MAT 131, Calculus I   Fall 2005

Midterm II Solutions

(1) (**10pts**) *For each of the questions below, indicate if the statement is true* (**T**) *or false* (**F**).

a	**F**	f	**T**
b	**T**	g	**T**
c	**F**	h	**F**
d	**T**	i	**T**
e	**F**	j	**T**

(a) *If* $f(x) = \pi^4$ *then* $f'(x) = 4\pi^3$.
F: Since f is a constant function, $f'(x)=0$.

(b) *The Chain Rule asserts that* $(f \circ g)'(x) = f'(g(x))\,g'(x)$, *provided that* $f(x)$ *and* $g(x)$ *are differentiable functions.*
T: This is precisely one of the versions of the Chain Rule.

(c) $f(x) = e^x$ *is the only function such that* $f'(x) = f(x)$.
F: If $f(x)=0$, $f'(x)=0=f(x)$. More generally, if A is any constant and $f(x)=Ae^x$, then $f'(x)=Ae^x=f(x)$.

(d) *There is no point on the curve* $xy^2 = 3$ *where the tangent line is horizontal.*
T: This curve looks like hyperbolas in the 1st and 4th quadrants. Alternatively, using implicit differentiation, we find that

$$\big(xy^2\big)' = x'y^2 + x(y^2)' = y^2 + x \cdot 2y \cdot y' = 0.$$

If $y'=0$, then $y=0$. However, for no point on the curve $xy^2=3$, the y-coordinate is 0.

(e) $\frac{d}{dx}(x^a) = \frac{d}{dx}(a^x)$ *for any* $a > 0$.
F: $(x^a)'=ax^{a-1}$, while $(a^x)'=(\ln a)a^x$.

(f) *Let* $f(x)$ *and* $f^{-1}(x)$ *be inverses of each other. Then,* $\frac{d}{dx}f^{-1}(x) = 1/f'(f^{-1}(x))$, *provided that* $f(x)$ *is differentiable.*
T: If $y=f^{-1}(x)$, then $f(y)=x$. Using Implicit Differentiation, we then find that

$$f'(y) \cdot y' = x' = 1 \qquad \Longrightarrow \qquad \frac{d}{dx}f^{-1}(x) = y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.$$

(g) $\frac{d^{47}}{dx^{47}}\sin(2x) = -2^{47}\,\cos(2x)$.
T: With $f(x){=}\sin(2x)$ and $f^{\langle k\rangle}$ denoting the kth derivative of f, we find that

$$\begin{aligned} f^{\langle 0\rangle}(x) &= f(x) = \sin 2x, \quad f^{\langle 1\rangle}(x) = f'(x) = \cos 2x\cdot 2 = 2\cos 2x \\ f^{\langle 2\rangle}(x) &= f''(x) = 2(-\sin 2x)\cdot 2 = -4\sin 2x = -4f(x) \\ \Longrightarrow \qquad f^{\langle 47\rangle}(x) &= \big(f^{\langle 2\rangle}(x)\big)^{\langle 45\rangle} = \big(-4f(x)\big)^{\langle 45\rangle} = (-1)2^2 f^{\langle 45\rangle}(x) = \dots \\ &= (-1)^{23}2^{46}f^{\langle 1\rangle}(x) = -1\cdot 2^{46}\cdot 2\cos 2x = -2^{47}\cos 2x. \end{aligned}$$

(h) *If* $y = 3x + e^{-2x}$ *then the differential* dy *is given by* $dy = 3 - 2e^{-2x}$.
F: $dy = y'dx = \big(3{-}2e^{-2x}\big)dx$.

(i) *If* $\frac{d}{dx}[f(2x)] = x^2$ *then* $f''(x) = x/4$.
T: Since $\frac{d}{dx}[f(2x)] = f'(2x)\cdot 2$,

$$f'(2x) = x^2/2 = (2x)^2/8 \quad\Longrightarrow\quad f'(x) = x^2/8 \quad\Longrightarrow\quad f''(x) = 2x/8 = x/4.$$

(j) *The curves* $2x^2 + 3y^2 = 5$ *and* $y^2 = x^3$ *are orthogonal.*
T: Using Implicit Differentiation, we find that

$$\begin{aligned} &2x^2+3y^2=5 &&\Longrightarrow\quad 2\cdot 2x + 3\cdot 2y\cdot y' = 0 &&\Longrightarrow\quad y' = -\frac{2}{3}\cdot\frac{x}{y} \\ &y^2 = x^3 &&\Longrightarrow\quad 2y\cdot y' = 3x^2 &&\Longrightarrow\quad y' = \frac{3}{2}\cdot\frac{x^2}{y} \end{aligned}$$

Thus, the product of the slopes of the tangent lines to the two curves at a point (x, y) shared by both curves is

$$\Big(-\frac{2}{3}\cdot\frac{x}{y}\Big)\Big(\frac{3}{2}\cdot\frac{x^2}{y}\Big) = -\frac{x^3}{y^2} = -1.$$

The last equality holds because (x, y) lies on the second curve. Since the product of the slopes of the tangent lines to the two curves at every shared point is -1, the tangent lines are orthogonal and thus so are the curves.

(2) (**15pts**) *Let f and g be differentiable functions and let $H(x) = f(g(x) + \sin x)$. Suppose that*

$$f(-1) = 10, \quad f'(-1) = 1/2, \quad g(0) = -1, \quad g'(0) = 5.$$

(a) (**10pts**) *Find $H'(0)$.*

By Chain Rule,

$$H'(x) = f'\big(g(x) + \sin x\big) \cdot \big(g(x) + \sin x\big)' = f'\big(g(x) + \sin x\big) \cdot \big(g'(x) + \cos x\big).$$

Plugging in $x=0$, we find that

$$H'(0) = f'\big(g(0) + \sin 0\big) \cdot \big(g'(0) + \cos 0\big) = f'(-1+0) \cdot (5+1) = (1/2) \cdot 6 = \boxed{3}$$

(b) (**5pts**) *Find an equation of the tangent line to the curve $y=H(x)$ at the point $(0, H(0))$.*

The line of slope m passing through a point (x_0, y_0) is given by

$$y - y_0 = m(x - x_0).$$

In this case, $x_0=0$, $m=H'(0)=3$, and

$$y_0 = H(0) = f\big(g(0) + \sin 0\big) = f(-1+0) = 10.$$

Thus, the line tangent to the curve $y=H(x)$ at the point $(0, H(0))$ is given by

$$y - 10 = 3(x - 0) \qquad \Longrightarrow \qquad \boxed{y - 10 = 3x} \quad \text{OR} \quad \boxed{y = 3x + 10}$$

(3) (**15 pts**)

(a) (**10 pts**) *Estimate the number* $\sqrt[4]{15}$ *by using a tangent line approximation (linear approximation).*

Since $2^4 = 16$ and 15 is close to 16, we will use the linear approximation of

$$f(x) = \sqrt[4]{x} = x^{1/4} \qquad \text{at} \qquad x_0 = 16.$$

It is given by

$$f(15) \approx f(16) + f'(16) \cdot (15 - 16).$$

Then,

$$f'(x) = \frac{1}{4}x^{-3/4} \quad \Longrightarrow \quad f'(16) = \frac{1}{4} \cdot 16^{-3/4} = \frac{1}{4} \cdot \frac{1}{16^{3/4}} = \frac{1}{4} \cdot \frac{1}{(\sqrt[4]{16})^3} = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}.$$

Thus,

$$\sqrt[4]{15} \approx \sqrt[4]{16} + \frac{1}{32} \cdot (-1) = 2 - \frac{1}{32} = \boxed{\frac{63}{32}} = \boxed{1\frac{31}{32}}$$

(b) (**5 pts**) *Is this estimate greater than the actual value of* $\sqrt[4]{15}$ *or smaller? Why?*

Since the graph of $y = \sqrt[4]{x}$ is concave down, all tangent lines to this graph lie *above* the graph. Thus, the linear estimate in part (a) is $\boxed{\text{greater}}$ than the actual value of $\sqrt[4]{15}$.

The fact that the graph is concave down can be seen by looking at the second derivative of f:

$$f''(x) = \left(\frac{1}{4}x^{-3/4}\right)' = \frac{1}{4}\left(-\frac{3}{4}\right)x^{-7/4} = -\frac{3}{16}x^{-7/4} < 0.$$

(4) (**35pts**) *Find* $y' = \frac{dy}{dx}$ *for the following functions.*

(a) (**5 pts**) $y = (x^{19} - 6x)\left(3 + \frac{1}{x} - \frac{5}{x^2}\right)$

Use Product Rule:

$$\begin{aligned} y' &= \big((x^{19}-6x)(3+x^{-1} - 5x^{-2})\big)' \\ &= \big(x^{19}-6x\big)'\big(3+x^{-1} - 5x^{-2}\big) + \big(x^{19}-6x\big)\big(3+x^{-1} - 5x^{-2}\big)' \\ &= \boxed{\big(19x^{18}-6\big)\big(3+x^{-1} - 5x^{-2}\big) + \big(x^{19}-6x\big)\big(-x^{-2} + 10x^{-3}\big)} \\ &= \boxed{\big(19x^{18}-6\big)\big(3+\tfrac{1}{x} - \tfrac{5}{x^2}\big) - \big(x^{18}-6\big)\big(\tfrac{1}{x} - \tfrac{10}{x^2}\big)} \end{aligned}$$

(b) (**10 pts**) $y = e^{\sin(7x)} + \tan\left(2\sqrt{x}\,\right) + \sqrt[3]{e}$

Use Chain Rule:

$$\begin{aligned} y' &= e^{\sin(7x)} \cdot \cos(7x) \cdot 7 + \frac{1}{\cos^2(2\sqrt{x})} \cdot 2 \cdot \big(x^{1/2}\big)' \\ &= 7\big(\cos(7x)\big)e^{\sin(7x)} + \frac{1}{\cos^2(2\sqrt{x})} \cdot 2 \cdot \frac{1}{2} \cdot x^{-1/2} \\ &= \boxed{7\big(\cos(7x)\big)e^{\sin(7x)} + \tfrac{1}{\sqrt{x}\cos^2(2\sqrt{x})}} \end{aligned}$$

(c) (**10pts**) $y = 2^{1+5x} + \ln(1+x^4) + \cos\left(\frac{3}{x}\right)$

Use Chain Rule:

$$y' = (\ln 2) \cdot 2^{1+5x} \cdot 5 + \frac{1}{1+x^4} \cdot 4x^3 - \sin\left(\frac{3}{x}\right) \cdot \left(-\frac{3}{x^2}\right)$$

$$= \boxed{\big(5\ln 2\big) \cdot 2^{1+5x} + \tfrac{4x^3}{1+x^4} + \tfrac{3}{x^2}\sin\left(\tfrac{3}{x}\right)}$$

(d) (**10pts**) $y = (\sin x)^x + x^{\sin x}$

Use Logarithmic Differentiation to differentiate $y_1 = (\sin x)^x$ and $y_2 = x^{\sin x}$:

$$\ln y_1 = \ln(\sin x)^x = x\ln(\sin x)$$

$$\Longrightarrow \quad \frac{y_1'}{y_1} = \big(x\ln(\sin x)\big)' = x'\ln(\sin x) + x\big(\ln(\sin x)\big)'$$

$$= \ln(\sin x) + x \cdot \frac{1}{\sin x} \cdot \cos x$$

$$\Longrightarrow \quad y_1' = \left(\ln(\sin x) + \frac{x}{\tan x}\right)y_1 = \left(\ln(\sin x) + \frac{x}{\tan x}\right)(\sin x)^x$$

$$\ln y_2 = \ln x^{\sin x} = \sin x \cdot \ln x$$

$$\Longrightarrow \quad \frac{y_2'}{y_2} = \big(\sin x \cdot \ln x\big)' = (\sin x)' \cdot \ln x + \sin x \cdot (\ln x)' = (\cos x)(\ln x) + \frac{\sin x}{x}$$

$$\Longrightarrow \quad y_2' = \left((\cos x)(\ln x) + \frac{\sin x}{x}\right)y_2 = \left((\cos x)(\ln x) + \frac{\sin x}{x}\right)x^{\sin x}$$

Putting the two results together, we obtain:

$$y' = y_1' + y_2' = \boxed{\left(\ln(\sin x) + \tfrac{x}{\tan x}\right)(\sin x)^x + \left((\cos x)(\ln x) + \tfrac{\sin x}{x}\right)x^{\sin x}}$$

(5) (**15 pts**) *Find the points on the curve* $3x^2 + 2xy + y^2 = 6$ *at which the tangent line is horizontal.*

By Implicit Differentiation:

$$3 \cdot 2x + 2(xy)' + 2yy' = 0 \qquad \Longrightarrow \qquad 3x + y + xy' + yy' = 0.$$

The tangent line is horizontal at the points (x, y) of the curve such that

$$y' = 0 \quad \Longrightarrow \quad 3x + y + x \cdot 0 + y \cdot 0 = 0 \quad \Longrightarrow \quad y = -3x.$$

Since the point (x, y) lies on the curve $3x^2+2xy+y^2=6$, we need to solve the system of equations

$$\begin{cases} 3x^2 + 2xy + y^2 = 6 \\ y = -3x \end{cases}$$

Plugging the second equation into the first, we obtain:

$$3x^2 + 2x \cdot (-3x) + (-3x)^2 = 6 \quad \Longrightarrow \quad 6x^2 = 6 \quad \Longrightarrow \quad x^2 = 1 \quad \Longrightarrow \quad x = \pm 1.$$

Using the second equation in the system, we find the corresponding y-coordinates:

$$\boxed{(1, -3), (-1, 3)}$$

(6) (**10 pts**) *If a snowball melts so that its surface area decreases at a rate of* 1 in^2/min, *find the rate at which the diameter decreases when the diameter is* 12 in.

The surface area of a sphere of radius r and diameter d is given by

$$A = 4\pi r^2 = 4\pi(d/2)^2 = \pi d^2.$$

Differentiating the left and right sides of this equation with respect to time, t, we obtain

$$A' = \pi \cdot 2dd'.$$

Plug in $A' = 1$ in^2/min and $d = 12$ in:

$$1 = 2\pi \cdot 12d' \quad \Longrightarrow \quad d' = \boxed{\frac{1}{24\pi} \text{ in/min}}$$

Alternatively, the surface area of a sphere of radius r is given by

$$A = 4\pi r^2.$$

Differentiating both sides of this equation with respect to time, t, we obtain

$$A' = 4\pi \cdot 2rr'.$$

Plug in $A' = 1$ in^2/min and $r = d/2 = 6$ in:

$$1 = 8\pi \cdot 6r' \quad \Longrightarrow \quad r' = \frac{1}{48\pi} \quad \Longrightarrow \quad d' = (2r)' = 2r' = \boxed{\frac{1}{24\pi} \text{ in/min}}$$