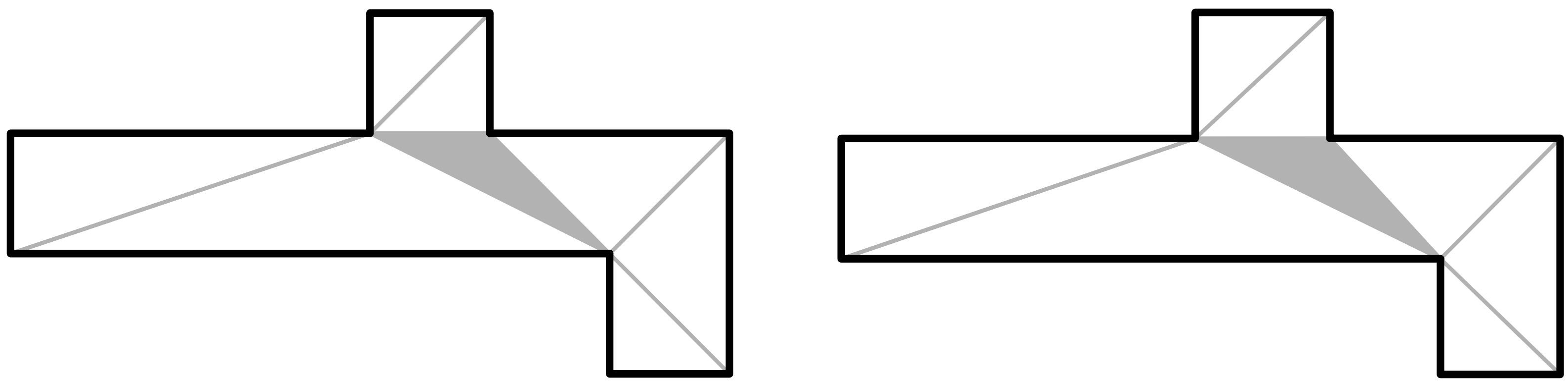


1-QC = conformal

QC-distance between n-gons = minimal K for map between polygons sending vertices to vertices.

Upper bound for QC-distance:

- Find compatible triangulation
- Compute affine map between triangles
- Take largest K.



The most distorted triangle is shaded. Here $K = 1.24$.

Theorem: Medial axis map is always K-QC, $K < 8$.

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MA-map extends to QC map of polygon interior to disk.

Why is this theorem true?

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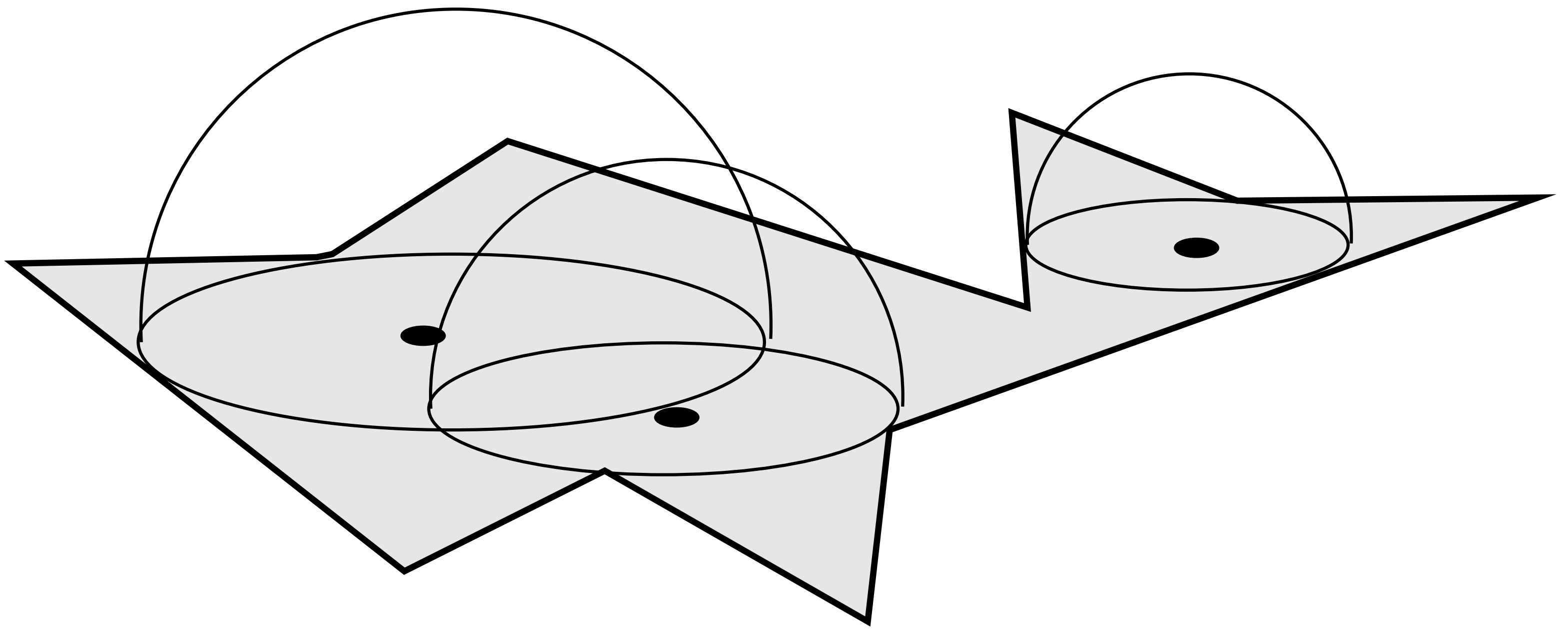
MA-map extends to QC map of polygon interior to disk.

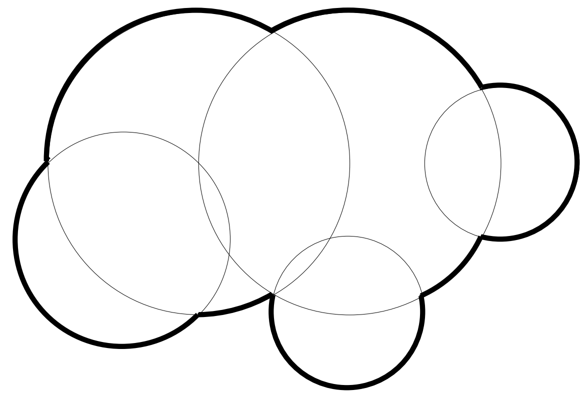
Why is this theorem true?

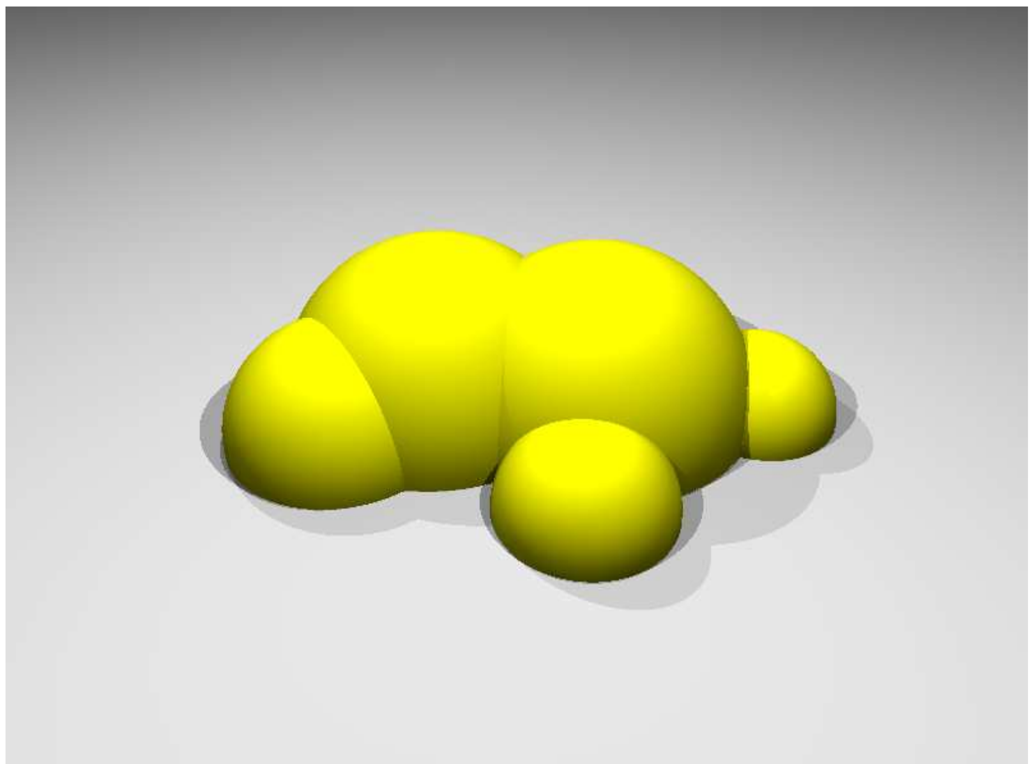
Because MA-map has conformal extension to different region with same boundary.

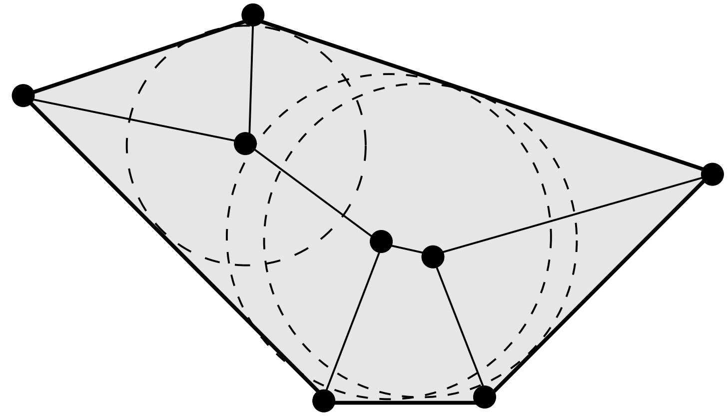
Other region is "bounded distortion" of the polygon.

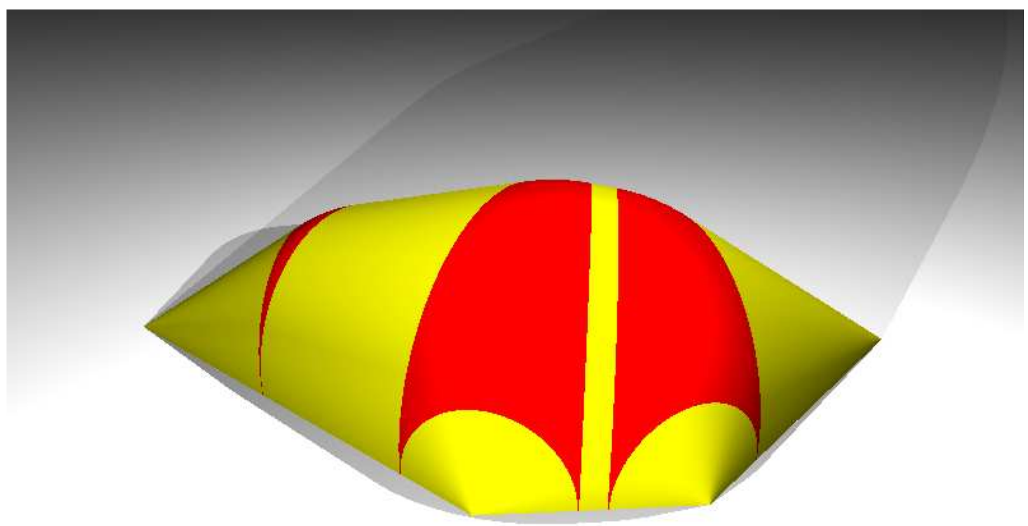
The **dome** of a domain is upper envelope of all hemispheres where base disk is a medial axis disk of Ω.

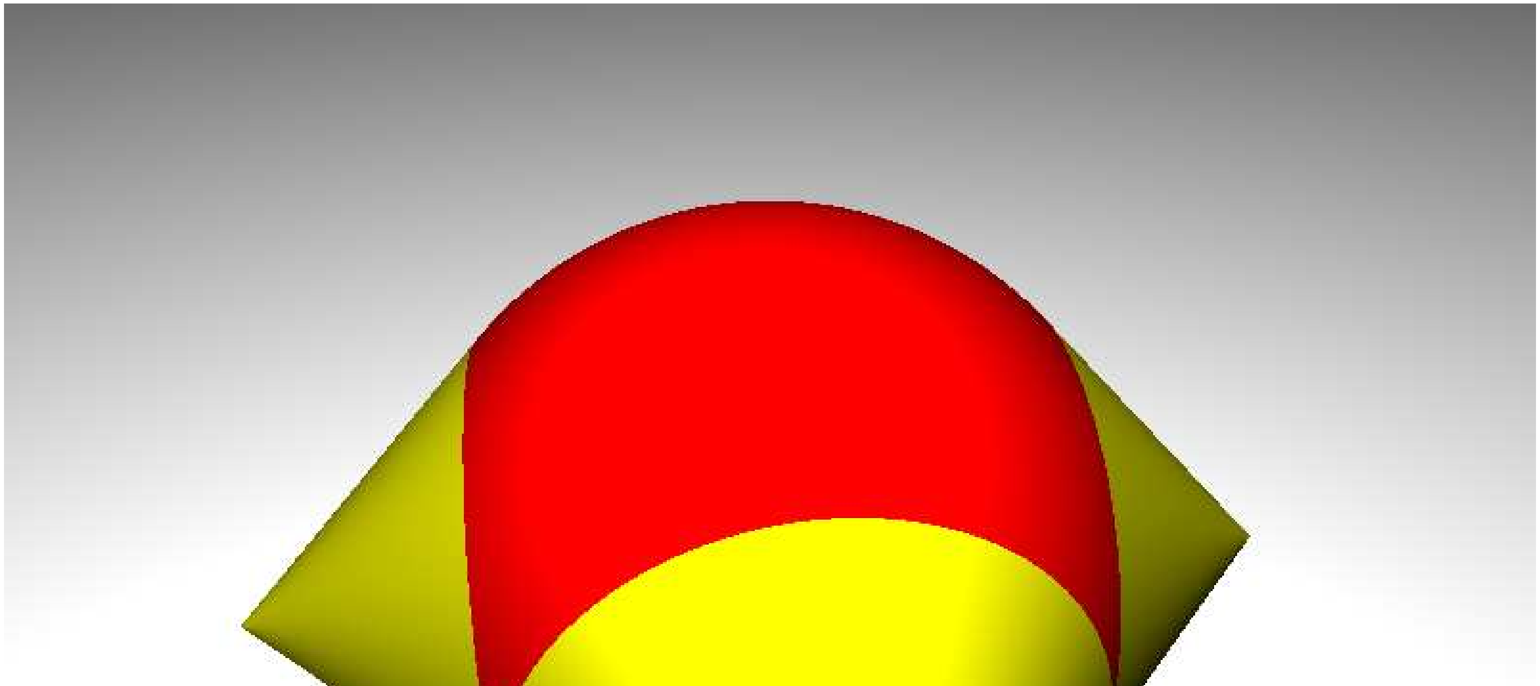


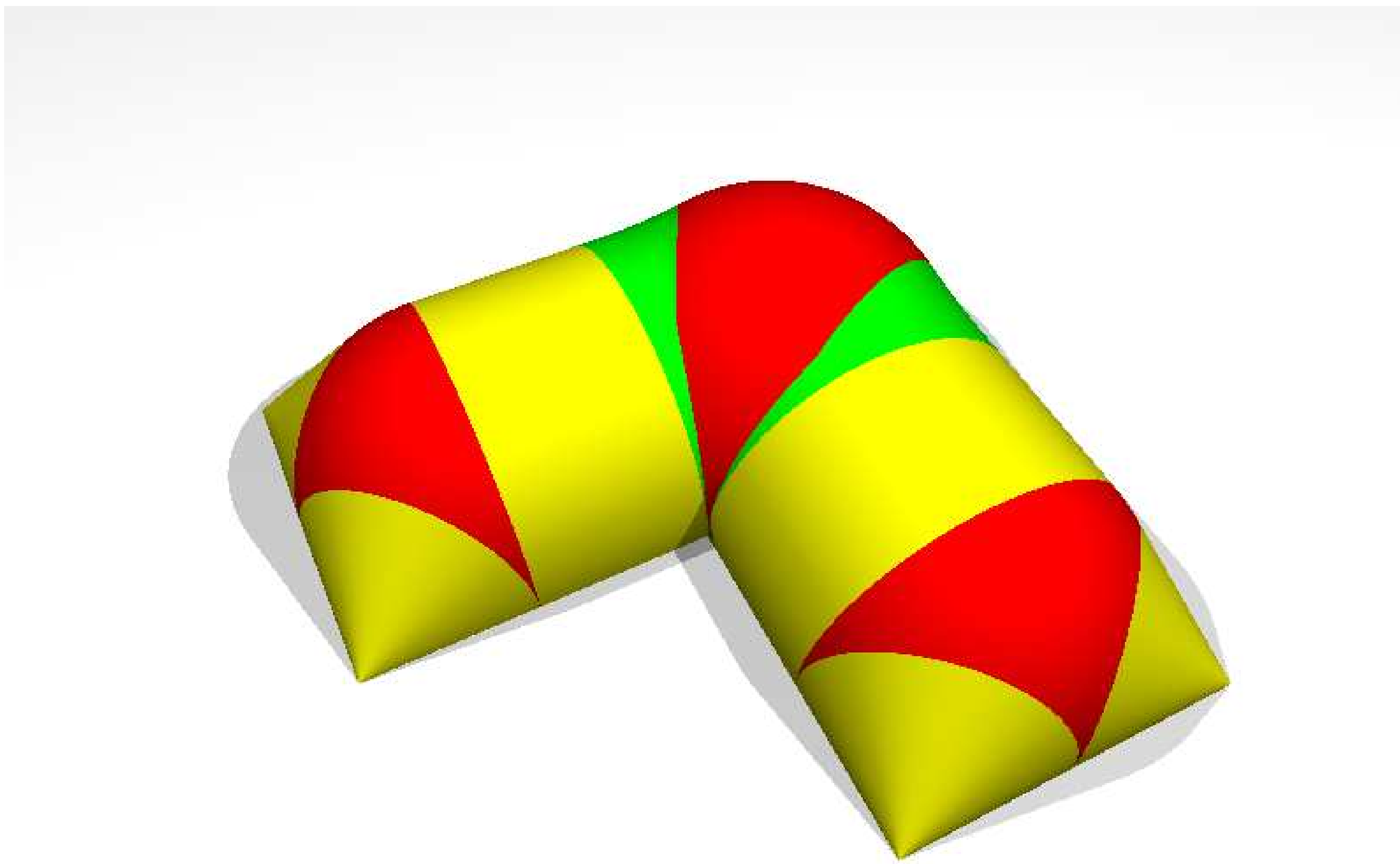


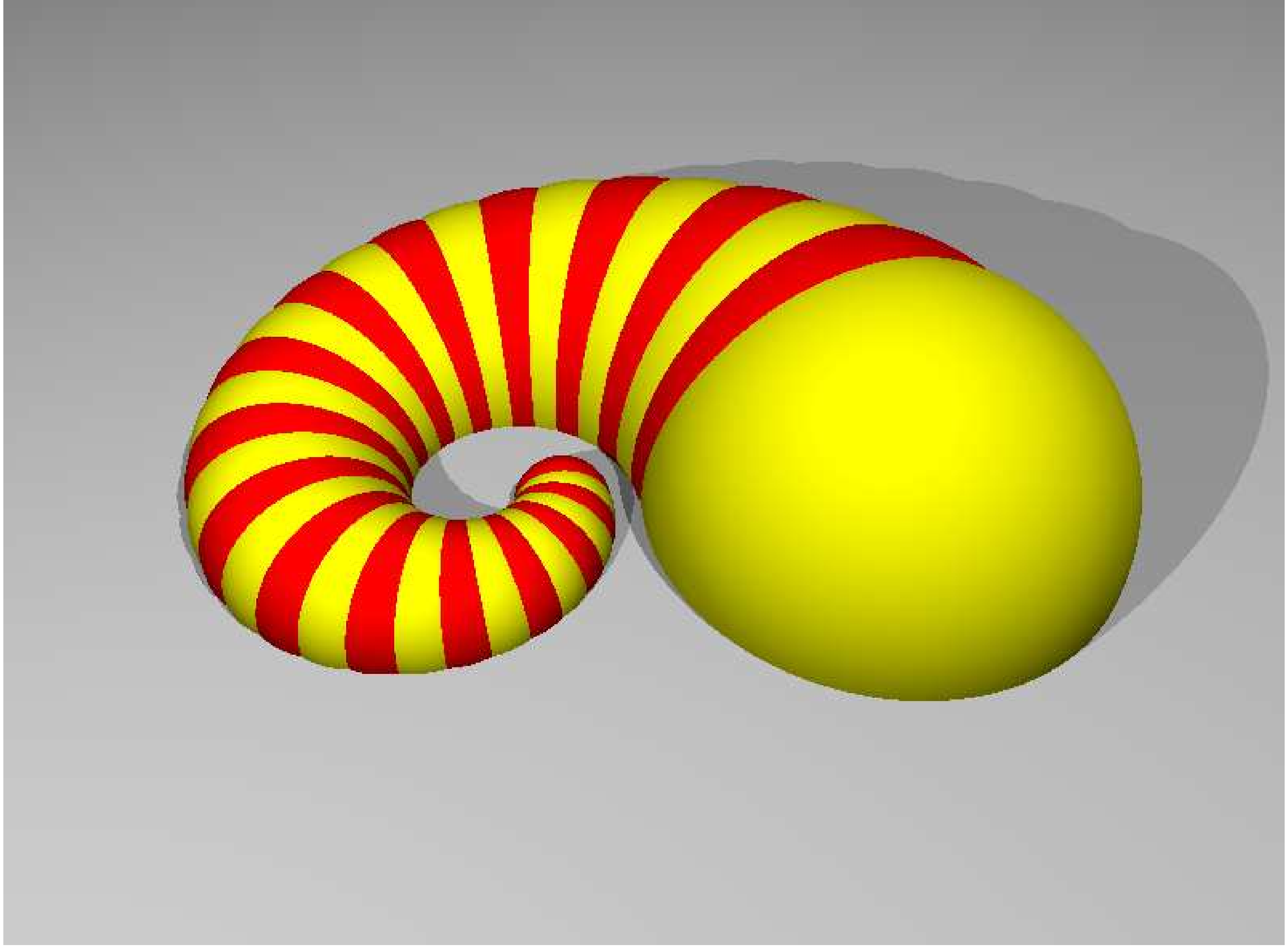




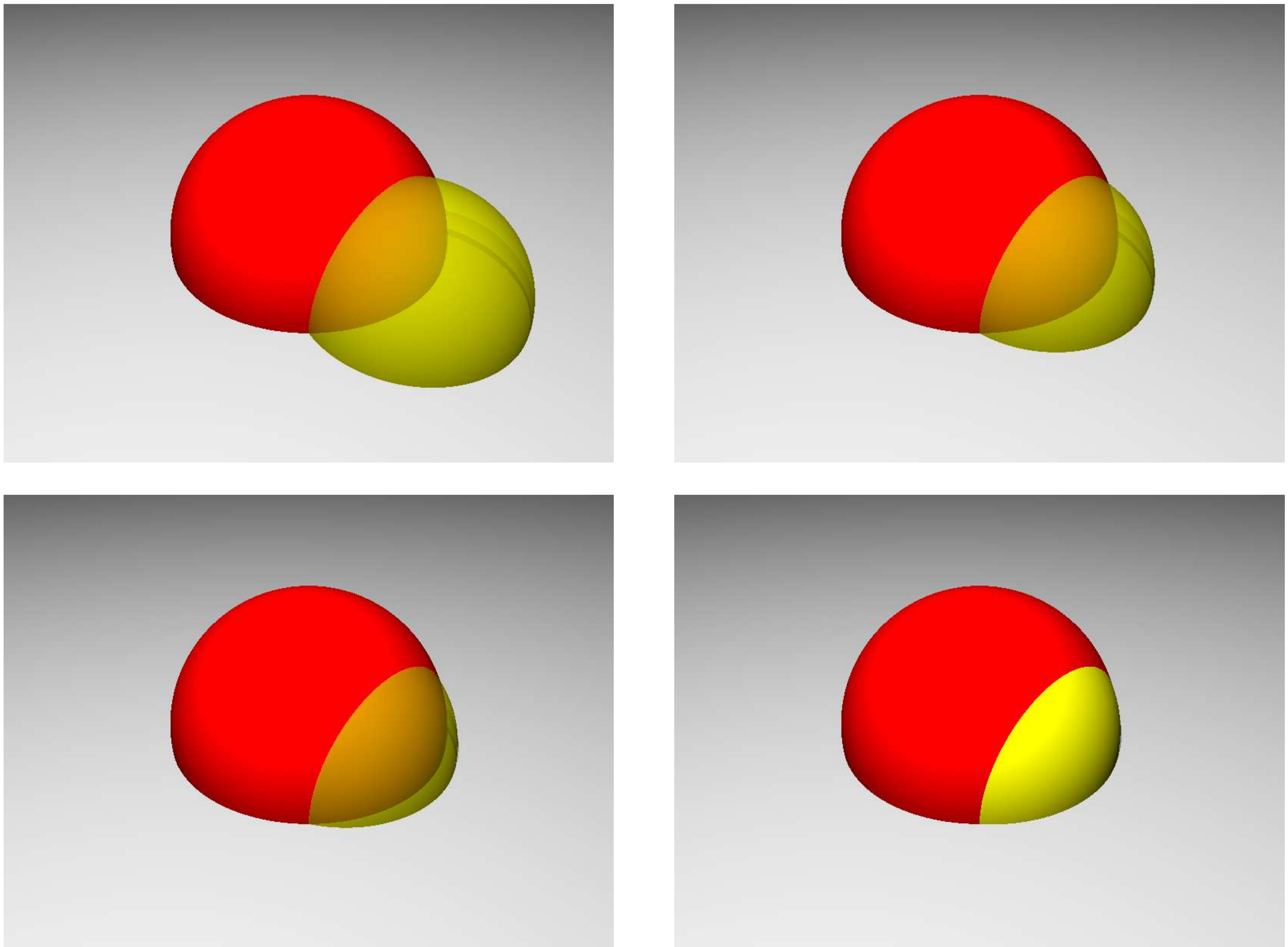


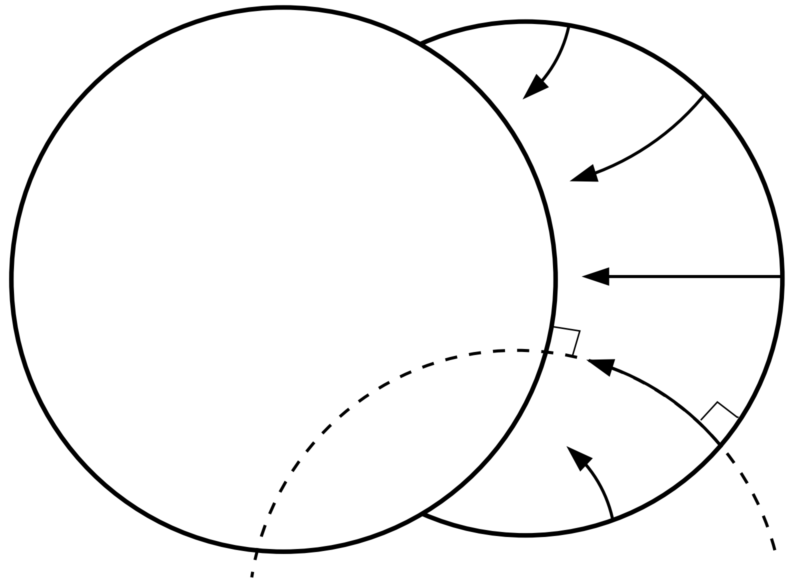






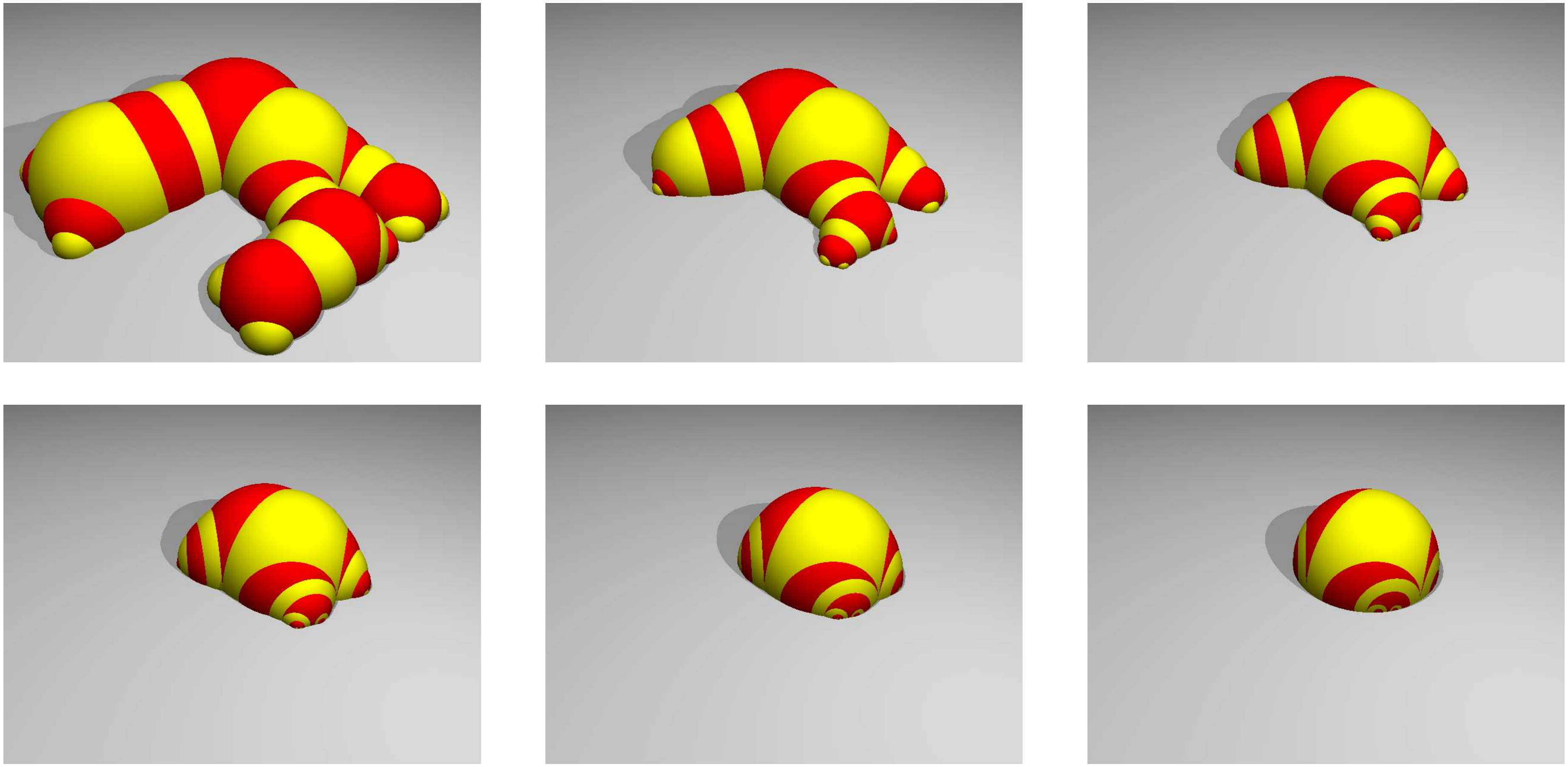
Every dome has conformal map to disk by "flattening".



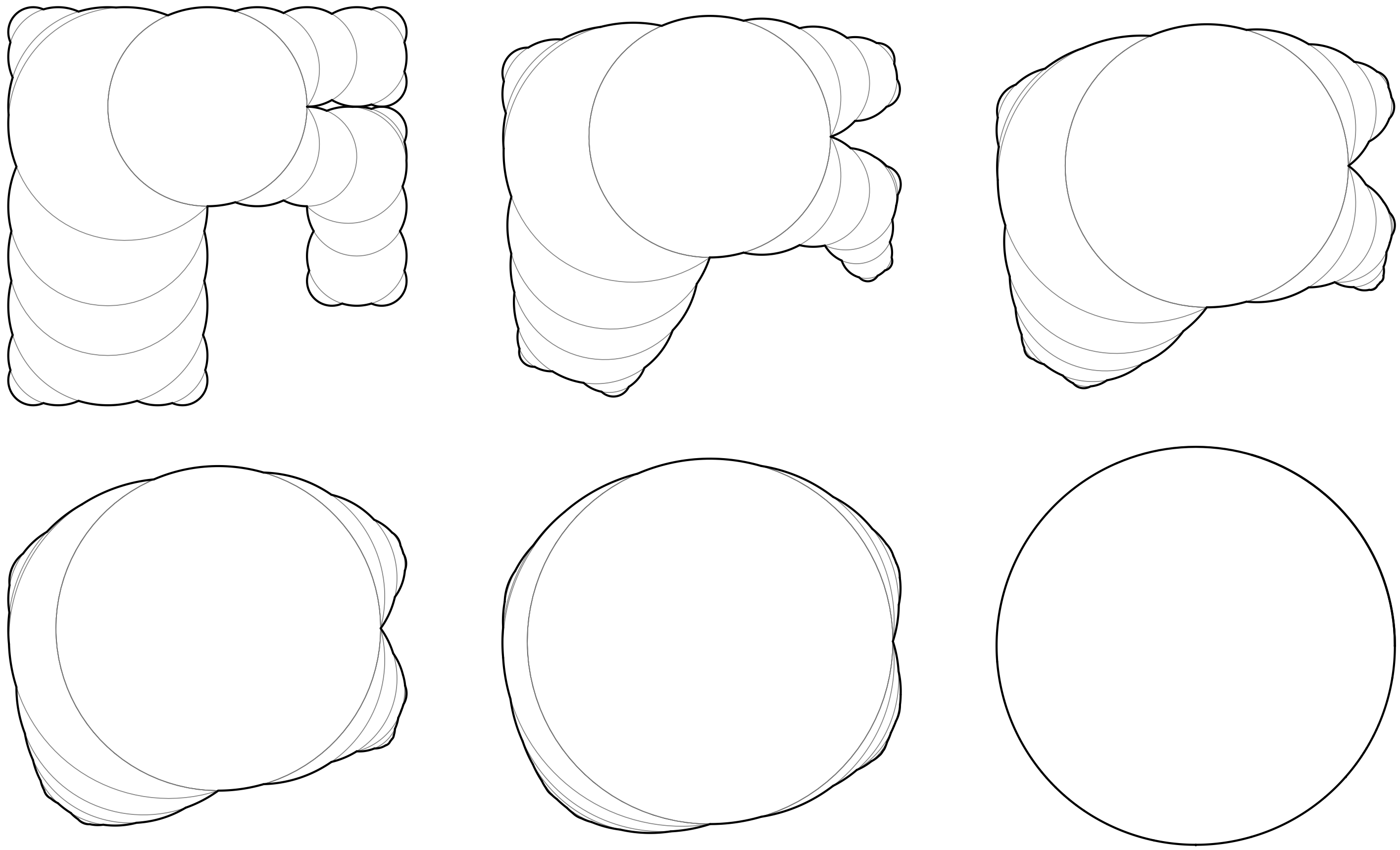


Medial axis map = boundary of flattening map (iota)

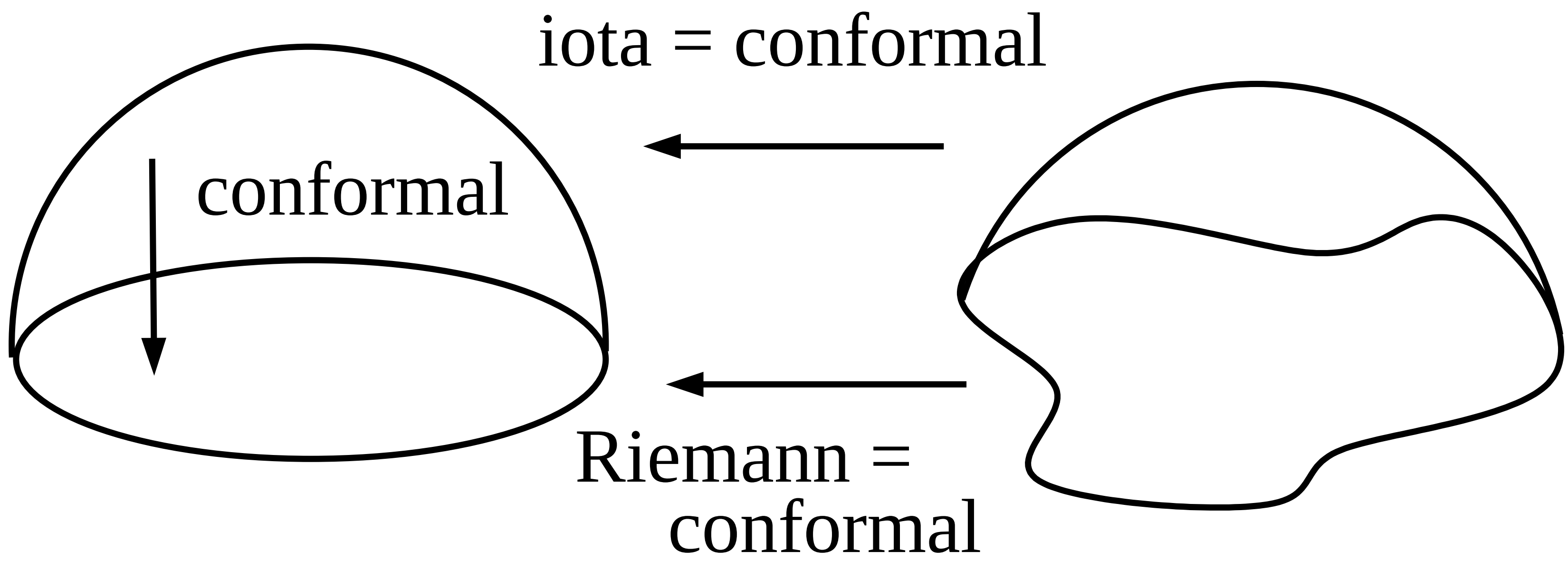
= boundary of conformal map of dome to hemisphere



Map dome to hemisphere by makings sides flush.



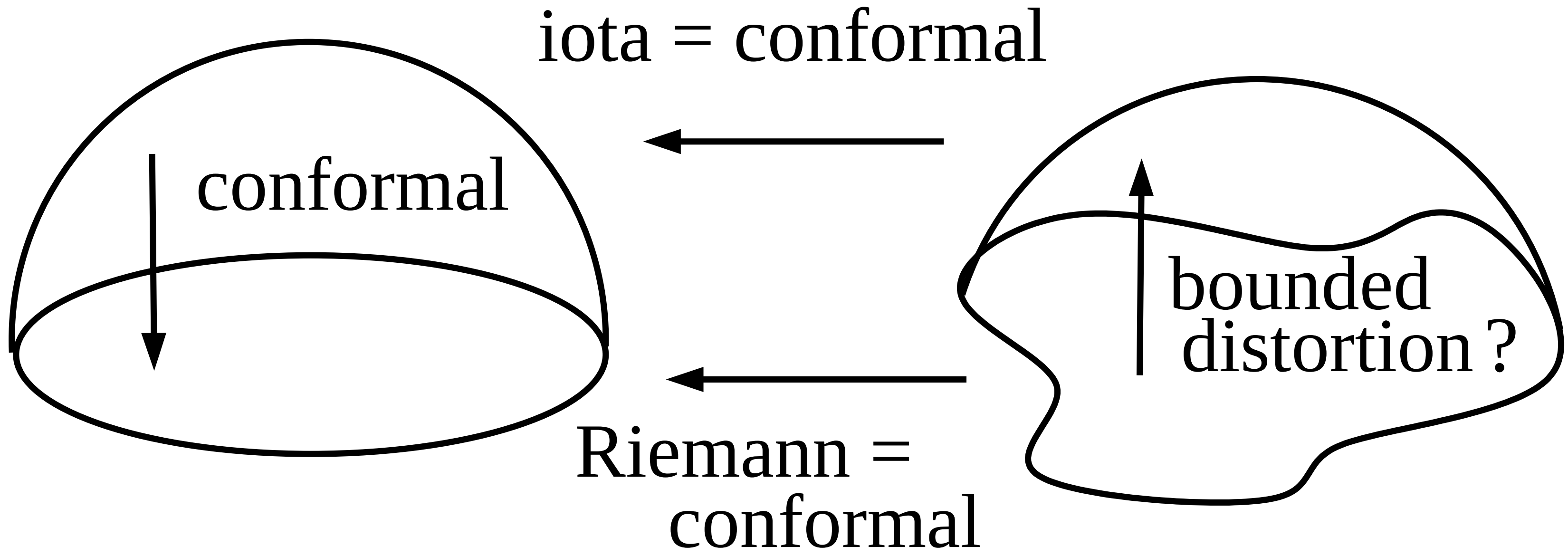
Angle scaling family: flatten bends simultaneously



Iota = conformal from dome to disk.

Medial axis flow = boundary values of iota

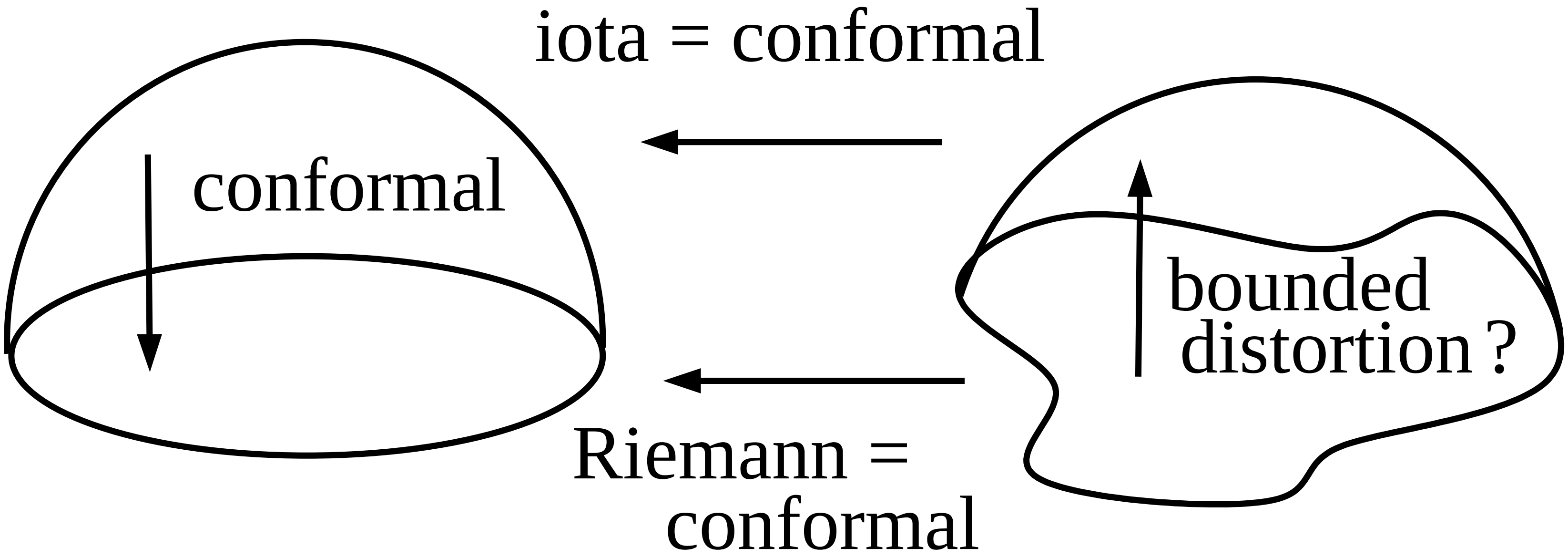
Riemann map = conformal from base to disk



MA-map from polygon to circle has QC extension

iff

there is QC map from base to dome fixing boundary.



MA-map from polygon to circle has QC extension
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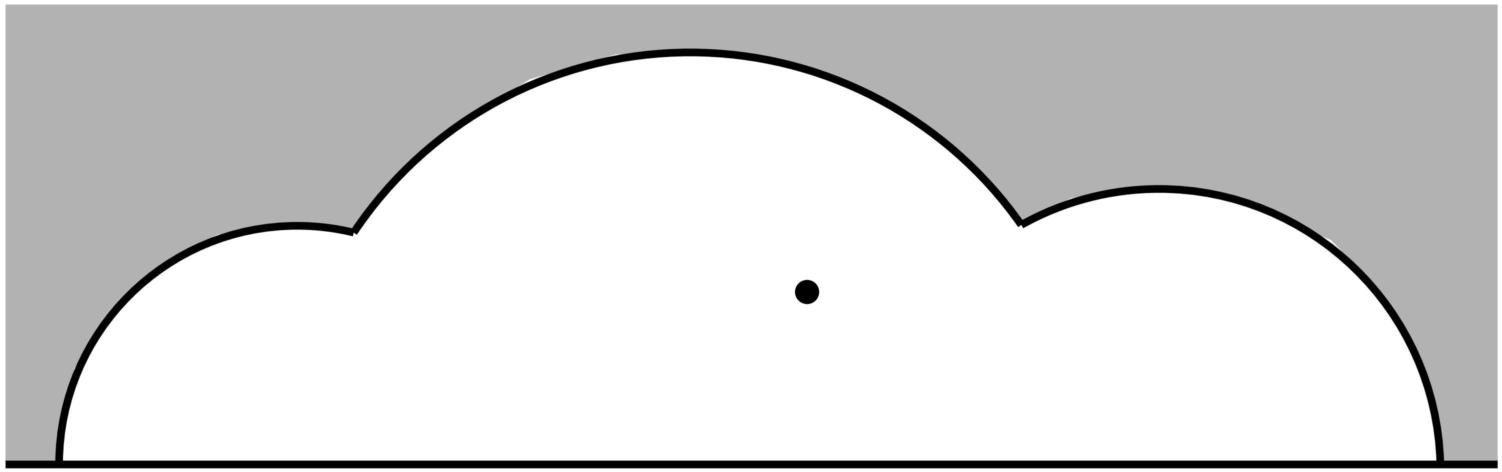
2nd claim is Dennis Sullivan's **convex hull theorem**.

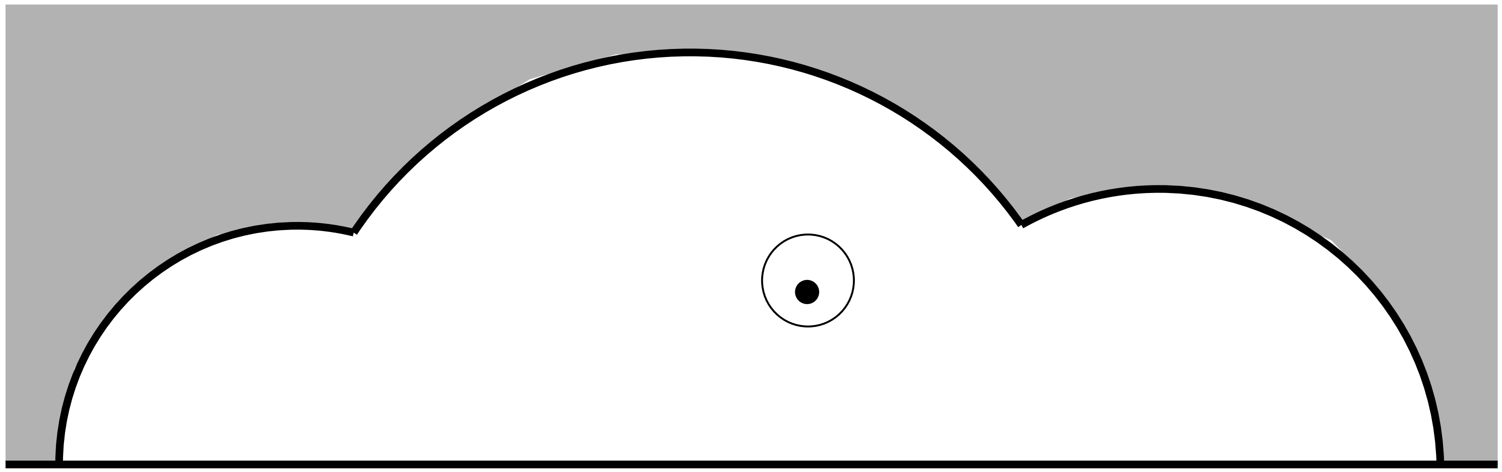


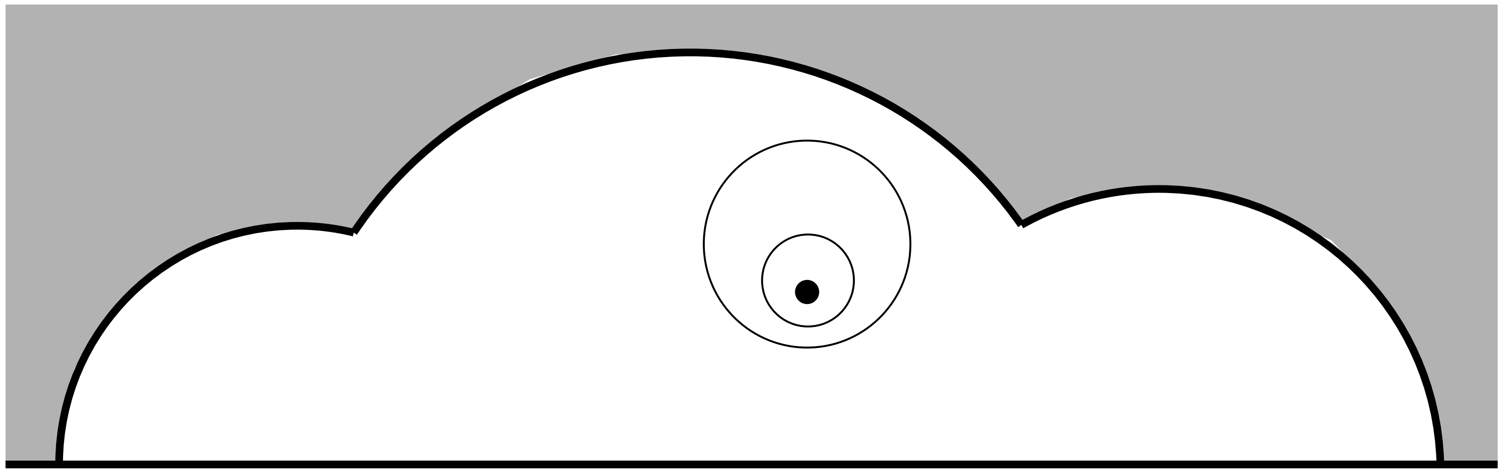
Region below dome is union of hemispheres

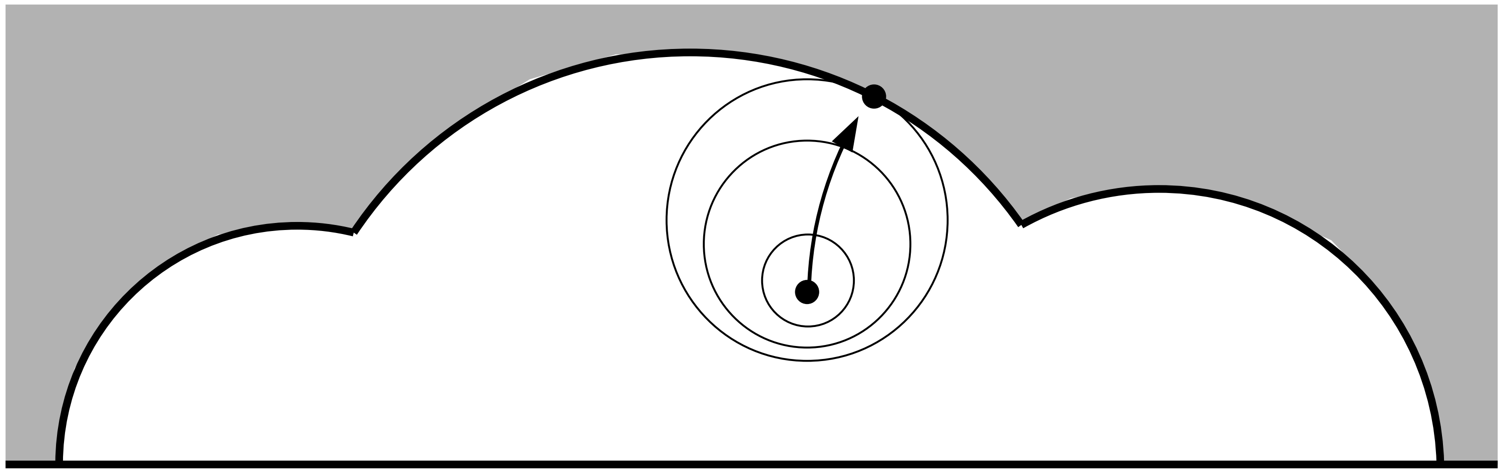
Hemispheres = hyperbolic half-spaces.

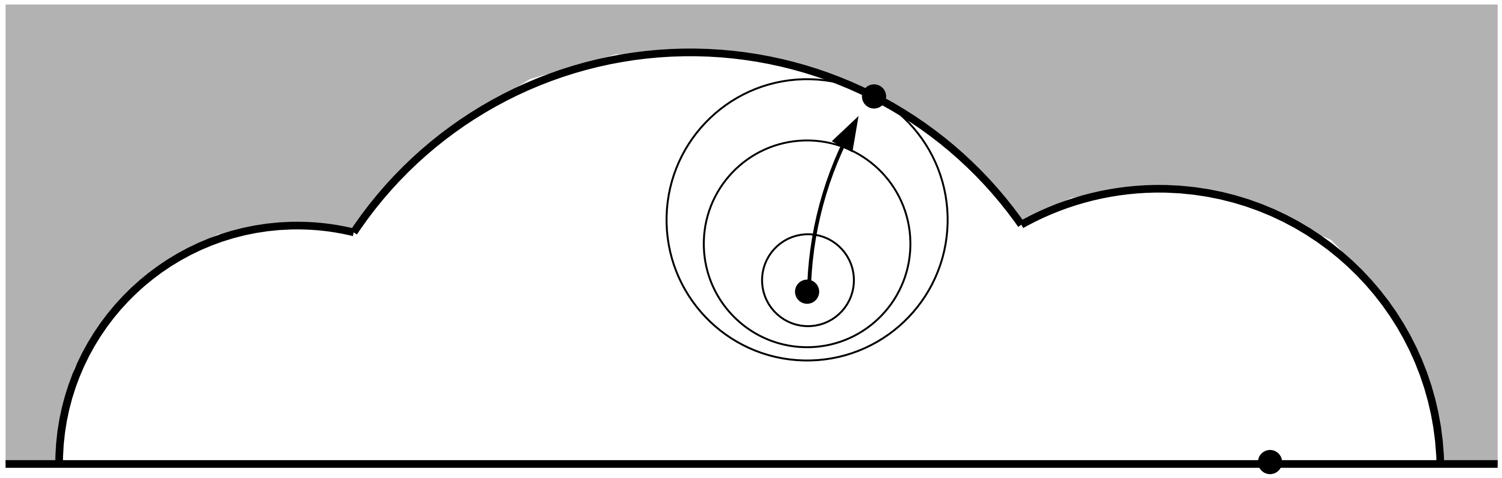
Region above dome is hyperbolically convex.

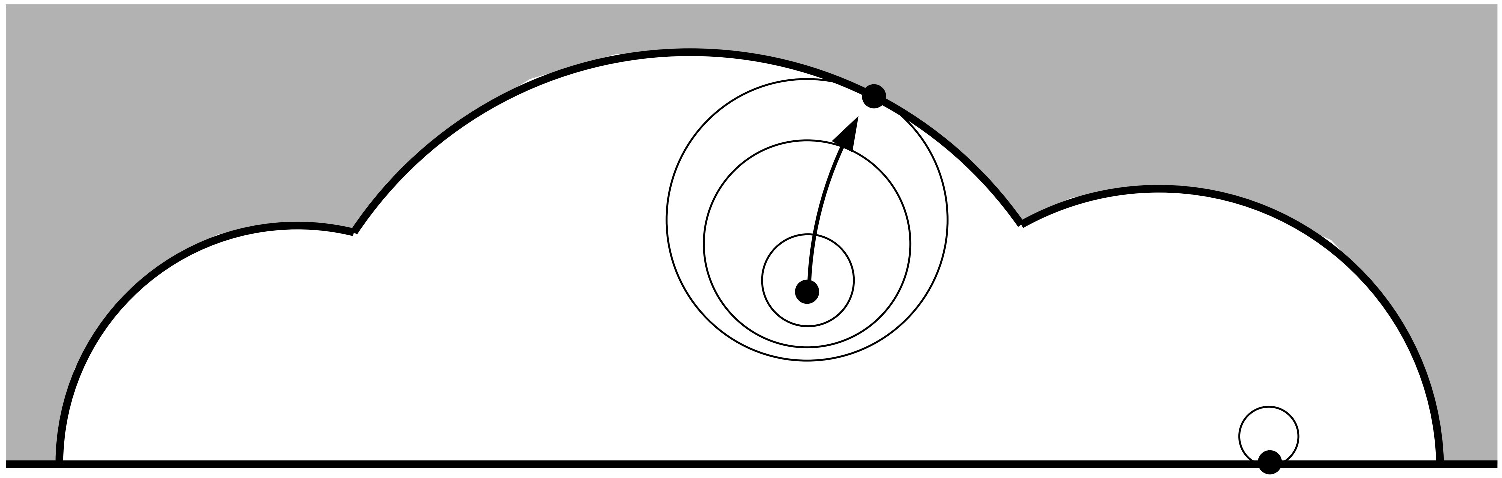
Consider nearest point retraction onto this convex set.

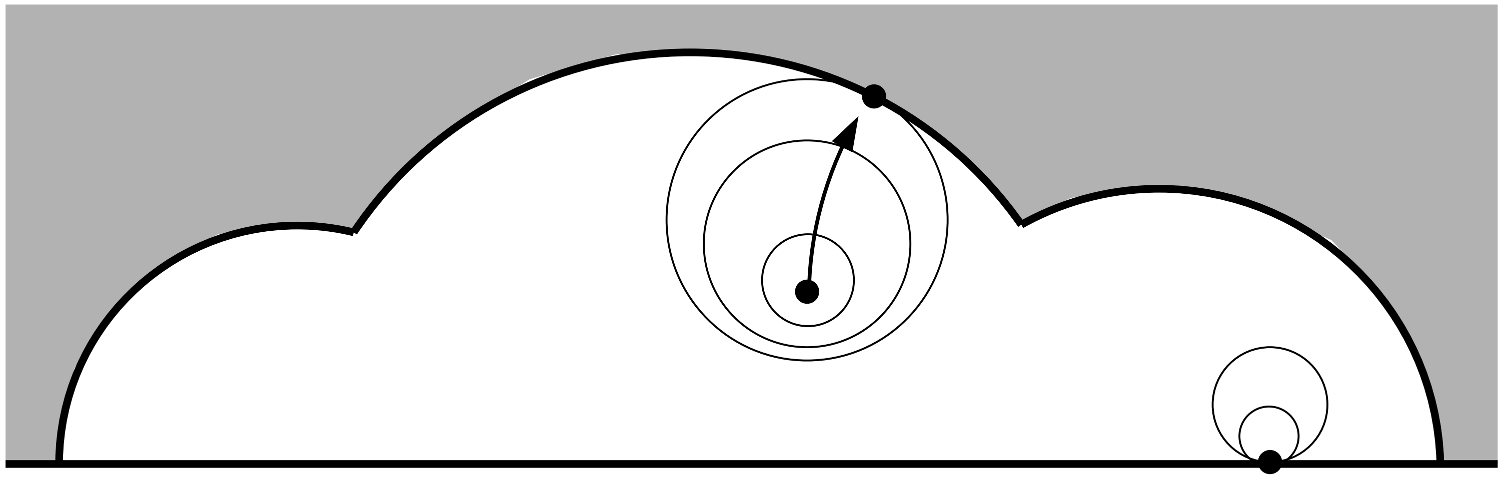


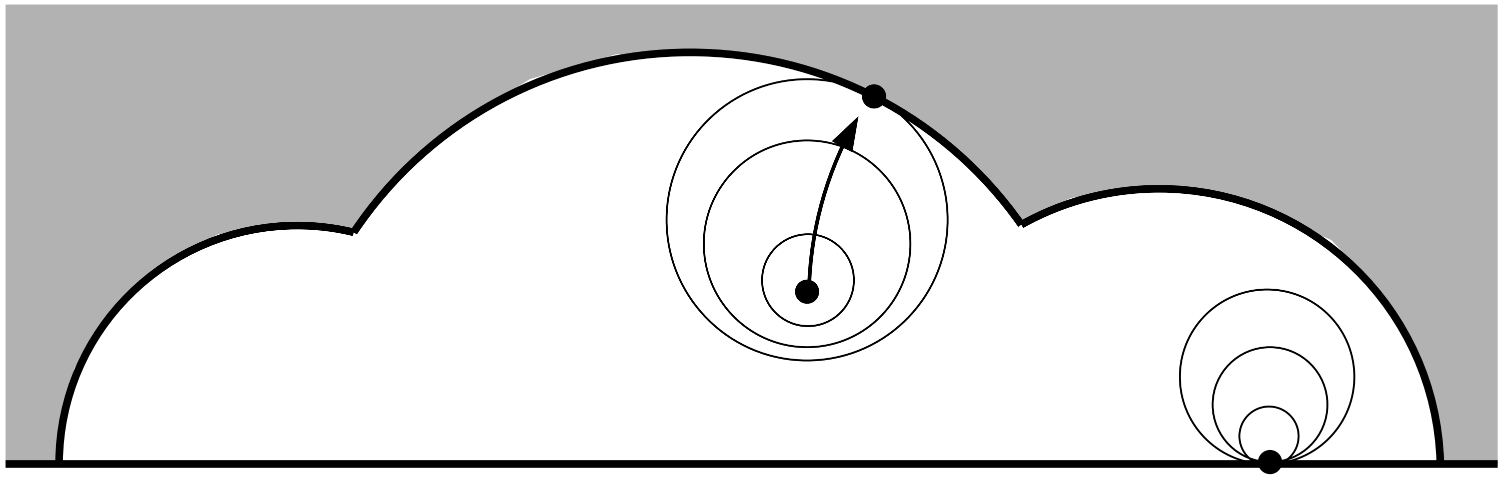


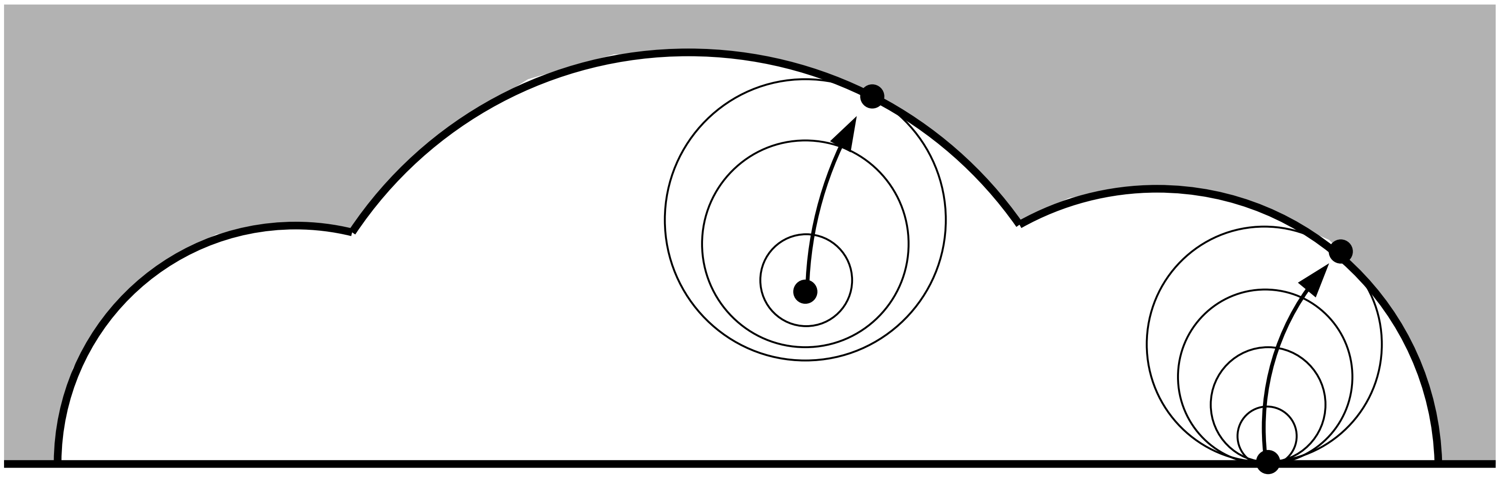


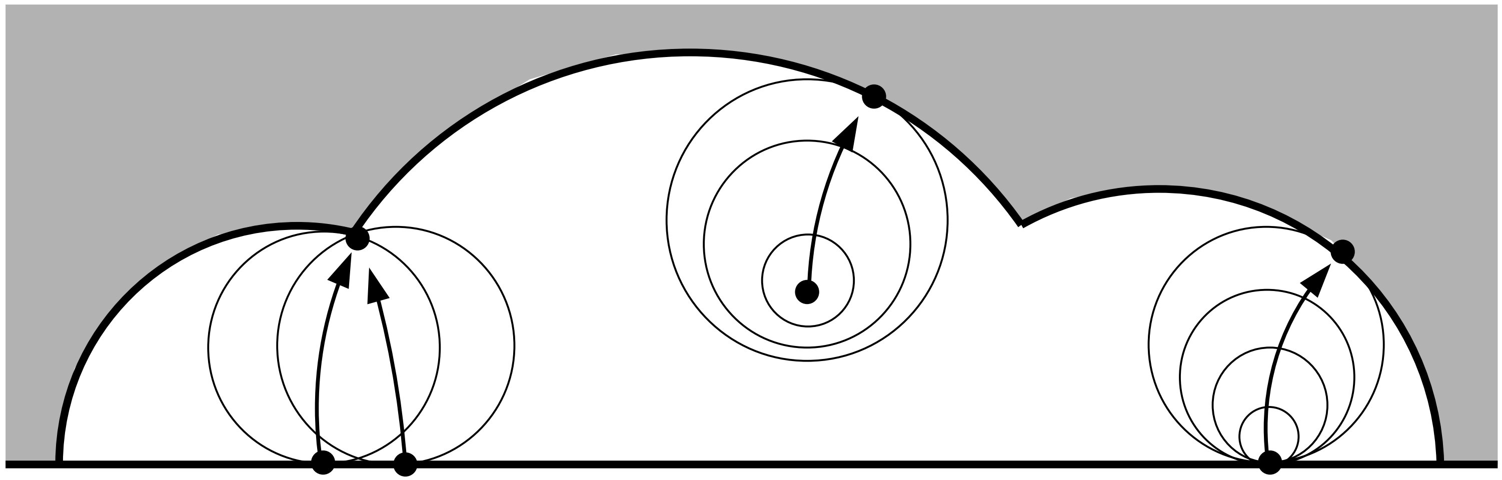


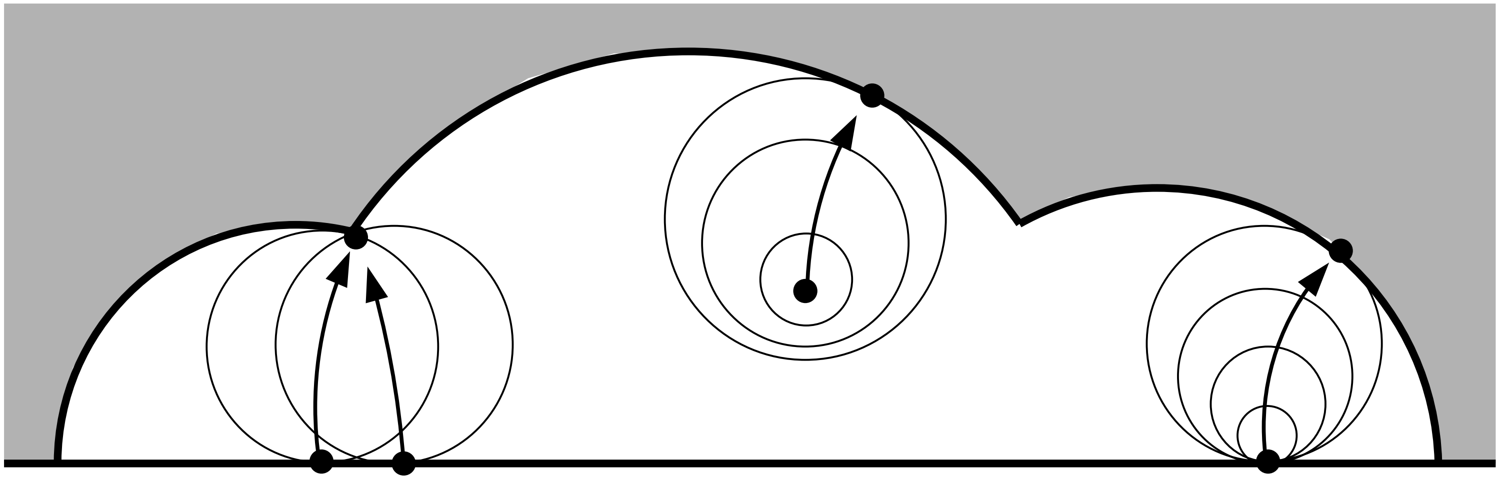










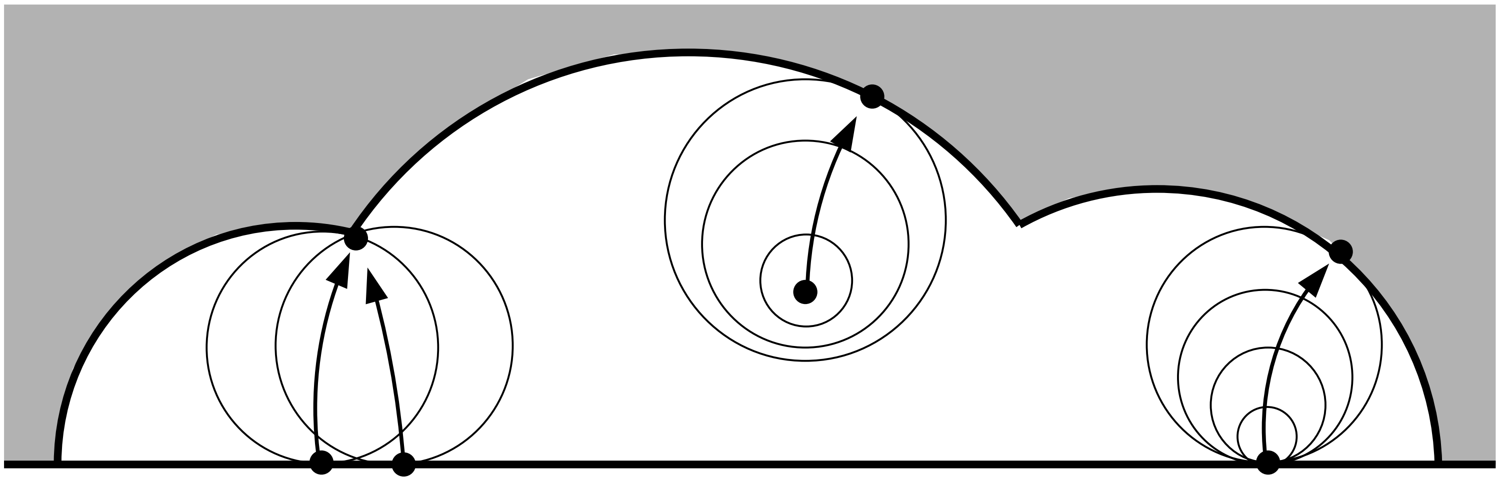


Nearest point retraction R is a **quasi-isometry**

$$\frac{1}{A} \le \frac{\rho(R(x), R(y))}{\rho(x,y)} \le A, \quad \text{if } \rho(x,y) \ge B.$$

i.e., R is bi-Lipschitz on large scales.

Metrics are hyperbolic metrics on Ω and S.



“Smoothing” gives QC map that fixes boundary points.

Hence medial axis map approximates Riemann map.

Dennis Sullivan, David E**p**stein and Al Marden, C.B.

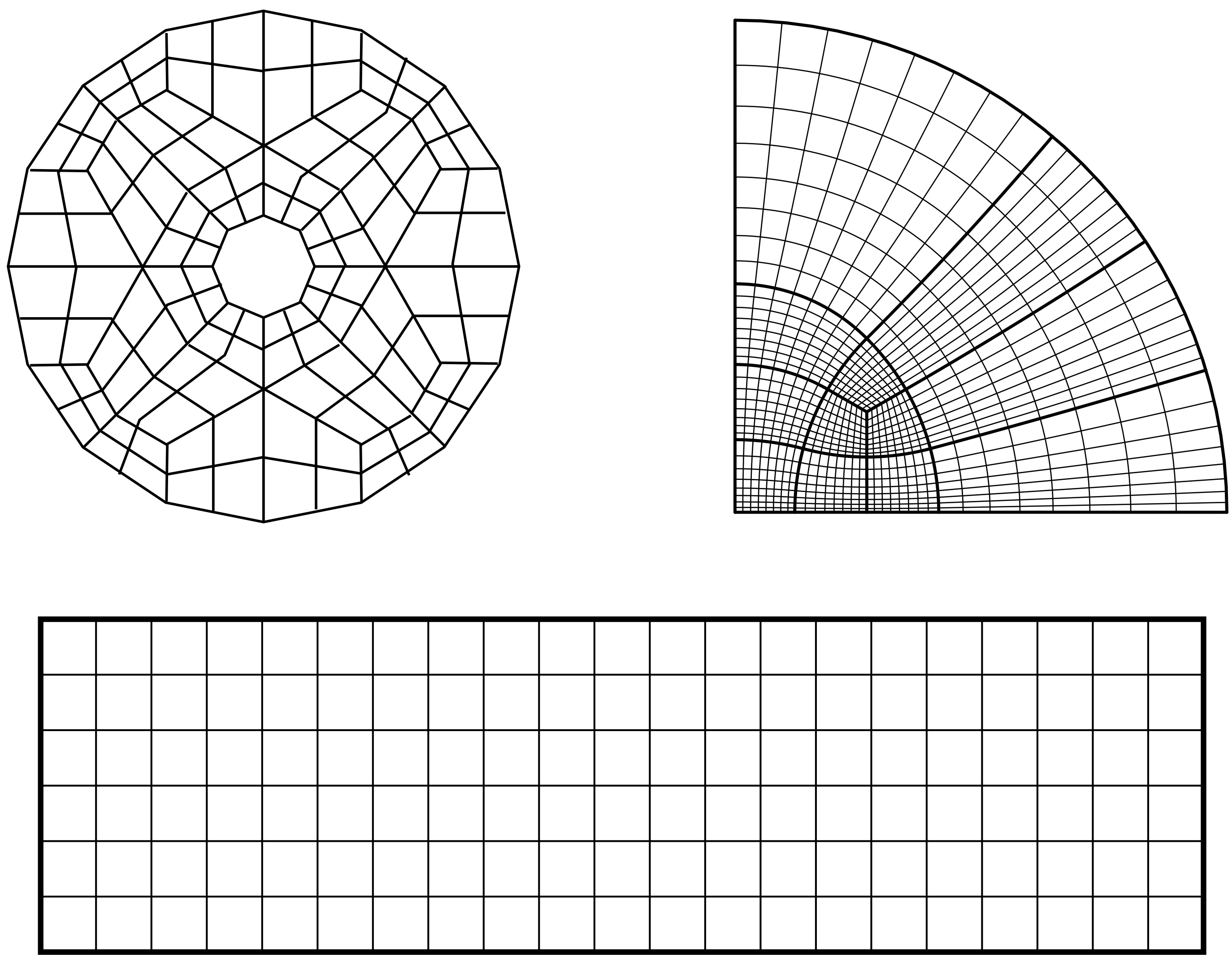
P

P^2

P = hyperbolic geometry, University of Warwick

P^2 = computational geometry, UC Irvine

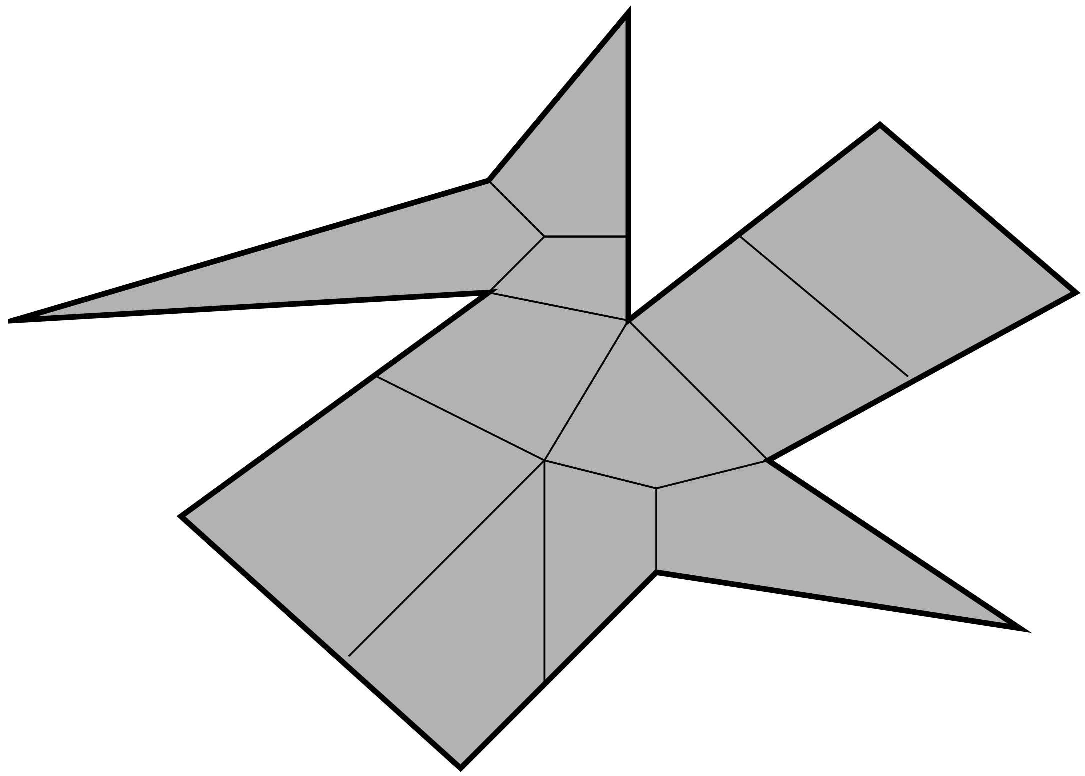
Application of conformal mapping to CG: Quadrilateral meshes



- n-gons have $O(n)$ quad mesh with angles $\leq 120°$.

Bern and E**pp**stein, 2000. $O(n \log n)$ work.

- Regular hexagon shows $120°$ is sharp.



- n-gons have $O(n)$ quad mesh with angles $\leq 120^\circ$.

Bern and E**pp**stein, 2000. $O(n \log n)$ work.

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Can we bound angles from below?

- n-gons have $O(n)$ quad mesh with angles $\leq 120°$.

Bern and E**pp**stein, 2000. $O(n \log n)$ work.

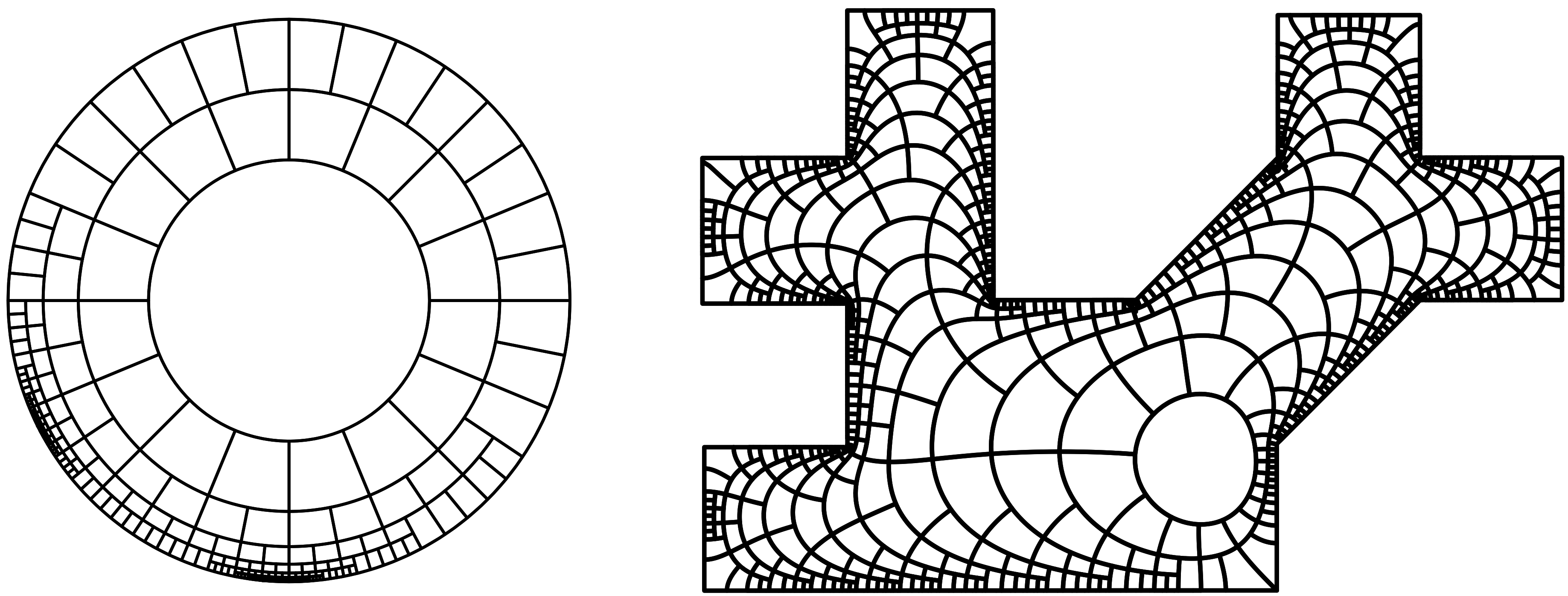
- Regular hexagon shows $120°$ is sharp.

Can we bound angles from below?

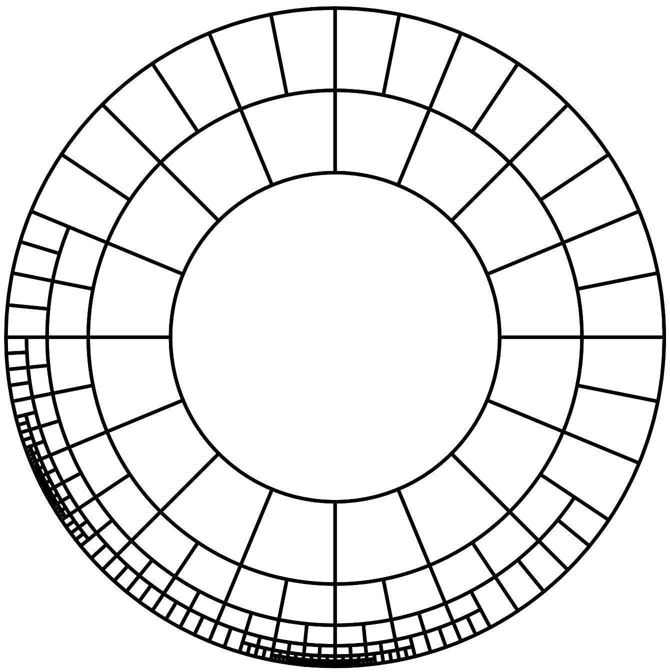
Theorem: Every n-gon has $O(n)$ quad mesh with all angles $\leq 120°$ and new angles $\geq 60°$. $O(n)$ work.

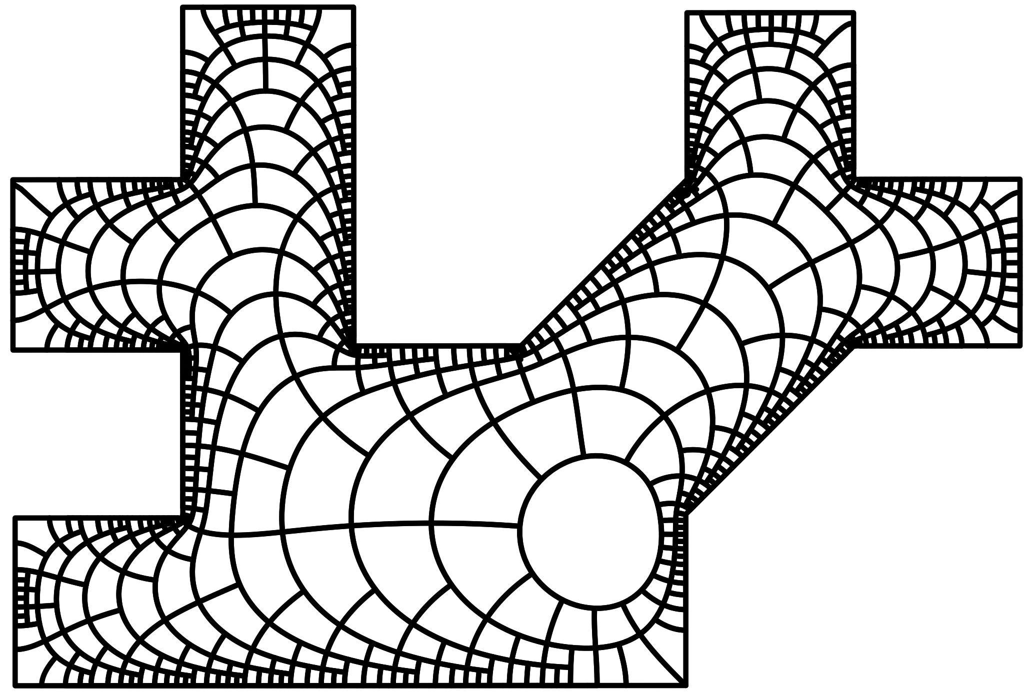
Original angles $< 60°$ remain unchanged. $60°$ is sharp.

Idea of proof: transfer mesh from disk

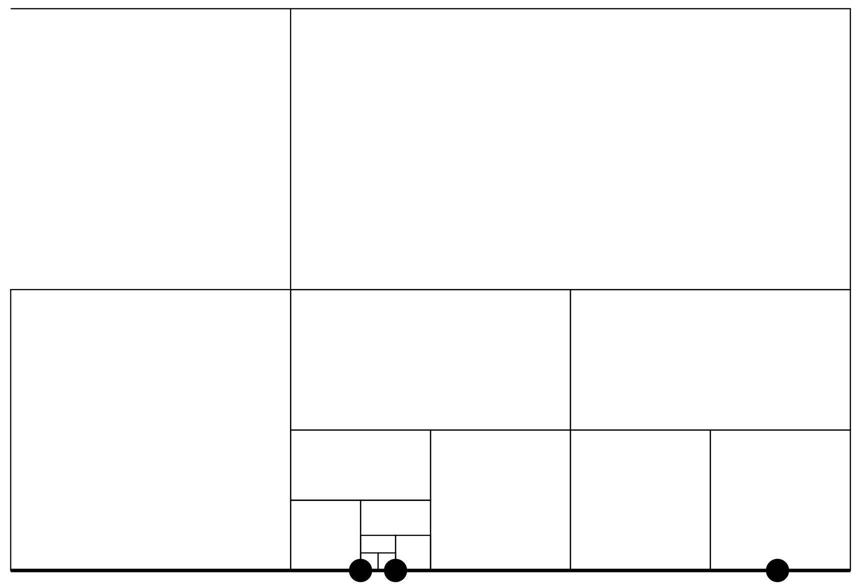


Idea of proof: transfer mesh from disk





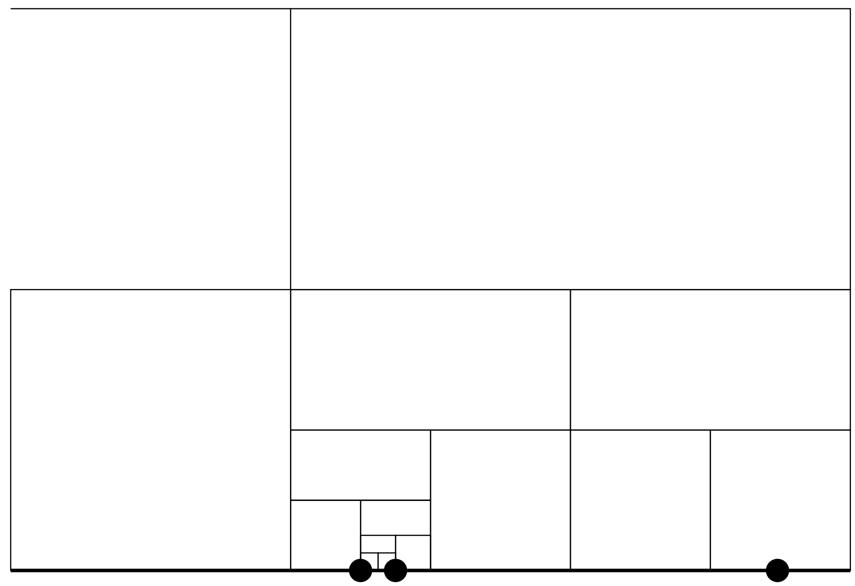
But number of boxes depends on geometry (not just n).



Subdivide until boxes separate points.

Gives $\gg n$ boxes if pre-vertices cluster tightly.

Cluster is $S \subset V$ so $\mathrm{diam}(S) \ll \mathrm{dist}(S, V \setminus S)$.



Subdivide until boxes separate points.

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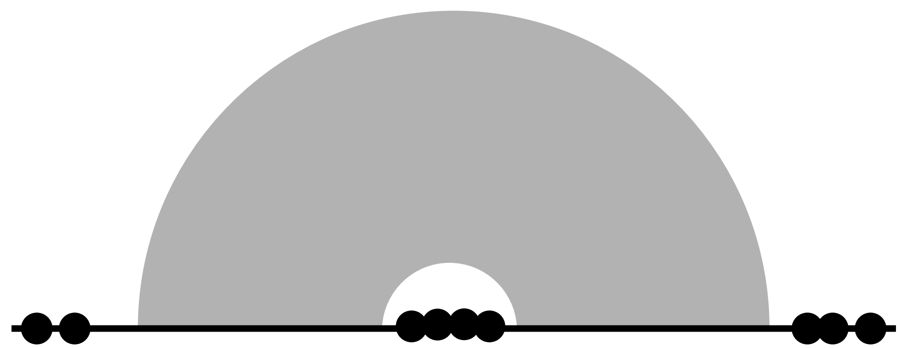
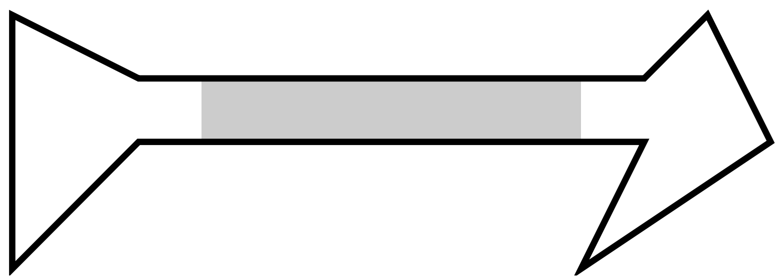
Cluster is $S \subset V$ so $\mathrm{diam}(S) \ll \mathrm{dist}(S, V \setminus S)$.

Fact: can find all clusters in linear time.

Fact: MA-clusters = conformal-clusters

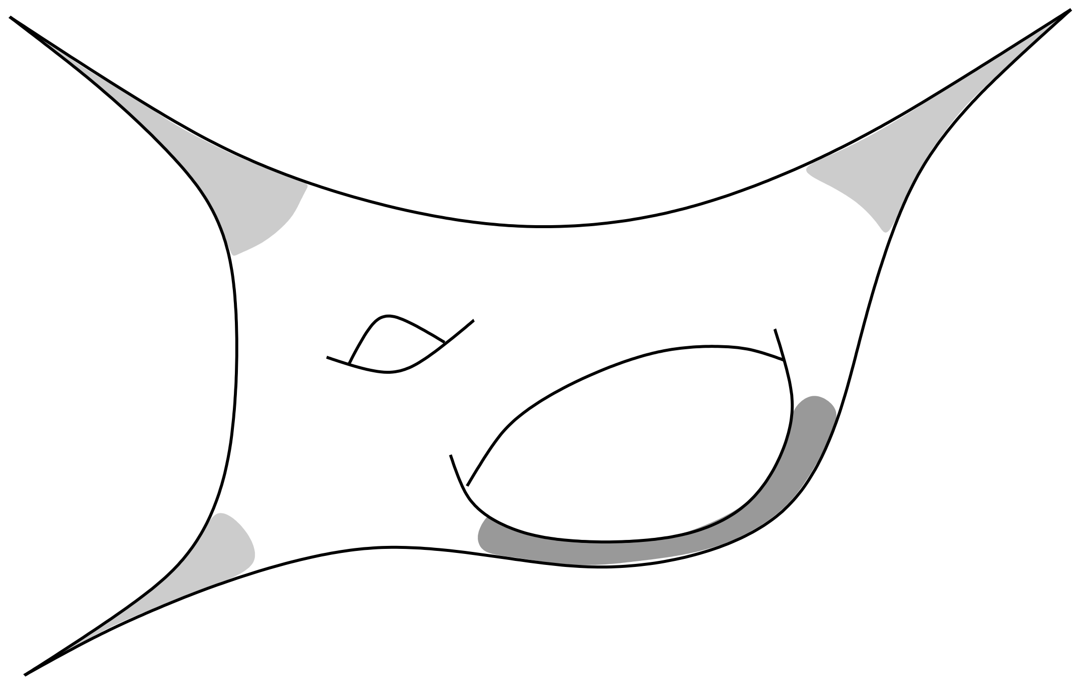
“Cover” each cluster by a half-annulus.





Maps to a **hyperbolic thin piece** dividing polygon.

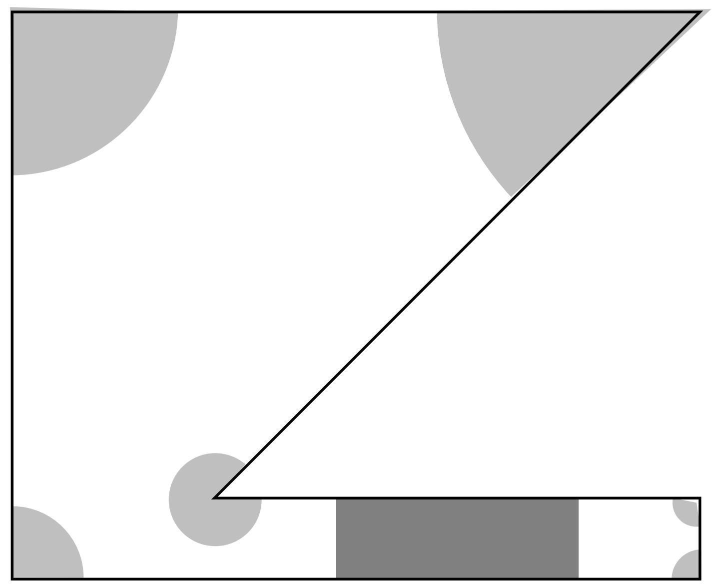
Surface **thin part** is union of short non-trivial loops.



parabolic = puncture, hyperbolic = handle

Thick and Thin parts of a polygon

Thin part is union of short curves between edges.

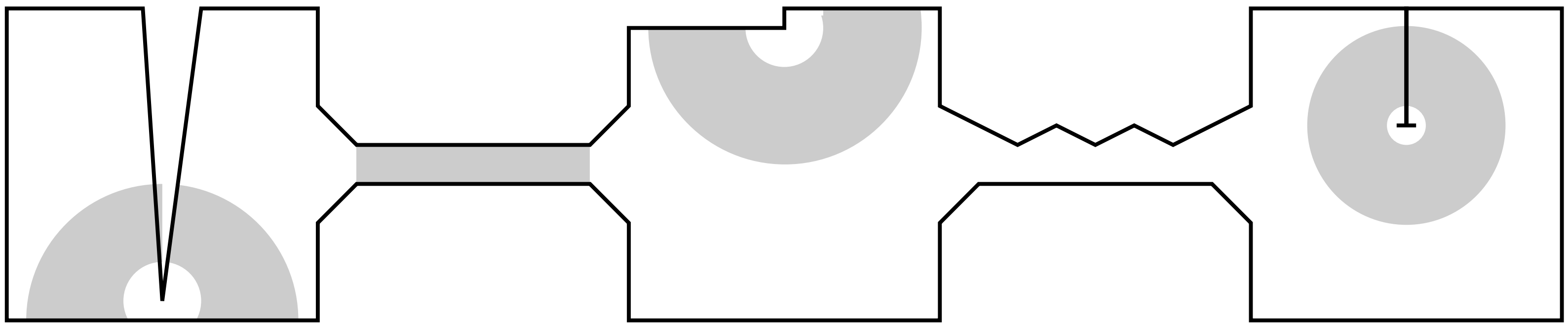


Parabolic = adjacent, Hyperbolic = non-adjacent

Rough idea: sides I, J so $\mathrm{dist}(I, J) \ll \min(|I|, |J|)$.

Thick parts = remaining components (white)

More examples of hyperbolic thin parts.

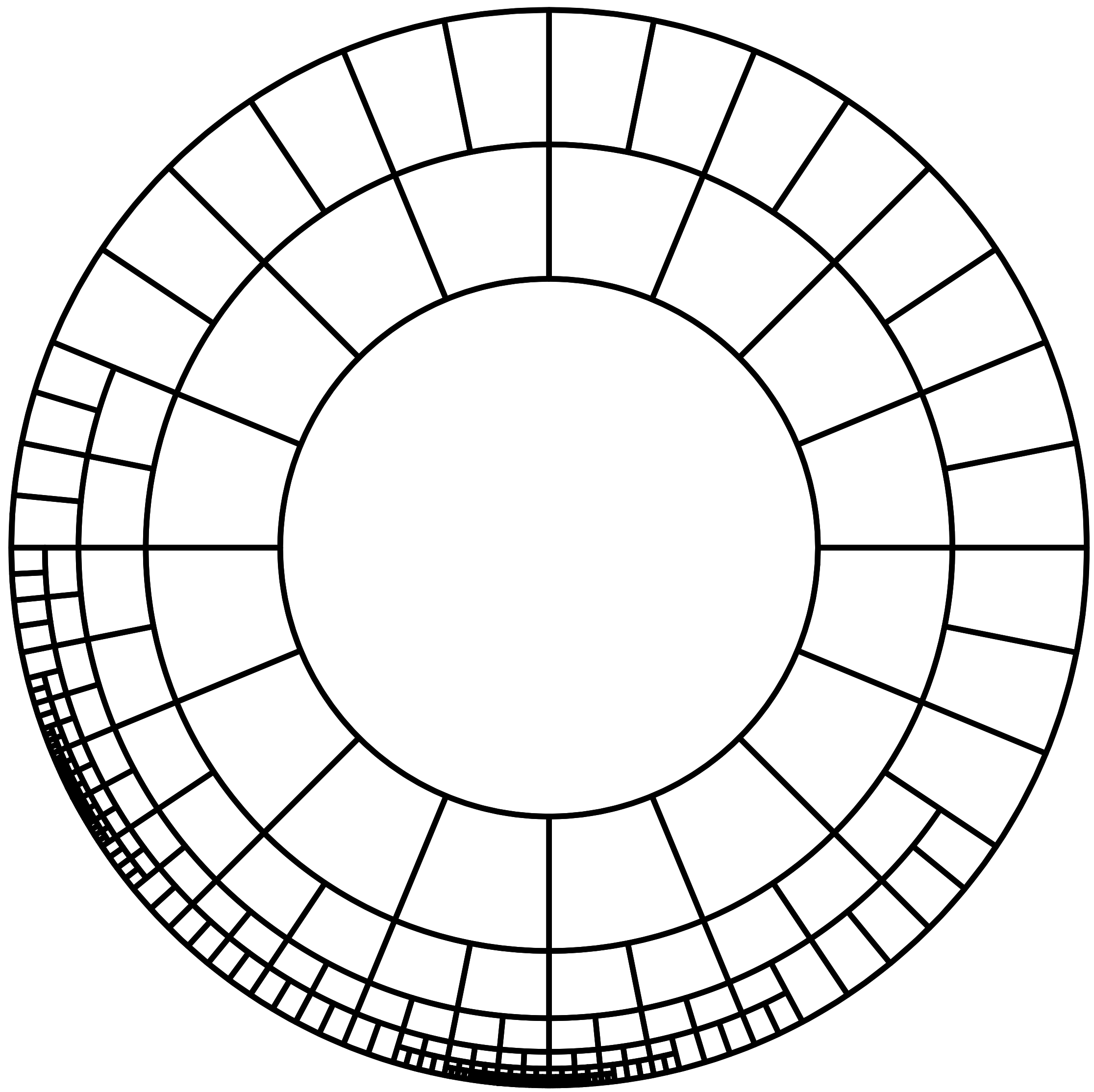


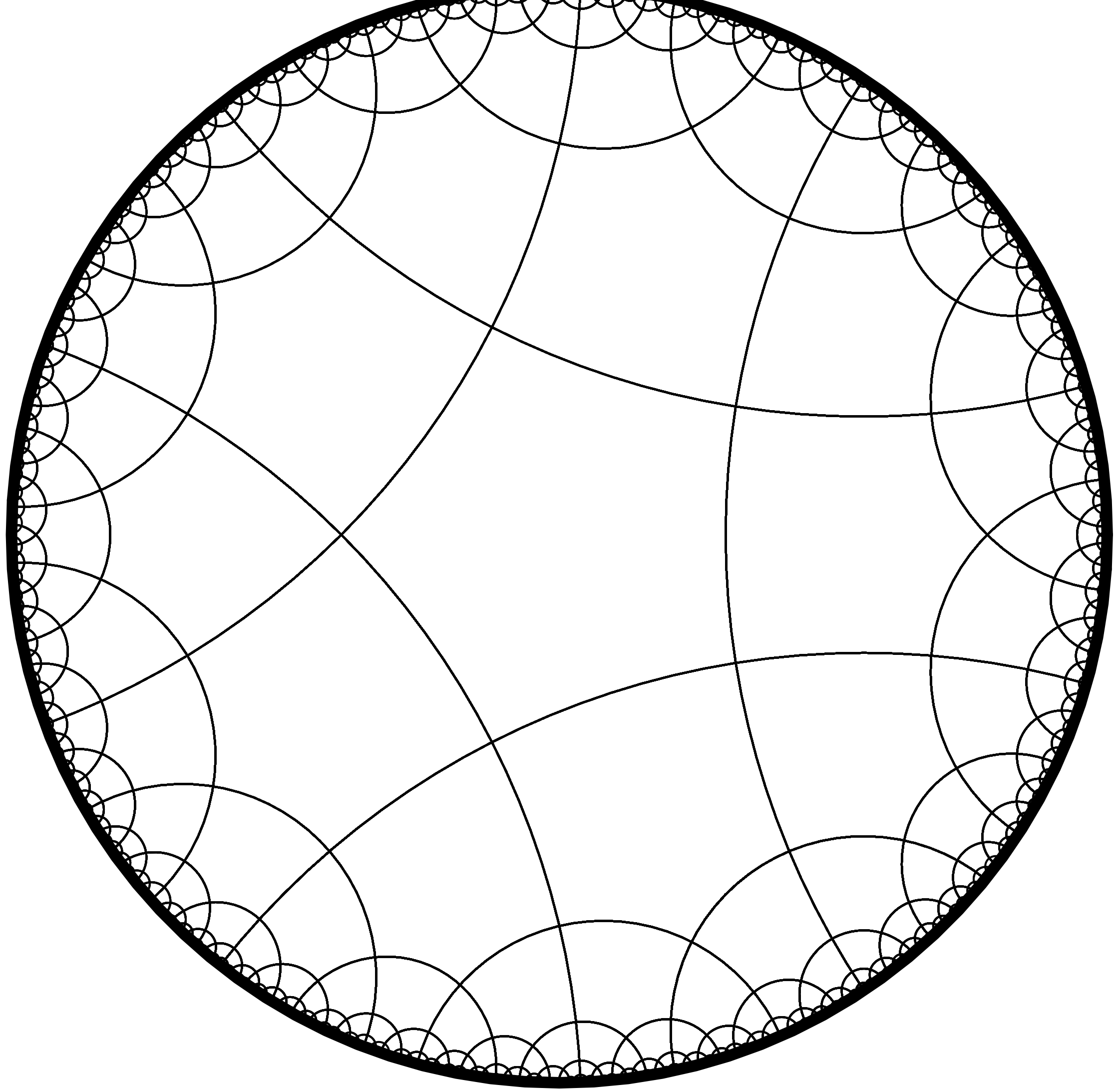
Idea for quad mesh theorem:

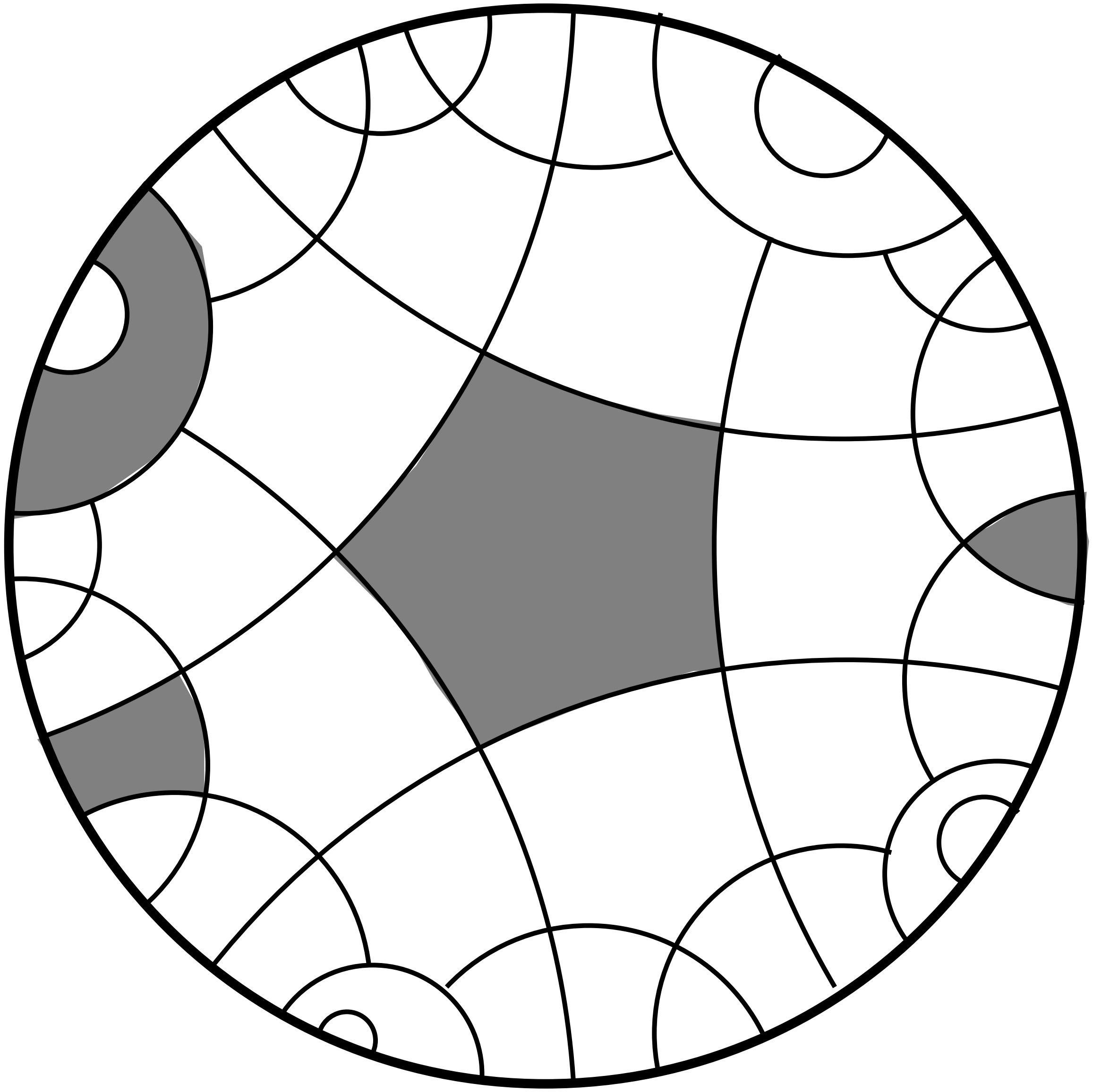
• Decompose polygon into $O(n)$ thick and thin parts.

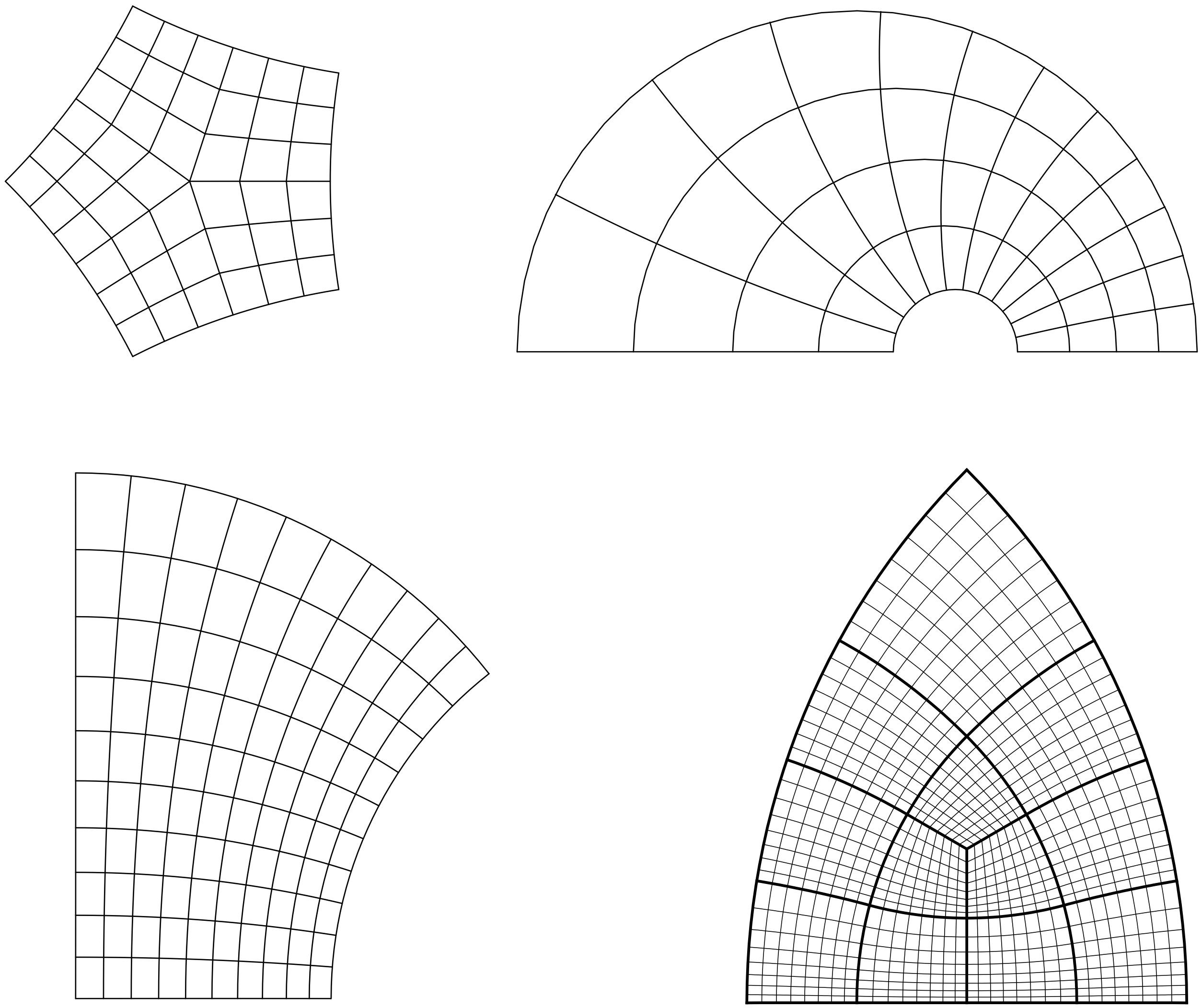
• Mesh thin parts "by hand"; explicit construction with $O(1)$ pieces each. Some pieces may be long and narrow.

• Transfer mesh on disk to thick parts by conformal map. $O(n)$ pieces. Every quad is "roundish".



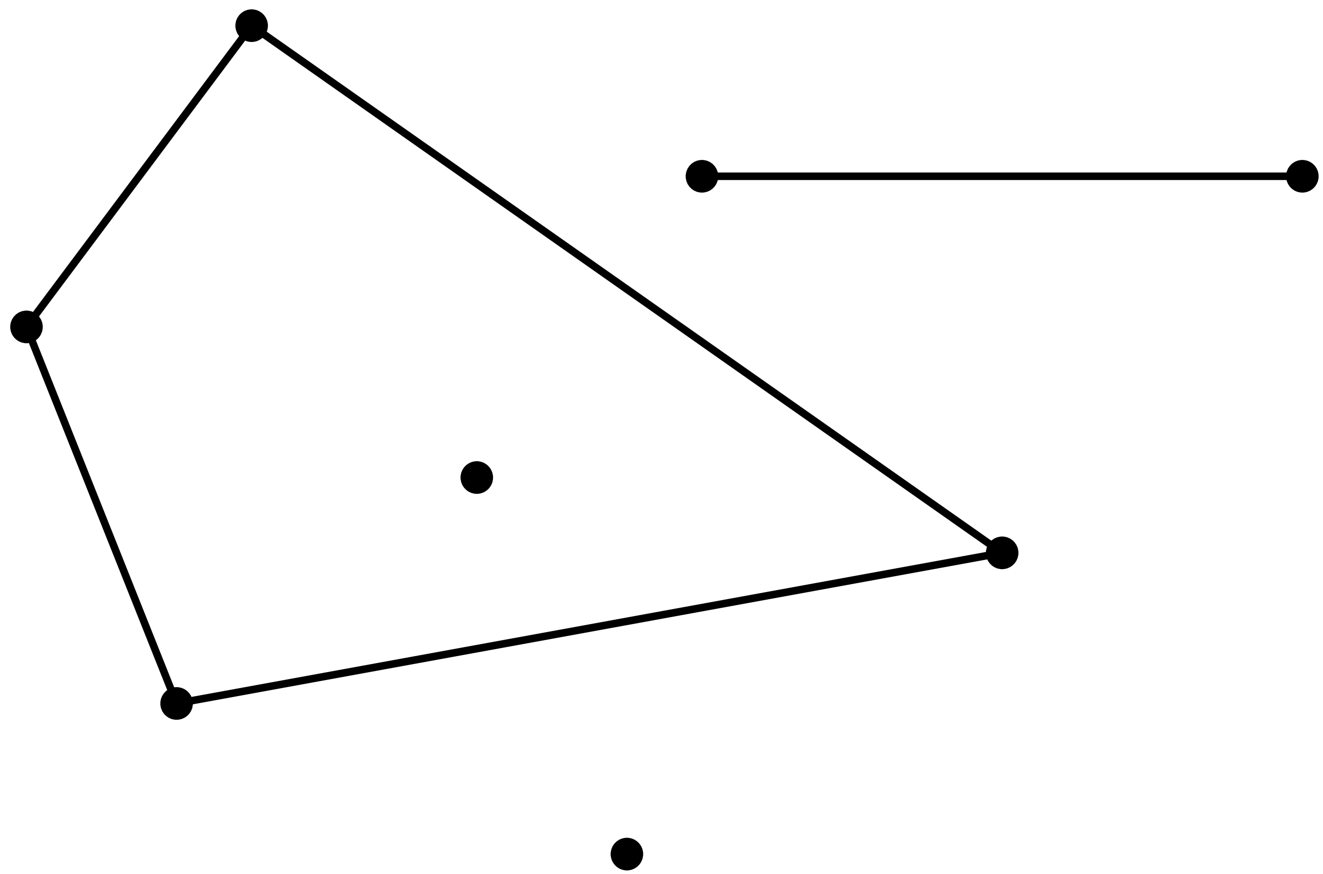






Meshes designed to match along common edges.

A **Planar Straight Line Graph** (PSLG) is a finite point set plus a set of disjoint edges between them.



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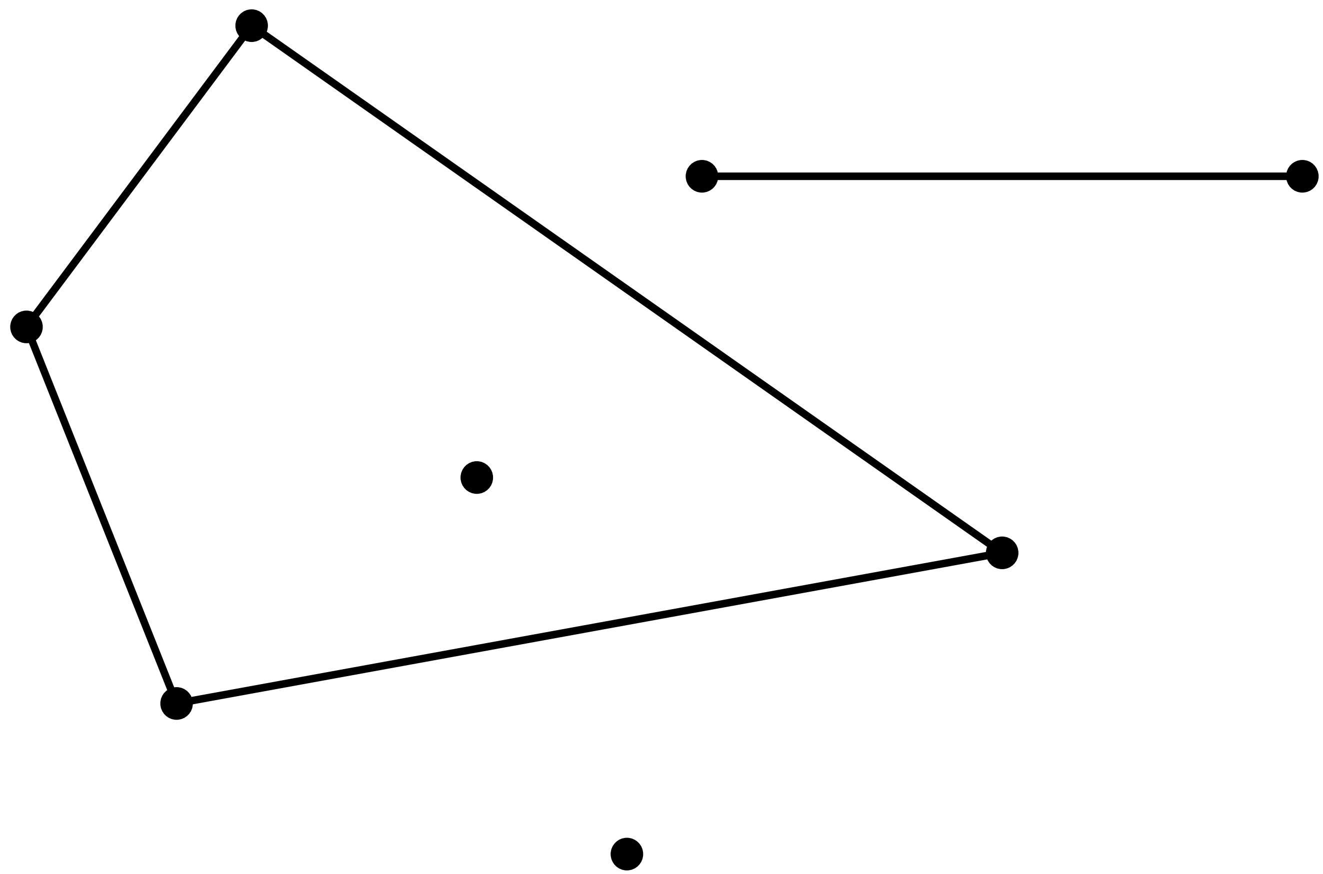
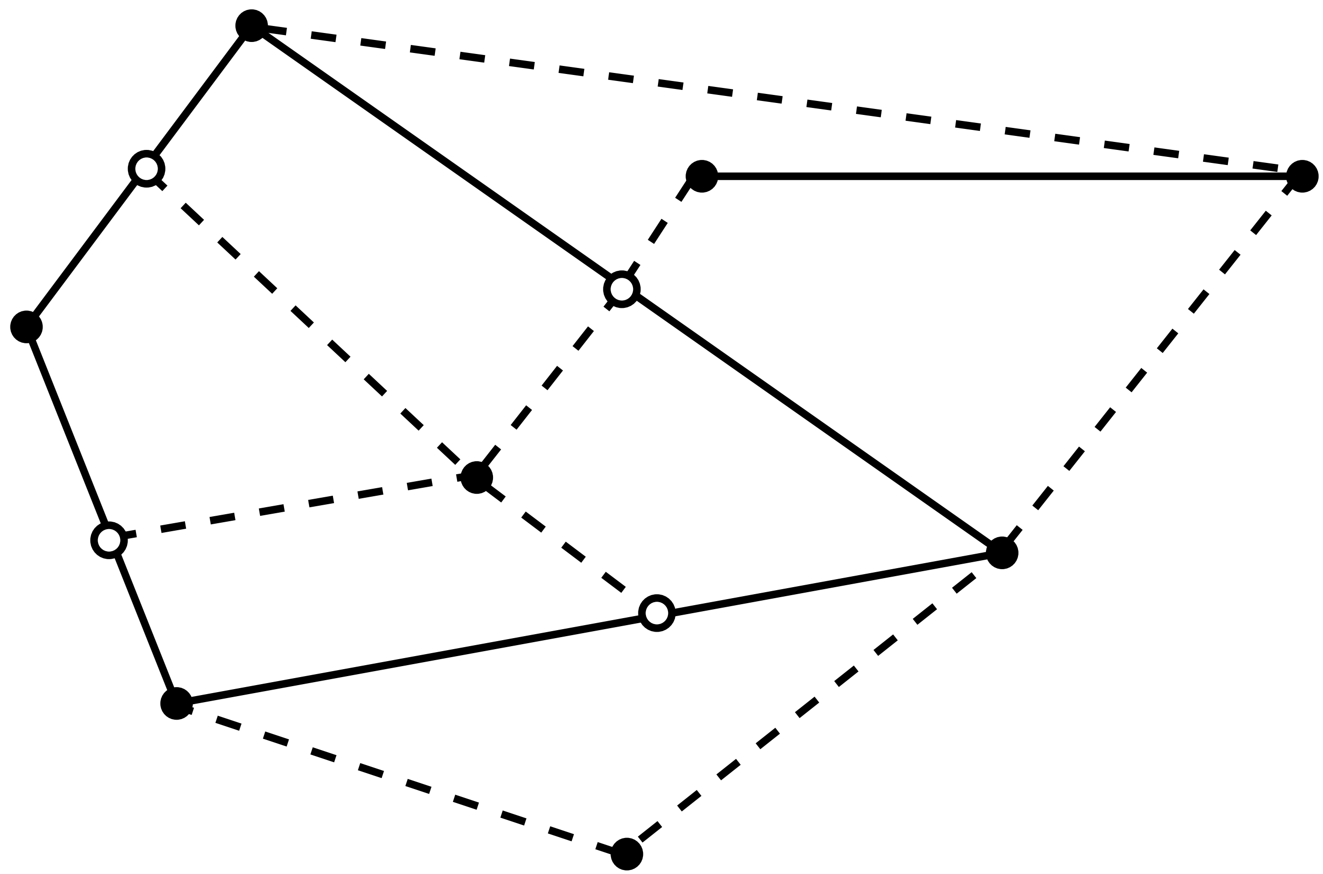
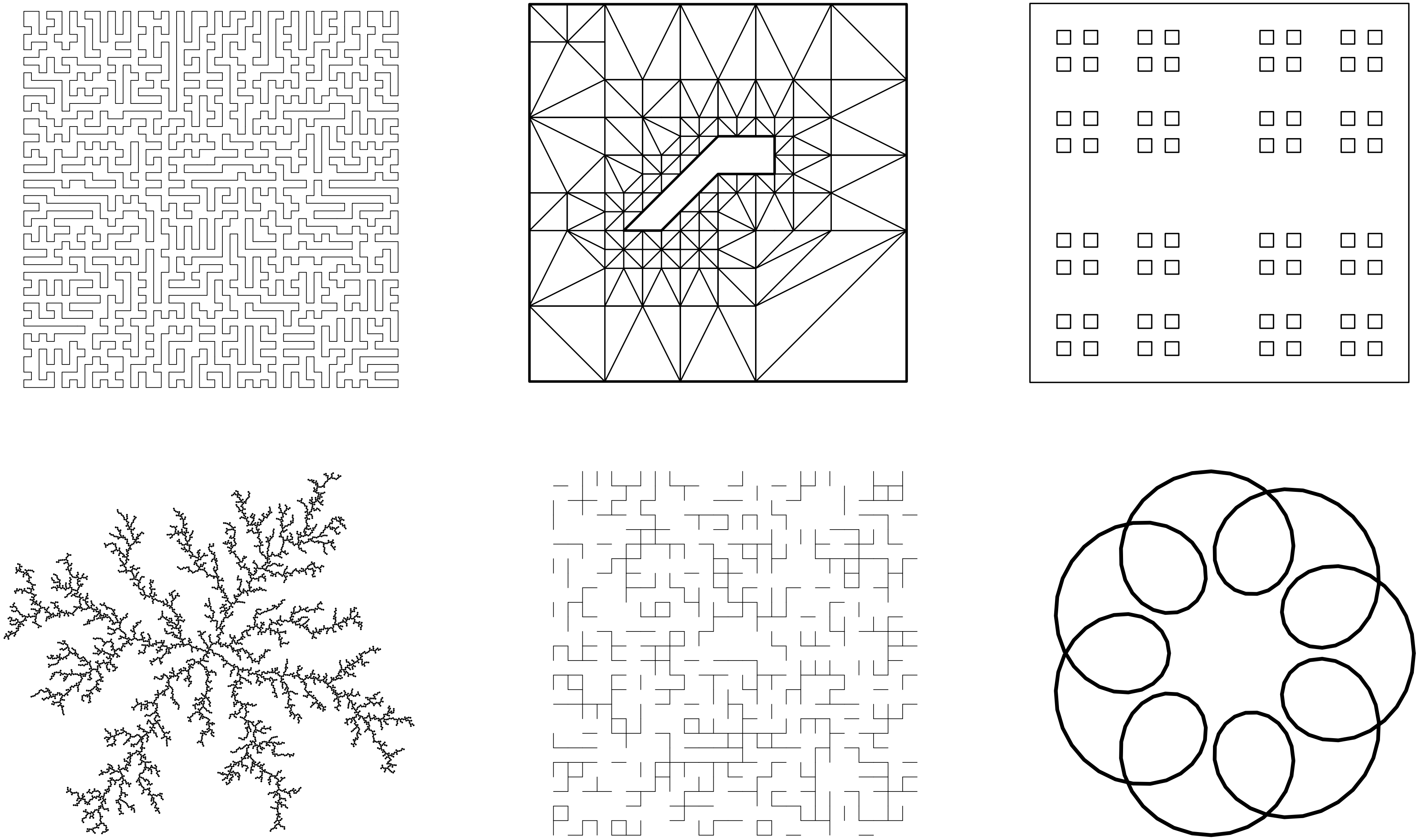


Size = number of vertices = n.

A **Planar Straight Line Graph** (PSLG) is a finite point set plus a set of disjoint edges between them.



Mesh convex hull conforming to PSLG.



More PSLGs

Theorem: Every PSLG of size n has $O(n^2)$ quad mesh with all angles $\leq 120°$ and all new angles $\geq 60°$.

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Angles and complexity sharp.

All but $O(n/\epsilon)$ vertices have angles in $[90° - \epsilon, 90° + \epsilon]$.

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Angles and complexity sharp.

All but $O(n/\epsilon)$ vertices have angles in $[90° - \epsilon, 90° + \epsilon]$.

All but $O(n)$ vertices are degree 4.

Can cut into $O(n)$ rectangular grids using motorcycle graphs (Eppstein, Goodrich, Kim, Tamstorf)

The $O(n^2)$ is sharp because any mesh with maximum angle $\leq \theta < 180°$ sometimes requires n^2 elements.

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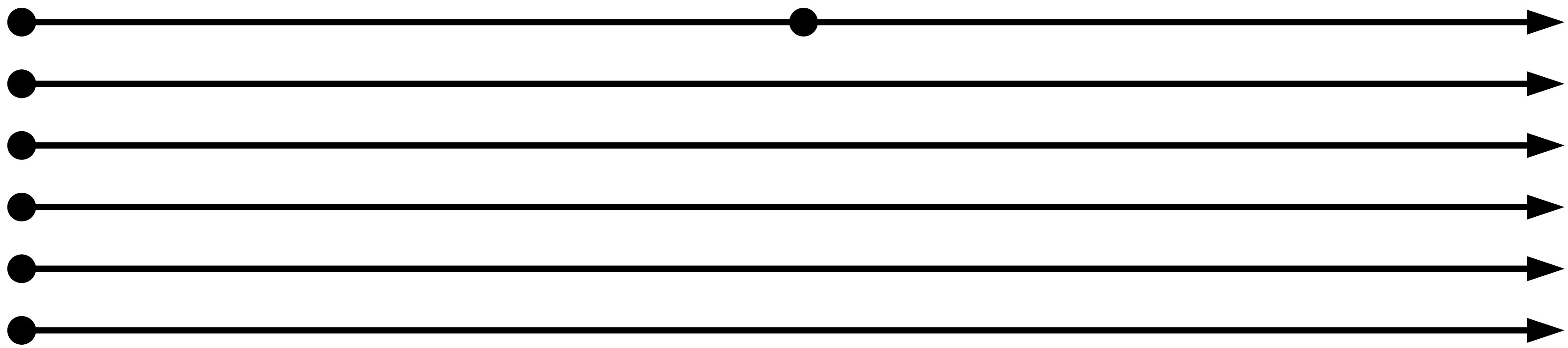

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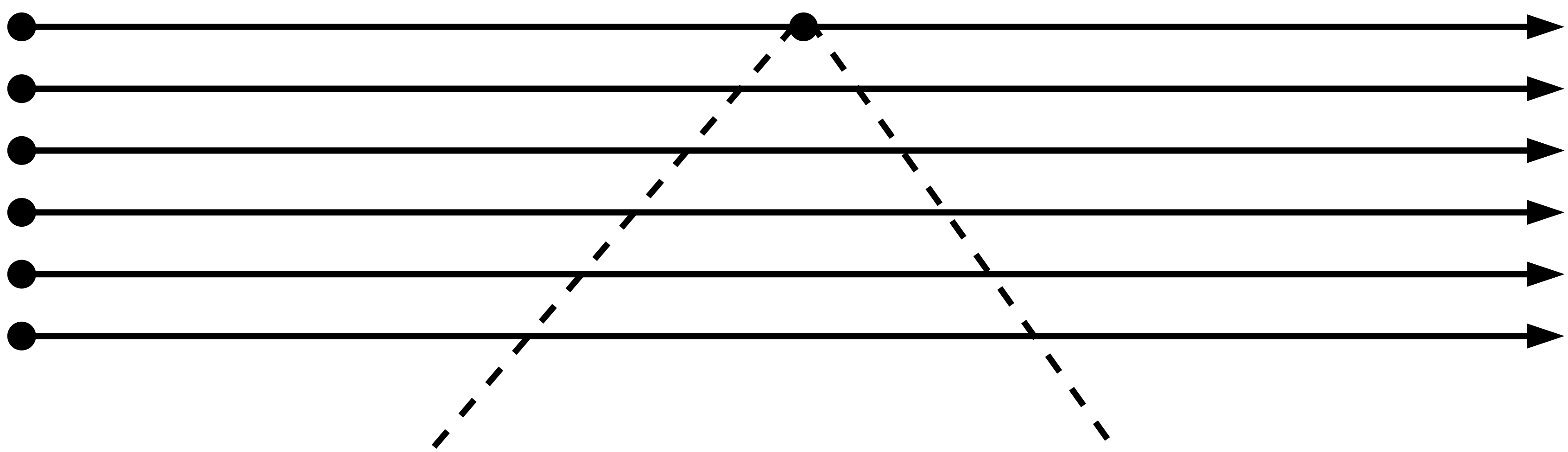

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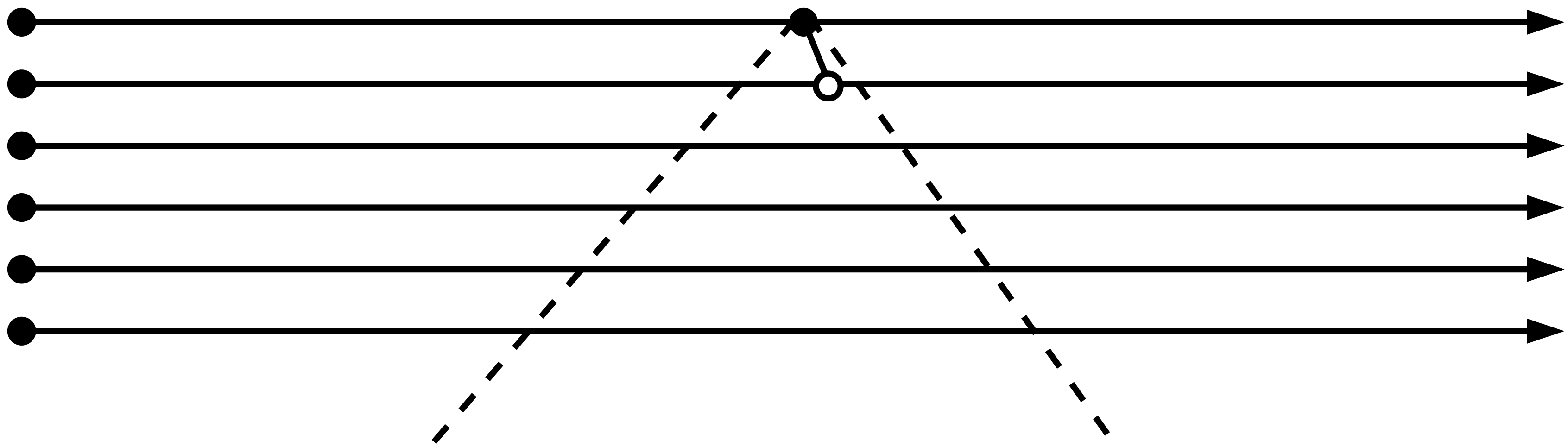

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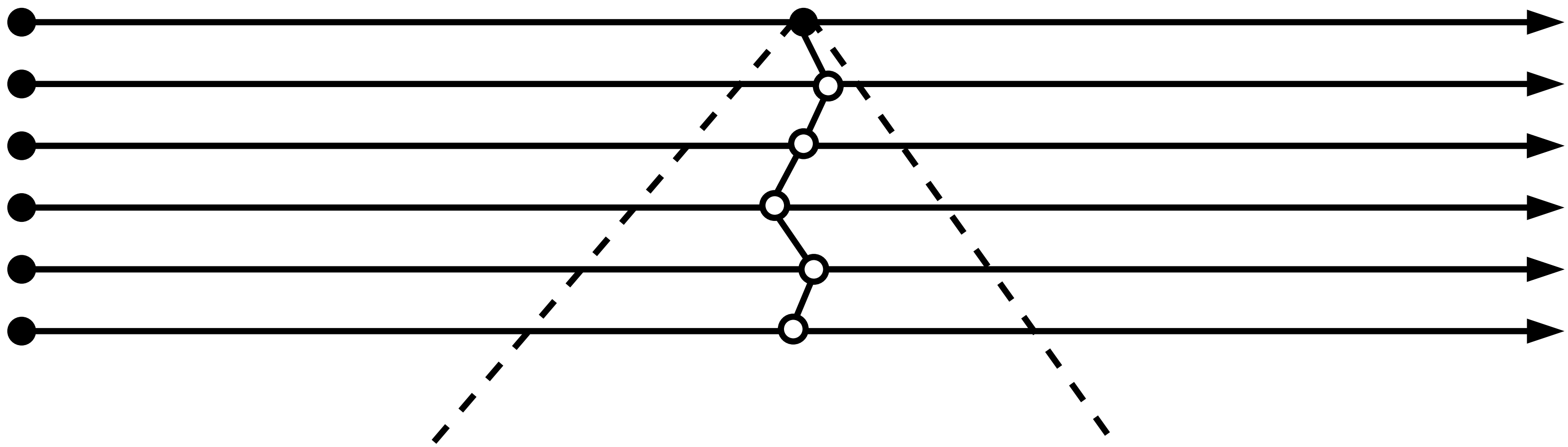
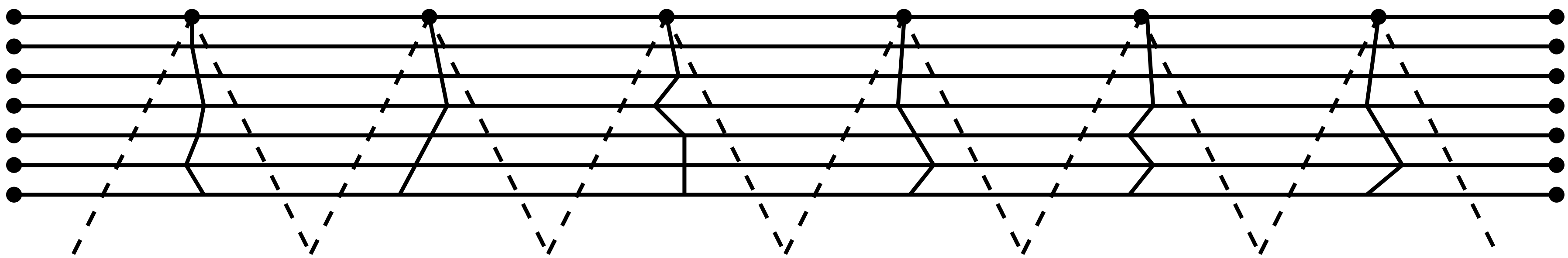


The $O(n^2)$ is sharp because any mesh with maximum angle $\leq \theta < 180°$ sometimes requires n^2 elements.



Convert quadrilaterals to triangles by adding diagonals.

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Corollary: Every PSLG has a $O(n^2)$ triangulation with maximum angle $\leq 120°$.

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Previous: S. Mitchell 1993 (157.5°), Tan 1996 (132°).

Can we improve the 120° upper bound?

Can we get a positive lower bound on angles?

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Can we improve the 120° upper bound? **Yes**

Can we get a positive lower bound on angles? **No**

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Previous: S. Mitchell 1993 (157.5°), Tan 1996 (132°).

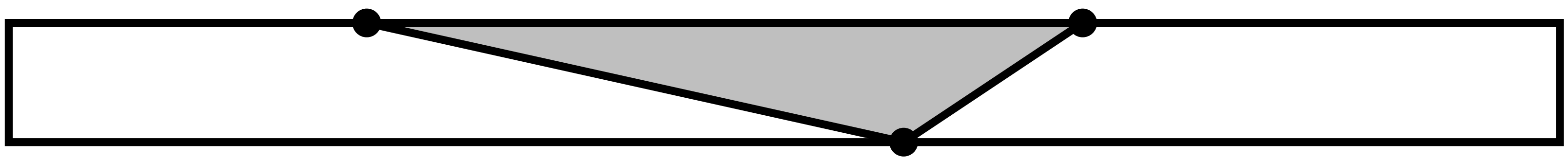
Can we improve the 120° upper bound? **Yes**

Can we get a positive lower bound on angles? **No**

(At least not with complexity depending only on n.)

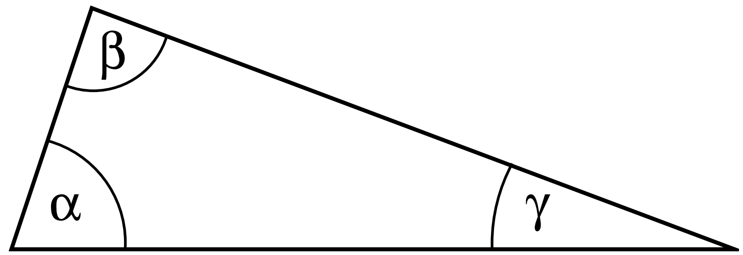
No lower angle bound. For $1 \times R$ rectangle

number of triangles $\gtrsim R \times$ (smallest angle)



So, bounded complexity $\Rightarrow$ no lower angle bound.

Upper bound $< 90^\circ$ implies lower bound > 0:



$$\alpha, \beta < (90^\circ - \epsilon) \Rightarrow \gamma = 180^\circ - \alpha - \beta \geq 2\epsilon.$$

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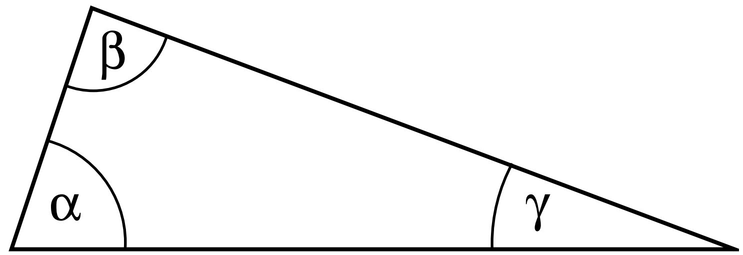
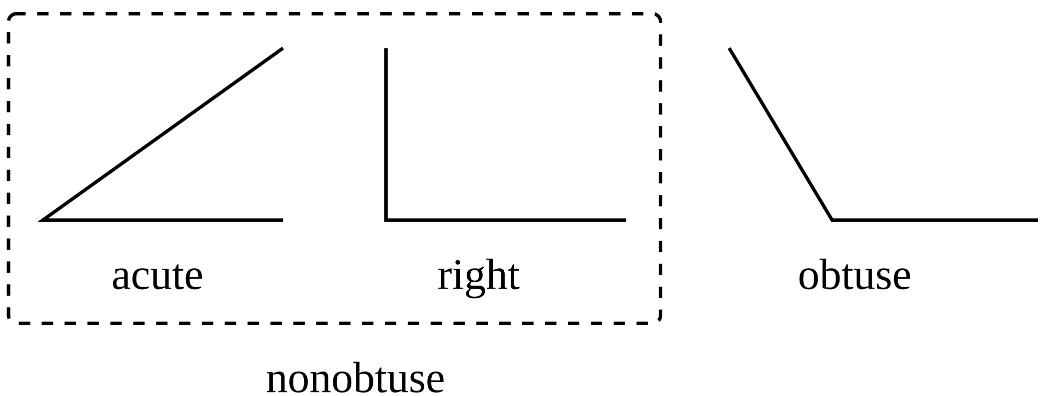


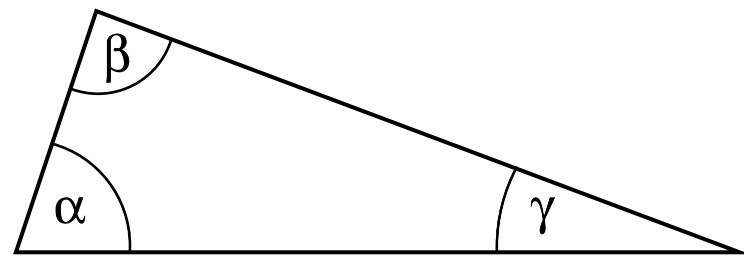
$$\alpha, \beta < (90^\circ - \epsilon) \Rightarrow \gamma = 180^\circ - \alpha - \beta \geq 2\epsilon.$$

So 90° is best uniform upper bound we can hope for.



Is NOT="non-obtuse triangulation" possible?

Upper bound $< 90^\circ$ **implies lower bound** > 0**:**



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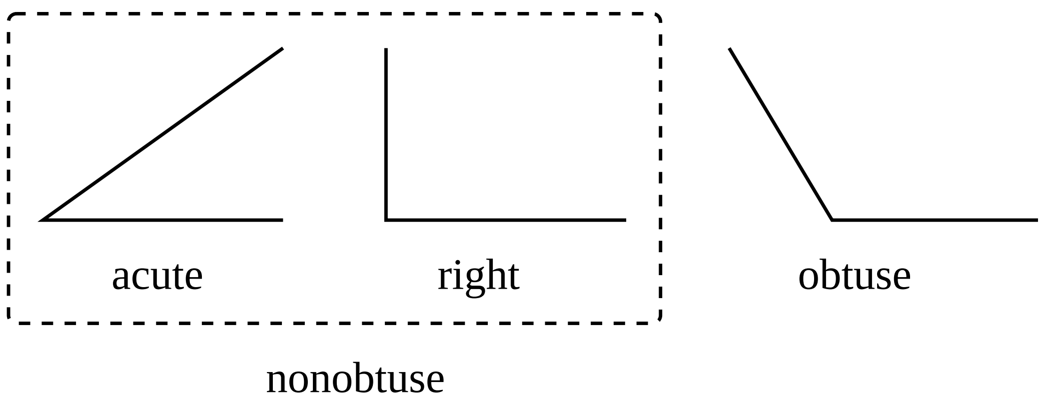

Is NOT="non-obtuse triangulation" possible? **Yes**

Brief history of NOTs:

- Always possible: Burago, Zalgaller 1960.
- Rediscovered: Baker, Grosse, Rafferty, 1988.
- $O(n)$ for points sets: Bern, Eppstein, Gilbert 1990
- $O(n^2)$ for polygons: Bern, Eppstein 1991
- $O(n)$ for polygons: Bern, Mitchell, Ruppert 1994

Numerous applications, heuristics.

Brief history of NOTs:

- Always possible: Burago, Zalgaller 1960.
- Rediscovered: Baker, Grosse, Rafferty, 1988.
- $O(n)$ for points sets: Bern, Eppstein, Gilbert 1990
- $O(n^2)$ for polygons: Bern, Eppstein 1991
- $O(n)$ for polygons: Bern, Mitchell, Ruppert 1994

Numerous applications, heuristics.

Is there a **polynomial** bound for PSLGs?

NOT-Theorem: Every PSLG has an $O(n^{2.5})$-NOT.

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$O(n^2)$ is best known lower bound, so a gap remains.

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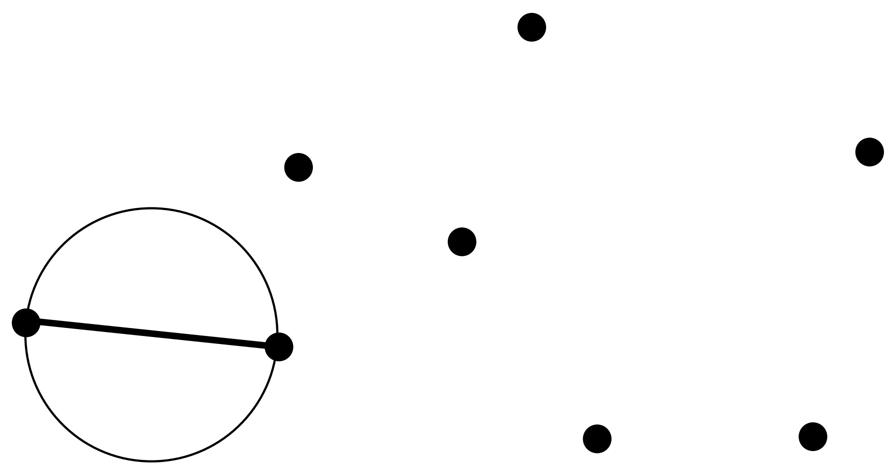
Follows from proof of NOT-theorem.

For m edges, proof gives $O(m^{2.5} + mn + n)$-NOT.

The segment $[v, w]$ is a **Gabriel** edge if it is the diameter of a disk containing no other points of V.



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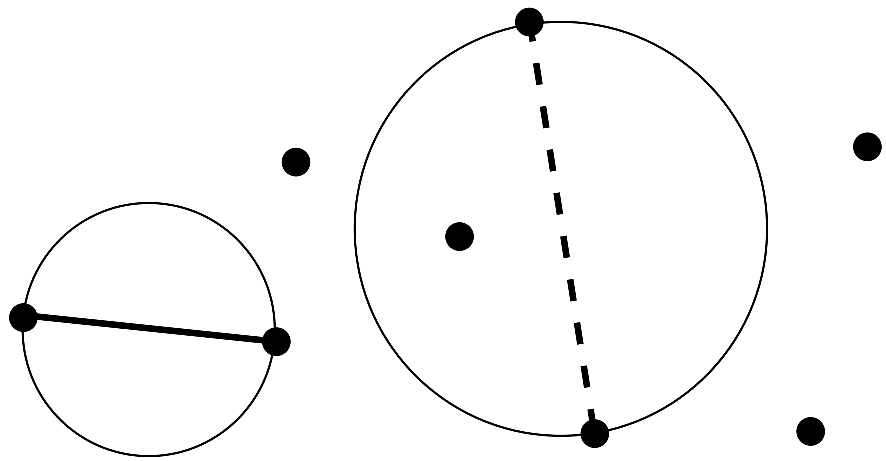
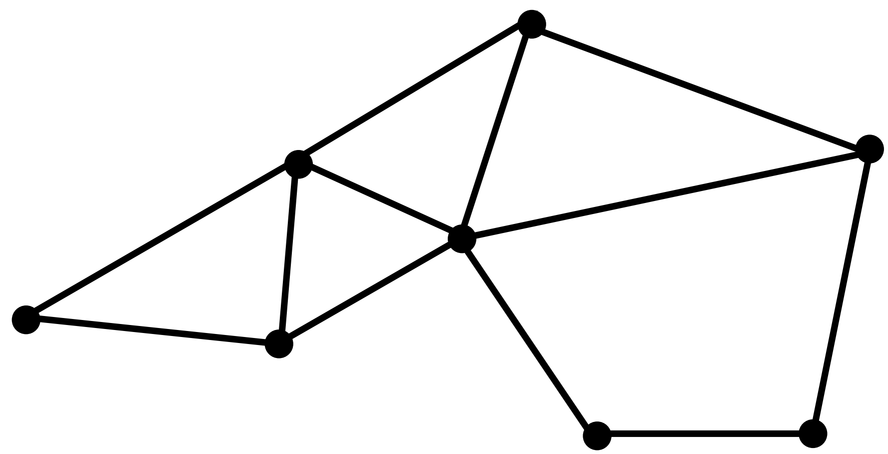


Not a Gabriel edge.

The segment $[v, w]$ is a **Gabriel** edge if it is the diameter of a disk containing no other points of V.



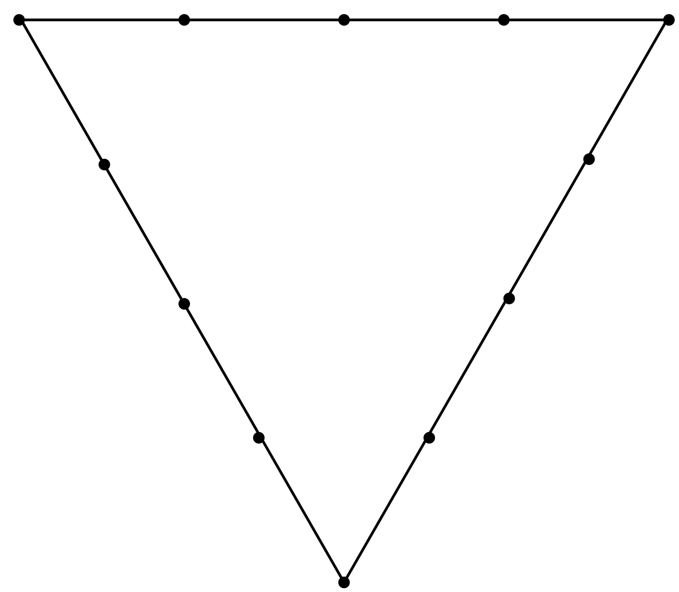
Gabriel graph contains the minimal spanning tree.

Gabriel and Sokol, *A new statistical approach to geographic variation analysis*, Systematic Zoology, 1969.

Gabriel edges $\Rightarrow$ non-obtuse triangulation

Bern-Mitchell-Ruppert (1994)

BMR Lemma: Add k vertices to sides of triangle (at least one per side) so all edges become Gabriel, then add all midpoints. Resulting polygon has a $O(k)$ NOT, with no additional vertices on boundary.



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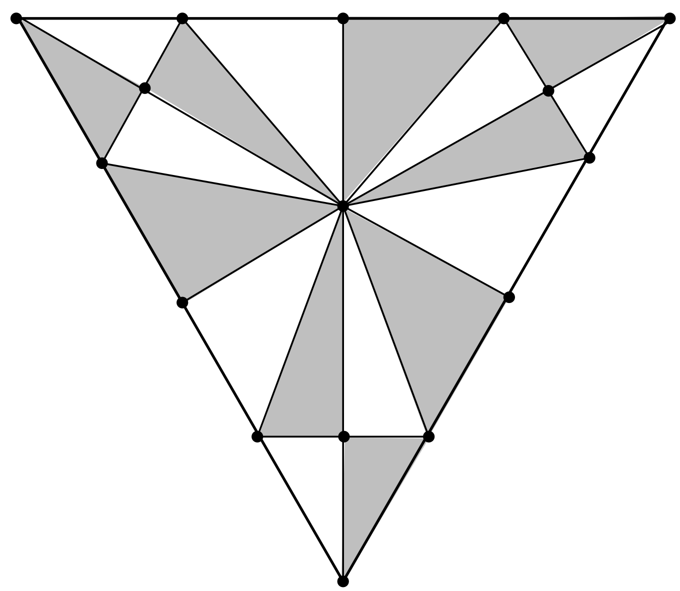

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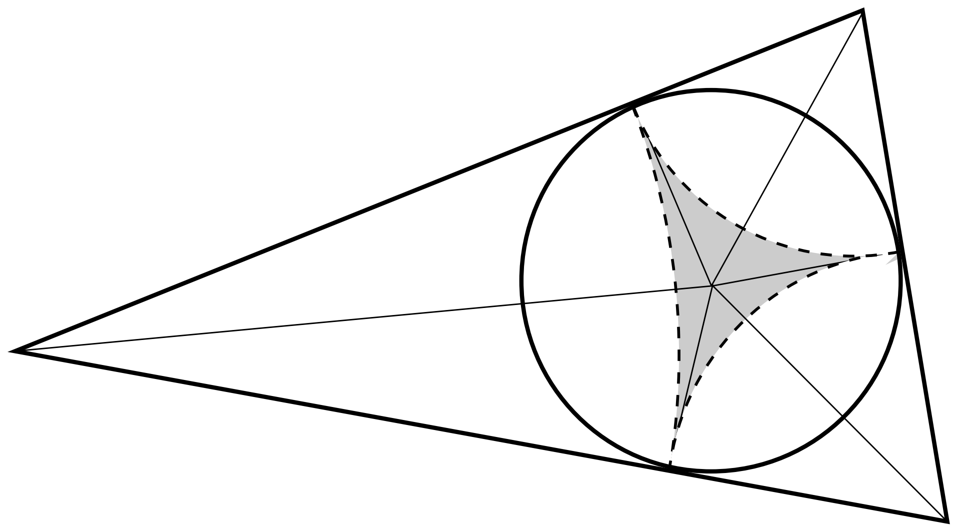
Bern-Mitchell-Ruppert (1994)

BMR Lemma: Add k vertices to sides of triangle (at least one per side) so all edges become Gabriel, then add all midpoints. Resulting polygon has a $O(k)$ NOT, with no additional vertices on boundary.

Building a NOT for a PSLG:

- Replace PSLG by triangulation of itself.
- Add vertices to make all edges Gabriel.
- Apply BMR lemma.

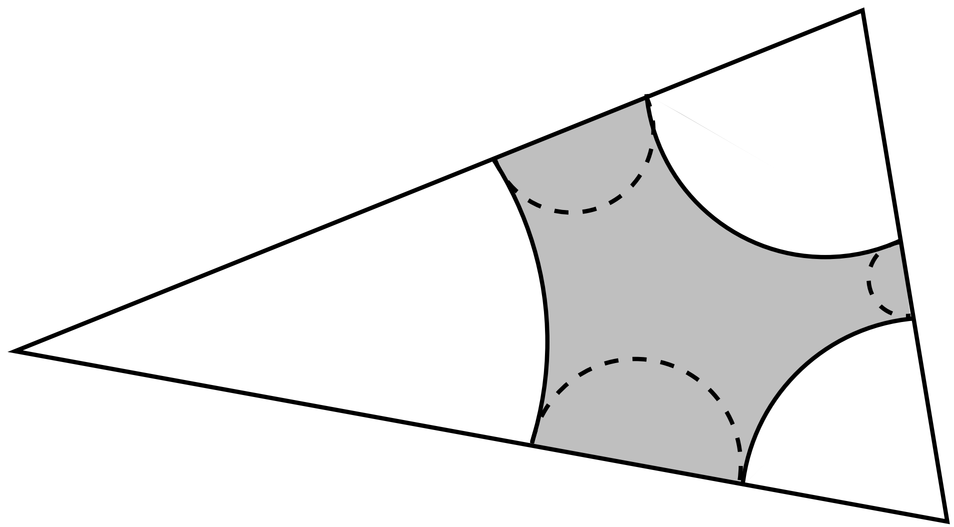
Construct Gabriel points:



Break every triangle into thick and thin parts.

Thin parts = corners, Thick part = central region

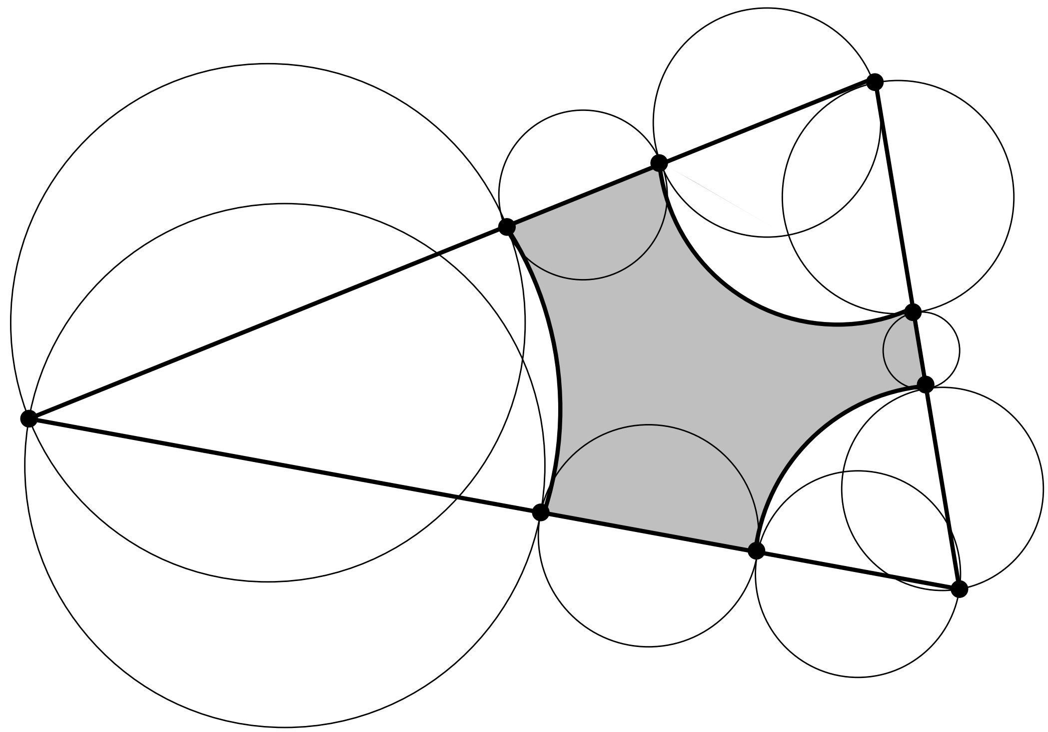
Construct Gabriel points:



Advantageous to increase thick part.

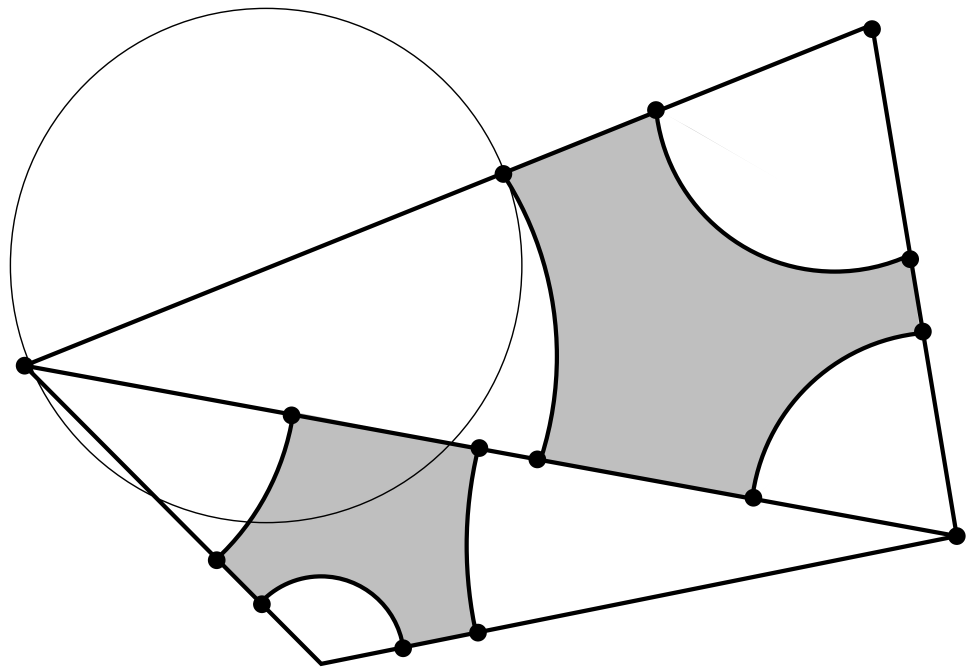
Thick sides are base of half-disk inside triangle.

Construct Gabriel points:



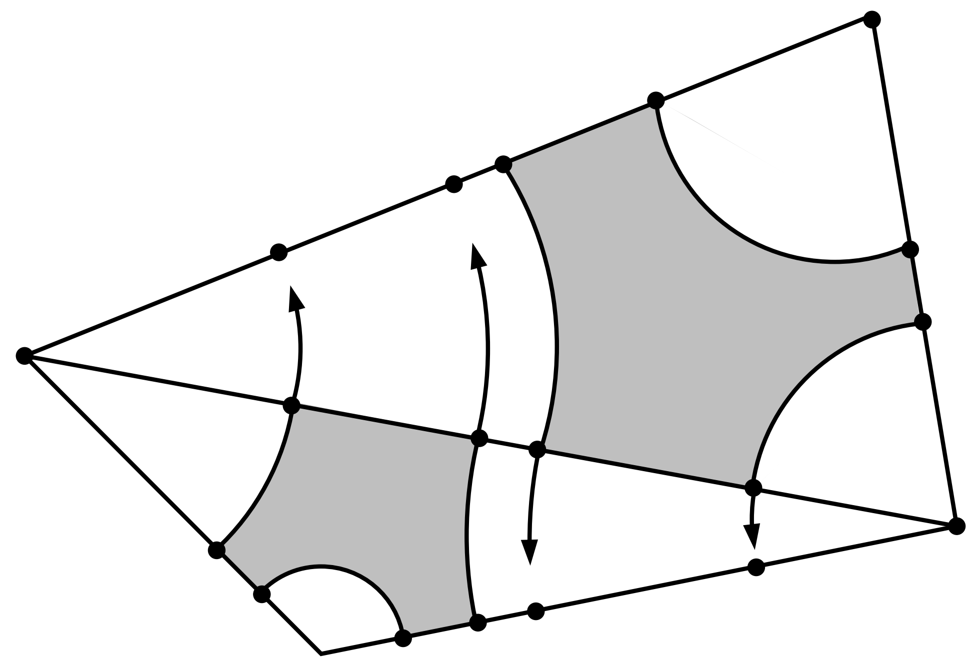
Then vertices of thick part give Gabriel edges.

Construct Gabriel points:



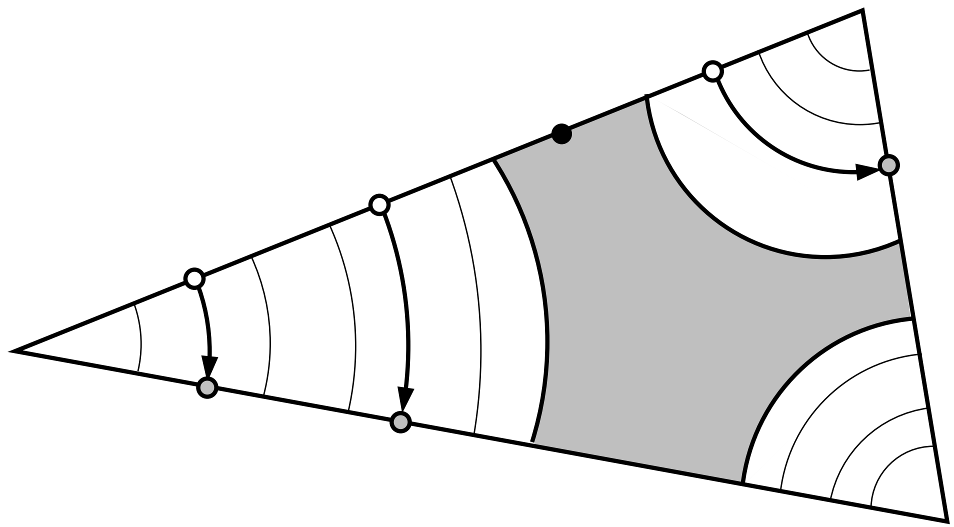
But, adjacent triangle can make Gabriel condition fail.

Construct Gabriel points:

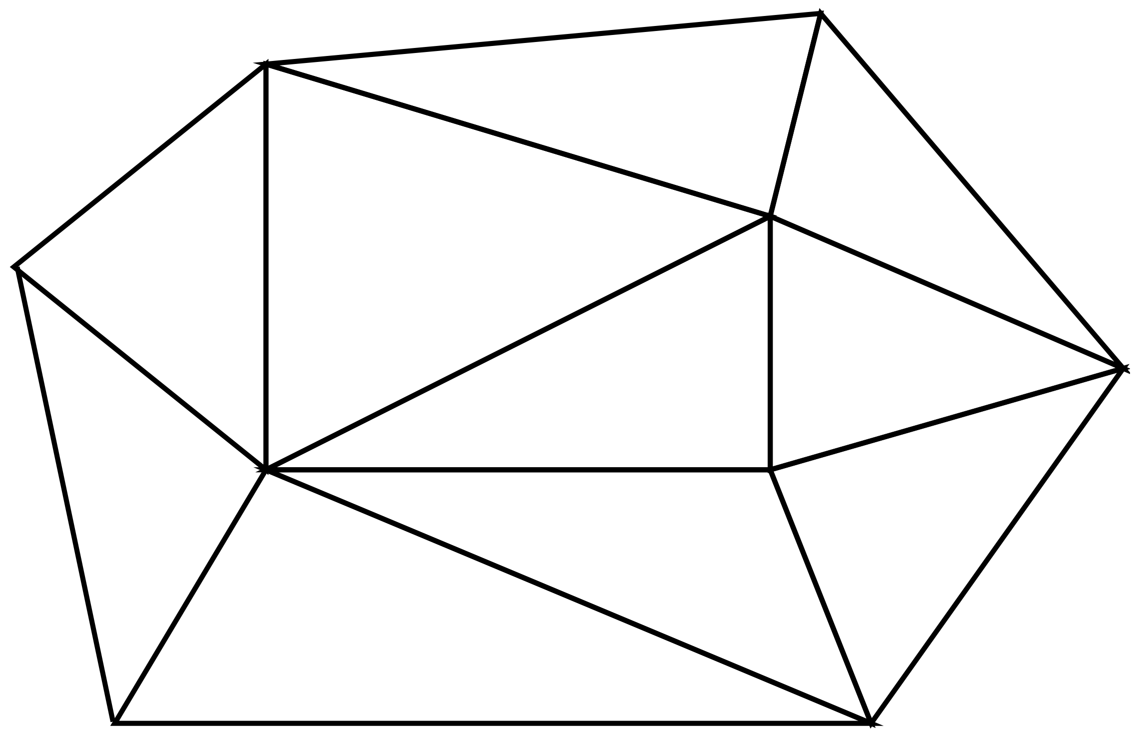


Idea: "Push" vertices across the thin parts.

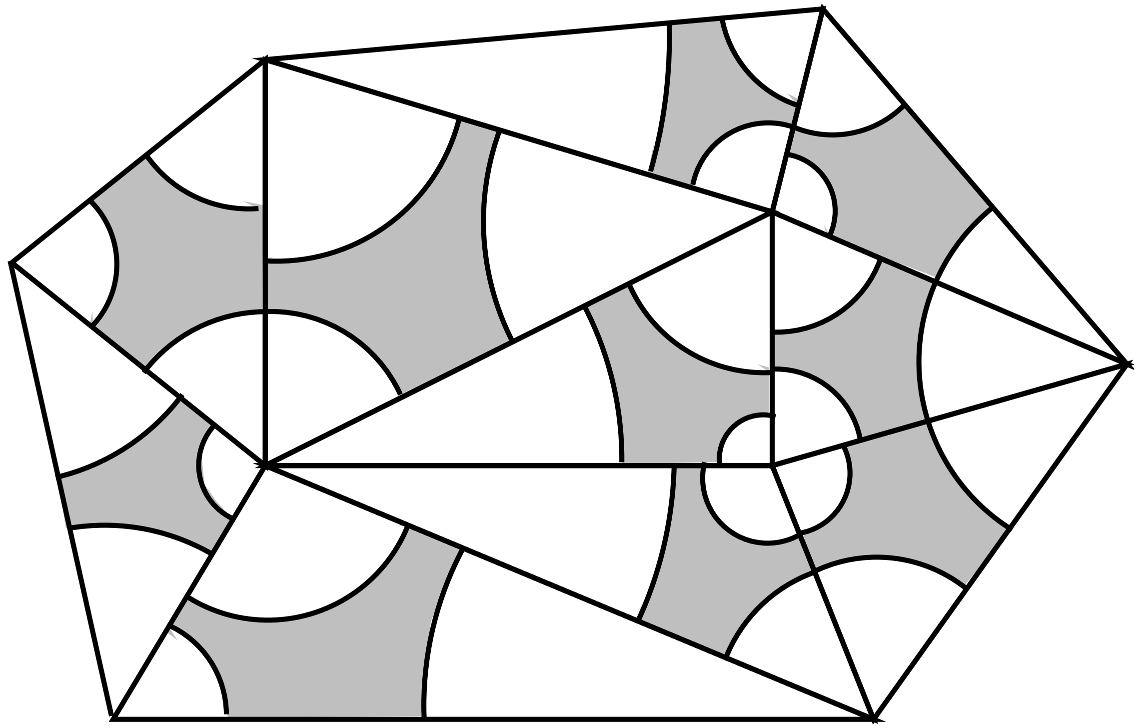
Construct Gabriel points:



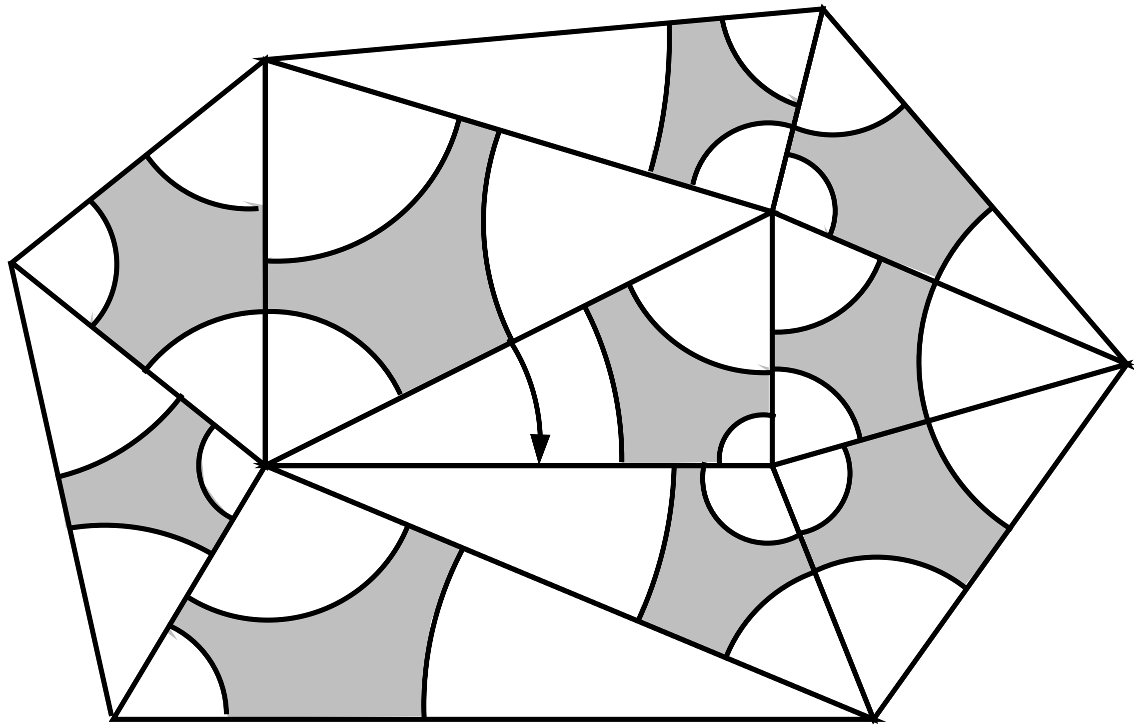
Thin parts foliated by circles centered at vertices.



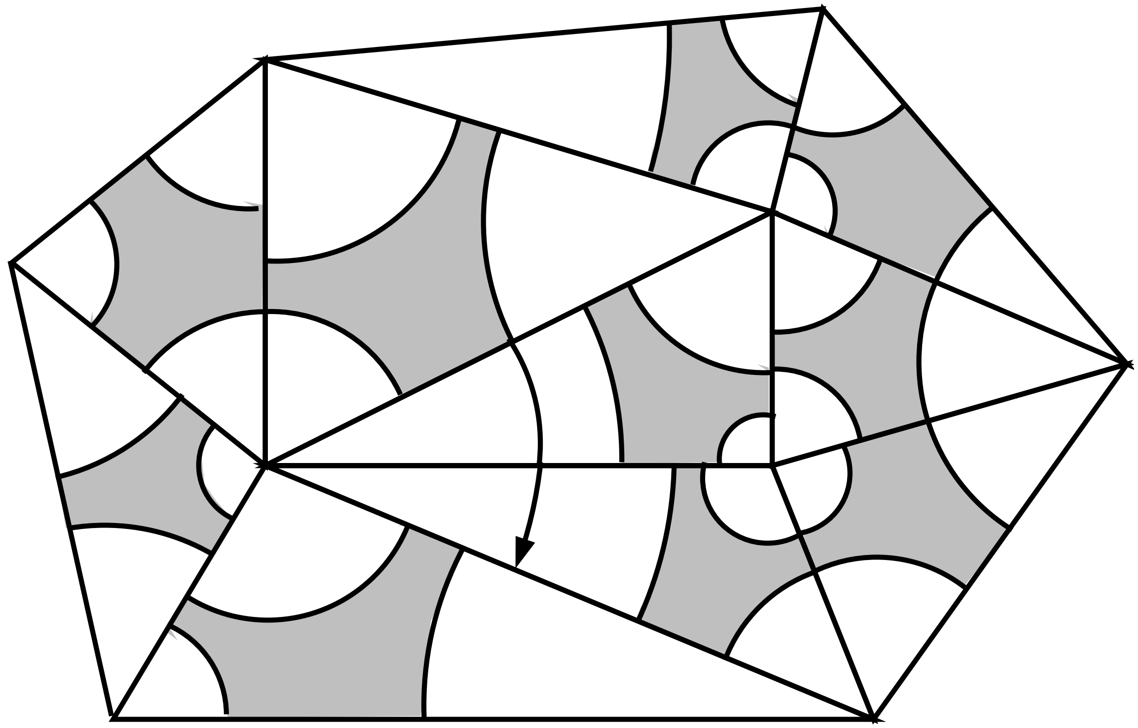
- Start with any triangulation.



- Start with any triangulation.
- Make thick/thin parts.



- Start with any triangulation.
- Make thick/thin parts.
- Propagate vertices until they leave thin parts.



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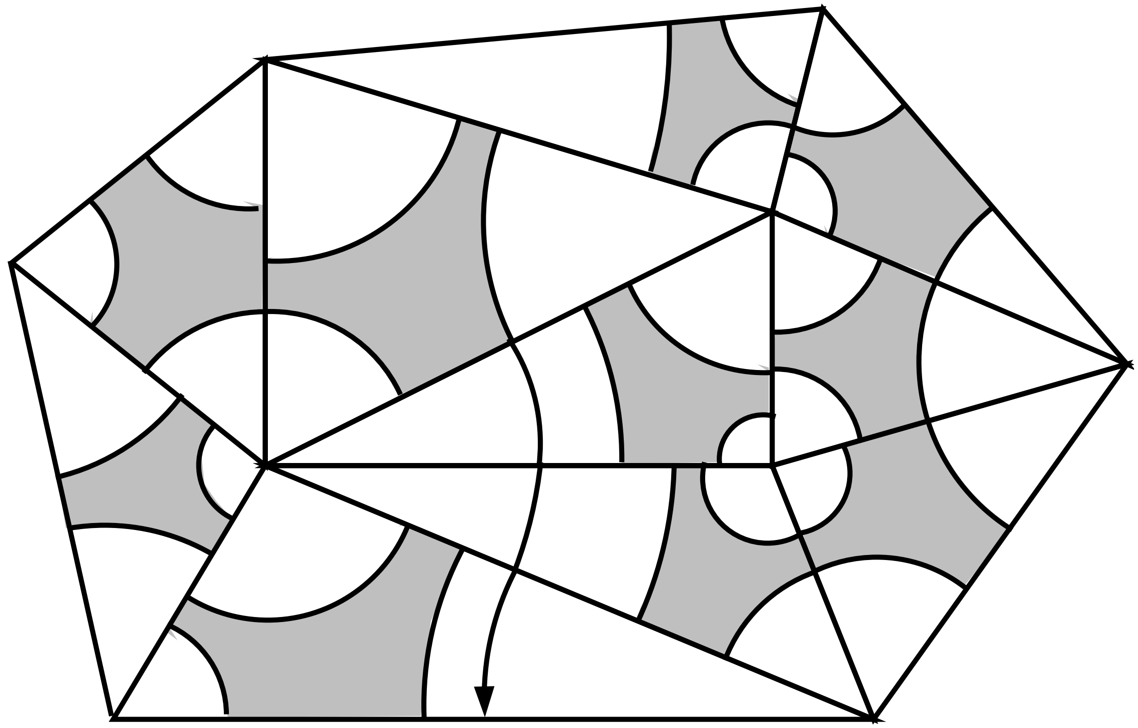

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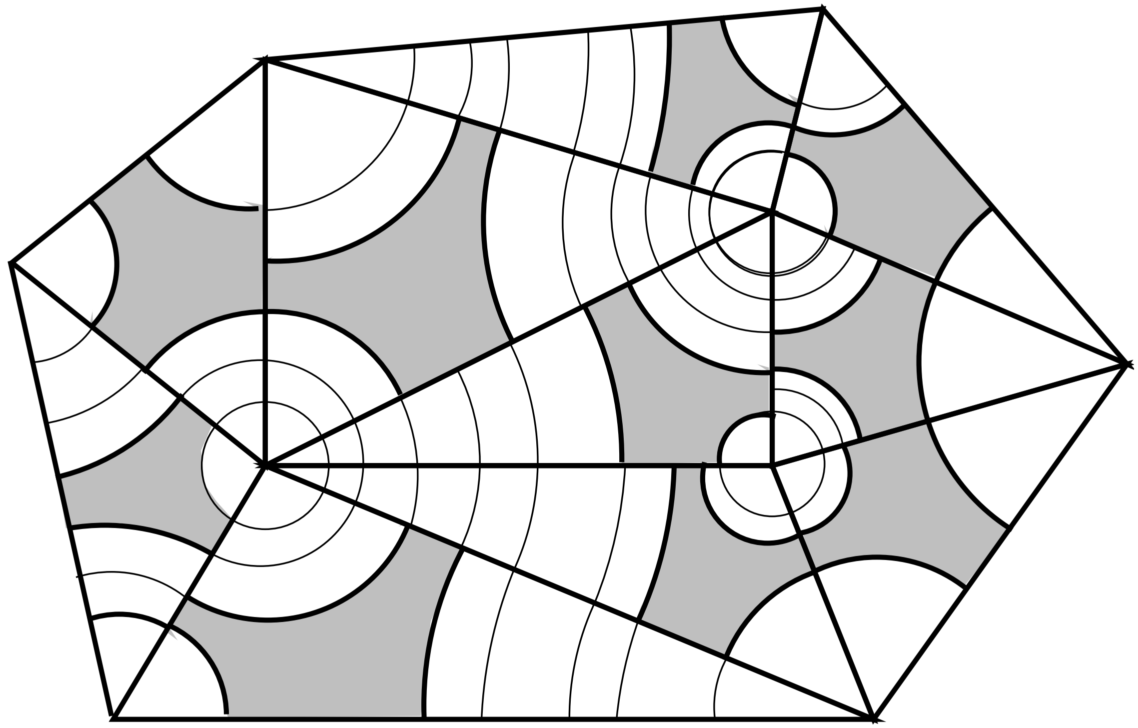

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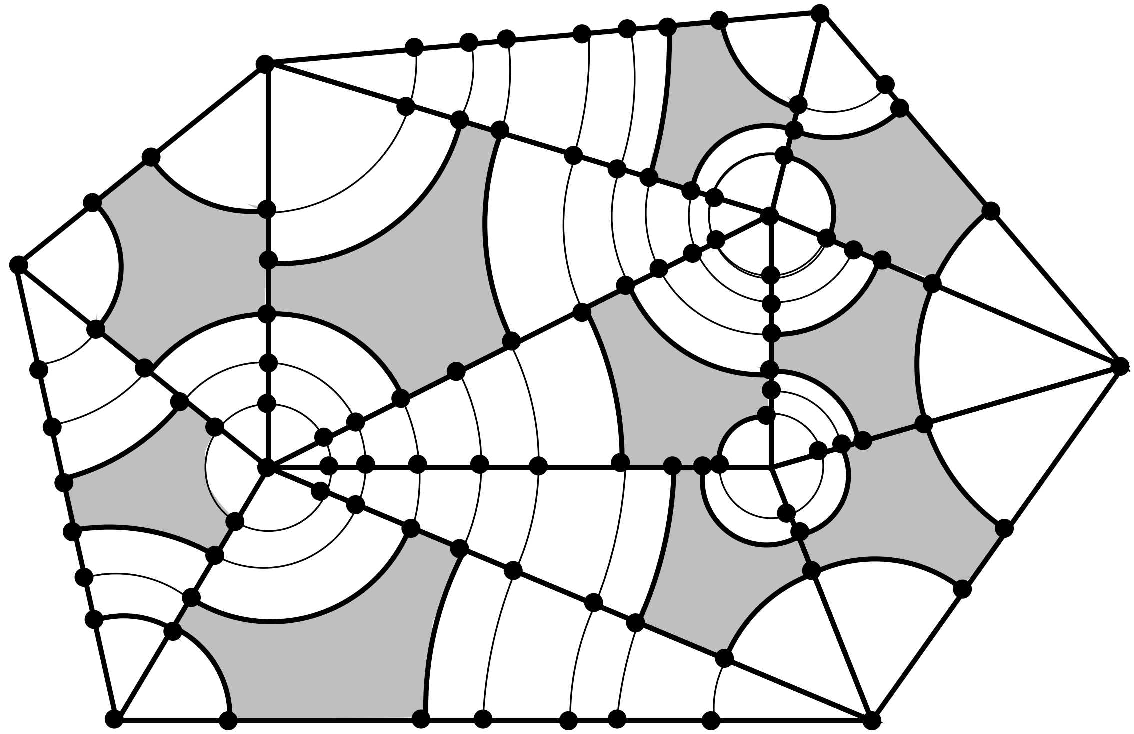
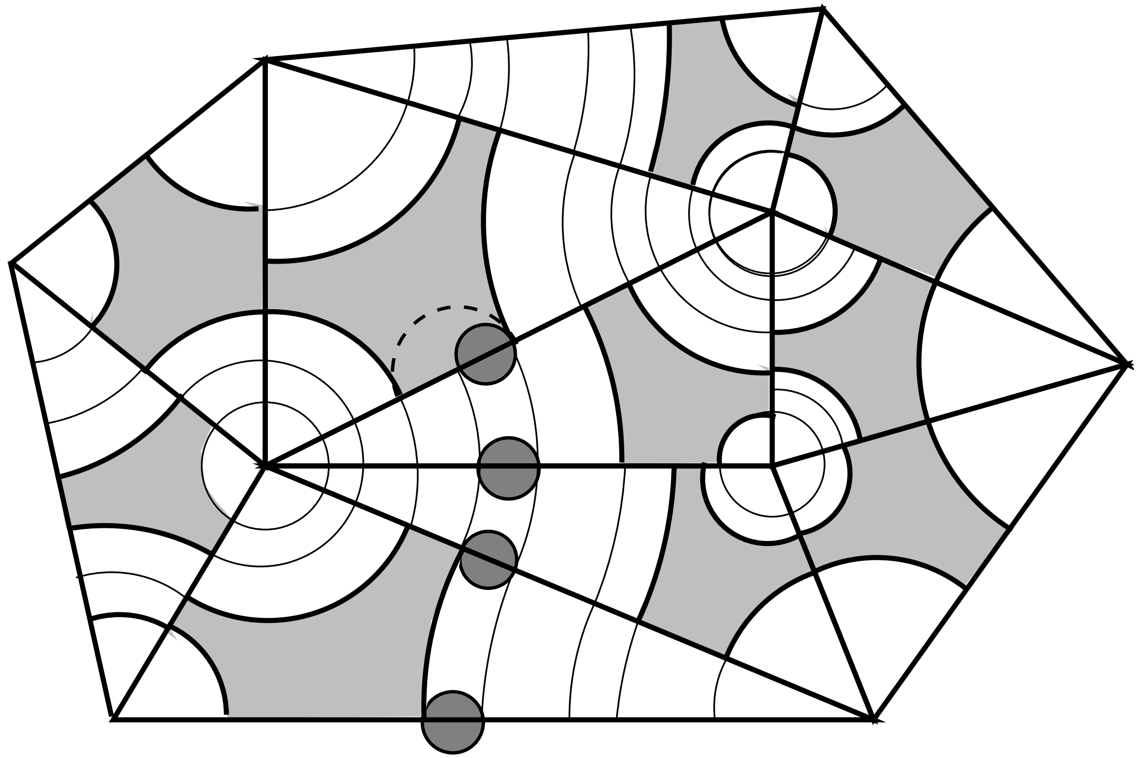


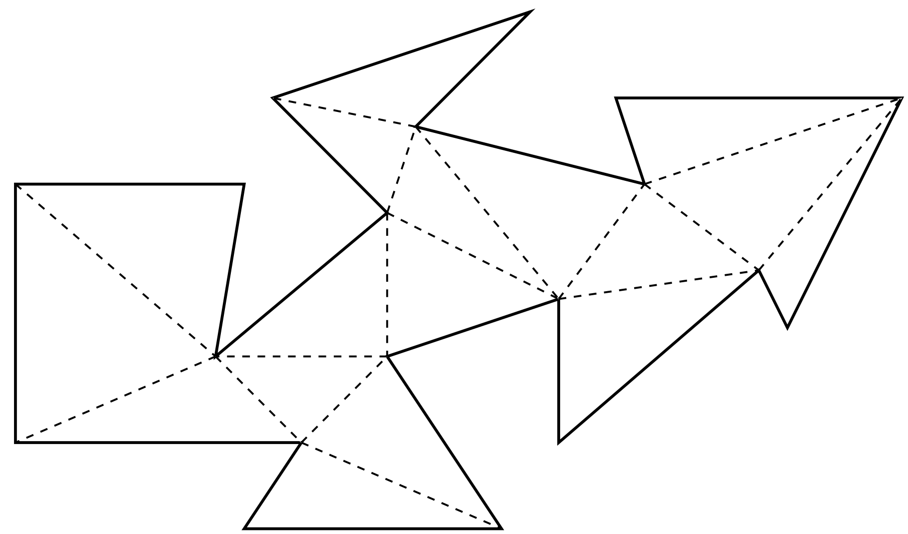
- Start with any triangulation.
- Make thick/thin parts.
- Propagate vertices until they leave thin parts.
- Intersections satisfy Gabriel condition. Why?



Tube is “swept out” by fixed diameter disk.

Disk lies inside tube or thick part or outside convex hull.

In triangulation of a n-gon, adjacent triangles form tree.



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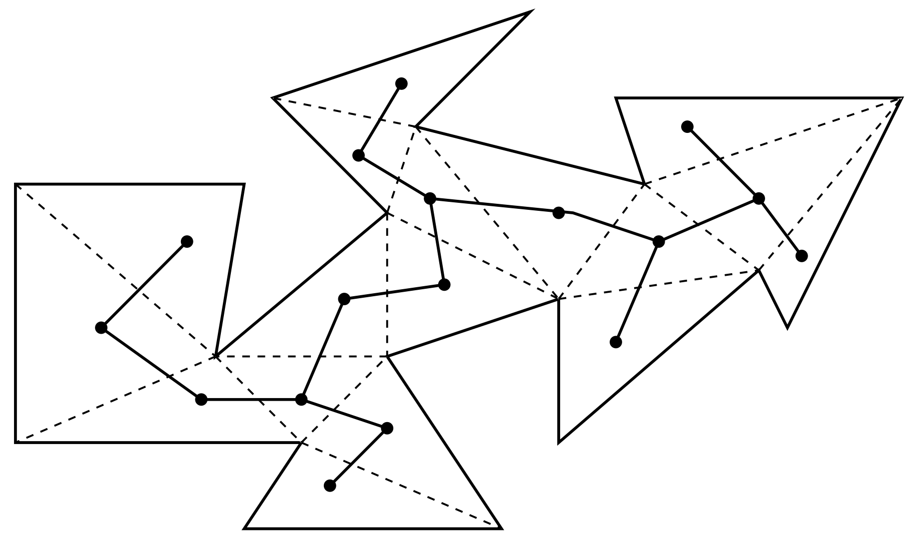

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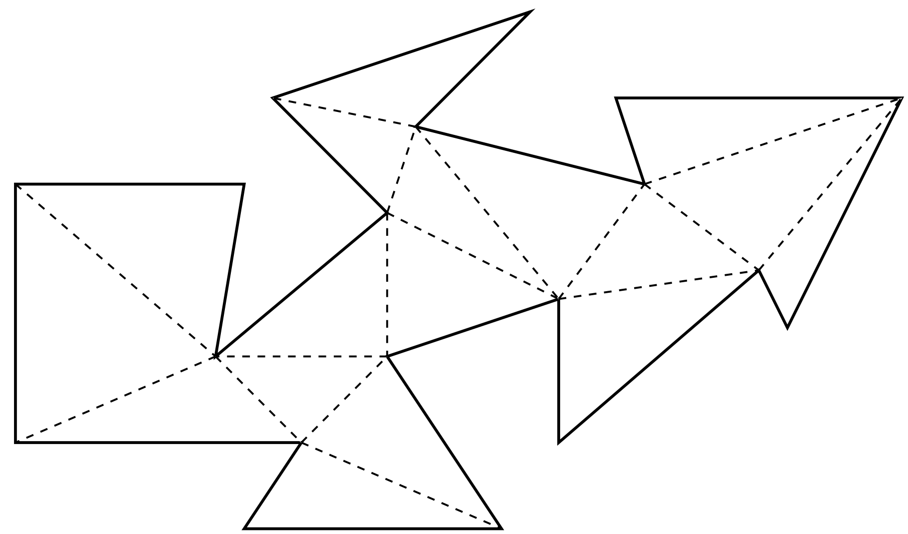

Hence paths never revisit a triangle.
$6n$ starting points, so $O(n^2)$ points are created.

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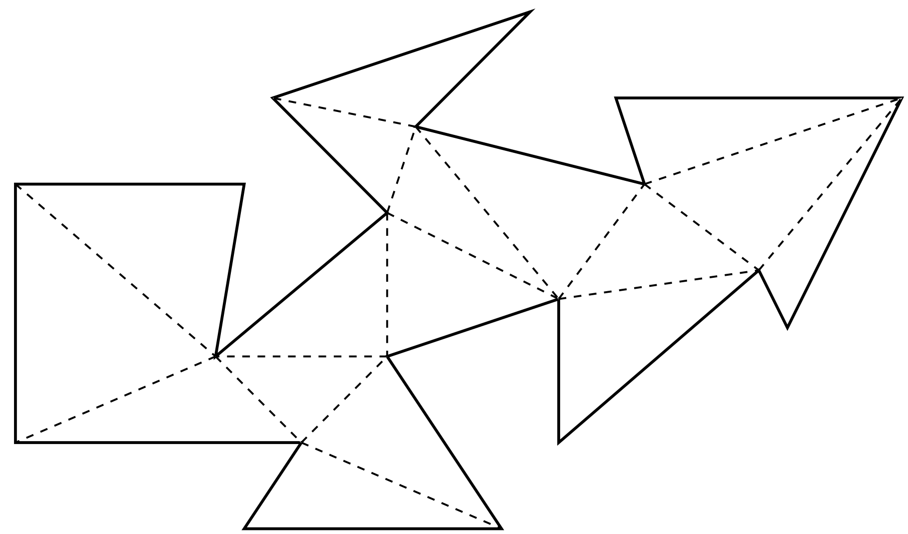

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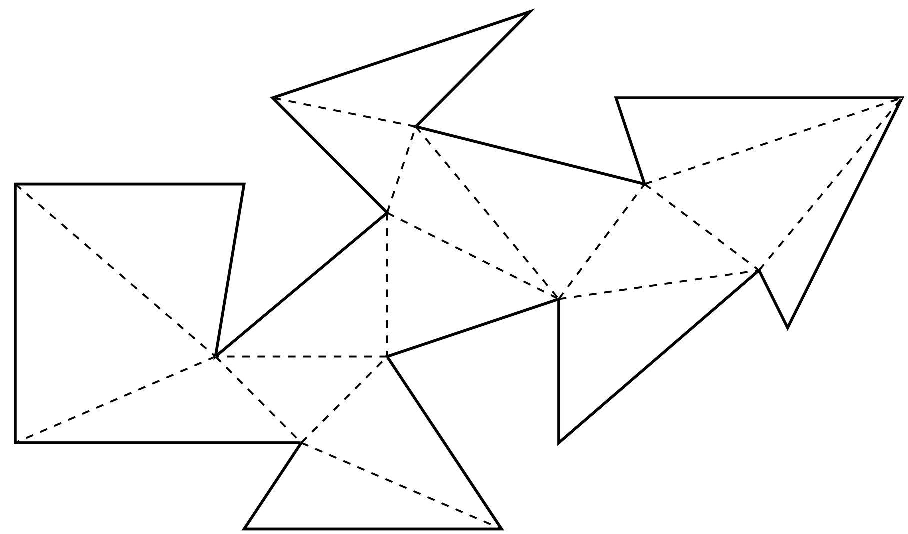

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$6n$ starting points, so $O(n^2)$ points are created.

Theorem: Any triangulation of a n-gon has a refinement into $O(n^2)$ non-obtuse triangles.

Improves $O(n^4)$ bound by Bern and Eppstein (1992).

Questions:

- Implementable?
- Average versus worst case bounds?
- Improve 2.5 to 2?
- 3-D meshes? The eightfold way? Ricci flow?
- Other applications for thick/thin pieces?
- Applications of Mumford-Bers compactness?
- Best K for medial axis map? $(2.1 < k < 7.82)$
- Can we do better than medial axis map?

Workshop on Analysis and CG

2:20-3:05 Raanan Schul (Stony Brook),
The Traveling Salesman Problem, Data Parameterization and Multi-resolution Analysis

3:05-3:50 Mauro Maggioni (Duke),
Multiscale SVD and Geometric Multi-Resolution Analysis for noisy point clouds in high dimensions

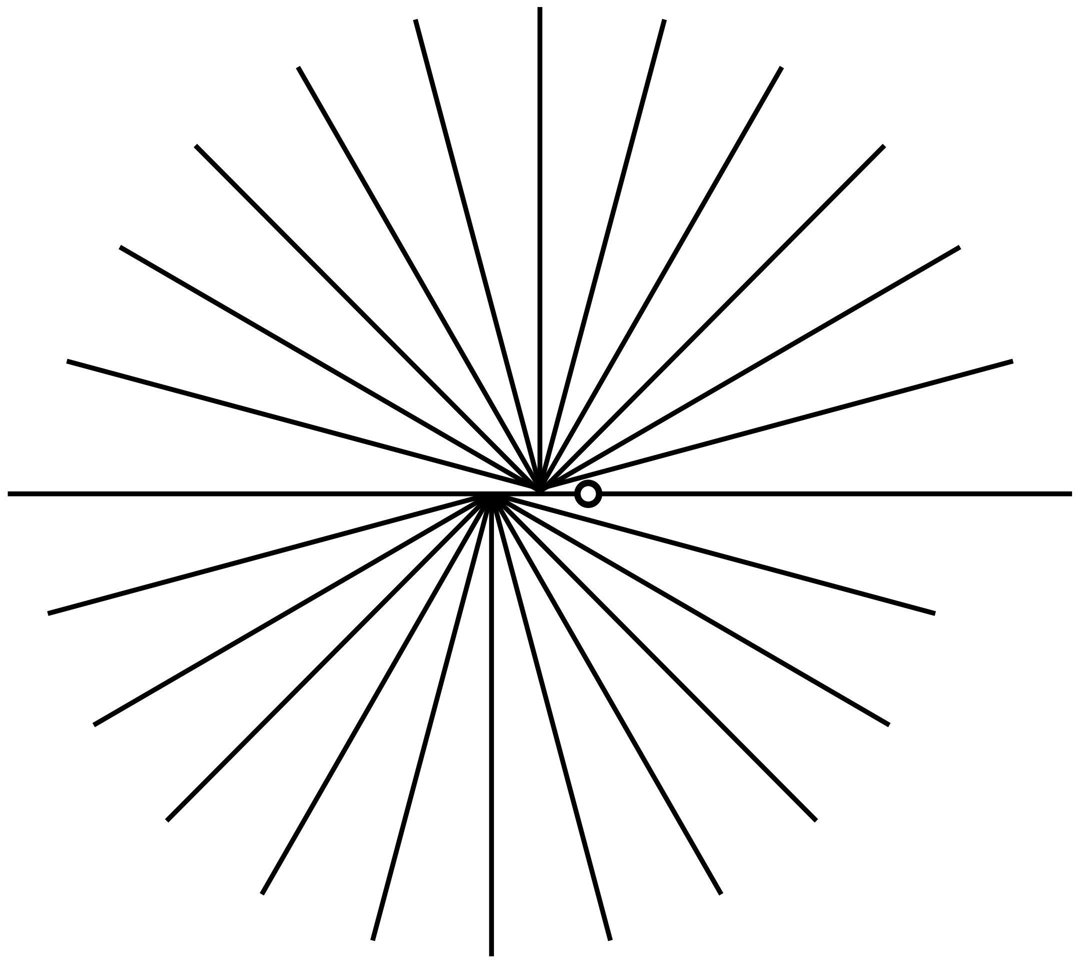
4:10-4:55 Arie Israel (Courant),
Interpolation by Sobolev functions

4:55- 5:40 Ken Stephenson (UT Knoxville)
Discrete Conformality and Graph Embedding

5:40-6:10 Open problems and discusion

How do we get 2.5 in the NOT-Theorem?

In general, path can hit same thin part many times.



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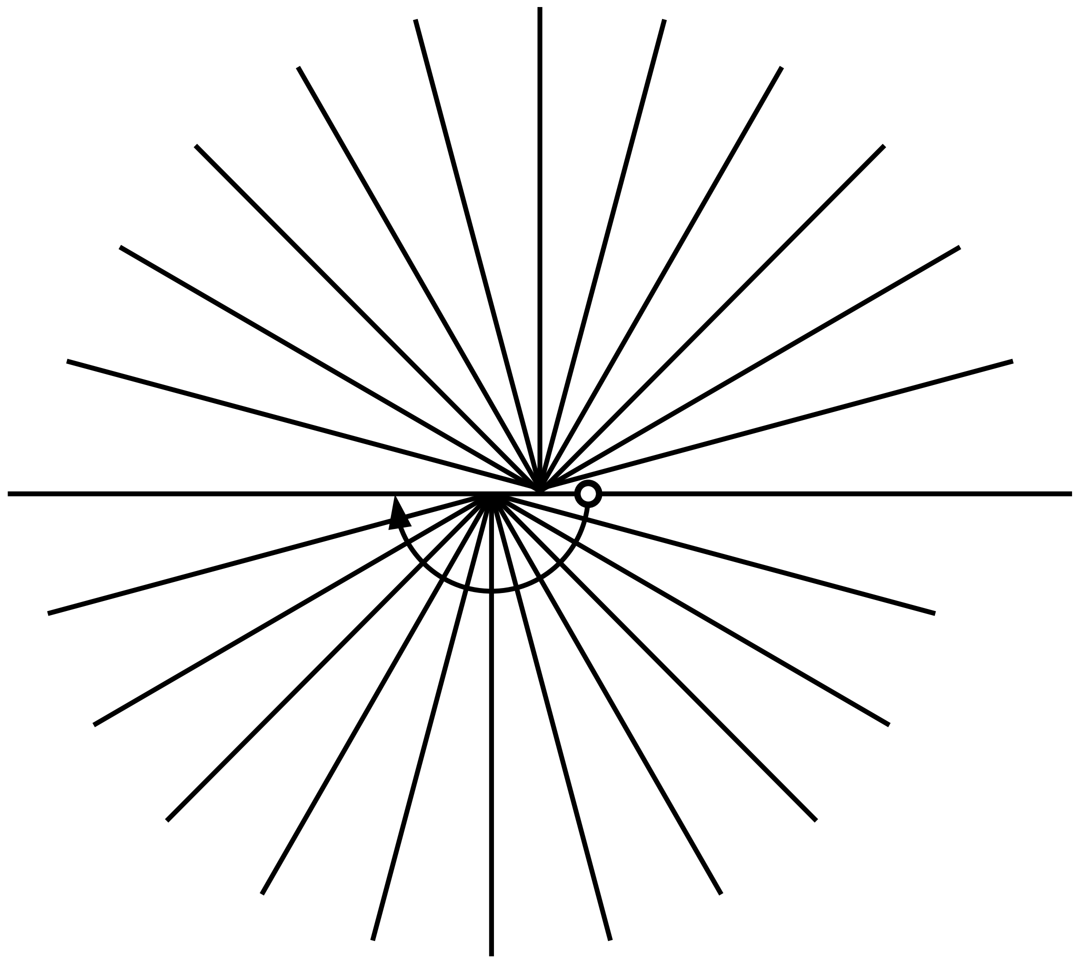

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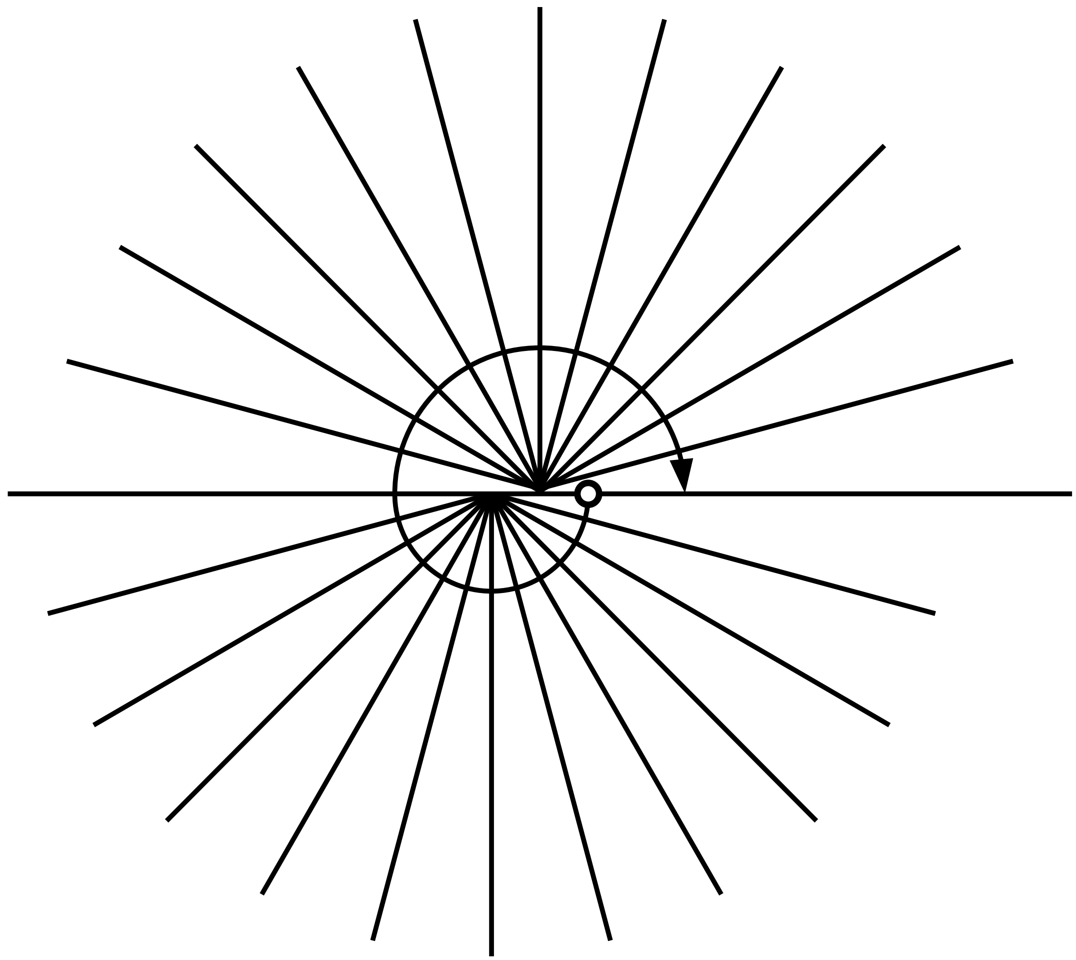

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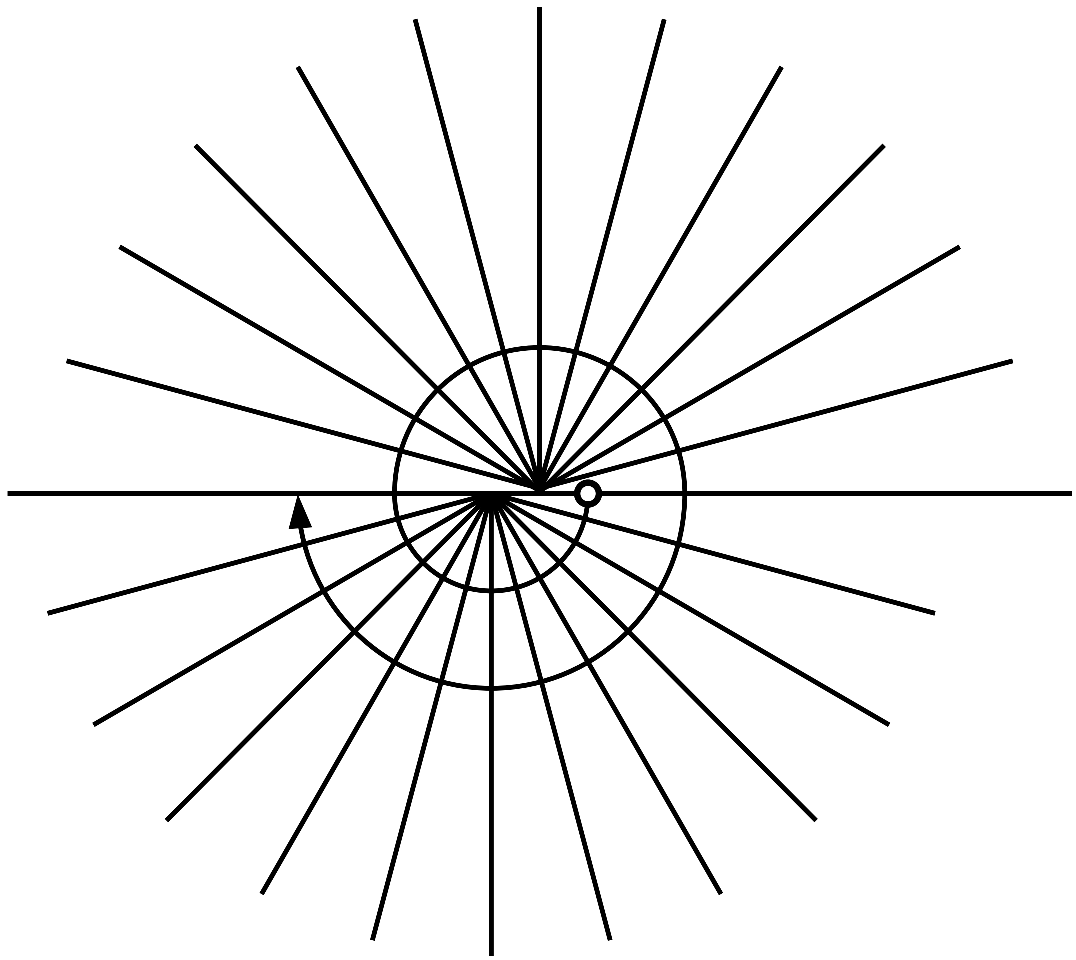

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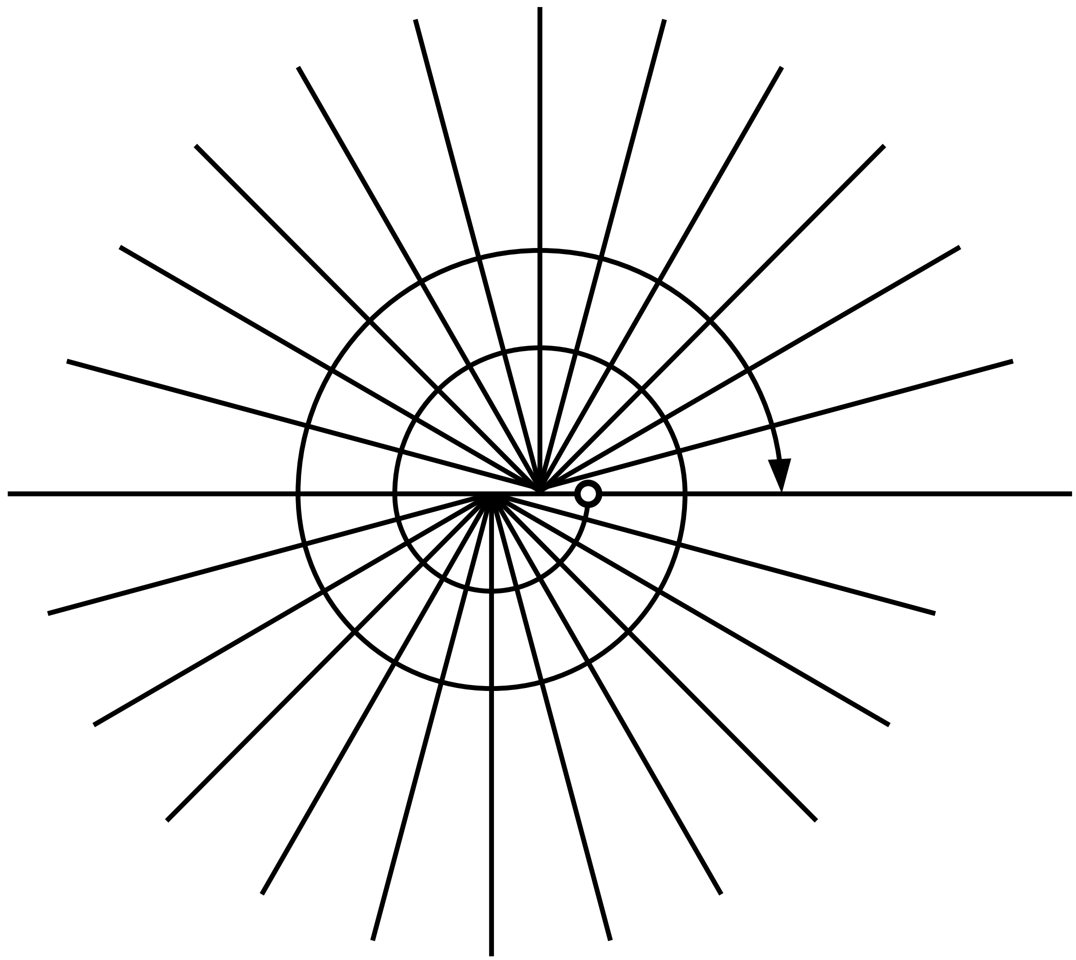

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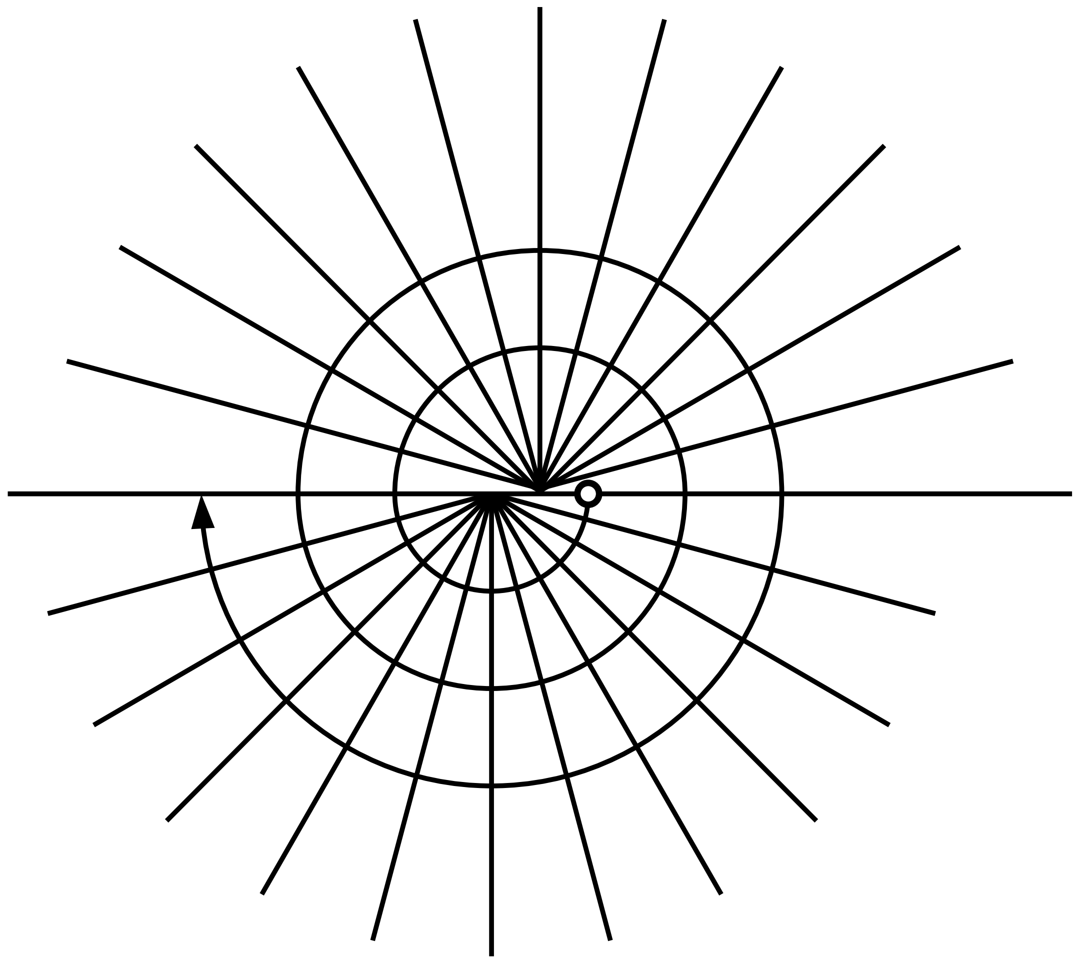

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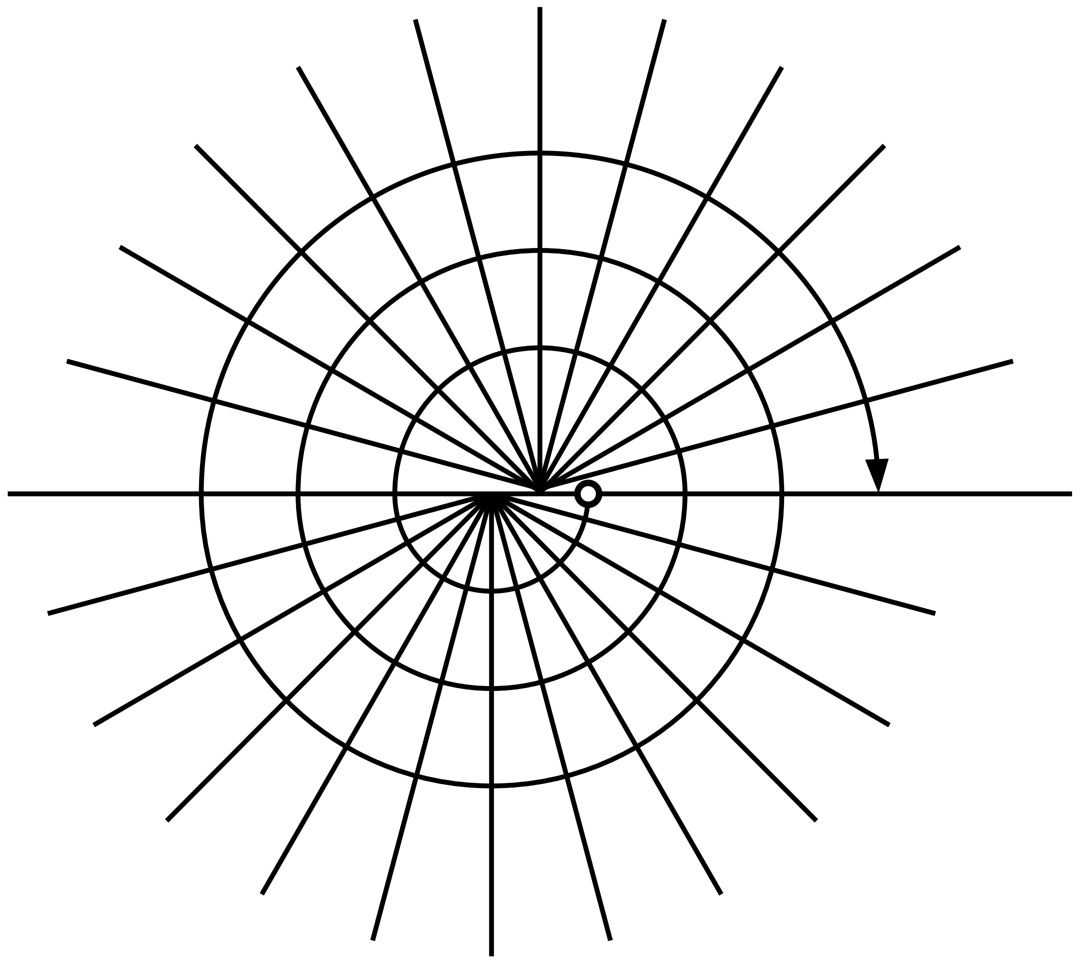
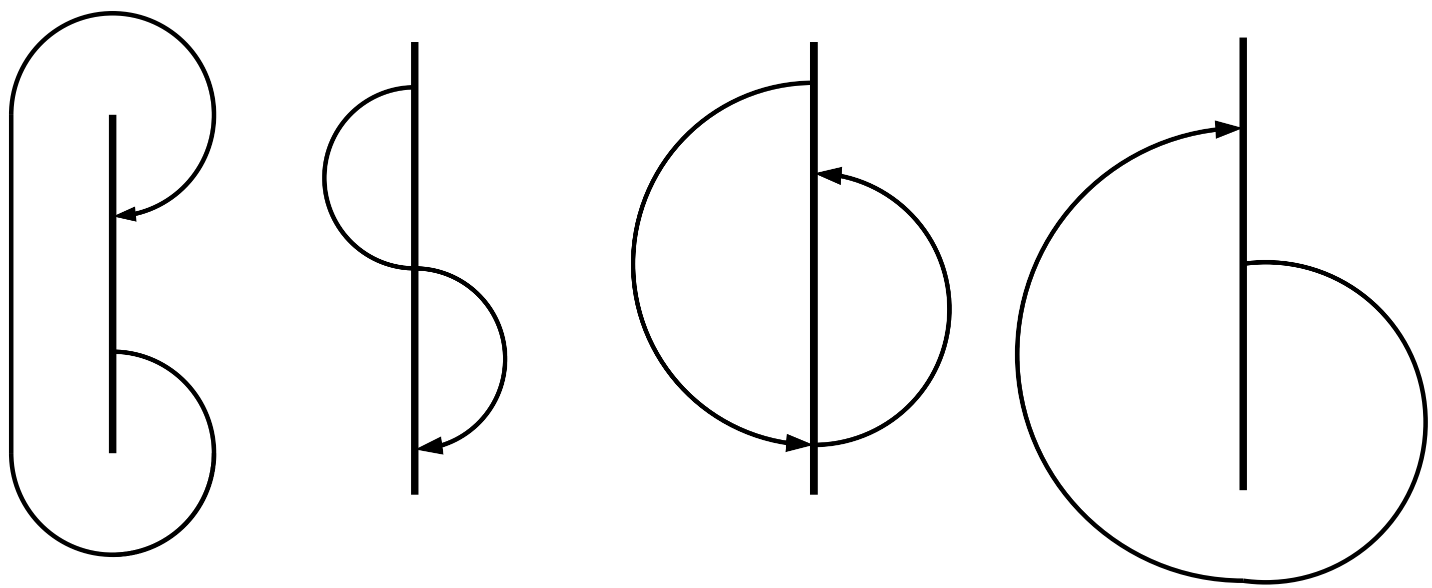
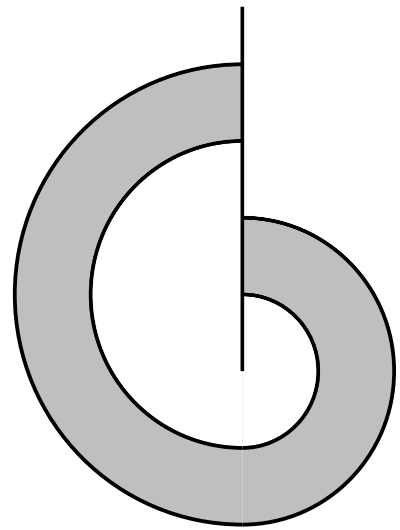


If a path returns to same thin edge at least 3 times it has a sub-path that looks like one of these:



C-curve, S-curve, G-curves

Return region consists of paths “parallel” to one of these.



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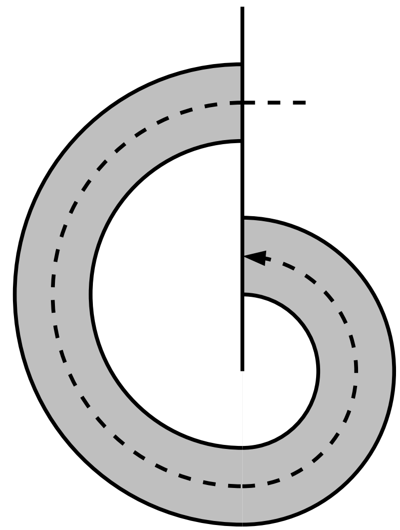

There are $O(n)$ return regions and every propagation path enters one after crossing at most $O(n)$ thin parts.

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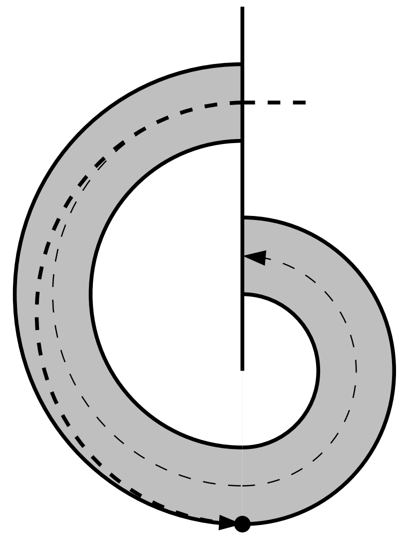
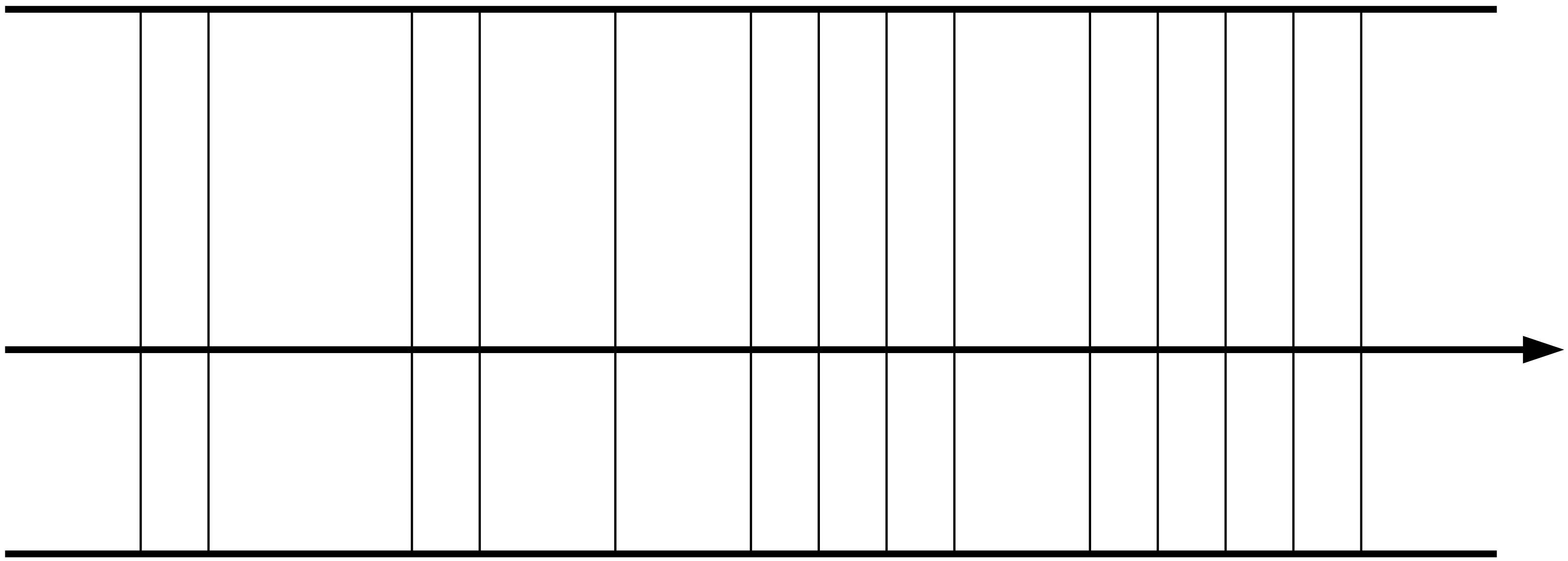


There are $O(n)$ return regions and every propagation path enters one after crossing at most $O(n)$ thin parts.

IDEA: bend paths to terminate before they exit.

Gives $O(n^2)$ if it works.

For simplicity, "straighten" region to rectangle.



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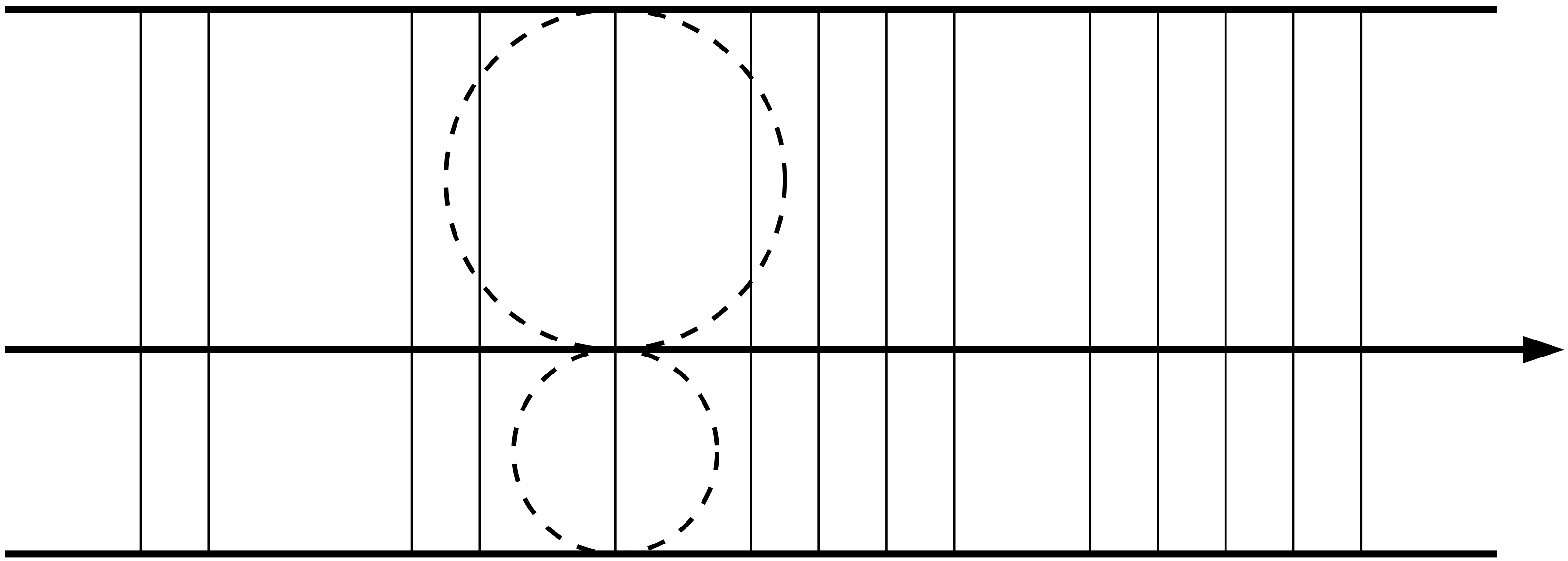

Gabriel condition is satisfied if path follows foliation.

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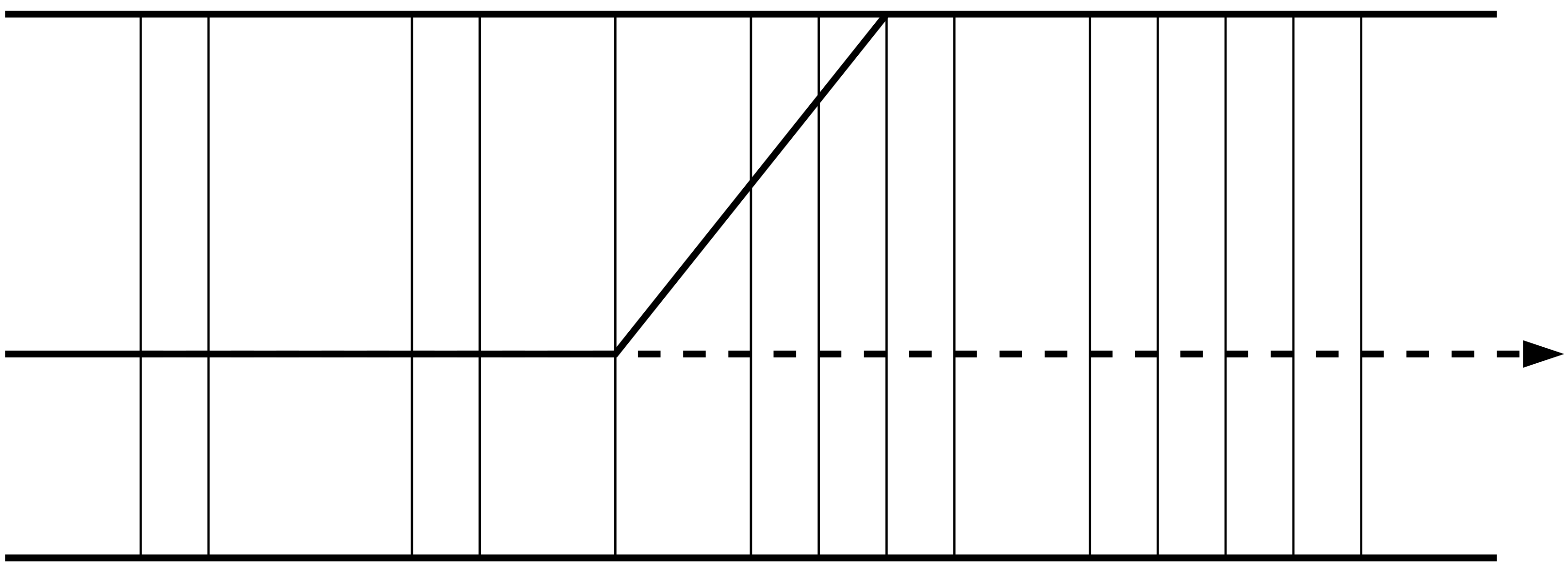
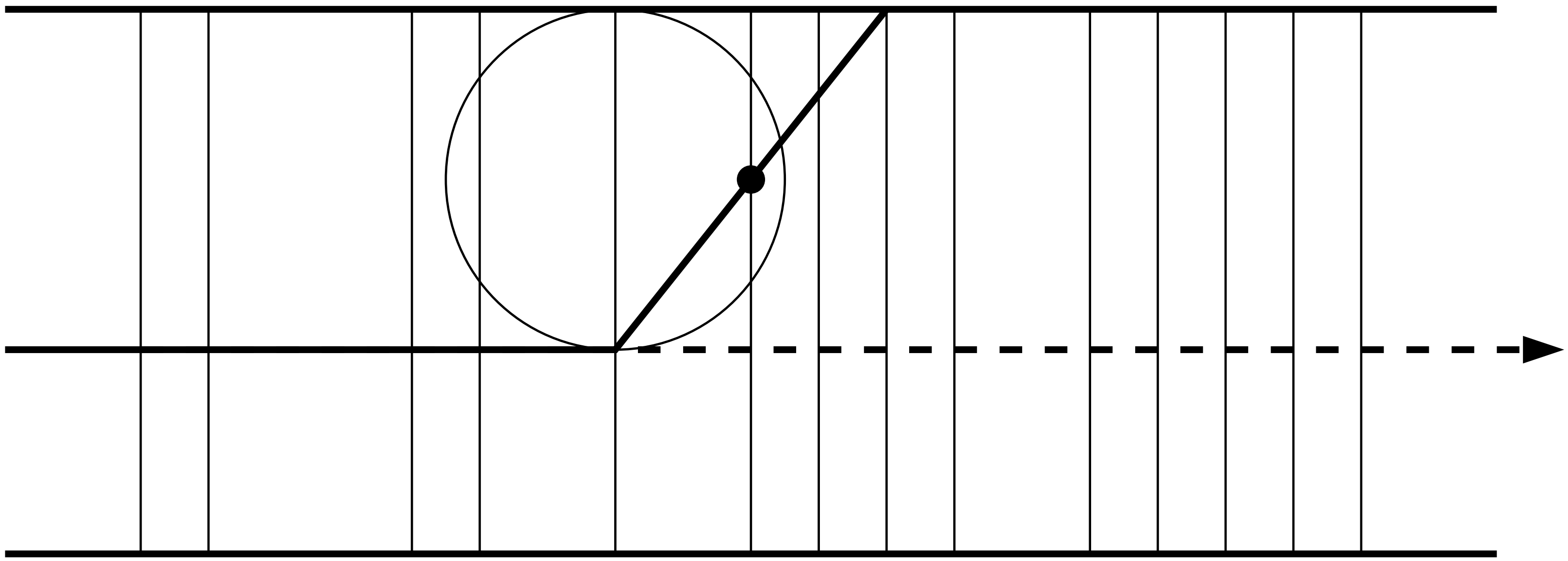


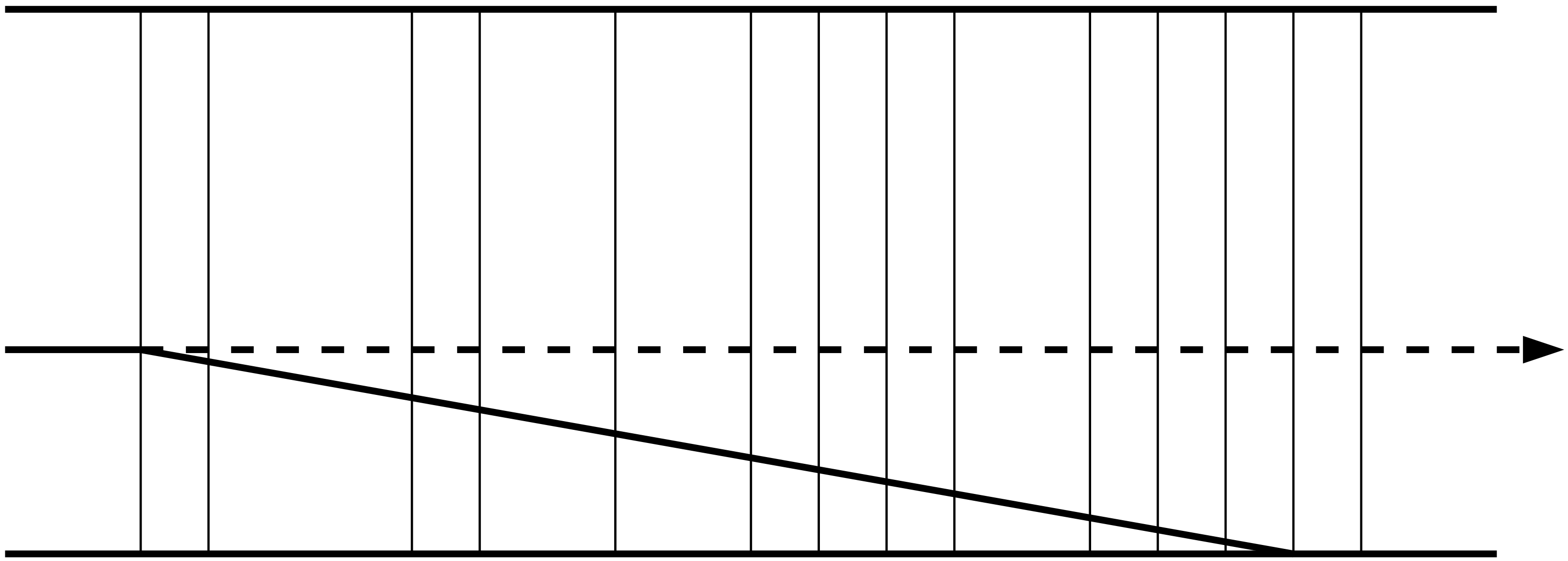
We want to bend path to hit side of tube. If it hits existing vertex, then path ends.

For simplicity, “straighten” region to rectangle.



If path bends too fast, Gabriel condition can fail.

For simplicity, “straighten” region to rectangle.



Bend slowly enough to satisfy Gabriel condition.

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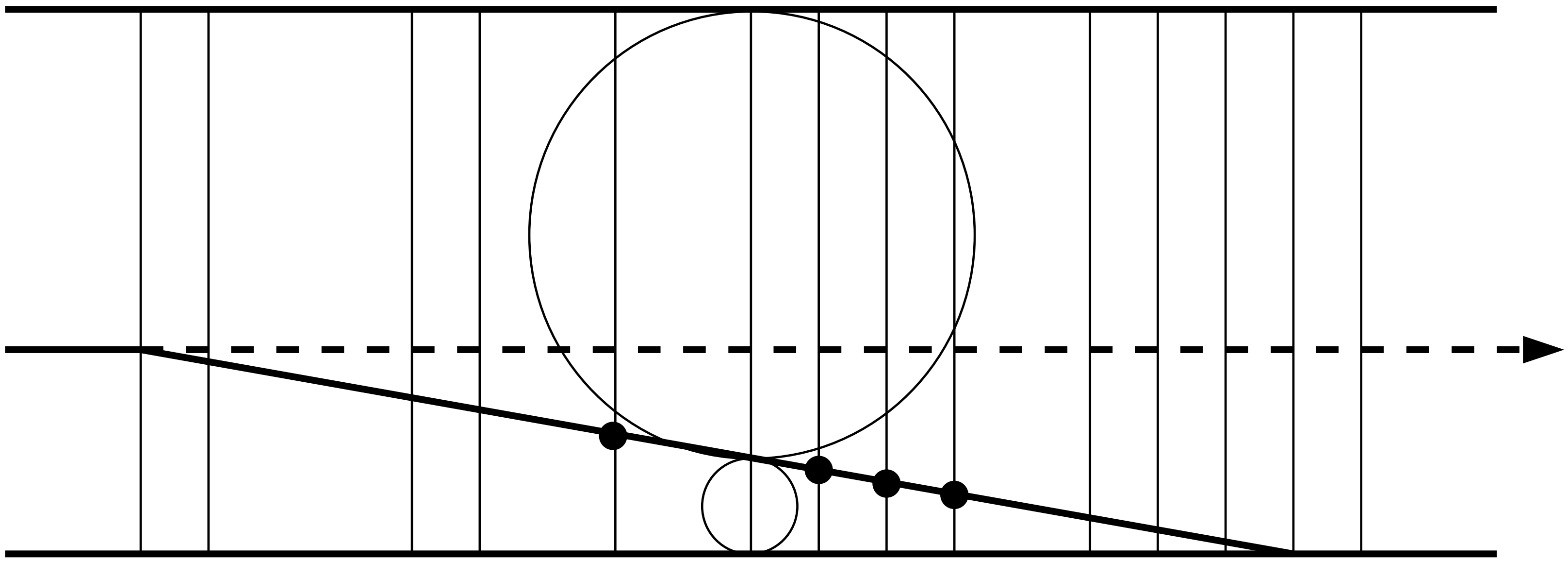

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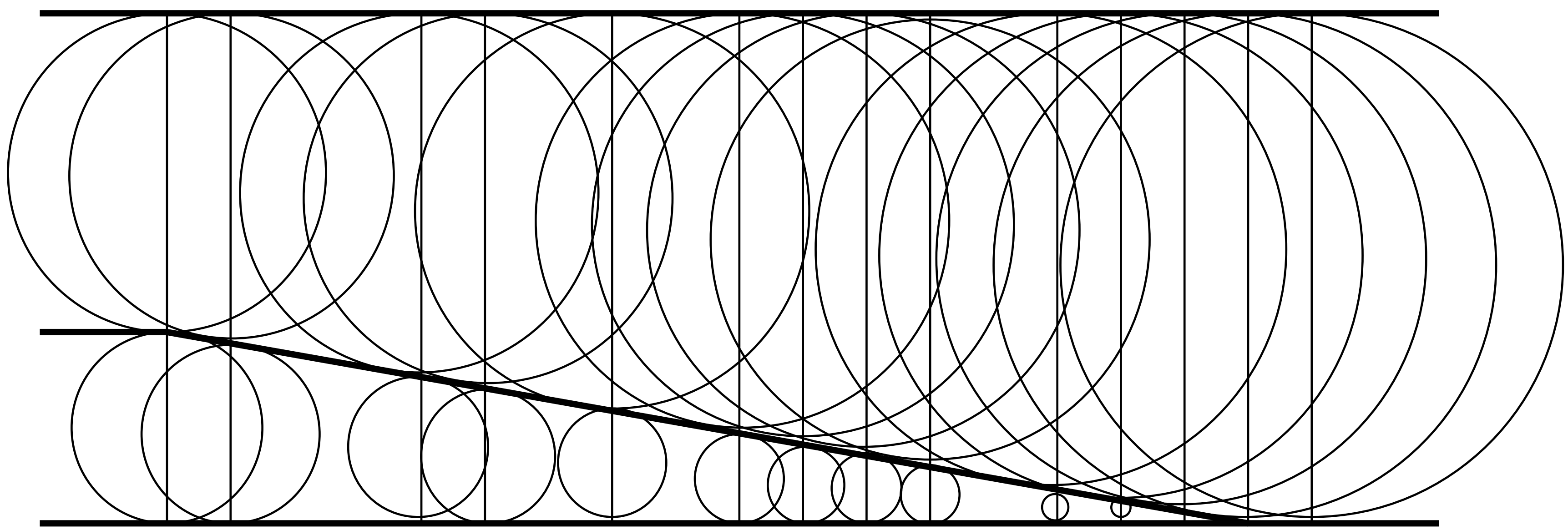
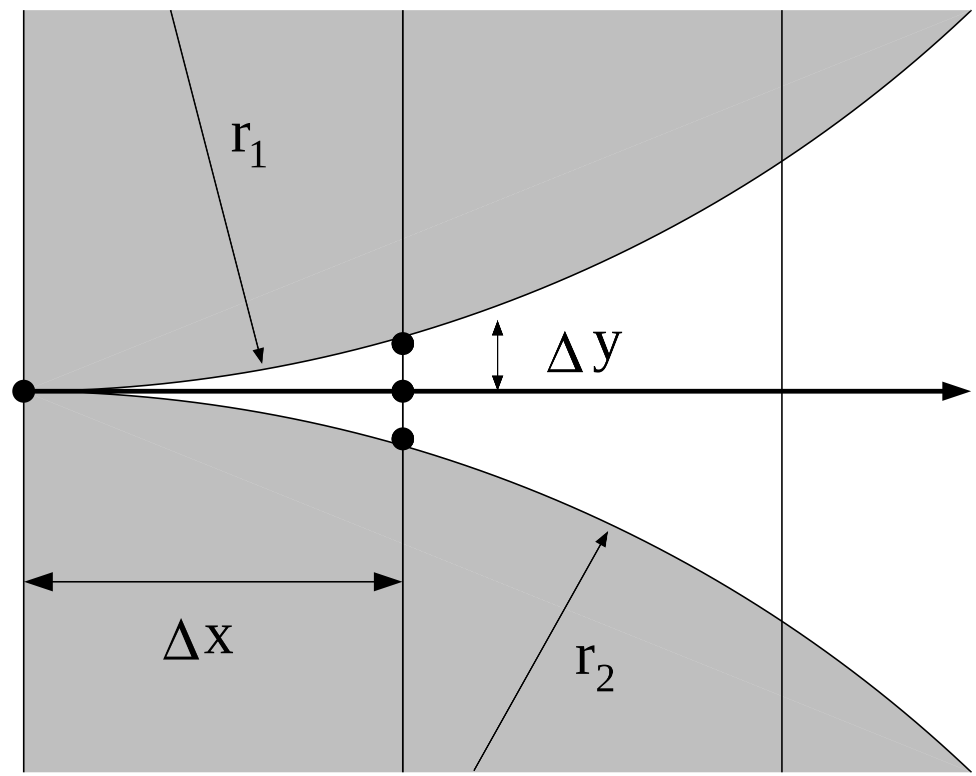


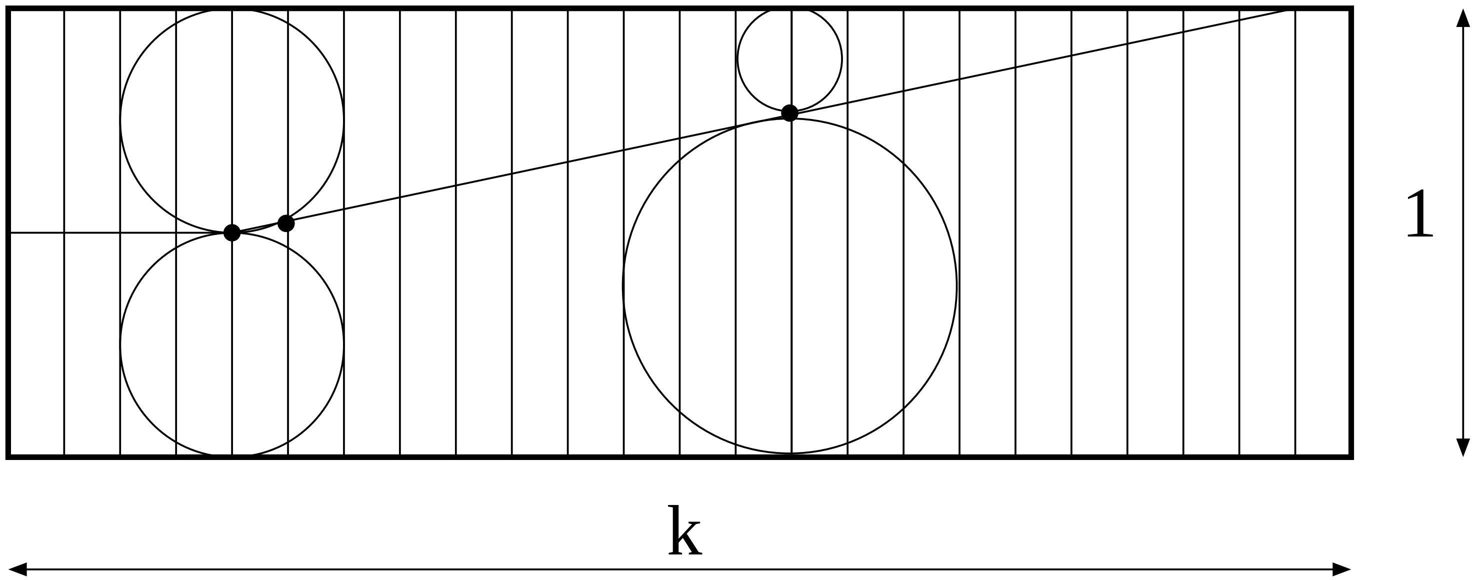
Bend slowly enough to satisfy Gabriel condition.

How far can we bend?



$$\text{Answer: } \Delta y \approx (\Delta x/r)^2 r = (\Delta x)^2/r.$$

$$r = \max(r_1, r_2).$$



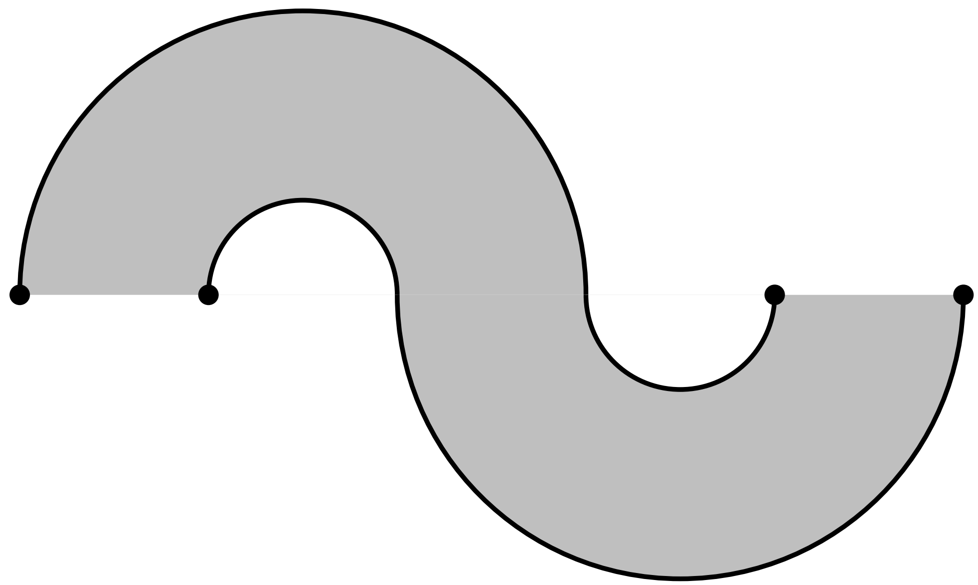
$k \times 1$ region crossing n (equally spaced) thin parts,

$r \approx 1, \quad \Delta x \approx k/n, \quad \Rightarrow \quad \Delta y \approx k^2/n^2$

Need $1 \leq \sum \Delta y = n\Delta y = k^2/n$.

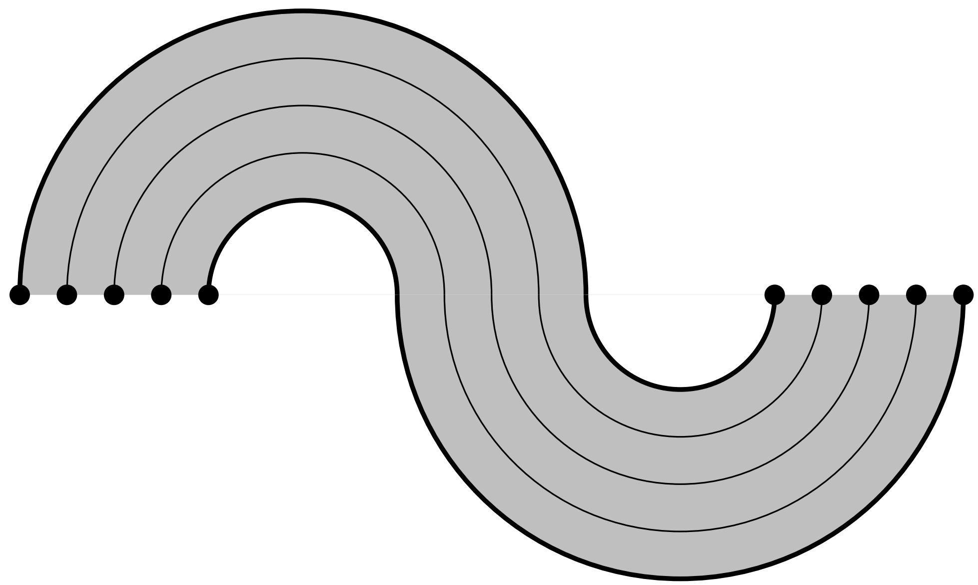
Bent path hits side of region if $k \gg \sqrt{n}$.

Easy case: return region length $>$ width.



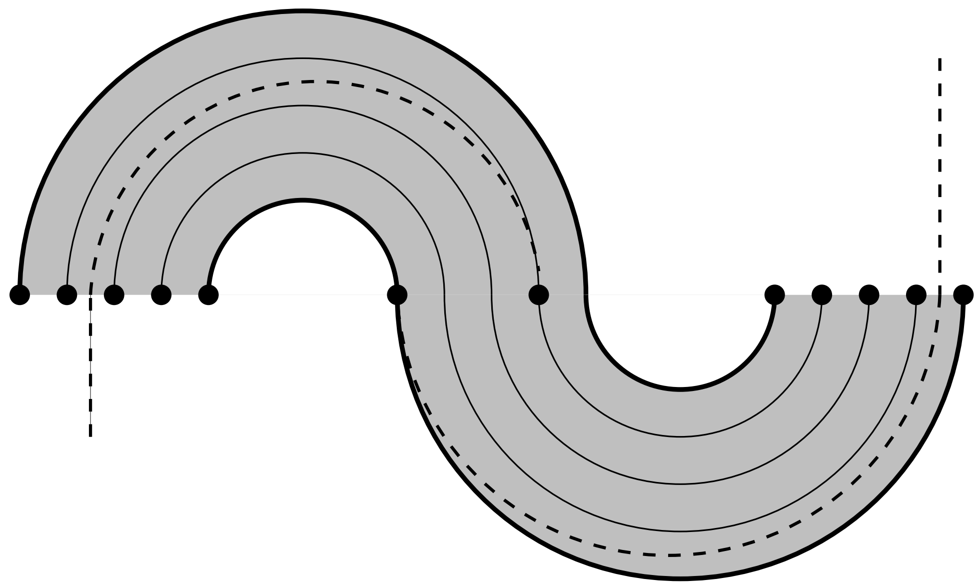
- Show there are $O(n)$ return regions.

Easy case: return region length > width.



- Show there are $O(n)$ return regions.
- Divide each region into $O(\sqrt{n})$ long parallel tubes.

Easy case: return region length > width.



- Show there are $O(n)$ return regions.
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- Entering paths can be bent and terminated.

  Total vertices created $= O(n^2)$, but …

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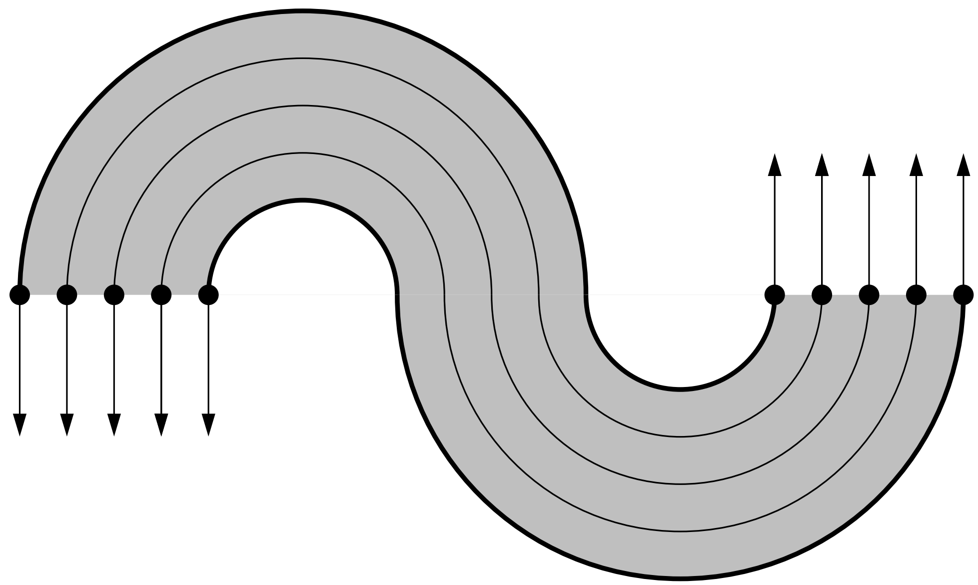
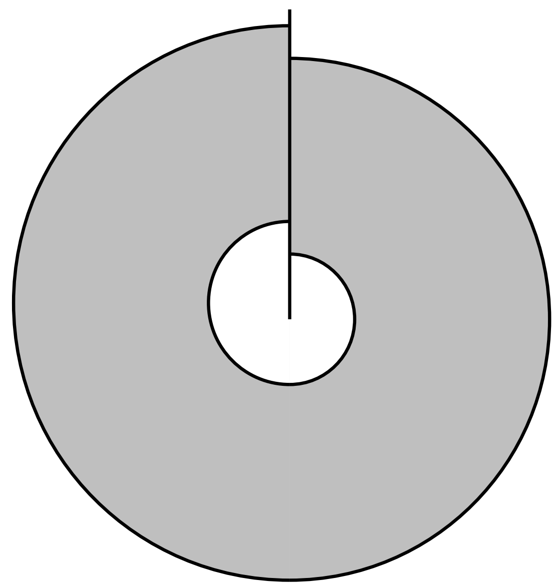


- Show there are $O(n)$ return regions.
- Divide each region into $O(\sqrt{n})$ long parallel tubes.
- Entering paths can be bent and terminated.
  Total vertices created $= O(n^2)$, but ...
- Each region has $O(\sqrt{n})$ new vertices to propagate.
  Vertices created is $O(\sqrt{n} \cdot n^2) = O(n^{2.5})$.

Hard case is spirals:



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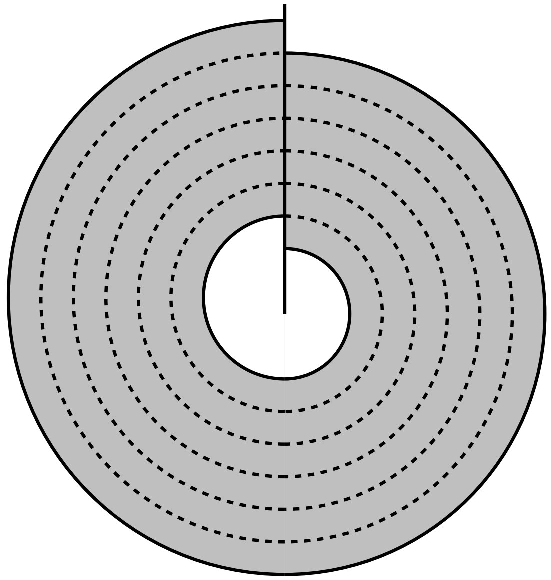
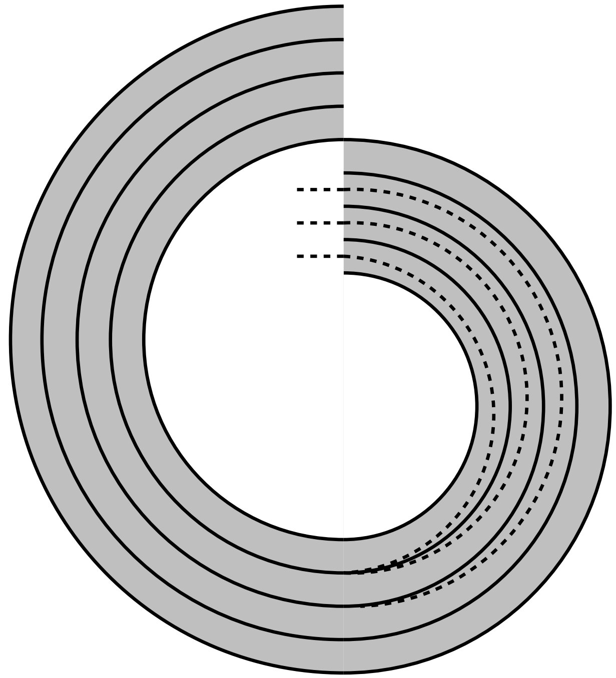


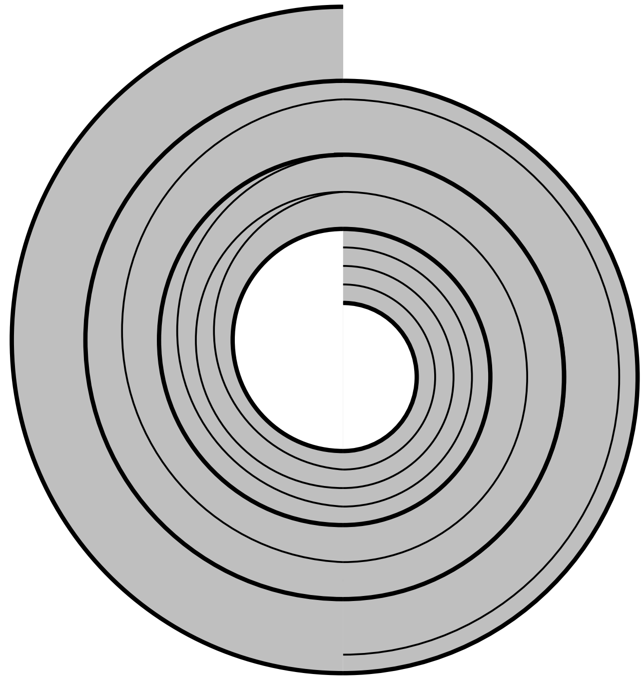
Curves may spiral arbitrarily often.

No curve can be allowed to pass all the way through the spiral. We stop them in a multi-stage construction.

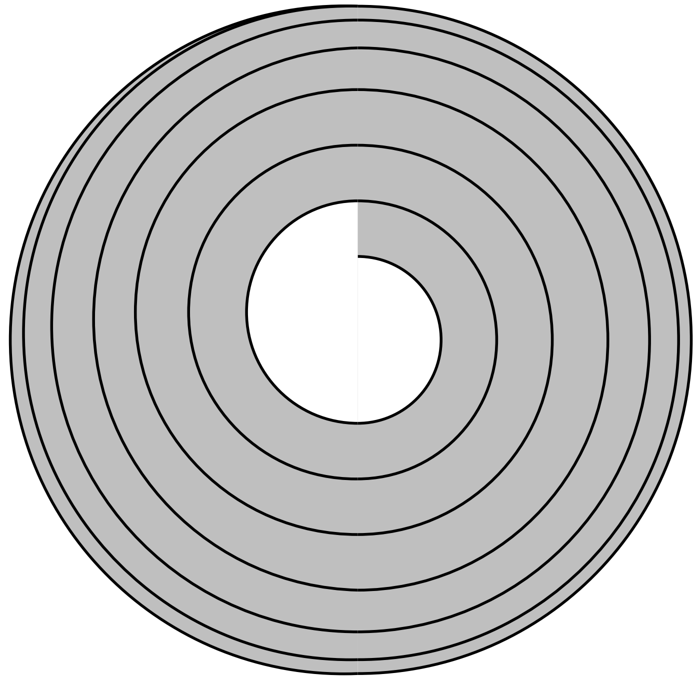
Normalize so “entrance” is unit width.



- Start with $\sqrt{n}$ parallel tubes at entrance of sprial. Terminate entering paths (1 spiral).



- Start with $\sqrt{n}$ parallel tubes at entrance of spiral. Terminate entering paths (1 spiral).
- Merge $\sqrt{n}$ tubes to single tube ($n^{1/3}$ spirals). (spirals get longer as we move out.)



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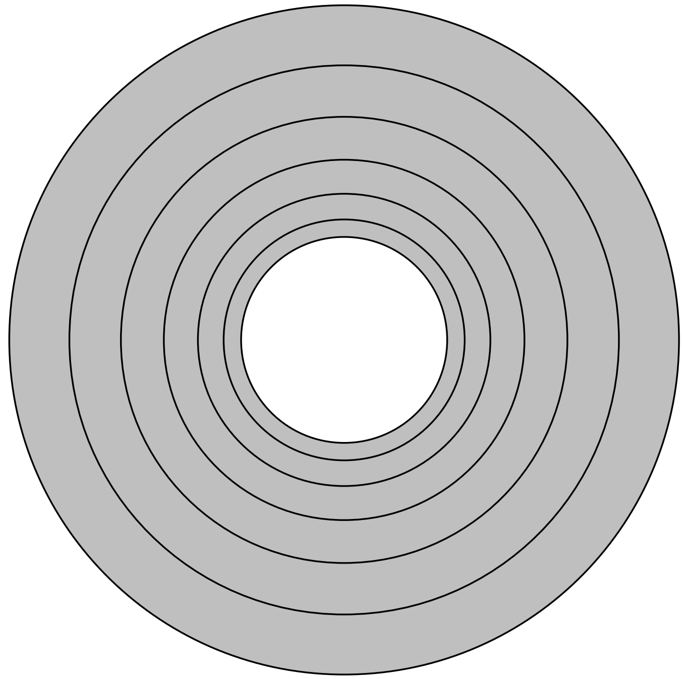

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- Make tube edge self-intersect ($n^{1/2}$ spirals)
- Loops with increasing gaps ($n^{1/2}$ loops, to radius n)
- Beyond radius n spiral is empty. Gives $O(n^{2.5})$.

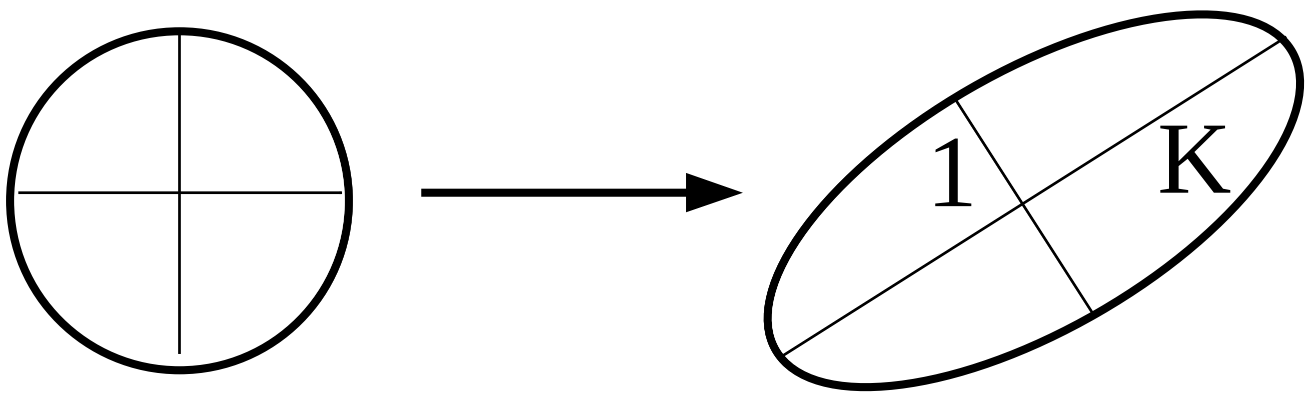
A CG problem in geometric function theory.

Quasiconformal maps: bounded angle distortion

$$f_z = \frac{1}{2}(f_x - i f_y), \quad f_{\bar z} = \frac{1}{2i}(f_x + i f_y).$$

Conformal means $f_{\bar z} = 0$ (Cauchy-Riemann equations).

Measure non-conformality by $\mu \equiv \sup |f_{\bar z}/f_z| \le 1$.



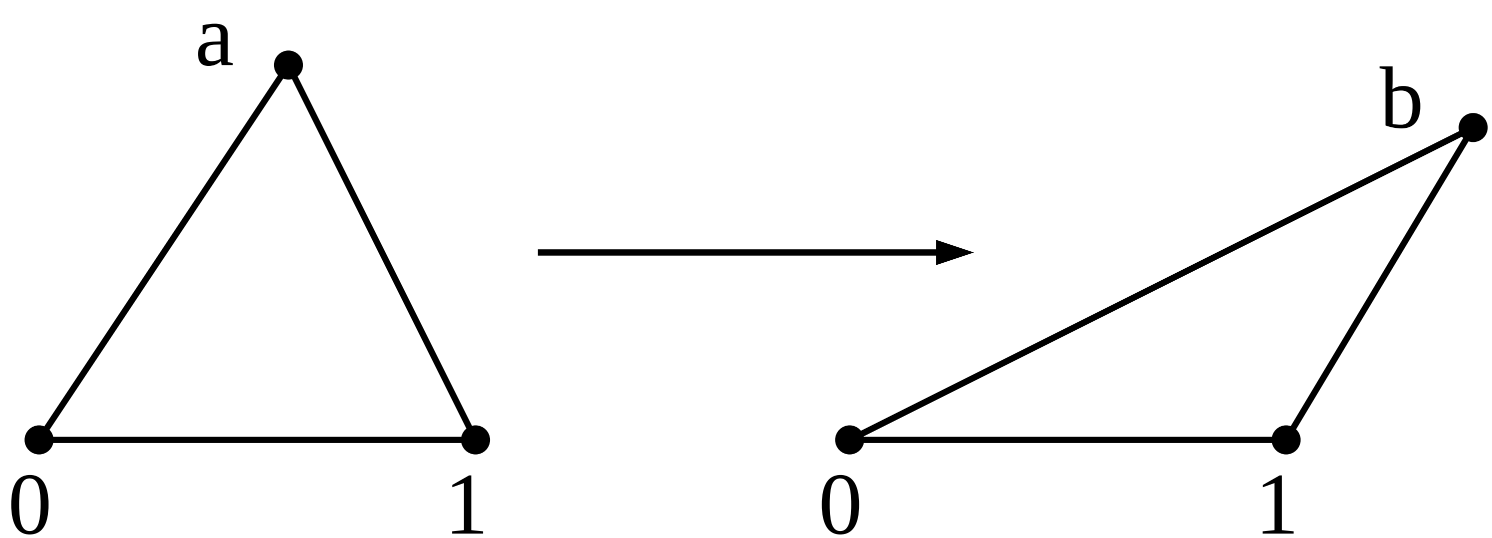
"small circles map to small ellipses," $K = \frac{1+\mu}{1-\mu} \ge 1$.

Affine map between triangles $\{0, 1, a\}$ and $\{0, 1, b\}$ is

$$f(z) \to \alpha z + \beta \bar{z}$$

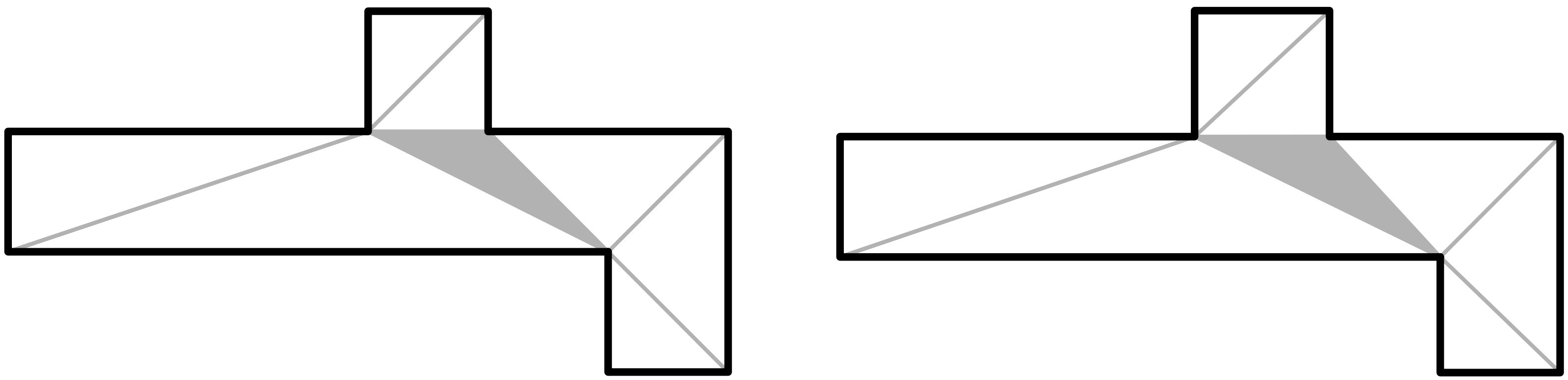
where $\alpha + \beta = 1$ and $\beta = (b - a)/(a - \bar{a})$. Then

$$\mu = \frac{f_{\bar{z}}}{f_z} = \frac{\beta}{\alpha} = \frac{b - a}{b - \overline{a}},$$



QC-distance between polygons:

- Find compatible triangulation
- Compute unique affine map for corresponding pairs
- Takes largest $|\mu|$.



The most distorted triangle is shaded. Here $|\mu| = .108$.

Theorem: MA-map always gives $\mu < .77$.

Conjecture: If Ω has inradius ≥ 1 then there is a 2-QC map $f : \Omega \to \mathbb{D}$ with $|f'| = O(1)$.

Conjecture: If Ω has inradius ≥ 1 then there is a 2-QC map $f : \Omega \to \mathbb{D}$ with $|f'| = O(1)$.

2-QC means $K \leq 2$ or $\mu \leq \frac{1}{3}$.

$\Rightarrow$ **Brennan's conjecture:** if $f : \Omega \to \mathbb{D}$ is conformal then $\iint_\Omega |f'|^p dxdy < \infty$ for all $2 \leq p < 4$.

- Main open problem of geometric function theory.
- Has been intensely studied for over 30 years.
- Connections to dynamics and statistical physics.